Lecture Note 1

I. Solution of One-Dimensional, One-Group Neutron Diffusion Equation

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1D 1G Neutron Diffusion Equation

- Neutron Diffusion Equation

$$\frac{dJ(x)}{dx} + \Sigma_a(x)\phi(x) = \lambda \cdot \nu\Sigma_f(x)\phi(x) , \qquad \lambda = \frac{1}{k_{eff}}$$

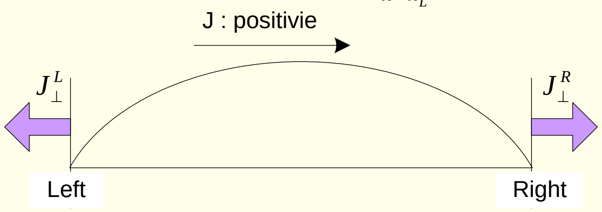
$$\text{Fick's Law: } J(x) = -D(x)\frac{d\phi(x)}{dx} \rightarrow \qquad -\frac{d}{dx}\left(D(x)\frac{d\phi(x)}{dx}\right) + \Sigma_a(x)\phi(x) = \lambda\nu\Sigma_f(x)\phi(x)$$

- Boundary condition with Albedo
  - Right ($x = x_R$)

$$1)\ \alpha_R = \frac{J_\perp^R}{\phi} \qquad 2)\ J_\perp^R = J\big|_{x=x_R} = -D\frac{d\phi}{dx}\bigg|_{x=x_R} \qquad \rightarrow D\frac{d\phi}{dx} + \alpha_R\phi = 0 \quad \text{(Robin condition)}$$

  - Left ($x = x_L$)

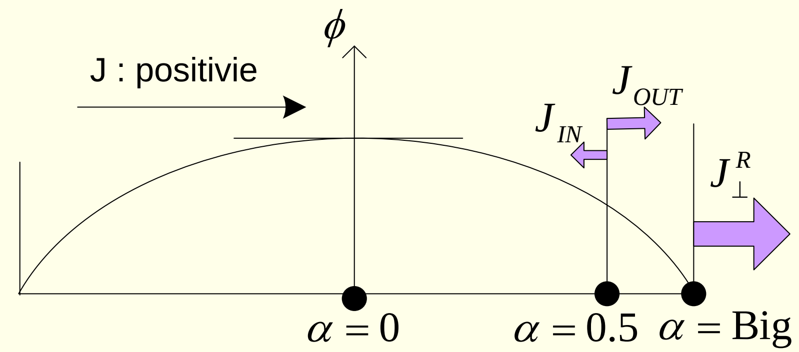
$$1)\ \alpha_L = \frac{J_\perp^L}{\phi} \qquad 2)\ J_\perp^L = -J\big|_{x=x_L} = -\left(-D\frac{d\phi}{dx}\right) = D\frac{d\phi}{dx}\bigg|_{x=x_L} \qquad \rightarrow -D\frac{d\phi}{dx} + \alpha_L\phi = 0$$



- Homogeneous D.E. with Homogeneous B.C $\rightarrow$ Homogeneous problem, Eigenvalue problem

Albedos for Various Boundary Conditions

$$\alpha = \frac{J_\perp}{\phi}$$



a) $\alpha = 0 \quad \rightarrow \quad J = -D\frac{d\phi}{dx} = 0$: Reflective BC

b) $\alpha = \infty \quad \rightarrow \quad \phi = 0$: Zero-flux BC

c) $\alpha = 0.5 \quad \rightarrow \quad J_{IN} = 0$: Zero-incoming current BC

$$\varphi(\mu) = \frac{1}{2}\phi + \frac{3}{2}J\mu \quad \Rightarrow \begin{cases} J_{IN} = \int_0^{-1} \varphi(\mu)\mu \, d\mu = \frac{1}{4}\phi - \frac{1}{2}J \\ J_{OUT} = \int_0^{1} \varphi(\mu)\mu \, d\mu = \frac{1}{4}\phi + \frac{1}{2}J \end{cases} \qquad (\mu = \cos\theta)$$

$$J_{IN} = 0 \quad \Leftrightarrow \quad \frac{1}{4}\phi - \frac{1}{2}J = 0 \quad \Leftrightarrow \quad \alpha = \frac{J}{\phi} = 0.5$$

Positiveness of Eigenvalue

- Neutron Diffusion Equation

$$-\frac{d}{dx}\left(D\frac{d\phi}{dx}\right)+\Sigma_a\phi=\lambda\cdot\nu\Sigma_f\phi$$

  - Multiply $\phi(x)$

$$-\frac{d}{dx}\left(D\frac{d\phi}{dx}\right)\cdot\phi \quad + \quad \Sigma_a\ \phi\cdot\phi \quad = \quad \lambda\cdot\nu\Sigma_f\phi\cdot\phi$$

