Mechanics in Energy Resources Engineering
- Chapter 9. Deflections of Beams

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summary

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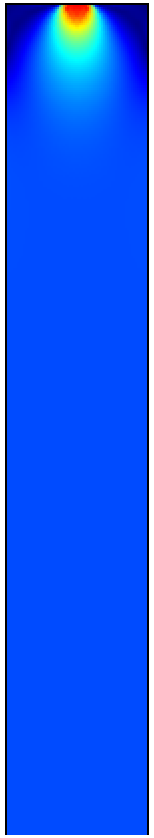
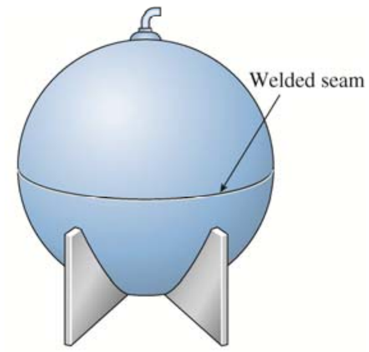
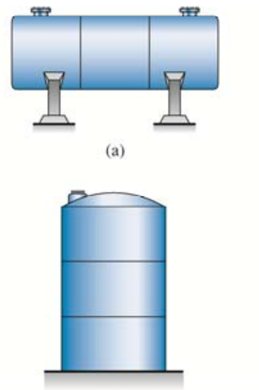
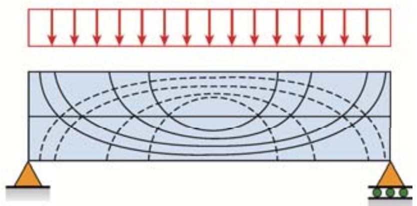
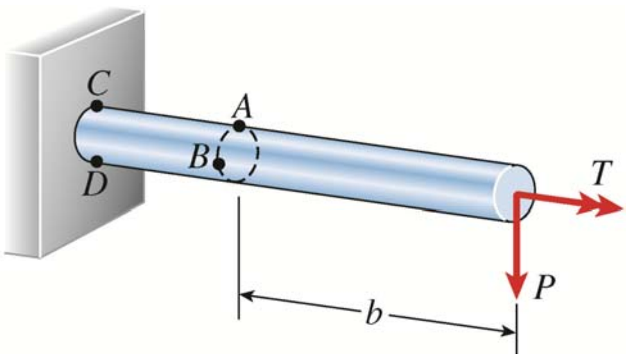
- Chapter 8: Practical Examples of plane stress or strain
  - Introduction
  - Spherical Pressure Vessels
  - Cylindrical Pressure Vessels
  - Maximum Stresses in Beams
  - Combined Loadings

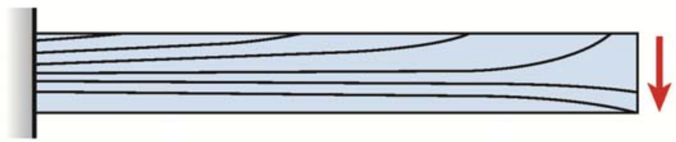












Useful Web Courses

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- http://ocw.mit.edu/OcwWeb/web/home/home/index.htm
- http://nptel.iitm.ac.in/
  - http://nptel.iitm.ac.in/video.php?courseId=1053: strength of materials

Deflections of Beams

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- Introduction
- Differential Equations of the Deflection Curve (처짐곡선의 미분방정식)
- Deflections by Integration of the Bending-Moment Equation (굽힘모멘트 방정식의 적분에 의한 처짐)
- Deflections by Integration of the Shear-Force and Load Equations (전단력과 하중방정식의 적분에 의한 처짐)
- Method of Superposition (중첩법)
- Moment-Area Method (모멘트-면적법)
- Nonprismatic Beams (불균일단면 보)

Introduction

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- A beam loaded by lateral forces → axis is deformed into a curve → deflection
- Chapter 5 Stresses in beams: curvature → normal stresses and strains. Not the deflection curve itself.
- Finding deflections
  - Serviceability requirement
  - Useful for analysis of statically indeterminate structure
  - Important for dynamic analyses

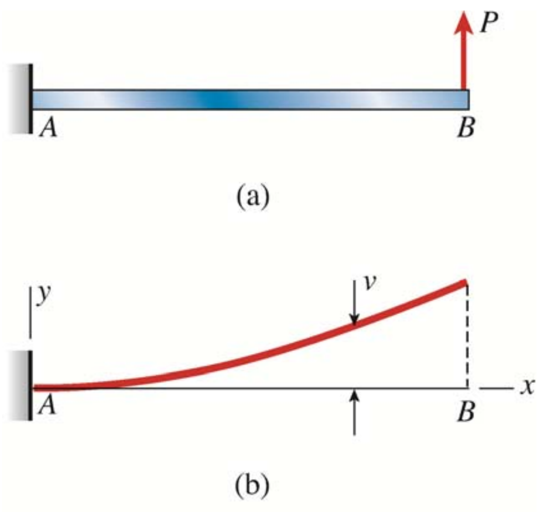

FIG. 9-1 Deflection curve of a cantilever beam

Differential Equations of the deflection curve

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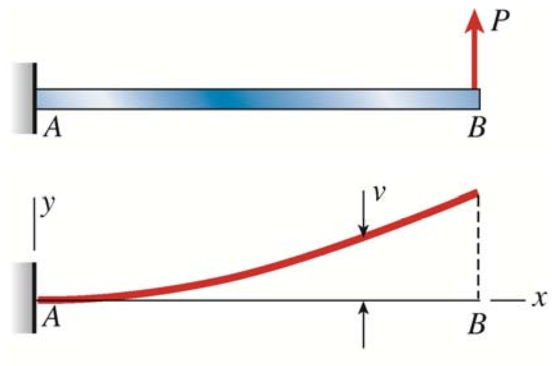
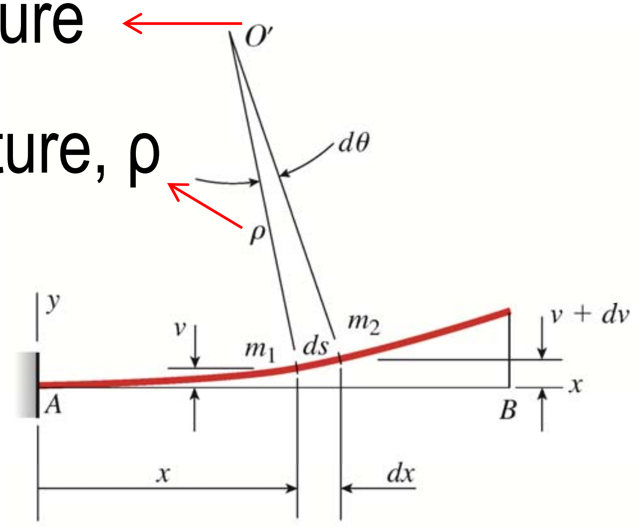
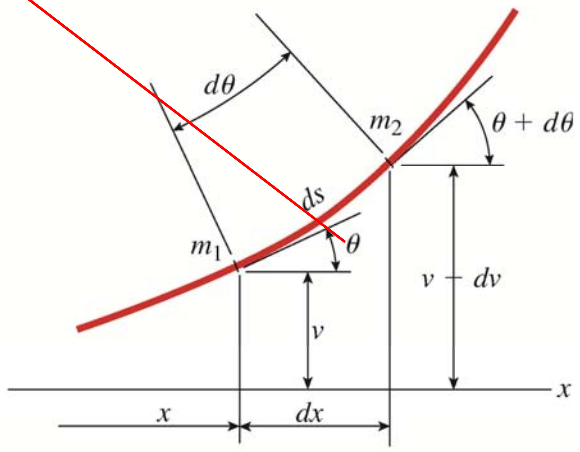


- Deflection (처 짐) v
  - displacement in y-direction. (+) ↑
- Angle of rotation (회 전 각), θ
  - angle between x-axis and the tangent to the deflection curve.
- Center of Curvature
- Radius of Curvature, ρ





