Lecture Note 3

Chebyshev Acceleration

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Chebyshev Acceleration Method

- Example of Large Dominance Ratio Cases (Causing Slow Convergence of Power Method)
  - Consider an eigenvalue problem in 1-D particle diffusion

$$-D\frac{d^2\phi}{dx^2}+\Sigma\phi=\lambda\nu\Sigma_f\phi\ ,\quad x\in[0,a],\quad \phi(0)=0,\quad \phi(a)=0$$

  - Discretization would lead to $A\phi=\lambda\phi$ $\rightarrow$ Eigenvalue of the diff. eqn. being the same as that of matrix equation
  - Rearrange after dividing by D, $\dfrac{d^2\phi}{dx^2}+\dfrac{\lambda\nu\Sigma_f-\Sigma}{D}\phi=0$
  - Let $B^2=\dfrac{\lambda\nu\Sigma_f-\Sigma}{D}\rightarrow\dfrac{d^2\phi}{dx^2}+B^2\phi=0$ $\rightarrow$ 0 Flux Boundary Condition $\rightarrow$ $B_n=\dfrac{n\pi}{a}$
  - Eigenvalue $\lambda_n=\dfrac{\Sigma+DB_n^2}{\nu\Sigma_f}=\dfrac{\Sigma+\dfrac{n^2\pi^2}{a^2}}{\nu\Sigma_f}$
  - As $a\rightarrow\infty,\ \sigma\rightarrow1.0$ (or $\Sigma\uparrow$ or $D\downarrow$)

▣ Dominance Ratio $\sigma=\dfrac{\dfrac{1}{\lambda_2}}{\dfrac{1}{\lambda_1}}=\dfrac{\Sigma+D\dfrac{\pi^2}{a^2}}{\Sigma+D\dfrac{4\pi^2}{a^2}}$ for smallest eigenvalue

- Diffusion problems for large domain have large dominance ratio

  $\rightarrow$ Slow convergence of power method

Properties of Chebyshev Polynomial (1/2)

- Chebyshev polynomial

$$T_m(x) = \cos(m\cos^{-1} x)$$
$$= \cos(m\theta)$$

$$\theta = \cos^{-1} x \quad x \in [-1,1] \quad \Leftrightarrow \quad x = \cos\theta \quad \theta \in [-\pi, 0]$$

- Examples of Chebyshev polynomial

$$T_0(x) = \cos(0) = 1$$

$$T_1(x) = \cos(\cos^{-1} x) = x$$

$$T_2(x) = \cos(2\cos^{-1} x) = ? \qquad T_2(x) = 2xT_1(x) - T_0(x) = 2x^2 - 1$$

$$T_3(x) = \cos(3\cos^{-1} x) = ? \qquad T_3(x) = 2xT_2(x) - T_1(x) = 2x(2x^2-1) - x = 4x^3 - 3x$$

- Recurrence Relation

$$\begin{array}{r|l} & T_{m+1}(x) = \cos\left((m+1)\theta\right) = \cos m\theta\cos\theta - \sin m\theta\sin\theta \\ + & T_{m-1}(x) = \cos\left((m-1)\theta\right) = \cos m\theta\cos\theta + \sin m\theta\sin\theta \\ \hline & \end{array}$$

$$T_{m+1}(x) + T_{m-1}(x) = 2\cos m\theta\cos\theta = 2xT_m(x)$$

$$\therefore \quad T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x)$$

Properties of Chebyshev Polynomial (2/2)

$$T_m(x) = \cos(m\theta)$$

- m Roots in $[-1,1]$

$$m\theta_{mk} = -\left(\frac{\pi}{2} + k\pi\right) \quad \text{for} \quad k = 0..(m-1)$$

- Maximum Absolute Value $= 1.0$

$$x_{mk} = \cos\theta_{mk} = \cos\left(-\left(\frac{1}{2} + k\right)\frac{\pi}{m}\right)$$

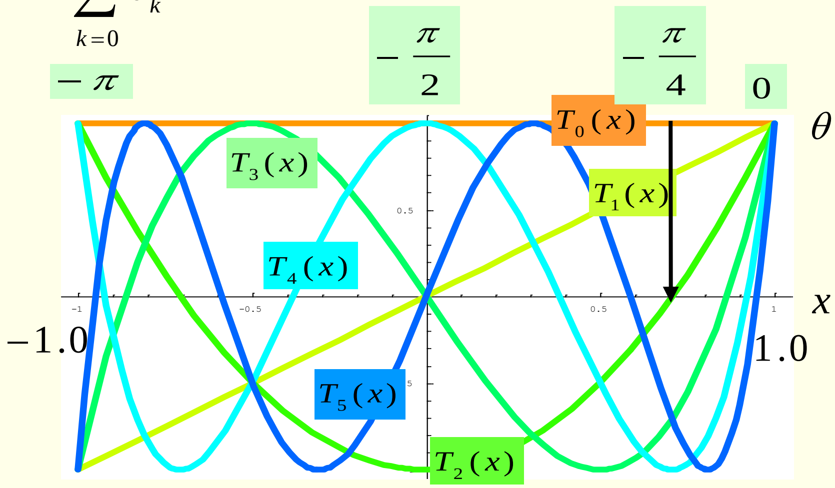
- $(\mathrm{m}+1)$ maximum values of 1.0 for $|T_m(x)|$ with alternating sign