  - Integrate above eqn. over $[x_L, x_R]$

$$A=\int_{x_L}^{x_R}\left[-\frac{d}{dx}\left(D\frac{d\phi}{dx}\right)\cdot\phi\right]dx = -D\frac{d\phi}{dx}\cdot\phi\Bigg|_{x_L}^{x_R}+\int_{x_L}^{x_R}D\frac{d\phi}{dx}\cdot\frac{d\phi}{dx}dx = \alpha_R\phi_R^2+\alpha_L\phi_L^2+\int_{x_L}^{x_R}D\left(\frac{d\phi}{dx}\right)^2dx>0$$

$$B=\int_{x_L}^{x_R}[\Sigma_a\ \phi\cdot\phi]dx \qquad >0$$

$$C=\int_{x_L}^{x_R}\left[\nu\Sigma_f\cdot\phi\cdot\phi\right]dx \quad >0$$

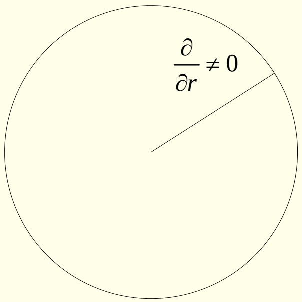
$$\lambda=\frac{A+B}{C}>0 \quad (\text{Always})$$

physically λ is an adjustment factor for nontrivial solution

1D Problem in Other Coordinates

- Extension of 1-D to other coordinates

$$\nabla \cdot J = -\nabla \cdot D\nabla\phi \quad = -D\nabla^2\phi \;\text{ if } D = const.$$

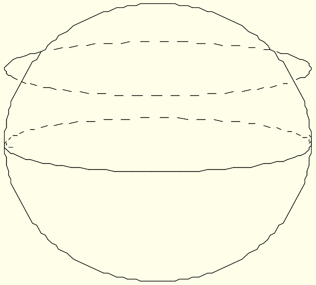


- Polar

$$-\frac{1}{r}\frac{d}{dr}\left(rD\frac{d\phi}{dr}\right) = -\frac{d}{dr}\left(D\frac{d\phi}{dr}\right) - \frac{1}{r}D\frac{d\phi}{dr} \quad = \frac{dJ}{dr} + \frac{J}{r}$$

$$\nabla_r^2 = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)$$

- Spherical

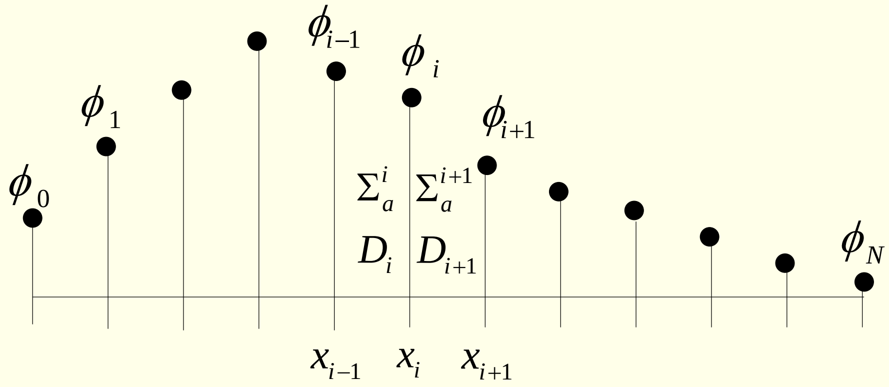


$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2 D\frac{d\phi}{dr}\right) = -\frac{d}{dr}\left(D\frac{d\phi}{dr}\right) - \frac{2}{r}D\frac{d\phi}{dr} \quad = \frac{dJ}{dr} + 2\frac{J}{r}$$

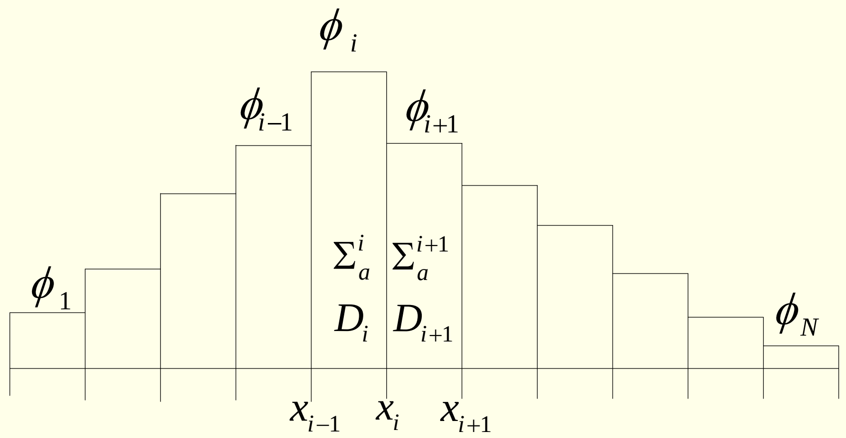
$$\nabla_r^2 = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)$$