Curvature of a Beam

- From the geometry of triangle $O'm_1m_2$,

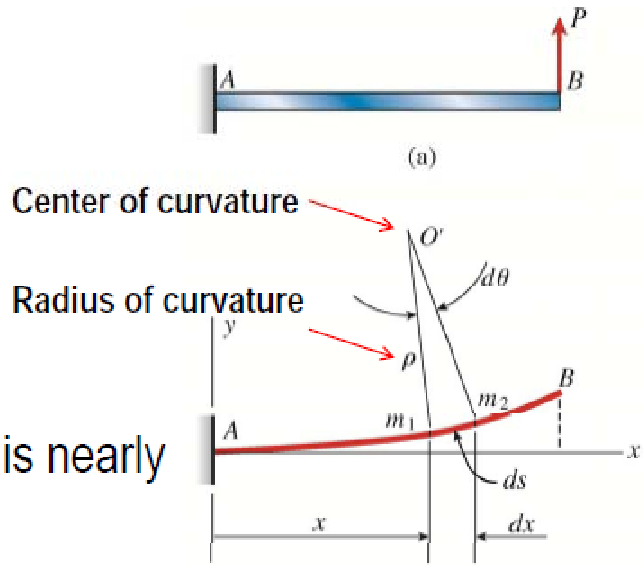
$$\rho d\theta = ds$$

- By rearranging,

$$\kappa = \frac{1}{\rho} = \frac{d\theta}{ds}$$

- Under the assumption of small deflections → deflection curve is nearly flat

$$\kappa = \frac{1}{\rho} = \frac{d\theta}{dx}$$



– From chapter 5 Stresses in beams

Differential Equations of the deflection curve

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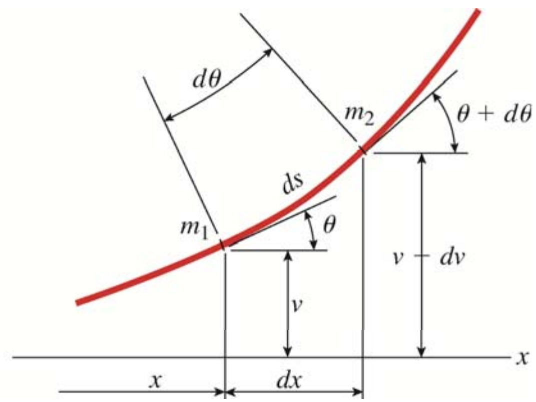
– Expression for Curvature κ & Angle of rotation



$$\kappa = \frac{1}{\rho} = \frac{d\theta}{ds}$$ $$\frac{dv}{dx} = \tan\theta$$

– With small θ, v & κ → ds ≈ dx, θ ≈ tanθ

$$\kappa = \frac{1}{\rho} = \frac{d\theta}{dx} \qquad \theta = \frac{dv}{dx} \longrightarrow \kappa = \frac{1}{\rho} = \frac{d^2v}{dx^2}$$

– If a material is linearly elastic and follows Hooke's law

$$\kappa = \frac{1}{\rho} = \frac{M}{EI}$$

Moment -curvature relationship

– Differential equation of the deflection curve of a beam

$$\frac{d^2v}{dx^2} = \frac{M}{EI} \quad \text{or} \quad EI\frac{d^2v}{dx^2} = M \quad \text{or} \quad EIv'' = M$$

Differential Equations of the deflection curve

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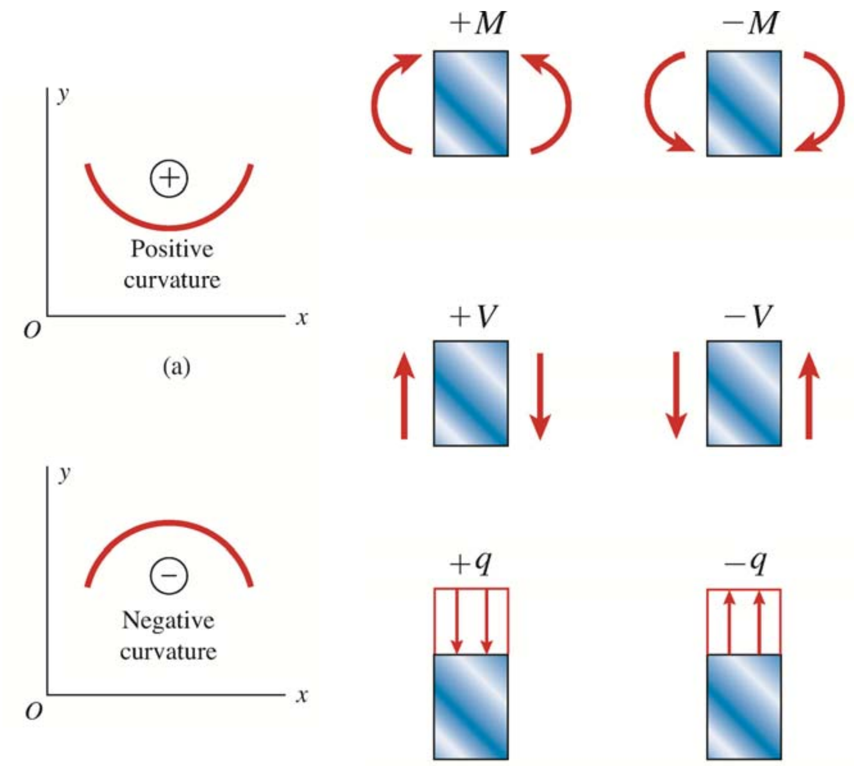
- Differential equation of the deflection curve of a beam
  - Deflection v can be found with known bending moment M and flexural rigidity EI as functions of x.

$$\frac{d^2v}{dx^2} = \frac{M}{EI}$$

  - Sign conventions:
    - v ↑ (+)
    - θ counterclockwise (+)
    - κ concave upward (+)
    - M compression upper part (+)



Additional equations for M, V and q

– Relations between bending moment M, shear force V, and intensity q of distributed load

$$\frac{dV}{dx} = -q \qquad \frac{dM}{dx} = V$$

– Through rearrangement

$$\frac{d}{dx}\left(EI_x \frac{d^2 v}{dx^2}\right) = \frac{dM}{dx} = V$$

$$\frac{d^2}{dx^2}\left(EI_x \frac{d^2 v}{dx^2}\right) = \frac{dV}{dx} = -q$$

$$\xrightarrow[\text{Constant EI}]{\text{Prismatic beam}}$$

$$\boxed{EI \frac{d^3 v}{dx^3} = V}$$

$$\boxed{EI \frac{d^4 v}{dx^4} = -q}$$

Differential equations of the deflection curve
Exact Expression for Curvature

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- With large slopes of deflection curve

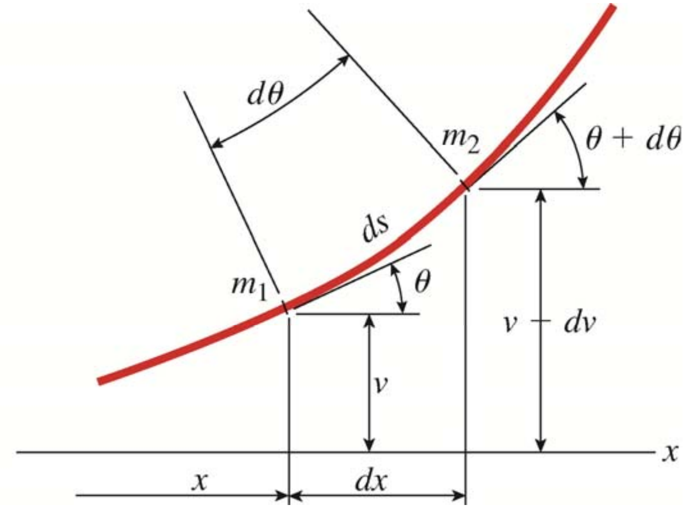
$$\kappa = \frac{1}{\rho} = \frac{d\theta}{ds} = \frac{d(\arctan v')}{dx}\frac{dx}{ds}$$

$$ds^2 = dx^2 + dv^2$$

$$\frac{ds}{dx} = \left[1 + \left(\frac{dv}{dx}\right)^2\right]^{1/2} = \left[1 + (v')^2\right]^{1/2}$$

$$\frac{d(\arctan v')}{dx} = \frac{v''}{1 + (v')^2}$$

$$\kappa = \frac{1}{\rho} = \frac{v''}{\left[1 + (v')^2\right]^{3/2}}$$



With small θ

$$\kappa = \frac{1}{\rho} = v''$$

Deflections by integration of the differential equation

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$$EI\frac{d^2v}{dx^2} = EIv'' = M$$

