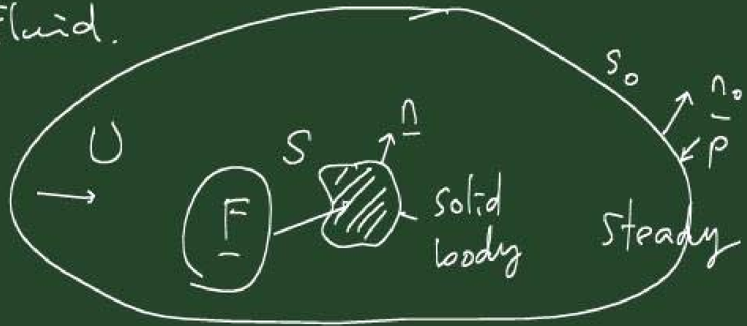
노트 제목 2010-04-29

- D'Alembert's paradox

If an arbitrary 3-dim. body is immersed in a uniform flow, the eq. of hydrodynamics predicts that there will be no force exerted on the body by the fluid.



$$\Sigma \underline{F} = -\underline{F} - \int_{S_0} p\, \underline{n}_0 \, dS = \int_{S_0} \rho\, \underline{u}(\underline{u}\cdot\underline{n}_0)\, dS$$

$$p = c - \frac{1}{2}\rho\, \underline{u}\cdot\underline{u} \qquad \int_{S_0} c\, \underline{n}_0\, ds = 0$$

$$\rightarrow \underline{F} = \rho \int_{S_0} \left[\frac{1}{2}(\underline{u}\cdot\underline{u})\,\underline{n}_0 - \underline{u}\,(\underline{u}\cdot\underline{n}_0)\right] ds$$

$$\left(\underline{u} = \underline{U} + \underline{u}' \quad \text{where } \underline{U} = U\underline{e}\right)$$

$$\rightarrow \underline{F} = \rho \int_{S_0} \left\{ \left(\frac{1}{2}U^2 + \underline{U}\cdot\underline{u}' + \frac{1}{2}\underline{u}'\cdot\underline{u}'\right)\underline{n}_0 - (\underline{U}+\underline{u}')\left[(\underline{U}+\underline{u}')\cdot\underline{n}_0\right] \right\} ds$$

$$\int_{S_0} \underline{u}\cdot\underline{n}_0\, ds = 0$$

Note that $\int_{S_0} \underline{U}\cdot\underline{n}_0\, ds = 0 \rightarrow \int_{S_0} \underline{u}'\cdot\underline{n}_0\, ds = 0$ ∵ continuity

$$= \rho \int_{S_0} \left\{ \left(\underline{U}\cdot\underline{u}' + \frac{1}{2}\underline{u}'\cdot\underline{u}' \right)\underline{n}_0 - \left[\underline{u}'(\underline{U}\cdot\underline{n}_0) + \underline{u}'(\underline{u}'\cdot\underline{n}_0) \right] \right\} ds$$

$$\left(\underline{U}\times(\underline{u}'\times\underline{n}_0) = \underline{u}'(\underline{U}\cdot\underline{n}_0) - \underline{n}_0(\underline{U}\cdot\underline{u}') \right)$$

$$= \rho \int_{S_0} \left[-\underline{U}\times(\underline{u}'\times\underline{n}_0) + \frac{1}{2}(\underline{u}'\cdot\underline{u}')\underline{n}_0 - \underline{u}'(\underline{u}'\cdot\underline{n}_0) \right] ds$$

$$\underline{u}' = \nabla\phi'$$

$$\phi' = \sum_{l=0}^{\infty} A_l \frac{P_l(\cos\theta)}{r^{l+1}} + \sum_{l=0}^{\infty} \cancel{B_l r^l} P_l(\cos\theta)$$

0

ϕ' is the perturbation by the existence of solid body

as $r\to\infty$, $\phi'\to 0$

$$\phi' = \frac{Q}{4\pi r} + \frac{\mu\cos\theta}{4\pi r^2} + O\left(\frac{1}{r^3}\right)$$

$$\underline{u}' = \nabla\phi' \rightarrow |\underline{u}'| = O\left(\frac{1}{r^2}\right)$$

we set S_0 as a sphere

$$\underline{u}'\times\underline{n}_0 = \nabla\left(\frac{Q}{4\pi r} + \frac{\mu\cos\theta}{4\pi r^2} + O\left(\frac{1}{r^3}\right) \right)\times\underline{e}_r$$

$$= \left(-\frac{Q}{4\pi r^2}\underline{e}_r + O\left(\frac{1}{r^3}\right) \right)\times\underline{e}_r = O\left(\frac{1}{r^3}\right)$$

$$\underline{e}_r\times\underline{e}_r = 0$$

$$\int ds = O(r^2)$$

$$\therefore \int \underline{U}\times(\underline{u}'\times\underline{n}_0)\, ds = O\left(\frac{1}{r}\right)$$

$$\int (\underline{u}'\cdot\underline{u}')\underline{n}_0\, ds = O\left(\frac{1}{r^2}\right)$$