Discetization

- Point scheme



- Box scheme scheme

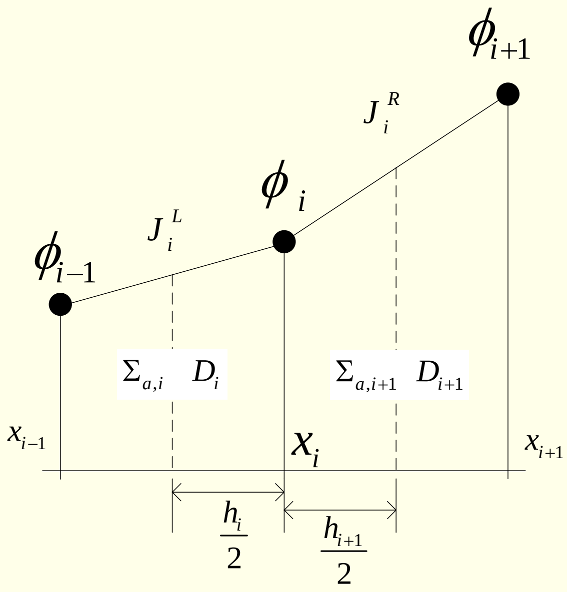


Discretization with Point Scheme

$$-\frac{d}{dx}\left(D(x)\frac{d\phi(x)}{dx}\right)+\Sigma_a(x)\phi(x)=\lambda\cdot\nu\Sigma_f(x)\phi(x)$$

$$\frac{dJ}{dx}+\Sigma_a\ \phi=\lambda\cdot\nu\Sigma_f\phi$$

$$\frac{dJ}{dx}\cong\frac{J_i^R-J_i^L}{\Delta x_i}=\frac{J_i^R-J_i^L}{\frac{1}{2}h_i+\frac{1}{2}h_{i+1}}$$



$$\begin{cases} J_i^L=-D_i\dfrac{\phi_i-\phi_{i-1}}{h_i} \\ J_i^R=-D_{i+1}\dfrac{\phi_{i+1}-\phi_i}{h_{i+1}} \end{cases}$$

$$a)\ \frac{dJ}{dx}=\frac{-D_{i+1}\dfrac{\phi_{i+1}-\phi_i}{h_{i+1}}-\left(-D_i\dfrac{\phi_i-\phi_{i-1}}{h_i}\right)}{\frac{1}{2}h_i+\frac{1}{2}h_{i+1}}=\frac{2}{h_i+h_{i+1}}\left\{-D_{i+1}\frac{\phi_{i+1}-\phi_i}{h_{i+1}}+D_i\frac{\phi_i-\phi_{i-1}}{h_i}\right\}$$

$$b)\ \overline{\Sigma}_{a,i}=\frac{h_i\Sigma_{a,i}+h_{i+1}\Sigma_{a,i+1}}{h_i+h_{i+1}}$$

$$c)\ \overline{\nu\Sigma}_{f,i}=\frac{h_i\nu\Sigma_{f,i}+h_{i+1}\nu\Sigma_{f,i+1}}{h_i+h_{i+1}}$$

Discretization with Point Scheme

$$\frac{dJ}{dx} = \frac{2}{h_i + h_{i+1}} \left\{ -D_{i+1} \frac{\phi_{i+1} - \phi_i}{h_{i+1}} + D_i \frac{\phi_i - \phi_{i-1}}{h_i} \right\}$$

– if $h_i = h_{i+1} = h$

$$a)\ \frac{dJ}{dx} = \frac{1}{h}\left(-D_{i+1}\frac{\phi_{i+1}-\phi_i}{h} + D_i \frac{\phi_i - \phi_{i-1}}{h} \right)$$

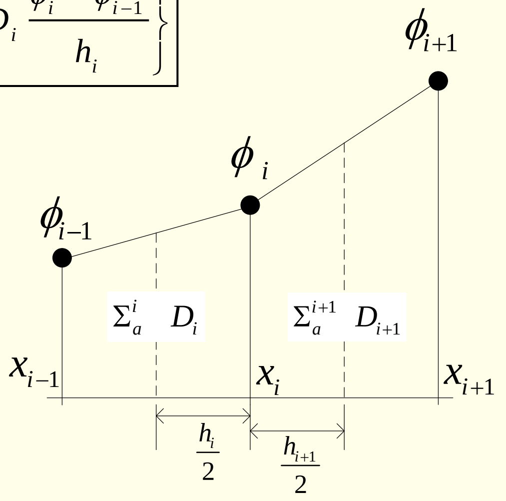
$$= -\frac{D_{i+1}}{h^2}(\phi_{i+1} - \phi_i) + \frac{D_i}{h^2}(\phi_i - \phi_{i-1})$$

$$= -\Sigma_{i+1}^D(\phi_{i+1} - \phi_i) + \Sigma_i^D(\phi_i - \phi_{i-1})$$

$$= -\Sigma_{i-1}^D \phi_{i-1} + \left(\Sigma_{i+1}^D + \Sigma_i^D\right)\phi_i - \Sigma_{i+1}^D \phi_{i+1}$$

$$\otimes\ \frac{D\ [cm]}{h^2\ [cm^2]} = \Sigma^D [cm^{-1}]$$

Diffusion Xsec!



$$b)\ \overline{\Sigma}_{a,i} = \frac{\Sigma_{a,i} + \Sigma_{a,i+1}}{2}$$

$$c)\ \overline{\nu\Sigma}_{f,i} = \frac{\nu\Sigma_{f,i} + \nu\Sigma_{f,i+1}}{2}$$

