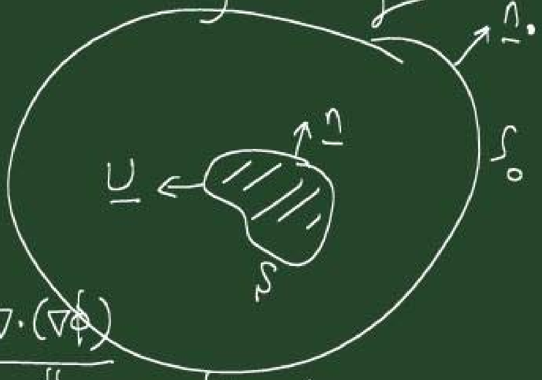
노트 제목 2010-05-04

- Kinetic energy of a moving fluid in a quiescent fluid by the moving body



$$T = \int_V \frac{1}{2}\rho(\underline{u}\cdot\underline{u})\,d\forall$$

$$= \frac{1}{2}\rho\int_V \nabla\phi\cdot\nabla\phi\,dV$$

$$\nabla\phi\cdot\nabla\phi = \nabla\cdot(\phi\nabla\phi) - \phi\,\underbrace{\nabla\cdot(\nabla\phi)}_{=0\ \because\ \text{continuity}}$$

$$= \frac{1}{2}\rho\int_V \nabla\cdot(\phi\nabla\phi)\,d\forall$$

$$= \frac{1}{2}\rho\int_\Sigma \phi\frac{\partial\phi}{\partial n}dS = \frac{1}{2}\rho\int_{S_0}\phi\frac{\partial\phi}{\partial n}ds - \frac{1}{2}\rho\int_S \phi\frac{\partial\phi}{\partial n}ds$$

$$\int_V \nabla\cdot\underline{u}\,dV = 0 \rightarrow \int_{S_0}\underline{u}\cdot\underline{n}\,ds - \int_S \underline{u}\cdot\underline{n}\,ds = 0$$

$$\rightarrow \int_{S_0}\frac{\partial\phi}{\partial n}ds - \underbrace{\int_S \underline{U}\cdot\underline{n}\,ds}_{=0}\ = 0 \qquad \underline{U} \to \text{const}$$

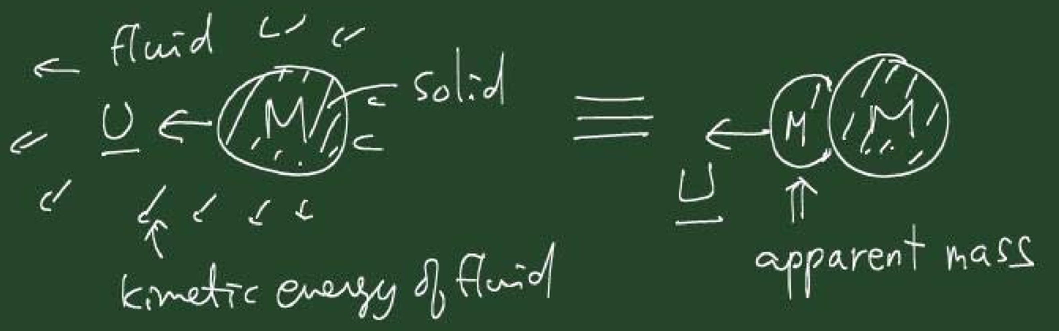
$$\rightarrow \int_{S_0}\frac{\partial\phi}{\partial n}ds = 0 \rightarrow \int_{S_0} C\frac{\partial\phi}{\partial n}ds = 0$$

$$= \frac{1}{2}\rho\int_{S_0}(\phi - c)\frac{\partial\phi}{\partial n}ds - \frac{1}{2}\rho\int_S \phi\frac{\partial\phi}{\partial n}ds$$

At far-field, $\underline{u} \to 0 \Rightarrow \phi$ is const

$$= -\frac{1}{2}\rho\int_S \phi\frac{\partial\phi}{\partial n}\,ds \quad //$$

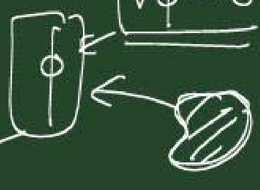
- Apparent mass : mass of fluid which, if it were moving with the same velocity as the body, would have the same kinetic energy as the entire fluid



$$T = \frac{1}{2}M'U^2 = -\frac{1}{2}\rho\int_S \phi\frac{\partial\phi}{\partial n}\,ds$$

$$\boxed{\nabla^2\phi = 0}$$

$$\rightarrow M' = -\frac{\rho}{U^2}\int_S \phi\frac{\partial\phi}{\partial n}\,ds$$



↳ ft. of shape of the body

ex) sphere : uniform flow around sphere − uniform flow

$$\phi(r,\theta) = U\left(r + \frac{1}{2}\frac{a^3}{r^2}\right)\cos\theta - Ur\cos\theta$$

$$= \frac{1}{2}U\frac{a^3}{r^2}\cos\theta$$

$$\frac{\partial\phi}{\partial n}(r,\theta) = \frac{\partial\phi}{\partial r}(r,\theta) = -U\frac{a^3}{r^3}\cos\theta$$

$$\phi \frac{\partial \phi}{\partial n}\Big|_{r=a} = -\frac{1}{2}U^2 a \cos^2\theta$$

fluid density

$$M' = -\frac{\rho}{U^2}\int_S \left(-\frac{1}{2}\right) U^2 a \cos^2\theta \, ds = \frac{2}{3}\pi a^3 \rho$$

$$M_{\text{solid body}} = \frac{4}{3}\pi a^3 \rho_s$$

one half of the mass of the same volume of the fluid.

this apparent mass may be added to the actual mass of the sphere, and the total mass may be used in the dynamic eqs. of the sphere for a

first-hand calculation.