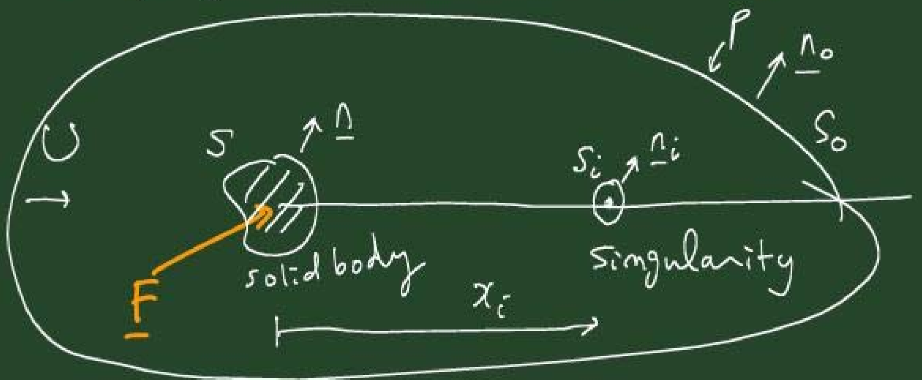
$$\int \underline{u}'(\underline{u}'\cdot\underline{n}_0)\, ds = O\left(\frac{1}{r^2}\right)$$

As $r \to \infty$, all RHS $\to 0$ $\Rightarrow$ $\underline{F} = 0$

D'Alembert paradox

- Forces induced by singularities

singularity inside the flow (but outside the solid body) produces a force on a solid body.



$$\Sigma \underline{F} = -\underline{F} - \int_{S_o} p\,\underline{n}_o\, ds + \int_{S_i} p\,\underline{n}_i\, ds$$

$$= \int_{S_o} \rho \underline{u}(\underline{u}\cdot\underline{n}_o)\,ds - \int_{S_i} \rho \underline{u}(\underline{u}\cdot\underline{n}_i)\,ds$$

↳ zero ($\Rightarrow$ D'Alembert paradox)

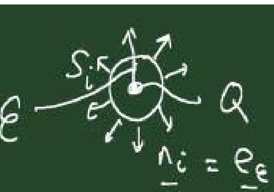
$$\rightarrow \underline{F} = \int_{S_i} [p\,\underline{n}_i + \rho \underline{u}(\underline{u}\cdot\underline{n}_i)]\,ds \qquad p = C - \tfrac{1}{2}\rho\,\underline{u}\cdot\underline{u}$$

$$= \rho \int_{S_i} \left(-\tfrac{1}{2}(\underline{u}\cdot\underline{u})\,\underline{n}_i + \underline{u}(\underline{u}\cdot\underline{n}_i)\right) ds$$

We have to specify the nature of the singularity. (source, sink, doublet, ...)

① Source at $x = x_i$ (Q)

vel. on the surface S_i

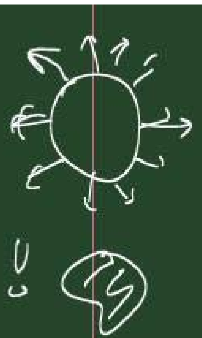


$$\underline{u} = \underbrace{\frac{Q}{4\pi\varepsilon^2}\,\underline{e}_\varepsilon}_{\text{radial direction}} + \underbrace{\underline{u}_i}_{\text{vel. induced by all means other than the source}}$$

Then $\underline{u}\cdot\underline{u} = \frac{Q^2}{16\pi^2\varepsilon^4} + \frac{Q}{2\pi\varepsilon^2}\,\underline{e}_\varepsilon\cdot\underline{u}_i + \underline{u}_i\cdot\underline{u}_i$

$$\underline{u}\cdot\underline{n}_i = \underline{u}\cdot\underline{e}_\varepsilon = \frac{Q}{4\pi\varepsilon^2} + \underline{u}_i\cdot\underline{e}_\varepsilon$$

$$\underline{F} = \rho\int_{S_i}\Big\{-\frac{1}{2}\Big(\frac{Q^2}{16\pi^2\varepsilon^4} + \frac{Q}{2\pi\varepsilon^2}\,\underline{e}_\varepsilon\cdot\underline{u}_i + \underline{u}_i\cdot\underline{u}_i\Big)\underline{e}_\varepsilon + \Big(\frac{Q}{4\pi\varepsilon^2}\underline{e}_\varepsilon + \underline{u}_i\Big)\Big(\frac{Q}{4\pi\varepsilon^2} + \underline{u}_i\cdot\underline{e}_\varepsilon\Big)\Big\}\,dS$$