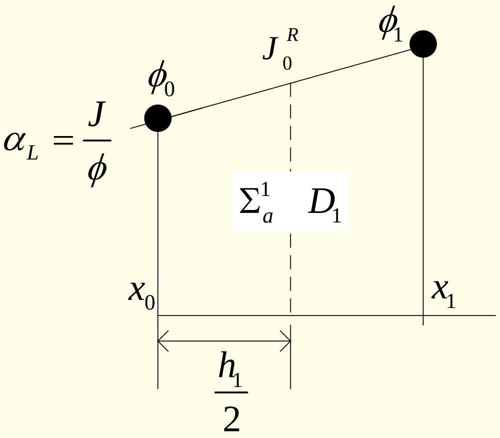
– At $x = x_i$

$$\frac{dJ}{dx} + \Sigma_a \phi = \lambda \cdot \nu\Sigma_f \phi \ \rightarrow \ -\Sigma_{i-1}^D \phi_{i-1} + \left(\Sigma_{i+1}^D + \Sigma_i^D\right)\phi_i - \Sigma_{i+1}^D \phi_{i+1} + \overline{\Sigma}_{a,i}\phi_i = \lambda \cdot \overline{\nu\Sigma}_{f,i}\phi_i$$

$$-\Sigma_{i-1}^D \phi_{i-1} + \left(\Sigma_{i+1}^D + \Sigma_i^D + \overline{\Sigma}_{a,i}\right)\phi_i - \Sigma_{i+1}^D \phi_{i+1} = \lambda \cdot \overline{\nu\Sigma}_{f,i}\phi_i$$

Boundary Condition with Point Scheme

– At left boundary $(x = 0)$



$$J_0 = -\alpha_L \phi_0 \qquad J_0^R = -D_1 \frac{\phi_1 - \phi_0}{h_1}$$

$$\frac{dJ}{dx} = \frac{J_0^R - J_0}{\frac{h_1}{2}} = \frac{-D_1 \frac{\phi_1 - \phi_0}{h_1} - (-\alpha_L \phi_0)}{\frac{h_1}{2}} = \left(\frac{2D_1}{h_1^2} + \frac{2\alpha_L}{h_1} \right) \phi_0 - \frac{2D_1}{h_1^2} \phi_1$$

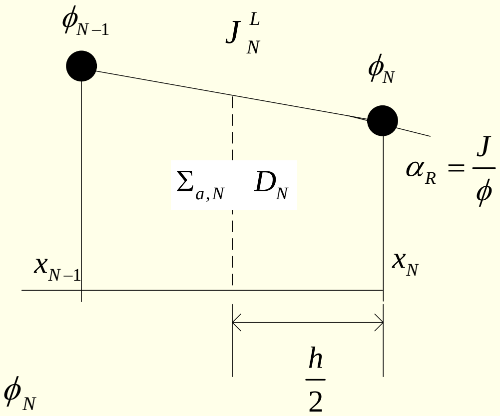
$$= \left(2\Sigma_1^D + \frac{2\alpha_L}{h_1} \right) \phi_0 - 2\Sigma_1^D \phi_1$$

$$\frac{dJ}{dx} + \Sigma_a \phi = \lambda \cdot \nu \Sigma_f \phi \ \rightarrow \ \left(2\Sigma_D^1 + \frac{2\alpha_L}{h_1} \right) \phi_0 - 2\Sigma_1^D \phi_1 + \Sigma_{a,1} \phi_0 = \lambda \cdot \nu \Sigma_{f,1} \phi_0$$

$$\left(\Sigma_1^D + \frac{\alpha_L}{h_1} + \frac{1}{2} \Sigma_{a,1} \right) \phi_0 - \Sigma_1^D \phi_1 = \lambda \cdot \frac{1}{2} \nu \Sigma_{f,1} \phi_0$$

Boundary Condition with Point Scheme

– At right boundary $(x = H)$



$$J_N = \alpha_R \phi_N \qquad J_N^L = -D_N \frac{\phi_N - \phi_{N-1}}{h_N}$$

$$\frac{dJ}{dx} = \frac{J_N - J_N^L}{\frac{h_N}{2}} = \frac{\alpha_R \phi_N - \left(-D_N \frac{\phi_N - \phi_{N-1}}{h_N} \right)}{\frac{h_N}{2}} = -\frac{2D_N}{h_N^2}\phi_{N-1} + \left(\frac{2D_N}{h_N^2} + \frac{2\alpha_R}{h_N} \right)\phi_N$$

$$= -2\Sigma_D^N \phi_{N-1} + \left(2\Sigma_D^N + \frac{2\alpha_R}{h_N} \right)\phi_N \qquad cf: \left(2\Sigma_1^D + \frac{2\alpha_L}{h_1} \right)\phi_0 - 2\Sigma_1^D \phi_1 \rightarrow \text{same as left}$$

$$\frac{dJ}{dx} + \Sigma_a \phi = \lambda \cdot \nu\Sigma_f \phi \rightarrow -2\Sigma_D^N \phi_{N-1} + \left(2\Sigma_D^N + \frac{2\alpha_R}{h_N} \right)\phi_N + \Sigma_{a,N}\phi_N = \lambda \cdot \nu\Sigma_{f,N}\phi_N$$

$$-\Sigma_D^N \phi_{N-1} + \left(\Sigma_D^N + \frac{\alpha_R}{h_N} + \frac{1}{2}\Sigma_{a,N} \right)\phi_N = \lambda \cdot \frac{1}{2}\nu\Sigma_{f,N}\phi_N$$