Ch. 6 Surface waves

↳ gravity on liquid surfaces

"free" surface

shear-stress free

potential flow

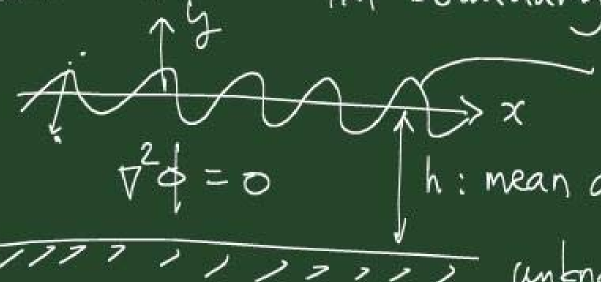
1. General surface-wave problem

gravity wave → free surface → irrotational motion

$\underline{u} = \nabla \phi$

no difficulty in governing eq. $\nabla^2 \phi = 0$

but " in boundary conditions.



$y = \eta(x, z, t)$: free surface

unknowns : ϕ, η

- kinematic boundary condition

  "a particle of fluid which is at some time on the free surface will always remain on the free surface."

$\rightarrow \frac{D}{Dt}(y-\eta) = 0 \quad @ \; y = \eta(x, z, t)$

$\rightarrow \frac{\partial}{\partial t}(y-\eta) + u_j \frac{\partial}{\partial x_j}(y-\eta) = 0$

$\rightarrow -\frac{\partial \eta}{\partial t} - u\frac{\partial \eta}{\partial x} + v - w\frac{\partial \eta}{\partial z} = 0$

$\rightarrow \boxed{\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial z}\frac{\partial \eta}{\partial z} = \frac{\partial \phi}{\partial y}}$ kinematic surface condition

- dynamic boundary condition on pressure
  unsteady Bernoulli eq. $(G = -gy)$

$$\boxed{\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}\nabla\phi\cdot\nabla\phi + g\eta = F(t)} \quad @ \; y = \eta$$

- $v = \boxed{\frac{\partial \phi}{\partial y} = 0} \quad @ \; y = -h$

Motion with gravity waves

$$\begin{cases} \nabla^2 \phi = 0 & \\ \frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial z}\frac{\partial \eta}{\partial z} = \frac{\partial \phi}{\partial y} & @ \; y = \eta \\ \frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}\nabla\phi\cdot\nabla\phi + g\eta = F(t) & @ \; y = \eta \\ \frac{\partial \phi}{\partial y} = 0 & @ \; y = -h \end{cases}$$

ϕ & η coupled

Nonlinear eq!

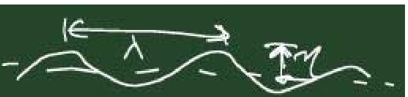
2. Small-amplitude plane waves

2-D waves $\leftrightarrow$ $\eta \ll h, \lambda$

λ wavelength of the waves

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

If $\eta \ll \lambda$, $\frac{\partial \eta}{\partial x} \ll 1$



$$u = \frac{\partial \phi}{\partial x} \ll 1 \qquad \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} \ll 1$$

then, the kinematic b.c

$$\rightarrow \frac{\partial \eta}{\partial t}(x,t) = \frac{\partial \phi}{\partial y}(x,\eta,t)$$

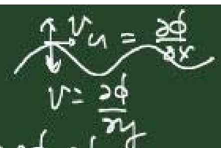
$$= \frac{\partial \phi}{\partial y}(x,0,t) + \eta \frac{\partial^2 \phi}{\partial y^2}(x,0,t) + O(\eta^2)$$

$$\rightarrow \boxed{\frac{\partial \eta}{\partial t}(x,t) = \frac{\partial \phi}{\partial y}(x,0,t)}$$

quiescent quadratically small.