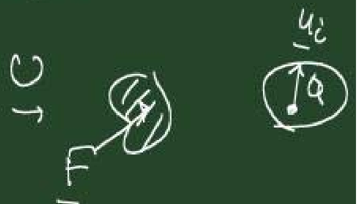


$$= \rho\int_{S_i}\Big[\underbrace{\frac{Q^2}{32\pi^2\varepsilon^4}\,\underline{e}_\varepsilon}_{\text{const}\;=\;0} - \underbrace{\frac{1}{2}(\underline{u}_i\cdot\underline{u}_i)\,\underline{e}_\varepsilon}_{\text{const } \oint\ \varepsilon\to 0\;=\;0} + \frac{Q}{4\pi\varepsilon^2}\,\underline{u}_i + \underbrace{(\underline{u}_i\cdot\underline{e}_\varepsilon)\,\underline{u}_i}_{+\ \text{const}\ -\;=\;0}\Big]\,dS$$

u_i

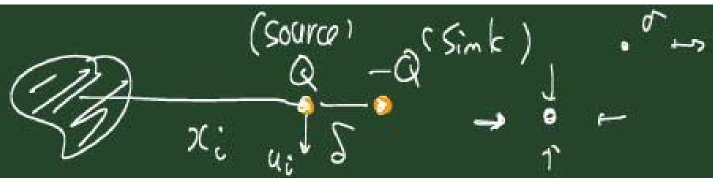
$$= \rho\int_{S_i}\frac{Q}{4\pi\varepsilon^2}\,\underline{u}_i\,dS = \rho\,\frac{Q}{4\pi\varepsilon^2}\,\underline{u}_i\,4\pi\varepsilon^2$$

$$\therefore\ \boxed{\underline{F} = \rho Q\,\underline{u}_i}$$



Sink $\Rightarrow$ $\underline{F} = -\rho Q\,\underline{u}_i$

② Doublet



$\underline{u}_i$ is the vel. at $x=x_i$ induced by all means other than source and sink.

vel. @ $x=x_i$; $\underline{u}_i' = \frac{Q}{4\pi\delta^2}\underline{e}_x + \underline{u}_i$ (except source)

$\underline{u}_i(x_i+\delta) \approx$

vel. @ $x=x_i+\delta$, $\underline{u}_i'' = \frac{Q}{4\pi\delta^2}\underline{e}_x + \left(\underline{u}_i + \delta\frac{\partial u_i}{\partial x} + \cdots\right)$ (except sink)

the force due to the source

$$\rho Q\,\underline{u}_i' = \rho Q\left(\frac{Q}{4\pi\delta^2}\underline{e}_x + \underline{u}_i\right)$$

the force due to the sink

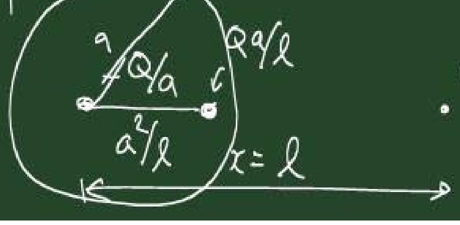
$$\rho Q\,\underline{u}_i'' = -\rho Q\left(\frac{Q}{4\pi\delta^2}\underline{e}_x + \underline{u}_i + \delta\frac{\partial u_i}{\partial x} + \cdots\right)$$

∴ total force

$$\underline{F} = \rho Q\,\underline{u}_i' + \rho Q\,\underline{u}_i'' = -\rho Q\delta\frac{\partial u_i}{\partial x} \qquad Q\delta \to \mu$$

$$\boxed{\therefore\ \underline{F} = -\rho\mu\frac{\partial u_i}{\partial x}}$$

③ Sphere in the flow field of a source



$\underline{u}_i$ @ $x = l$

$$\underline{u}_i = \frac{Qa/l}{4\pi}\frac{1}{\left(l - a^2/l\right)^2}\underline{e}_x - \int_0^{a^2/l}\frac{Q/a}{4\pi}\frac{\underline{e}_x}{(l-x)^2}\,dx$$

$$= \frac{Qa^3}{4\pi l\left(l^2 - a^2\right)^2}\underline{e}_x$$

$$\therefore \underline{F} = \rho Q \underline{u}_i = \frac{\rho Q^2 a^3}{4\pi l\left(l^2 - a^2\right)^2}\underline{e}_x .$$

F : positive

The sphere is attracted to the source.