Theory of Poroelasticity
7. Anisotropic material

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Generalized Hooke's Law
Tensor & Matrix Form

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$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl}$$

$\downarrow$ **Contracted form**

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix}$$

- Compliance matrix has 21 independent parameters
  (By the symmetry of stress tensor, strain tensor and consideration of strain energy)

More explicit expression - Lekhnitskii(1963), Hudson (1997)

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Coupling of normal in different directions

Coupling of normal in the same directions

Coupling of normal & Shear

$$
\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{pmatrix}
=
\begin{pmatrix}
\frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & \frac{\eta_{x,yz}}{G_{yz}} & \frac{\eta_{x,xz}}{G_{xz}} & \frac{\eta_{x,xy}}{G_{xy}} \\
-\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & \frac{\eta_{y,yz}}{G_{yz}} & \frac{\eta_{y,xz}}{G_{xz}} & \frac{\eta_{y,xy}}{G_{xy}} \\
-\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & \frac{\eta_{z,yz}}{G_{yz}} & \frac{\eta_{z,xz}}{G_{xz}} & \frac{\eta_{z,xy}}{G_{xy}} \\
\frac{\eta_{yz,x}}{E_x} & \frac{\eta_{yz,y}}{E_y} & \frac{\eta_{yz,z}}{E_z} & \frac{1}{G_{yz}} & \frac{\mu_{yz,xz}}{G_{xz}} & \frac{\mu_{yz,xy}}{G_{xy}} \\
\frac{\eta_{xz,x}}{E_x} & \frac{\eta_{xz,y}}{E_y} & \frac{\eta_{xz,z}}{E_z} & \frac{\mu_{xz,yz}}{G_{yz}} & \frac{1}{G_{xz}} & \frac{\mu_{xz,xy}}{G_{xy}} \\
\frac{\eta_{xy,x}}{E_x} & \frac{\eta_{xy,y}}{E_y} & \frac{\eta_{xy,z}}{E_z} & \frac{\mu_{xy,yz}}{G_{yz}} & \frac{\mu_{xy,xz}}{G_{xz}} & \frac{1}{G_{xy}}
\end{pmatrix}
\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix}
$$

Coupling of shear in different directions

Coupling of shear in the same directions

Monoclinic
One plane of elastic symmetry

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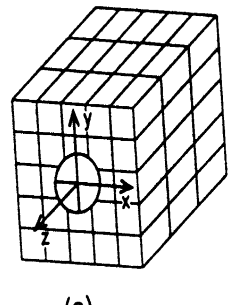
$$
\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{pmatrix}
=
\begin{pmatrix}
\frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & \frac{\eta_{x,xy}}{G_{xy}} \\
-\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & 0 & 0 & \frac{\eta_{y,xy}}{G_{xy}} \\
-\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & 0 & 0 & \frac{\eta_{z,xy}}{G_{xy}} \\
0 & 0 & 0 & \frac{1}{G_{yz}} & \frac{\mu_{yz,xz}}{G_{xz}} & 0 \\
0 & 0 & 0 & \frac{\mu_{xz,yz}}{G_{yz}} & \frac{1}{G_{xz}} & 0 \\
\frac{\eta_{xy,x}}{E_x} & \frac{\eta_{xy,y}}{E_y} & \frac{\eta_{xy,z}}{E_z} & 0 & 0 & \frac{1}{G_{xy}}
\end{pmatrix}
\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix}
$$

- With a plane of symmetry normal to z-axis
- 13 independent constants

Orthotropic

Three orthogonal planes of elastic symmetry

$$
\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{pmatrix} =
\begin{pmatrix}
\frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & 0 \\
-\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & 0 & 0 & 0 \\
-\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G_{yz}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G_{xz}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G_{xy}}
\end{pmatrix}
\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix}
$$



- Three orthogonal planes elastic symmetry
- 9 independent constants

Transversely Isotropic
One axis of elastic symmetry of rotation

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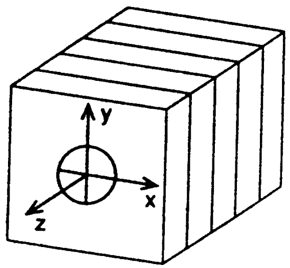
$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu'}{E'} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu'}{E'} & 0 & 0 & 0 \\ -\frac{\nu'}{E'} & -\frac{\nu'}{E'} & \frac{1}{E'} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G'} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G'} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix}$$