Bending-moment equation

$$EI\frac{d^3v}{dx^3} = EIv''' = V$$

Shear-force equation

$$EI\frac{d^4v}{dx^4} = EIv'''' = -q$$

Load equation

- Start from any equation that you want
- Deflections can be obtained by integrating above equations
  - Boundary condition
  - Continuity condition
  - Symmetry condition

Deflections by integration of the Bending-Moment Equation

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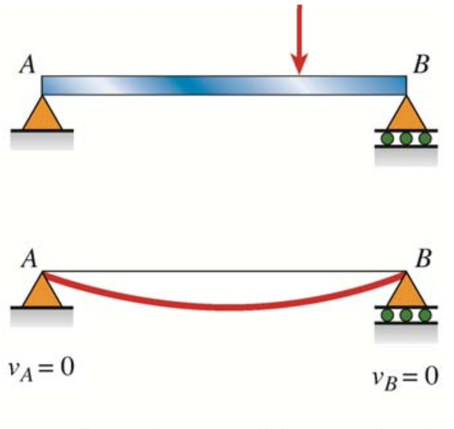
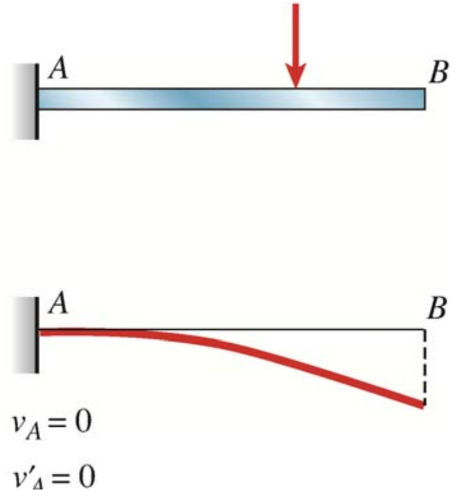
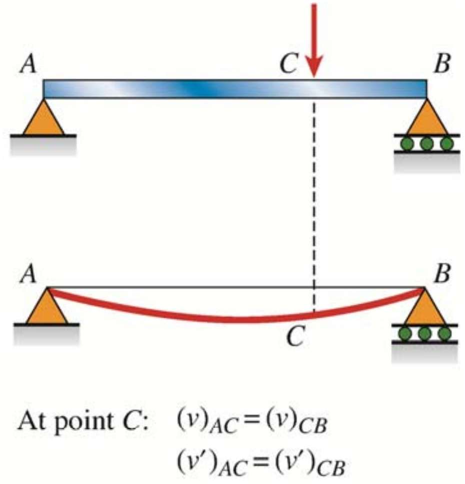
- Boundary, Continuity & Symmetry conditions



Boundary condition at a simple support



Boundary condition at a fixed support



Continuity conditions at point C

  – Symmetry Conditions: additional equation by inspection

    e.g.) simple beam under uniform load throughout its length

Example 9-1
Deflection curve for a simple beam under a uniform load

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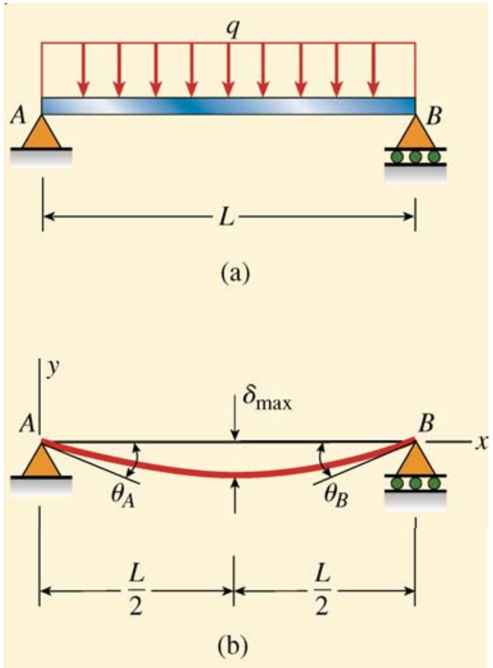
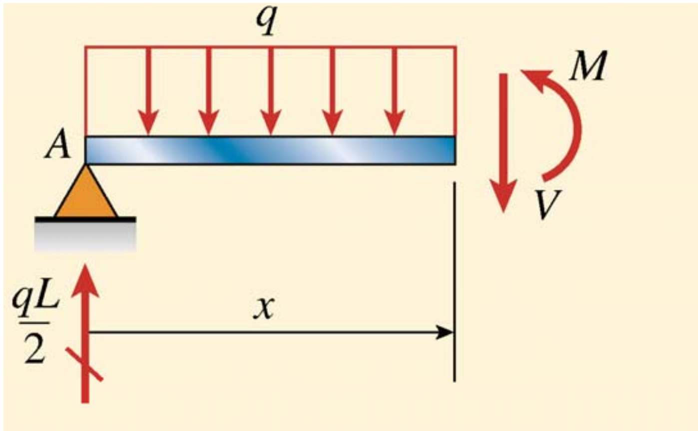
- Equation of deflection curve?
- Maximum deflection at the midpoint?
- Angles of rotation at the supports?





Example 9-2

Deflection curve for a cantilever beam under a uniform load

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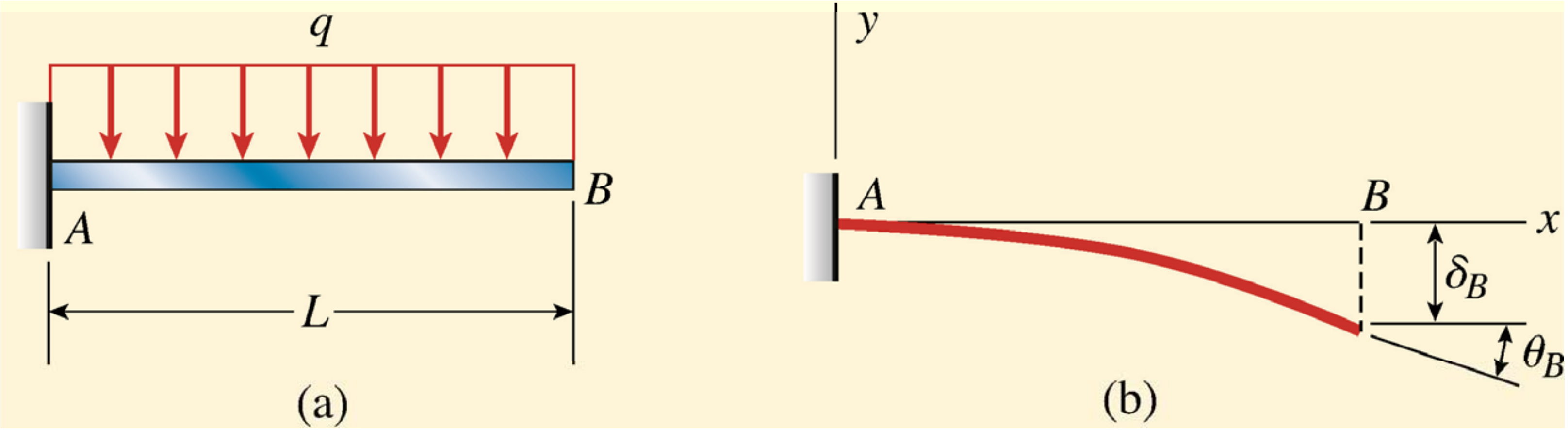

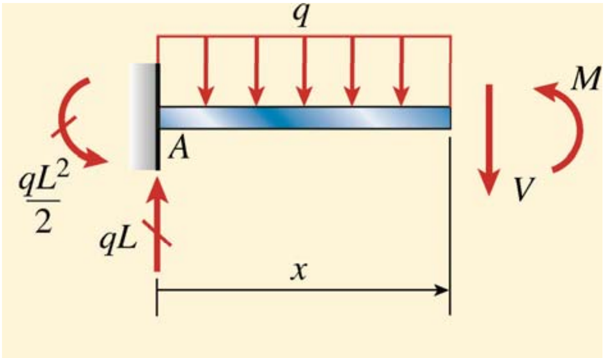

FIG. 9-11 Free-body diagram used in determining the bending moment M (Example 9-2)