Linear System for Point Scheme

- Point scheme – Result

$$\left(\Sigma_1^D + \frac{\alpha_L}{h_1} + \frac{1}{2}\Sigma_{a,1}\right)\phi_0 - \Sigma_1^D\phi_1 = \lambda\cdot\frac{1}{2}\nu\Sigma_{f,1}\phi_0 \quad (x=0)$$

$$-\Sigma_{i-1}^D\phi_{i-1} + \left(\Sigma_{i+1}^D + \Sigma_i^D + \overline{\Sigma}_{a,i}\right)\phi_i - \Sigma_{i+1}^D\phi_{i+1} = \lambda\cdot\nu\overline{\Sigma}_{f,i}\phi_i \quad (x_1 \sim x_{N-1})$$

$$-\Sigma_D^N\phi_{N-1} + \left(\Sigma_D^N + \frac{\alpha_R}{h_N} + \frac{1}{2}\Sigma_{a,N}\right)\phi_N = \lambda\cdot\frac{1}{2}\nu\Sigma_{f,N}\phi_N \quad (x=H)$$

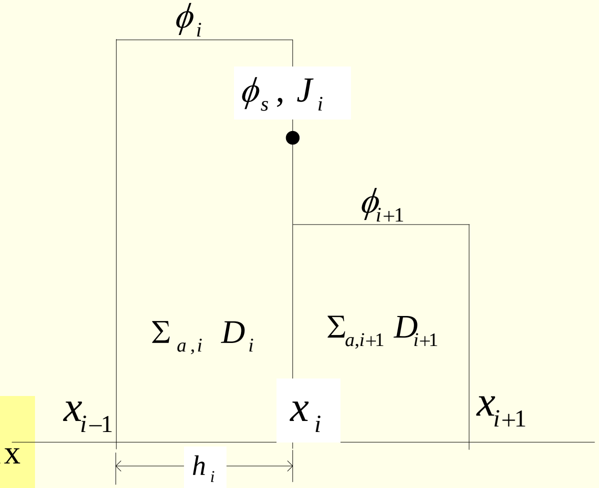
Matrix Form of Tridiagonal Linear System

$$
\begin{bmatrix}
\Sigma_1^D + \frac{\alpha_L}{h_1} + \frac{1}{2}\Sigma_{a,1} & -\Sigma_1^D & 0 & & & & 0 \\
-\Sigma_1^D & \Sigma_1^D + \Sigma_2^D + \overline{\Sigma}_{a,1} & -\Sigma_2^D & 0 & & 0 & \\
0 & \ddots & \ddots & \ddots & 0 & & \\
 & 0 & \ddots & \ddots & \ddots & 0 & \\
 & & 0 & \ddots & \ddots & \ddots & 0 \\
 & 0 & & 0 & -\Sigma_{N-1}^D & \Sigma_{N-1}^D + \Sigma_N^D + \overline{\Sigma}_{a,N-1} & -\Sigma_N^D \\
0 & & & & 0 & -\Sigma_N^D & \Sigma_N^D + \frac{\alpha_R}{h_N} + \frac{1}{2}\Sigma_{a,N}
\end{bmatrix}
\begin{bmatrix}
\phi_0 \\ \phi_1 \\ \vdots \\ \vdots \\ \vdots \\ \phi_{N-1} \\ \phi_N
\end{bmatrix}
= \lambda
\begin{bmatrix}
\frac{1}{2}\nu\Sigma_{f,1} & & & & & & \\
 & \overline{\nu\Sigma}_{f,1} & & & & & \\
 & & \ddots & & & & \\
 & & & \ddots & & & \\
 & & & & \ddots & & \\
 & & & & & \overline{\nu\Sigma}_{f,N-1} & \\
 & & & & & & \frac{1}{2}\nu\Sigma_{f,N}
\end{bmatrix}
\begin{bmatrix}
\phi_0 \\ \phi_1 \\ \vdots \\ \vdots \\ \vdots \\ \phi_{N-1} \\ \phi_N
\end{bmatrix}
$$

Box Scheme

- To derive conservation form

$$\frac{dJ}{dx} + \Sigma_a \phi = \lambda \cdot \nu\Sigma_f \phi \quad \cdots \quad (1)$$



– Integrate (1) over $[x_{i-1}, x_i]$

$$A = \int_{x_{i-1}}^{x_i} \frac{dJ}{dx} dx = J_i - J_{i-1}$$

$$B = \int_{x_{i-1}}^{x_i} \Sigma_{a,i} \phi(x) dx = \Sigma_{a,i} \cdot \bar{\phi}_i \cdot h_i$$

$$\leftarrow \bar{\phi}_i \equiv \frac{1}{h_i} \int_{x_{i-1}}^{x_i} \phi(x) dx : \text{average flux}$$