(d)

$$E_x = E_y = E$$

$$E_z = E'$$

$$\nu_{xy} = \nu_{yx} = \nu$$

$$\nu_{zx} = \nu_{zy} = \nu'$$

$$G_{xz} = G_{yz} = G'$$

$$\nu_{xz} = \nu_{yz} = \nu' \frac{E}{E'}$$

- 5 independent constants

Isotropic
Complete symmetry

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$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix}$$

$$E' = E$$

$$\nu' = \nu$$

$$G' = G$$

Bounds of elastic constants

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$$W = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} \qquad W = \frac{1}{2}\sigma^T S\sigma$$

- the 6×6 matrices of elastic constants must be positive definite (Ting, 1996)
- A necessary and sufficient condition for the quadratic form to be positive definite is that all principal minors of matrix (that is all minor determinants in the matrix having diagonal elements coincident with the principal diagonal of the matrix) are positive (Amadei et al 1987).

Bounds of elastic constants
Orthogonal

$$E_x, E_y, E_z, G_x, G_y, G_z > 0$$

$$\begin{vmatrix} \dfrac{1}{E_x} & -\dfrac{\nu_{yx}}{E_y} \\ -\dfrac{\nu_{xy}}{E_x} & \dfrac{1}{E_y} \end{vmatrix} = \frac{1}{E_x E_y} - \frac{\nu_{yx}\nu_{xy}}{E_x E_y} \rangle 0 \quad \longrightarrow \quad 1 - \nu_{yx}\nu_{xy} \rangle 0$$

$$\begin{vmatrix} \dfrac{1}{E_y} & -\dfrac{\nu_{zy}}{E_z} \\ -\dfrac{\nu_{yz}}{E_y} & \dfrac{1}{E_z} \end{vmatrix} = \frac{1}{E_y E_z} - \frac{\nu_{zy}\nu_{yz}}{E_y E_z} \rangle 0 \quad \longrightarrow \quad 1 - \nu_{zy}\nu_{yz} \rangle 0$$

$$\begin{vmatrix} \dfrac{1}{E_x} & -\dfrac{\nu_{zx}}{E_z} \\ -\dfrac{\nu_{xz}}{E_x} & \dfrac{1}{E_z} \end{vmatrix} = \frac{1}{E_x E_z} - \frac{\nu_{zx}\nu_{xz}}{E_x E_z} \rangle 0 \quad \longrightarrow \quad 1 - \nu_{zx}\nu_{xz} \rangle 0$$

$$\begin{vmatrix} \dfrac{1}{E_x} & -\dfrac{\nu_{yx}}{E_y} & -\dfrac{\nu_{zx}}{E_z} \\ -\dfrac{\nu_{xy}}{E_x} & \dfrac{1}{E_y} & -\dfrac{\nu_{zy}}{E_z} \\ -\dfrac{\nu_{xz}}{E_x} & -\dfrac{\nu_{yz}}{E_y} & \dfrac{1}{E_z} \end{vmatrix} = \frac{1}{E_x E_y E_z} - \frac{\nu_{zy}\nu_{yz}}{E_x E_y E_z} - \frac{\nu_{yx}\nu_{xy}}{E_x E_y E_z} - \frac{\nu_{xz}\nu_{zx}}{E_x E_y E_z} - \frac{\nu_{yx}\nu_{xz}\nu_{zy}}{E_x E_y E_z} - \frac{\nu_{zx}\nu_{xy}\nu_{yz}}{E_x E_y E_z} \rangle 0$$

$$\longrightarrow \quad 1 - \nu_{zy}\nu_{yz} - \nu_{yx}\nu_{xy} - \nu_{xz}\nu_{zx} - \nu_{yx}\nu_{xz}\nu_{zy} - \nu_{zx}\nu_{xy}\nu_{yz} \rangle 0$$