Example 9-3

Deflection curve for a simple beam under a concentrated load

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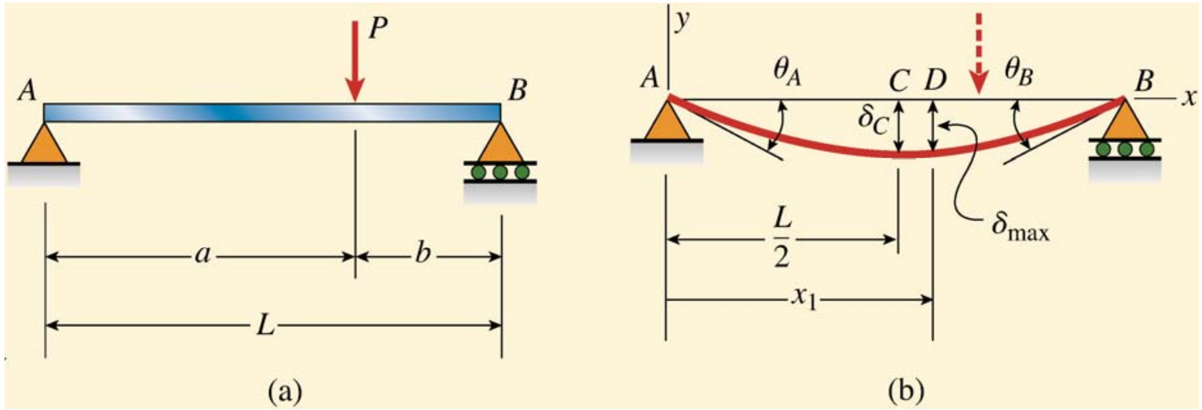

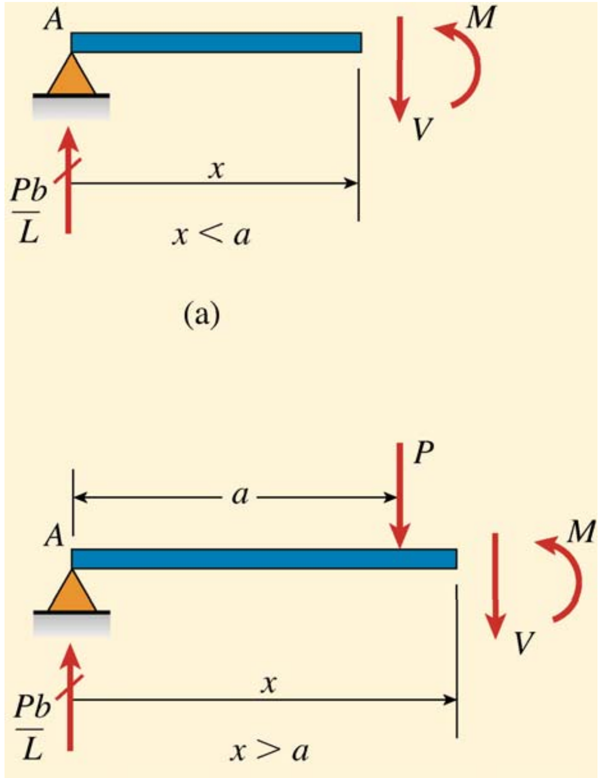

Example 9-3

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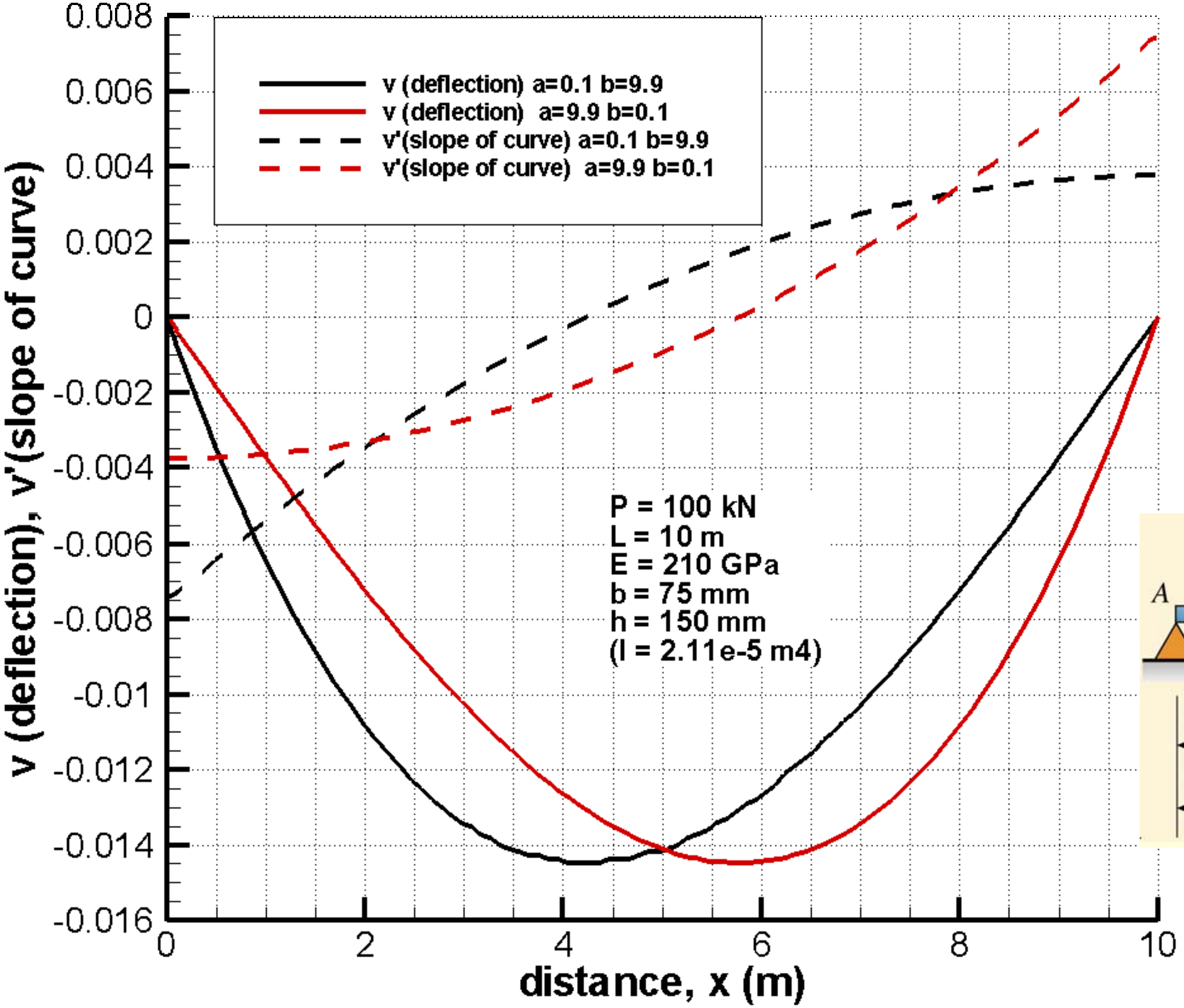

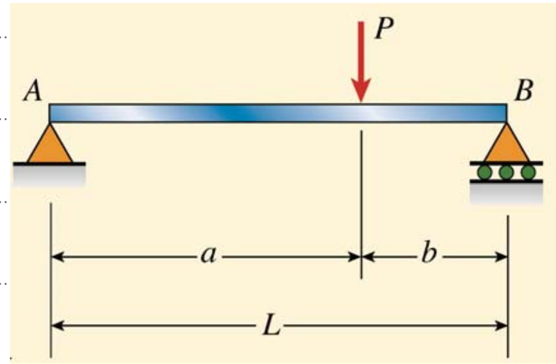

Schedule

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• Week 12	17 May	19 May	: (ch.7) + ch. 8
• Week 13	24 May	26 May (O)	: ch.9 (deflection)
• Week 14	31 May		: ch.10 (indeterminate)
• Week 15	7 June	9 June	: ch.10 (indeterminate)