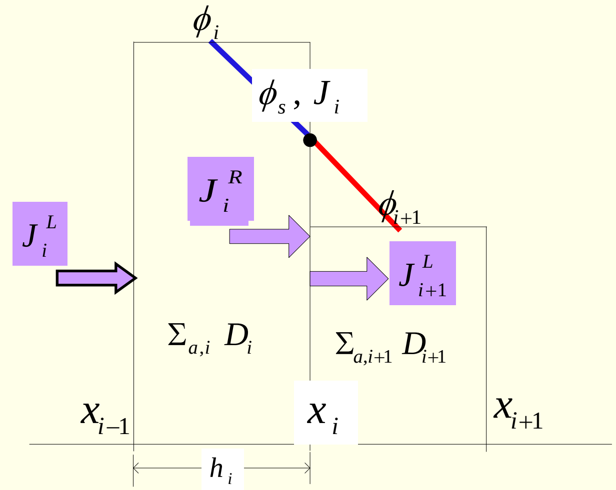
$$C = \int_{x_{i-1}}^{x_i} \lambda \cdot \nu\Sigma_{f,i} \phi(x) dx = \lambda \cdot \nu\Sigma_{f,i} \cdot \bar{\phi}_i \cdot h_i$$

$$J_i - J_{i-1} + \Sigma_{a,i} \cdot \bar{\phi}_i \cdot h_i = \lambda \cdot \nu\Sigma_{f,i} \cdot \bar{\phi}_i \cdot h_i \quad \leftarrow \text{Mesh Balance Equation}$$

$$\bar{\phi} = \phi \quad (\text{drop the overbar from now on})$$

Current in Box Scheme

– Current in terms of surface flux



$$J_i^R = -D_i \frac{\phi_s - \phi_i}{\frac{h_i}{2}} \qquad J_{i+1}^L = -D_{i+1} \frac{\phi_{i+1} - \phi_s}{\frac{h_{i+1}}{2}}$$

$$J_i^R = J_{i+1}^L$$

$$-D_i \frac{\phi_s - \phi_i}{\frac{h_i}{2}} = -D_i \frac{\phi_{i+1} - \phi_s}{\frac{h_{i+1}}{2}}$$

$$\phi_s = \frac{\frac{D_i}{h_i}\phi_i + \frac{D_{i+1}}{h_{i+1}}\phi_{i+1}}{\frac{D_i}{h_i} + \frac{D_{i+1}}{h_{i+1}}} = \omega_i\ \phi_i + (1-\omega_i)\ \phi_{i+1}$$

$$\beta_i \leftarrow \text{Diffusivity}$$

$$\leftarrow \omega_i = \frac{\frac{D_i}{h_i}}{\frac{D_i}{h_i} + \frac{D_{i+1}}{h_{i+1}}} = \frac{\beta_i}{\beta_i + \beta_{i+1}}$$

$$J_{i+1}^L = J_i^R = -D_i \frac{\omega_i\ \phi_i + (1-\omega_i)\ \phi_{i+1} - \phi_i}{h_i / 2} = -2\beta_i (1-\omega_i)\ (\ \phi_{i+1} - \phi_i)$$

$$= -\frac{2\beta_i\beta_{i+1}}{\beta_i + \beta_{i+1}}(\phi_{i+1} - \phi_i) = -\tilde{D}_i (\phi_{i+1} - \phi_i)$$

Coupling Coefficient

$$\tilde{D}_i = \frac{2\beta_i\beta_{i+1}}{\beta_i + \beta_{i+1}}$$

(Surface Quantity)

Mesh Balance Equation

– In the i-th node, $x_{i-1} \le x \le x_i$

$$J_i = -\tilde{D}_i(\phi_{i+1} - \phi_i) \quad \left(\tilde{D}_i = \frac{2\beta_i\beta_{i+1}}{\beta_i + \beta_{i+1}} \right)$$

$$J_{i-1} = -\tilde{D}_{i-1}(\phi_i - \phi_{i-1}) \quad \left(\tilde{D}_{i-1} = \frac{2\beta_{i-1}\beta_i}{\beta_{i-1} + \beta_i} \right)$$

$$J_i - J_{i-1} + \Sigma_{a,i} \cdot \phi_i \cdot h_i = \lambda \cdot \nu\Sigma_{f,i} \cdot \phi_i \cdot h_i$$

$$-\tilde{D}_i(\phi_{i+1} - \phi_i) + \tilde{D}_{i-1}(\phi_i - \phi_{i-1}) + \Sigma_a \cdot \phi_i \cdot h_i = \lambda \cdot \nu\Sigma_f \cdot \phi_i \cdot h_i$$

$$-\tilde{D}_{i-1}\phi_{i-1} + (\tilde{D}_{i-1} + \tilde{D}_i + \Sigma_a \cdot h_i)\phi_i - \tilde{D}_i\phi_{i+1} = \lambda \cdot \nu\Sigma_f \cdot h_i \cdot \phi_i$$

Boundary Condition with Box Scheme

– At left boundary

$$J_0 = -\alpha_L \phi_s = -D_1 \frac{\phi_1 - \phi_s}{\frac{h_1}{2}} \quad \rightarrow \quad \phi_s = \frac{\frac{D_1}{h_1}}{\frac{D_1}{h_1} + \frac{\alpha_L}{2}} \phi_1 = \frac{\beta_1}{\beta_1 + \underbrace{\frac{\alpha_L}{2}}_{\beta_0}} \phi_1 = \frac{\beta_1}{\beta_0 + \beta_1} \phi_1$$

$$J_0 = -\alpha_L \frac{\beta_1}{\beta_0 + \beta_1} \phi_1 = -\frac{2\frac{\alpha_L}{2}\beta_1}{\beta_0 + \beta_1} \phi_1 = -\tilde{D}_0 \phi_1$$

$$\tilde{D}_0 = \frac{2\beta_0\beta_1}{\beta_0 + \beta_1} \text{ with } \beta_0 = \frac{\alpha_L}{2}$$

$$J_1 - J_0 + \Sigma_a \cdot \phi_1 \cdot h_1 = \lambda \cdot \nu\Sigma_f \cdot \phi_1 \cdot h_1$$

$$-\tilde{D}_1(\phi_2 - \phi_1) + \tilde{D}_0\phi_1 + \Sigma_{a,1} \cdot \phi_1 \cdot h_1 = \lambda \cdot \nu\Sigma_f \cdot h_1 \cdot \phi_1$$

$$(\tilde{D}_0 + \tilde{D}_1 + \Sigma_{a,1} \cdot h_1)\phi_1 - \tilde{D}_1\phi_2 = \lambda \cdot \nu\Sigma_{f,1} \cdot h_1 \cdot \phi_1$$