Bounds of elastic constants
Orthogonal

$$E_x, E_y, E_z, G_x, G_y, G_z > 0$$

$$-\sqrt{\frac{E_x}{E_y}} \langle \nu_{xy} \langle \sqrt{\frac{E_x}{E_y}}$$

$$-\sqrt{\frac{E_y}{E_z}} \langle \nu_{yz} \langle \sqrt{\frac{E_y}{E_z}}$$

$$-\sqrt{\frac{E_x}{E_z}} \langle \nu_{xz} \langle \sqrt{\frac{E_x}{E_z}}$$

$$1 - \frac{E_z}{E_y} \nu_{yz}{}^2 - \frac{E_y}{E_x} \nu_{xy}{}^2 - \frac{E_z}{E_x} \nu_{xz}{}^2 - 2 \frac{E_z}{E_x} \nu_{xy} \nu_{xz} \nu_{yz} \rangle 0$$

Bounds of elastic constants
Transversely Isotropic

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$$E, E', G' > 0$$

$$-1 < \nu < 1$$

$$-\sqrt{\frac{E'}{E}\frac{(1-\nu)}{2}} < \nu' < \sqrt{\frac{E'}{E}\frac{(1-\nu)}{2}}$$

Bounds of elastic constants
Isotropic

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$$E > 0, G > 0$$

$$-1 < \nu < 1 \longrightarrow$$

$$-1 < \nu < \frac{1}{2}$$

$$E > 0$$

$$-\sqrt{\frac{(1-\nu)}{2}} < \nu < \sqrt{\frac{(1-\nu)}{2}}$$

Application to fractured rock masses - Amadei (1981)

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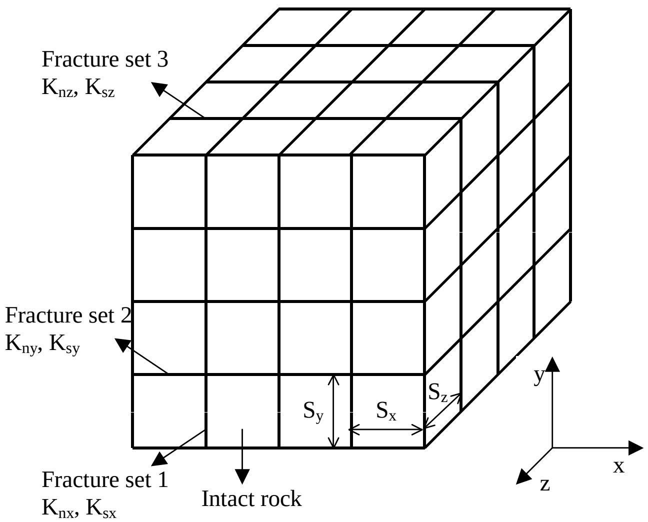
Rock masses with three perpendicular fracture sets can modelled as orthogonally isotropic rock

$$
\begin{pmatrix}
\frac{1}{E_x}+\frac{1}{K_{nx}S_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & 0 \\
-\frac{\nu_{yx}}{E_y} & \frac{1}{E_y}+\frac{1}{K_{ny}S_y} & -\frac{\nu_{zy}}{E_z} & 0 & 0 & 0 \\
-\frac{\nu_{zx}}{E_z} & -\frac{\nu_{zy}}{E_z} & \frac{1}{E_z}+\frac{1}{K_{nz}S_z} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G_{yz}}+\frac{1}{K_{sy}S_y}+\frac{1}{K_{sz}S_z} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G_{xz}}+\frac{1}{K_{sx}S_x}+\frac{1}{K_{sz}S_z} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G_{xy}}+\frac{1}{K_{sx}S_x}+\frac{1}{K_{sy}S_y}
\end{pmatrix}
$$



Transformation of compliance tensor under the transformation of axis

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- 0th order tensor (scalar) : no need to transform, independent of coordinate
- 1th order tensor (vector) :

$$x_i' = \beta_{ij} x_j$$

- 2nd order tensor :
  - i.e. stress, strain, permeability

$$\sigma'_{ij} = \beta_{im}\beta_{jn}\sigma_{mn}$$

- 4th order tensor :
  - Compliance tensor

$$S'_{ijkl} = \beta_{im}\beta_{jn}\beta_{kp}\beta_{lp}S_{mnpq}$$

$$\beta_{ij} = \begin{pmatrix} \cos(x',x) & \cos(x',y) & \cos(x',z) \\ \cos(y',x) & \cos(y',y) & \cos(y',z) \\ \cos(z',x) & \cos(z',y) & \cos(z',z) \end{pmatrix} \qquad \beta_{ij} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