- 9 June : Final Exam + beer party

- Finalized schedule

On Monday

Deflections by integration of the differential equation

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$$EI\frac{d^2v}{dx^2} = EIv'' = M$$

Bending-moment equation

$$EI\frac{d^3v}{dx^3} = EIv''' = V$$

Shear-force equation

$$EI\frac{d^4v}{dx^4} = EIv'''' = -q$$

Load equation

- Start from any equation that you want
- Deflections can be obtained by integrating above equations
  - Boundary condition
  - Continuity condition
  - Symmetry condition

Deflections by integration of the shear-force & Load equations

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- We may start from shear force V or load q
  - This may be more convenient to some of you.

$$EI\frac{d^3v}{dx^3} = EIv''' = V$$

Shear-force equation

$$EI\frac{d^4v}{dx^4} = EIv'''' = -q$$

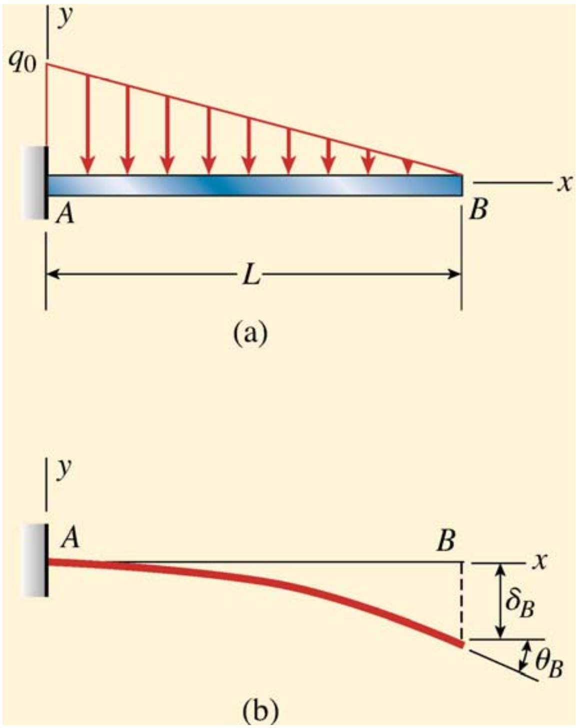
Load equation

Deflections by integration of the shear-force & Load equations

Example 9-4

– Equations of the Deflection curve?

– Deflection and angle of rotation at the free end?



Deflections by integration of the shear-force & Load equations
Example 9-5

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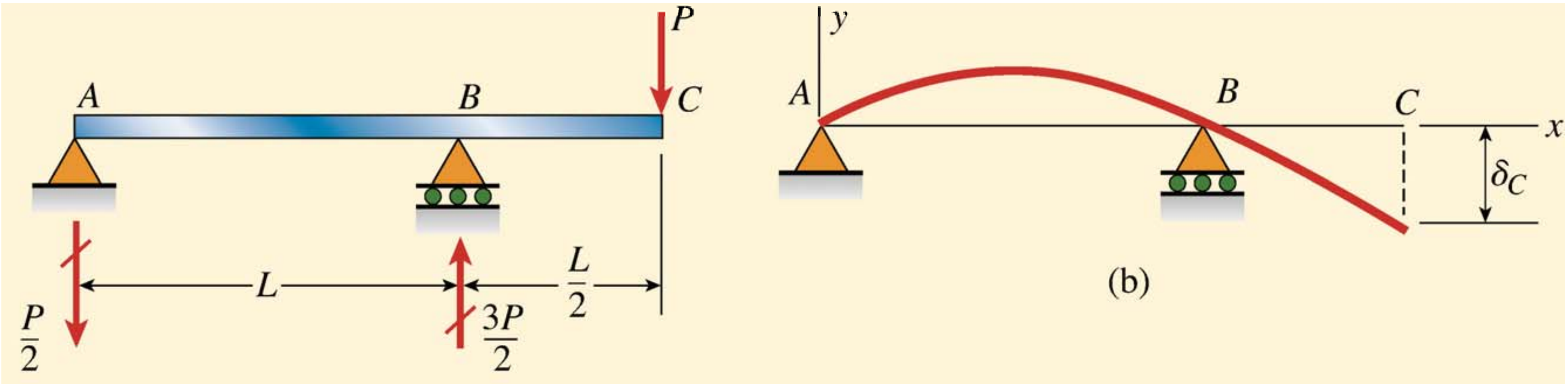
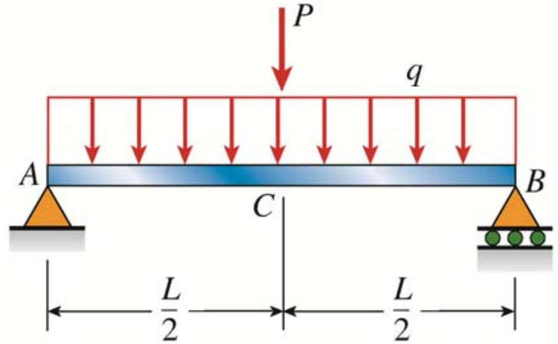
– Equations of the Deflection curve?

– Deflection and angle of rotation at the free end?

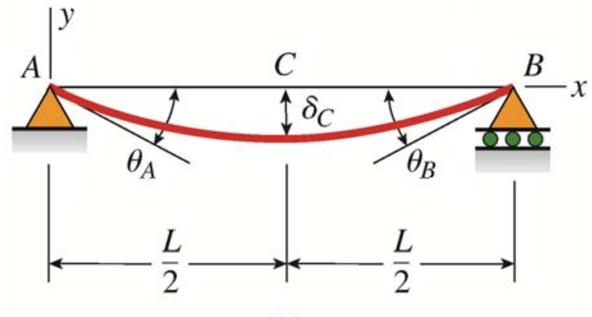


Method of Superposition

- Method of Superposition:
  - Deflections by different load can be found by superposition.
  - E.g.) v1 due to q1, v2 due to q2
    q1 + q2 → v1 +v2
  - Condition: *linear* differential equation



(a)



Contribution from concentrated loading

$$\delta_c = (\delta_c)_1 + (\delta_c)_2$$

$$\theta_A = (\theta_A)_1 + (\theta_A)_2$$

Contribution from distributed loading

Method of Superposition

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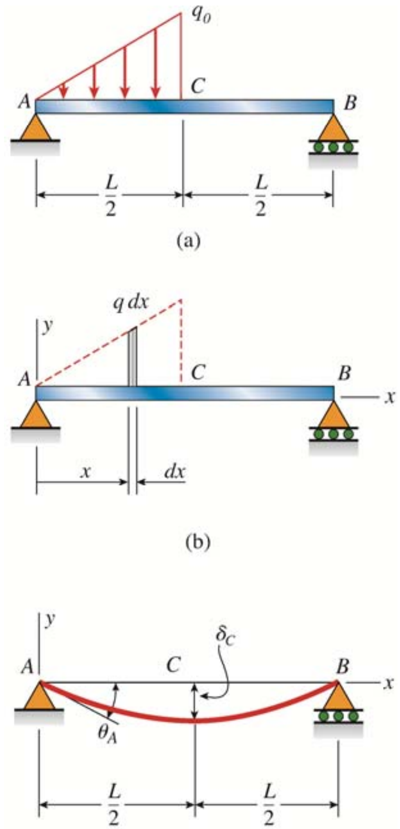
- Table is useful for calculation of deflection & rotation angles
- Superposition may be used for a type that is not available in the table
  - qdx may be seen as a concentrated load
  - Deflection of concentrated load

$$\boxed{\delta_C = \frac{Pa(3L^2 - 4a^2)}{48EI}}$$



$$qdx \leftarrow P \qquad a \leftarrow x \qquad q = \frac{2q_0 x}{L}$$

$$d\delta_C = \frac{q_0 x^2}{24LEI}(3L^2 - 4x^2)dx$$

Appendix G. Table G-2

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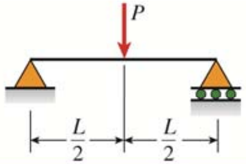
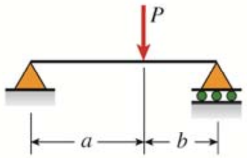
4



$$v = -\frac{Px}{48EI}(3L^2 - 4x^2) \qquad v' = -\frac{P}{16EI}(L^2 - 4x^2) \qquad \left(0 \le x \le \frac{L}{2}\right)$$

$$\delta_C = \delta_{max} = \frac{PL^3}{48EI} \qquad \theta_A = \theta_B = \frac{PL^2}{16EI}$$