– At Left Boundary $(x = 0)$

$$-\tilde{D}_{N-1}\phi_{N-1} + (\tilde{D}_{N-1} + \tilde{D}_N + \Sigma_{a,N} \cdot h_N)\phi_N = \lambda \cdot \nu\Sigma_{f,N} \cdot h_N \cdot \phi_N$$

– At Right Boundary $(x = H)$

Tridiagonal Linear System for Box Scheme

- Box scheme – Result

$$(\tilde{D}_0 + \tilde{D}_1 + \Sigma_{a,1} \cdot h_1)\phi_1 \quad - \tilde{D}_1\phi_2 \qquad = \lambda \cdot \nu\Sigma_{f,1} \cdot h_1 \cdot \phi_1 \quad (x_0)$$

$$-\tilde{D}_{i-1}\phi_{i-1} \quad + (\tilde{D}_{i-1} + \tilde{D}_i + \Sigma_a \cdot h_i)\phi_i \quad - \tilde{D}_i\phi_{i+1} \qquad = \lambda \cdot \nu\Sigma_f \cdot h_i \cdot \phi_i \quad (x_1 \sim x_{N-1})$$

$$-\tilde{D}_{N-1}\phi_{N-1} \quad + (\tilde{D}_{N-1} + \tilde{D}_N + \Sigma_{a,N} \cdot h_N)\phi_N \quad = \lambda \cdot \nu\Sigma_{f,N} \cdot h_N \cdot \phi_N \quad (x_N)$$

$$\begin{bmatrix} d_1 & u_1 & 0 & & & & 0 \\ l_2 & d_2 & u_2 & 0 & & 0 & \\ 0 & \ddots & \ddots & \ddots & 0 & & \\ & & l_i & d_i & u_i & 0 & \\ & & 0 & l_{l+1} & \ddots & \ddots & 0 \\ & 0 & & 0 & l_{N-1} & d_{N-1} & u_{N-1} \\ 0 & & & & 0 & l_N & d_N \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \vdots \\ \vdots \\ \phi_{N-1} \\ \phi_N \end{bmatrix} = \lambda \begin{bmatrix} \nu\Sigma_{f,1} & 0 & & & & & 0 \\ 0 & \nu\Sigma_{f,2} & 0 & & & 0 & \\ & 0 & \ddots & 0 & & & \\ & & 0 & \ddots & 0 & & \\ & & & 0 & \ddots & 0 & \\ & 0 & & & 0 & \nu\Sigma_{f,N-1} & 0 \\ 0 & & & & & 0 & \nu\Sigma_{f,N} \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \vdots \\ \vdots \\ \phi_{N-1} \\ \phi_N \end{bmatrix}$$

$$M \ \Phi = \lambda \ F \ \Phi$$

$$d_i = \tilde{D}_{i-1} + \tilde{D}_i + \Sigma_{a,i} \cdot h_i$$

$$u_i = -\tilde{D}_i$$

$$l_i = -\tilde{D}_{i-1}$$

$$l_{i+1} = u_i = -\tilde{D}_{i-1} \quad \Rightarrow \quad \text{Symmetric} \ \left(M = M^T\right)$$

1. *Generalized Form*
   – Same form every where
2. *Conservation Form*
   – Nodal balance assured because of starting from integration

Analytic Solution for Matrix Eigenvalue Problem

- For a single region

$$-D\frac{d^2\phi}{dx^2}+\Sigma_a\phi=\lambda\cdot\nu\Sigma_f\phi$$

– Point scheme with $\phi_0=0,\phi_N=0$

$$-\Sigma_{i-1}^D\phi_{i-1}+\left(\Sigma_{i+1}^D+\Sigma_i^D+\overline{\Sigma}_{a,i}\right)\phi_i-\Sigma_{i+1}^D\phi_{i+1}=\lambda\cdot\overline{\nu\Sigma}_{f,i}\phi_i$$

$$L=\sqrt{\frac{D}{\Sigma_a}}:\ \text{Diffusion Length}$$

–if $\Sigma_i^D=\Sigma_{i+1}^D=\dfrac{D}{h^2},\quad \Sigma_{a,i}=\Sigma_{a,i+1}=\Sigma_a,\quad \nu\Sigma_{f,i}=\nu\Sigma_{f,i+1}=\nu\Sigma_f$

$$-\frac{D}{h^2}\left(\phi_{k-1}-2\phi_k+\phi_{k+1}\right)+\Sigma_a\phi_k=\lambda\cdot\nu\Sigma_f\phi_k$$