the dynamic b.c

$$\underline{u}\cdot\underline{u} = \nabla\phi\cdot\nabla\phi \ll 1 \quad \because \text{ fluid is essentially}$$



$$u^2 + v^2\Big|_{\eta} \doteq \left(\frac{\partial \phi}{\partial y}\right)^2\Big|_{y=\eta} \ll 1.$$

$$\rightarrow \frac{\partial \phi}{\partial t}(x,\eta,t) + \frac{P(x,t)}{\rho} + g\eta(x,t) = F(t)$$

$$= \frac{\partial \phi}{\partial t}(x,0,t) + \underbrace{\eta}_{\ll 1}\frac{\partial^2 \phi}{\partial t \partial y} + \cdots$$

$$\rightarrow \frac{\partial \phi}{\partial t}(x,0,t) + \frac{P(x,t)}{\rho} + g\eta(x,t) = F(t)$$

$$\phi'(x,y,t) = \phi(x,y,t) + \int F(t)\,dt$$

$$\rightarrow \frac{\partial \phi'}{\partial t}(x,0,t) + \frac{P(x,t)}{\rho} + g\eta(x,t) = 0$$

$$\frac{\partial}{\partial t} \rightarrow \boxed{\frac{\partial^2 \phi'}{\partial t^2}(x,0,t) + \frac{1}{\rho}\frac{\partial P(x,t)}{\partial t} + g\frac{\partial \phi'}{\partial y}(x,0,t) = 0}$$

$$\frac{\partial \phi}{\partial y}(x,-h,t) = 0 \rightarrow \boxed{\frac{\partial \phi'}{\partial y}(x,-h,t) = 0}$$

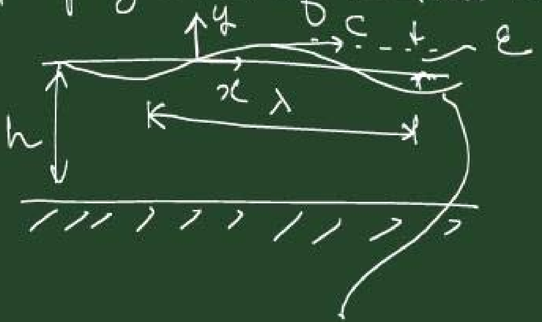
$$\nabla^2 \phi = 0 \rightarrow \boxed{\nabla^2 \phi' = 0}$$

$\eta \ll \lambda, h$

$u \,\&\, v \lesssim O(1)$

much more tractable.

3. Propagation of surface waves



A small-amplitude plane wave is traveling along the surface of this liquid with velocity c.

$$y = \eta(x,t) = \varepsilon \sin\frac{2\pi}{\lambda}(x - ct)$$

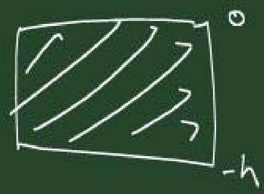
$$c = fn(\varepsilon, \lambda, h)$$

Assumption: neglect surface tension

$$\rightarrow P_{liquid} = P_{gas(air)} = constant$$

For small amplitude ε,

$$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}=0 \\ \dfrac{\partial\phi}{\partial y}(x,0,t)=\dfrac{\partial\eta}{\partial t} = -\varepsilon\dfrac{2\pi c}{\lambda}\cos\dfrac{2\pi}{\lambda}(x-ct) \\ \dfrac{\partial^2\phi}{\partial t^2}(x,0,t)+g\dfrac{\partial\phi}{\partial y}(x,0,t)=0 \\ \dfrac{\partial\phi}{\partial y}(x,-h,t)=0 \end{cases}$$



Use separation of variables

$$\phi = X(x,t)Y(y) \rightarrow \nabla^2\phi = 0$$

$$\rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -k^2 \Rightarrow k = \frac{2\pi}{\lambda}$$

$$\rightarrow \phi(x,y,t) = \cos\frac{2\pi}{\lambda}(x-ct)\left\{c_1\sinh\frac{2\pi y}{\lambda} + c_2\cosh\frac{2\pi y}{\lambda}\right\}$$

b.c. $\dfrac{\partial\phi}{\partial y} = 0$ @ $y=-h$

$$\rightarrow c_1 = c_2\tanh\frac{2\pi h}{\lambda}$$

$$\therefore \phi(x,y,t) = c_2\cos\frac{2\pi}{\lambda}(x-ct)\left\{\tanh\frac{2\pi h}{\lambda}\sinh\frac{2\pi y}{\lambda} + \cosh\frac{2\pi y}{\lambda}\right\}$$

b.c. $\dfrac{\partial^2\phi}{\partial t^2} + g\dfrac{\partial\phi}{\partial y} = 0$ @ $y=0$

$$\rightarrow \boxed{\frac{c^2}{gh} = \frac{\lambda}{2\pi h}\tanh\frac{2\pi h}{\lambda}} \quad \text{for } \varepsilon \ll \lambda, h$$

In case of deep liquids ($h \gg \lambda$)

$$\tanh\frac{2\pi h}{\lambda} = \frac{e^{2\pi h/\lambda} - e^{-2\pi h/\lambda}}{\;'' \; + \; ''\;} \to 1 \quad \text{as} \quad \frac{h}{\lambda} \gg 1$$

$$\therefore \quad \frac{c^2}{gh} = \frac{\lambda}{2\pi h} \quad \text{for} \quad \varepsilon \ll \lambda \ll h$$

↳ c is indep. of h. $c = fn(\lambda)$.

In case of shallow liquids ($h \ll \lambda$)

$$\tanh\frac{2\pi h}{\lambda} \simeq \frac{2\pi h}{\lambda} \qquad c = fn(h)$$

$$\therefore \quad \frac{c^2}{gh} = 1 \quad \text{for} \quad \varepsilon \ll h \ll \lambda .$$