5



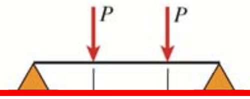
$$v = -\frac{Pbx}{6LEI}(L^2 - b^2 - x^2) \qquad v' = -\frac{Pb}{6LEI}(L^2 - b^2 - 3x^2) \qquad (0 \le x \le a)$$

$$\theta_A = \frac{Pab(L + b)}{6LEI} \qquad \theta_B = \frac{Pab(L + a)}{6LEI}$$

$$\text{If } a \ge b, \quad \delta_C = \frac{Pb(3L^2 - 4b^2)}{48EI} \qquad \text{If } a \le b, \quad \delta_C = \frac{Pa(3L^2 - 4a^2)}{48EI}$$

$$\text{If } a \ge b, \quad x_1 = \sqrt{\frac{L^2 - b^2}{3}} \quad \text{and} \quad \delta_{max} = \frac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}\,LEI}$$

6



$$v = -\frac{Px}{6EI}(3aL - 3a^2 - x^2) \qquad v' = -\frac{P}{2EI}(aL - a^2 - x^2) \qquad (0 \le x \le a)$$

Pa … $- 3x^2 - a^2)$ $\quad v' = -\frac{Pa}{2EI}$ …

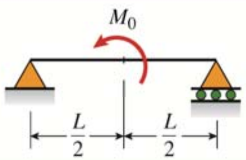
… $(3L^2 - 4a^2)$ $\quad \theta_A = \theta_B = Pa$ …

… $^2 - 3Lx + x^2)$ $\quad v' = -\frac{M}{6L}$ …

$$\delta_C = \frac{M_0L^2}{16EI} \qquad \theta_A = \frac{M_0L}{3EI} \qquad \theta_B = \frac{M_0L}{6EI}$$

$$x_1 = L\left(1 - \frac{\sqrt{3}}{3}\right) \quad \text{and} \quad \delta_{max} = \frac{M_0L^2}{9\sqrt{3}\,EI}$$

8



$$v = -\frac{M_0x}{24LEI}(L^2 - 4x^2) \qquad v' = -\frac{M_0}{24LEI}(L^2 - 12x^2) \qquad \left(0 \le x \le \frac{L}{2}\right)$$

$$\delta_C = 0 \qquad \theta_A = \frac{M_0L}{24EI} \qquad \theta_B = -\frac{M_0L}{24EI}$$

$$\boxed{\theta_A = \frac{Pab(L + b)}{6LEI}}$$

$$\boxed{\text{If } a \le b, \quad \delta_C = \frac{Pa(3L^2 - 4a^2)}{48EI}}$$

Method of Superposition

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- Total deflection can be obtained by superposing all the contribution of concentrated load

$$\delta_C = \int d\delta_C = \int_0^{L/2} \frac{q_0 x^2}{24LEI}(3L^2 - 4x^2)dx = \frac{q_0}{24LEI}\int_0^{L/2}(3L^2 - 4x^2)x^2 dx = \frac{q_0 L^4}{240LEI}$$

- Similarly rotation of angle can be obtained.

$$\boxed{\theta_A = \frac{Pab(L+b)}{6LEI}}$$

$$qdx \leftarrow P \qquad a \leftarrow x$$

$$q = \frac{2q_0 x}{L} \qquad b \leftarrow (L - x)$$

$$\theta_A = \int_0^{L/2} \frac{q_0}{3L^2 EI}(L - x)(2L - x)x^2 dx = \frac{41 q_0 L^3}{2880EI}$$

Method of Superposition
Example 9-6

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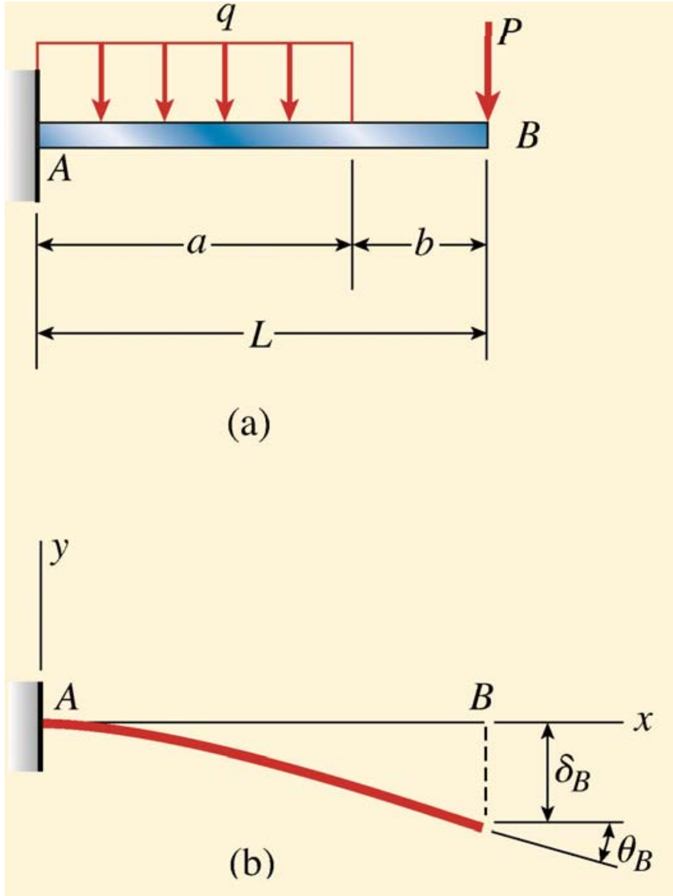
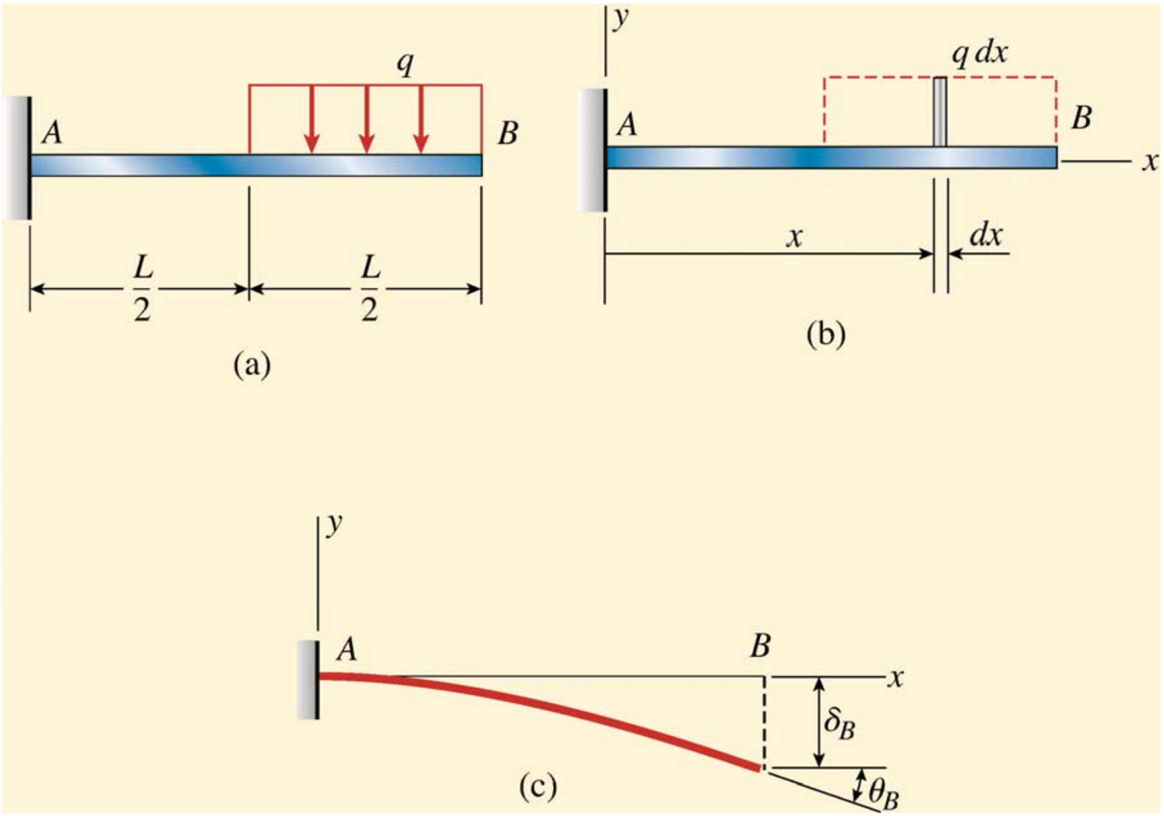
- θ_B & δ_B?
- Refer to Appendix G. Table G-2



Method of Superposition
Example 9-7

- θ_B & δ_B?