$$\phi_{k-1}-2\phi_k+\phi_{k+1}-\frac{h^2}{D}\left(\Sigma_a-\lambda\cdot\nu\Sigma_f\right)\phi_k=0$$

$$\phi_{k-1}-2\phi_k+\phi_{k+1}+\tilde{B}^2\phi_k=0$$

1) $$-\frac{h^2}{D}\left(\Sigma_a-\lambda\cdot\nu\Sigma_f\right)$$

$$=-h^2\frac{\Sigma_a}{D}\left(1-\lambda\cdot\frac{\nu\Sigma_f}{\Sigma_a}\right)$$

$$=-h^2\frac{1}{L^2}\left(1-\frac{1}{k_{eff}}\cdot k_\infty\right)$$

$$=h^2B^2$$

$$=\tilde{B}^2$$

2) $$k_{eff}=\frac{k_\infty}{1+L^2B^2}$$

$$B^2=\frac{1}{L^2}\left(\frac{k_\infty}{k_{eff}}-1\right)>0\ \because\ k_\infty>k_{eff}$$

or

$$=\frac{\lambda\cdot\nu\Sigma_f-\Sigma_a}{D}$$

Analytic Solution for Matrix Eigenvalue Problem

$$\phi_{k-1} - 2\phi_k + \phi_{k+1} + \tilde{B}^2 \phi_k = 0$$

– Let $\phi_k = e^{mk}$

$$e^{m(k-1)} - 2e^{mk} + e^{m(k+1)} + \tilde{B}^2 e^{mk} = 0$$

$$e^{-m} - 2 + e^{m} + \tilde{B}^2 = 0$$

$$\frac{e^{m} + e^{-m}}{2} = 1 - \frac{\tilde{B}^2}{2}$$

$$\text{Cosh } m(\geq 1) = 1 - \frac{\tilde{B}^2}{2}(<1) \qquad \text{Not possible}$$

– Let $\phi_k = e^{imk}$

$$\frac{e^{im} + e^{-im}}{2} = 1 - \frac{\tilde{B}^2}{2}$$

$$\cos m = 1 - \frac{\tilde{B}^2}{2}$$

$\phi_k = e^{-imk}$ as well

$$\phi_k = C_1 e^{imk} + C_2 e^{-imk} = \tilde{C}_1 \cos mk + \tilde{C}_2 \sin mk$$

Buckling Eigenvalue

$$\phi_k = \tilde{C}_1 \cos mk + \tilde{C}_2 \sin mk$$

$$\text{B.C}\begin{cases} \phi_0 = \tilde{C}_1 = 0 \\ \phi_N = \tilde{C}_2 \sin mN = 0, \qquad mN = l\pi \quad (l = 1, \ldots, n-1) \end{cases} \qquad \rightarrow \quad m = \frac{l\pi}{N}$$

$$\text{Fundamental Model Only } (l = 1); \quad \cos\left(\frac{\pi}{N}\right) = 1 - \frac{\tilde{B}^2}{2}$$

$$\leftarrow \cos m = 1 - \frac{\tilde{B}^2}{2}$$

$$\tilde{B}^2 = 2\left[1 - \cos\left(\frac{\pi}{N}\right)\right] = 2\left[1 - \left\{1 - \frac{1}{2}\left(\frac{\pi}{N}\right)^2 + \frac{1}{4!}\left(\frac{\pi}{N}\right)^4\right\} + \ldots.\right]$$

$$= \left(\frac{\pi}{N}\right)^2 \left(1 - \frac{1}{12}\left(\frac{\pi}{N}\right)^2\right) + O\left(\frac{\pi}{N}\right)^6$$

$$N = \frac{H}{h}, \quad \tilde{B} = hB$$

$$= \left(\frac{\pi}{H}\right)^2 h^2 \left(1 - \frac{1}{12}\left(\frac{\pi}{H}h\right)^2\right) + O(h^6) \qquad \rightarrow B^2 = \left(\frac{\pi}{H}\right)^2 \left(1 - \frac{1}{12}\left(\frac{\pi}{H}h\right)^2\right) + O(h^4)$$

Eigenvalue

$$B^2 = \frac{\lambda \cdot \nu\Sigma_f - \Sigma_a}{D} = \left(\frac{\pi}{H}\right)^2 \left(1 - \frac{1}{12}\left(\frac{\pi}{H}h\right)^2\right) + O(h^4)$$

$$\rightarrow \lambda = \frac{1}{\nu\Sigma_f}\left[\Sigma_a + D\left\{\left(\frac{\pi}{H}\right)^2 \left(1 - \frac{1}{12}\left(\frac{\pi}{H}h\right)^2\right) + O(h^4)\right\}\right]$$

– Analytic Solution for differential eqn.

$$B^2 = \left(\frac{\pi}{H}\right)^2 \quad vs \quad \left(\frac{\pi}{H}\right)^2 - \frac{1}{12}\left(\frac{\pi}{H}\right)^4 h^2$$

$$\lambda^* = \frac{1}{\nu\Sigma_f}\left(\Sigma_a + D\left(\frac{\pi}{H}\right)^2\right)$$

$$Error \quad \Delta\lambda = \frac{1}{\nu\Sigma_f} \cdot \frac{D}{12}\left(\frac{\pi}{H}\right)^4 h^2 \propto h^2$$