General transformation   Rotation

Compliance matrix
Transformation

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$$S'_{ijkl} = \beta_{im}\beta_{jn}\beta_{kp}\beta_{lp}S_{mnpq}$$

$$\downarrow$$

$$S'_{ij} = S_{mn}q_{mi}q_{nj}$$

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix}$$

	X	Y	z
X′	α_1	β_1	γ_1
Y′	α_2	β_2	γ_2
Z′	α_3	β_3	γ_3

	1	2	3	4	5	6
1	α_1^2	α_2^2	α_3^2	$2\alpha_2\alpha_3$	$2\alpha_3\alpha_1$	$2\alpha_1\alpha_2$
2	β_1^2	β_2^2	β_3^2	$2\beta_2\beta_3$	$2\beta_3\beta_1$	$2\beta_1\beta_2$
3	γ_1^2	γ_2^2	γ_3^2	$2\gamma_2\gamma_3$	$2\gamma_3\gamma_1$	$2\gamma_1\gamma_2$
4	$\beta_1\gamma_1$	$\beta_2\gamma_2$	$\beta_3\gamma_3$	$\beta_2\gamma_3+\beta_3\gamma_2$	$\beta_1\gamma_3+\beta_3\gamma_1$	$\beta_1\gamma_2+\beta_2\gamma_1$
5	$\gamma_1\alpha_1$	$\gamma_2\alpha_2$	$\gamma_3\alpha_3$	$\gamma_2\alpha_3+\gamma_3\alpha_2$	$\gamma_1\alpha_3+\gamma_3\alpha_1$	$\gamma_1\alpha_2+\gamma_2\alpha_1$
6	$\alpha_1\beta_1$	$\alpha_2\beta_2$	$\alpha_3\beta_3$	$\alpha_2\beta_3+\alpha_3\beta_2$	$\alpha_1\beta_3+\alpha_3\beta_1$	$\alpha_1\beta_2+\alpha_2\beta_1$

Transformation of compliance tensor

Elastic modulus and Poisson's ratio (Min & Jing, 2004)