Moment-Area method
First Moment-Area Theorem

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- From geometry (assuming a small angle of rotation)

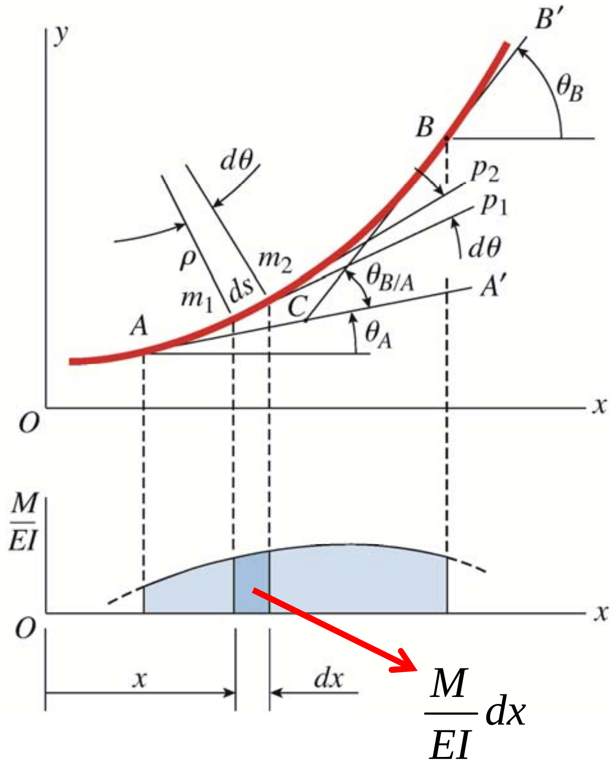
$$\kappa = \frac{1}{\rho} = \frac{d\theta}{dx} \qquad \kappa = \frac{1}{\rho} = \frac{M}{EI}$$

  - Two points m_1 & m_2 small distance apart

$$d\theta = \frac{M}{EI}dx$$

Area of the M/EI diagram between points A and B

$$\int_A^B d\theta = \theta_B - \theta_A = \theta_{B/A} = \int_A^B \frac{M}{EI}dx$$



<u>First Moment-Area Theorem</u>:
The angle $\theta_{B/A}$ = the area of the M/EI Diagram

Moment-Area method
<u>Second</u> Moment-Area Theorem

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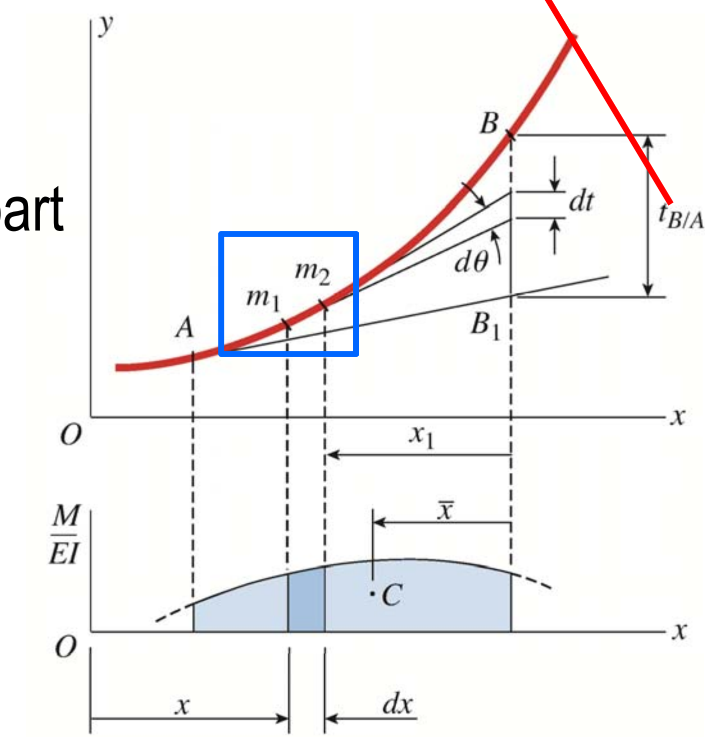
- Tangential deviation
  - Vertical deviation of B on the deflection curve from the tangent at A
  - (+) when B is above
  - Two points m_1 & m_2 small distance apart

$$dt = x_1 d\theta = x_1 \frac{M}{EI} dx$$

Deviation due to bending of element m_1m_2
= first moment of the area of the shaded strip

Tangential deviation of B with respect to A



Moment-Area method
Second Moment-Area Theorem

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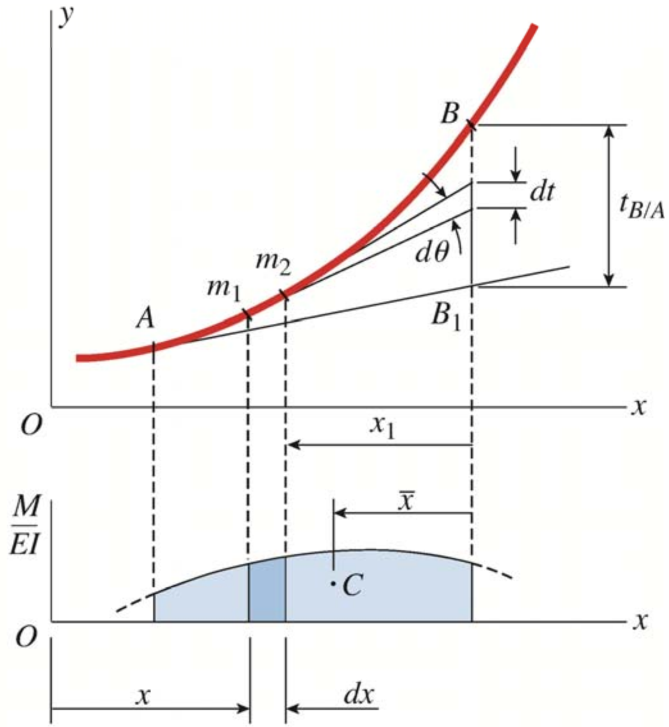
– By integrating

$$\int_A^B dt = t_{B/A} = \int_A^B x_1 \frac{M}{EI} dx$$

Second Moment-Area Theorem:
Tangential deviation $t_{B/A}$ = first moment of the area of M/EI diagram between A and B



– (+) M → B is above A
(-) M → B is below A

– First moment of the area of the M/EI diagram: area x centroid C

Moment-Area method
Example 9-10

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- θ_B & δ_B?

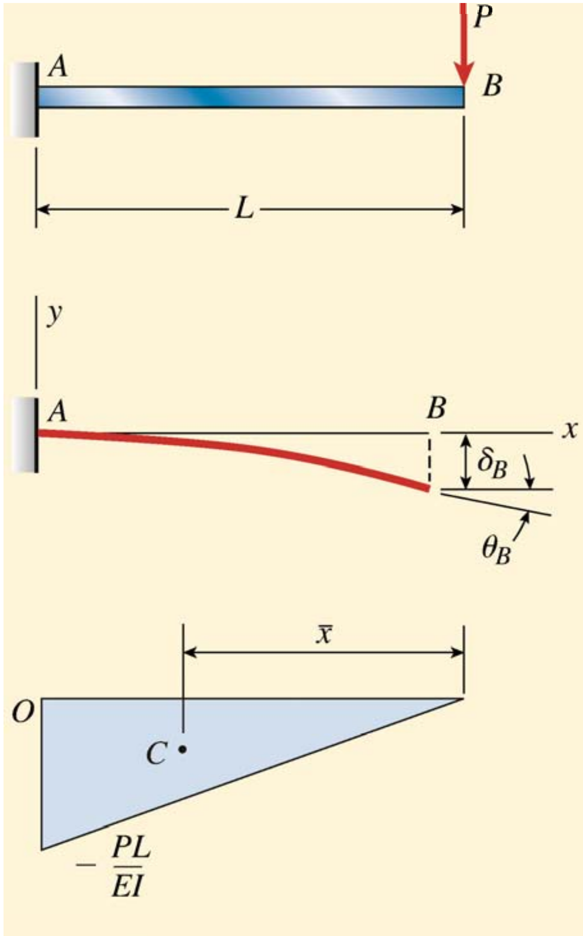

FIG. 9-24 Example 9-10. Cantilever beam with a concentrated load

Moment-Area method
Example 9-11

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- θ_B & δ_B?

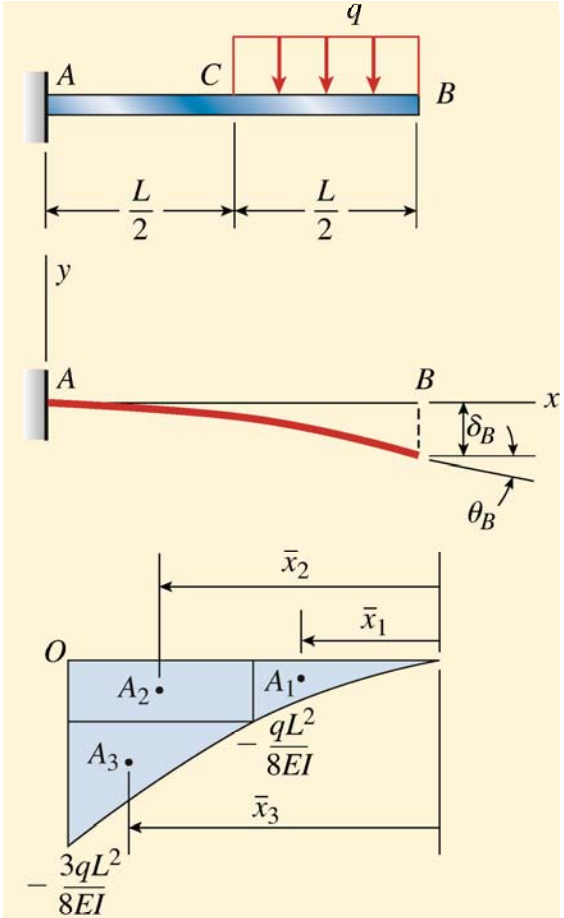

FIG. 9-25 Example 9-11. Cantilever beam supporting a uniform load on the right-hand half of the beam

Moment-Area method
Example 9-12

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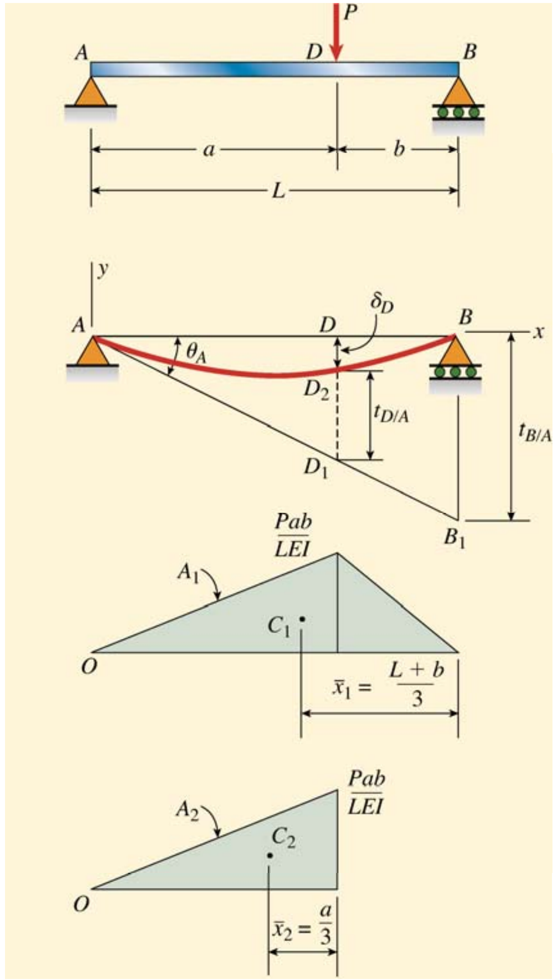
- θ_A & δ_D?



Nonprismatic beams

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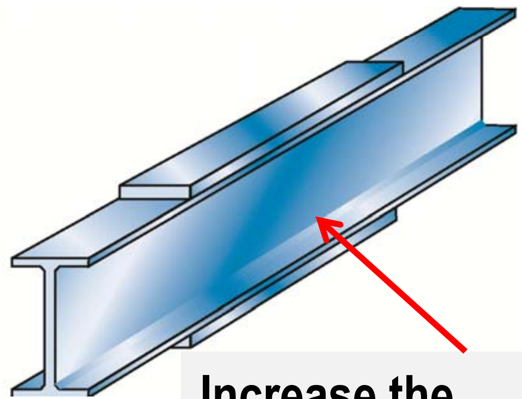
- Beams having varying moments of inertia
  - No new concept is needed
  - Not always work by analytical method
  - Analysis could be more complex



Increase the moment of inertia ← M is largest

Nonprismatic beams

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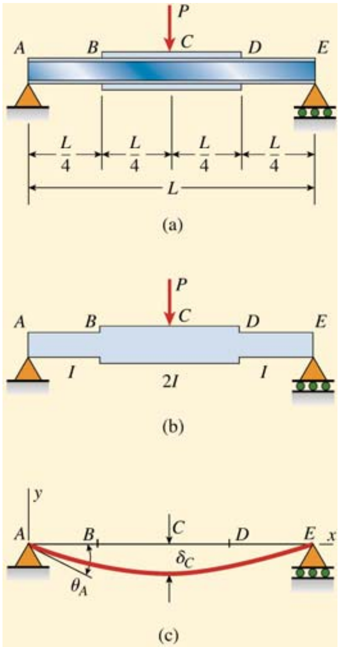
- Example 9-13
  - Cover plate doubles the moment of inertia
  - θ_A & δ_C?



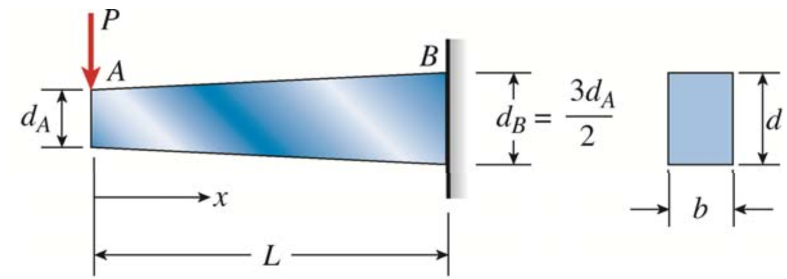
- Prob 9.7-9

$$d = \frac{d_A}{2L}(2L + x)$$

$$I = \frac{bd^3}{12} = \frac{bd_A^3}{96L^3}(2L + x)^3 = \frac{I_A}{8L^3}(2L + x)^3$$

  - θ_A & δ_C?

$$M = -Px$$



PROB. 9.7-9

Deflections of Beams

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- Introduction
- Differential Equations of the Deflection Curve (처짐곡선의 미분방정식)
- Deflections by Integration of the Bending-Moment Equation (굽힘모멘트 방정식의 적분에 의한 처짐)
- Deflections by Integration of the Shear-Force and Load Equations (전단력과 하중방정식의 적분에 의한 처짐)
- Method of Superposition (중첩법)
- Moment-Area Method (모멘트-면적법)
- Nonprismatic Beams (불균일단면 보)