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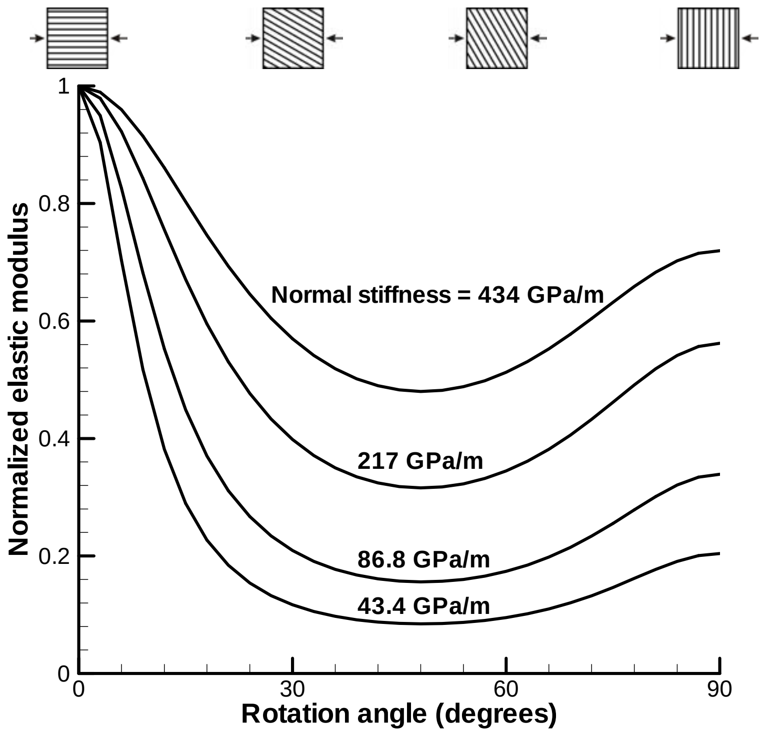
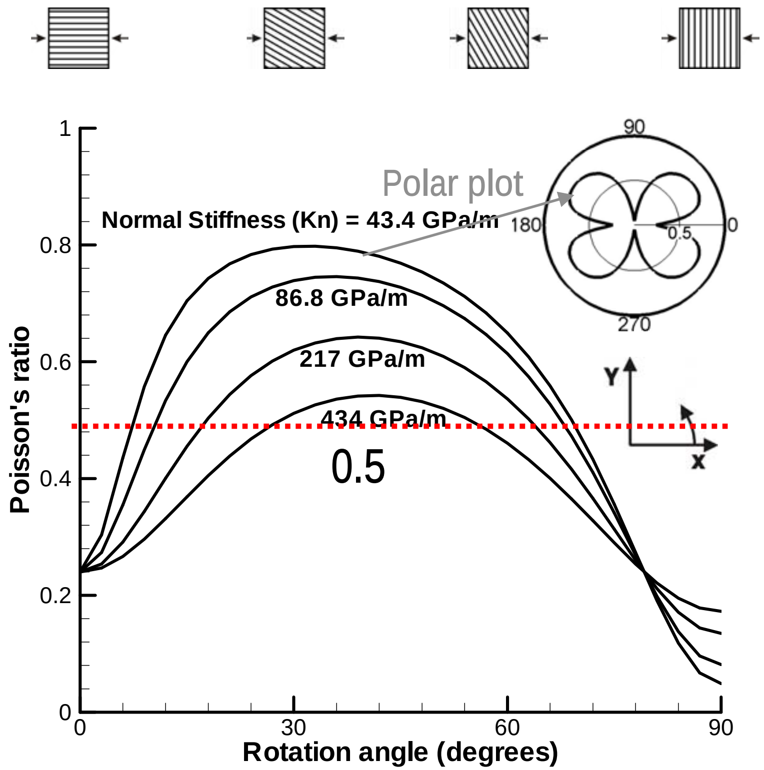
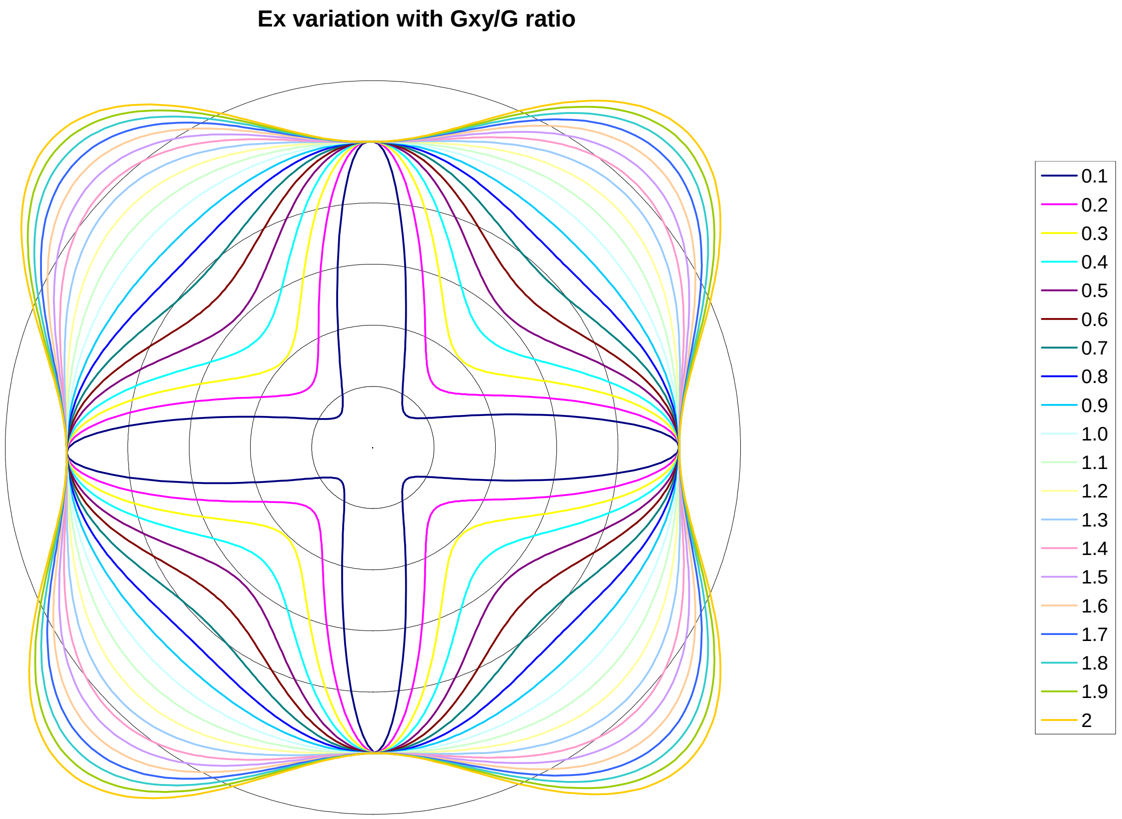




Transversely Isotropic rock

Transformation of compliance tensor
Elastic modulus

Ex=Ey
Nu = 0.25
Gxy/G varies



- Orthotropic rock

Transformation of compliance tensor
Elastic modulus

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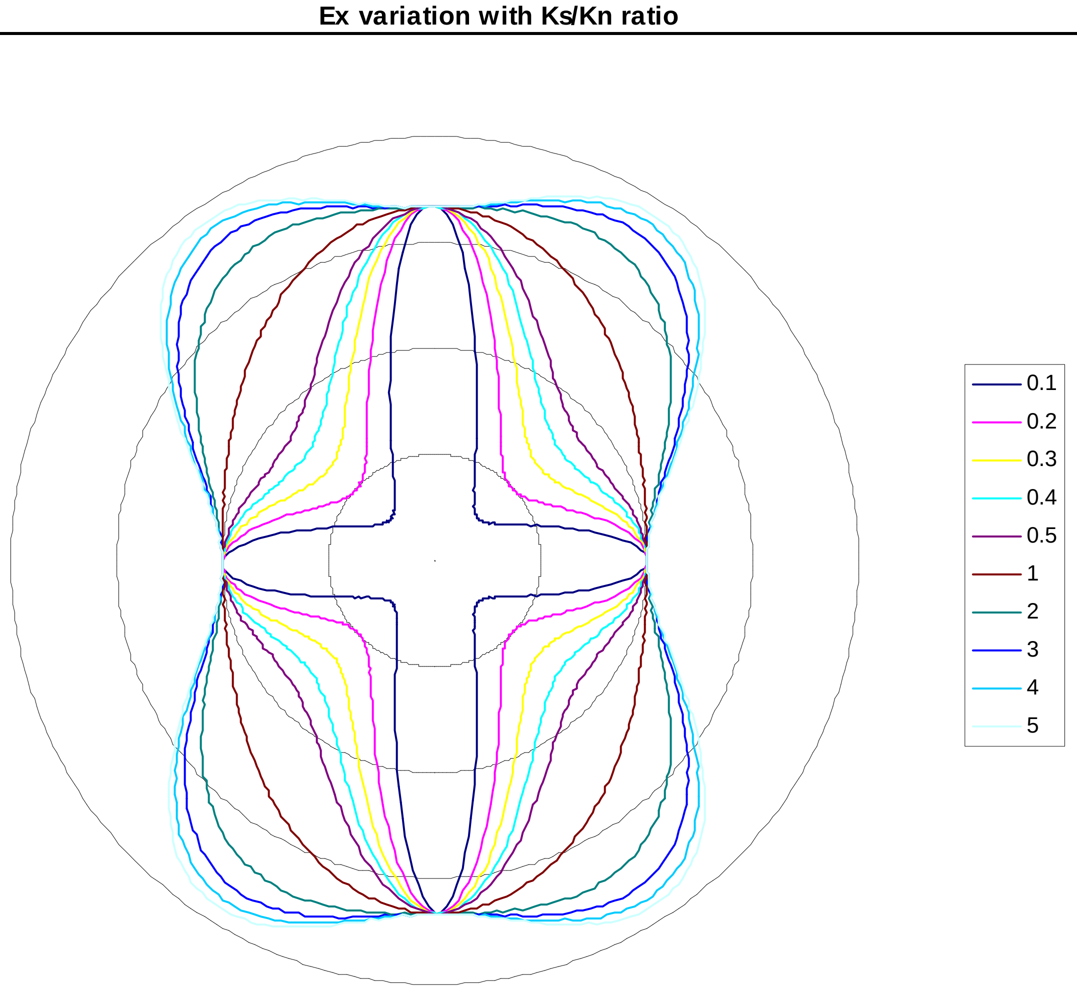
K_n for set 1

$= 2^* K_n$ for set 2



Fractured Rock Masses