- Coefficient of the Highest order term : 2^{m-1}

$\rightarrow$ Normalized Chebyshev $\phi_m^T(x) = \frac{1}{2^{m-1}} T_m(x) = x^m + \sum_{k=0}^{m-1} c_k x^k$

- Orthogonality

$$\int_{-1}^{1} \frac{T_m(x) T_n(x)}{\sqrt{1-x^2}} dx = 0 \quad \text{for } m \neq n$$



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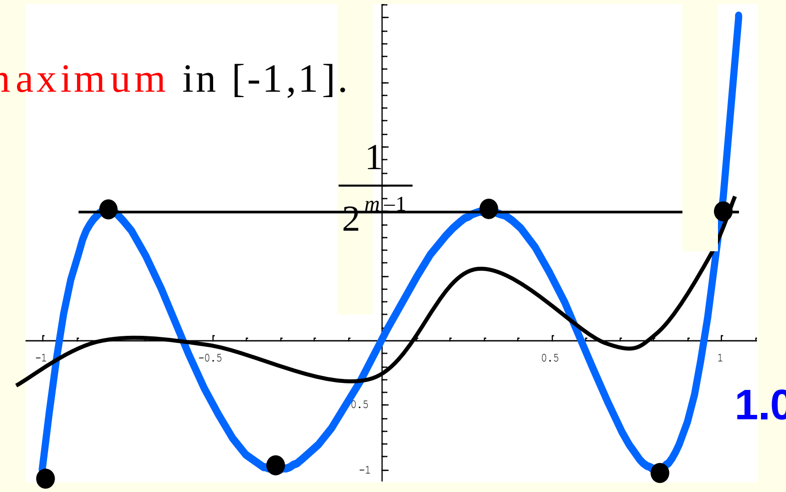
Significance of Chebyshev Polynomial (1/2)

- Among the various normalized polynomials of order m (coefficient of the m-th order term being 1.0), normalized Chebyshev Polynomial has the smallest maximum in [-1,1]. And the maximum is $\dfrac{1}{2^{m-1}}$



- Proof
  - Assume there is a polynomial of order m

$$\min_{-1\le x\le 1}(\max|p_m(x)|) = \|p_m(x)\|_\infty < \frac{1}{2^{m-1}}$$

  - Define $E_m(x) = \phi_m^T(x) - p_m(x)$ which becomes an (m-1)th order function(x^m *cancelled*)
  - Among the (m+1) maximal points, $E_m(x)$ is positive at the odd numbered points from the right ($x=1$) whereas negative at even numbered points $\leftarrow$ Assumption.
  - This means $E_m(x)$ has m zeros, or m-th order polynomial which is a contradiction. (not m -1 th order) $\rightarrow$ Chebyshev has the smallest maximum.

Significance of Chebyshev Polynomial (2/2)

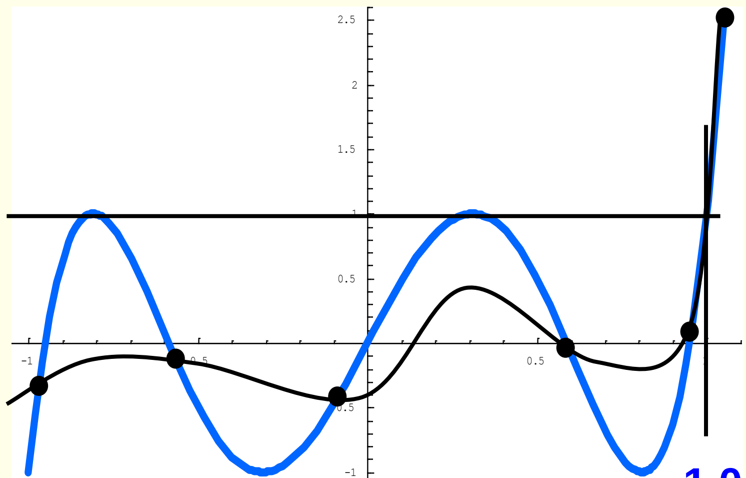
- Other interpretation: Among the various polynomials of order m passing through (1+a,b) (a,b>0), Chebyshev Polynomial has the smallest maxima in [-1,1]. And the maxima is 1.0.

$$x > 1, \quad T_m(x) = \cosh(m \cosh^{-1} x),$$

- Proof
  - Under the same assumption as the above, E(x) has m zeros in [-1,1] since there are m+1 alternating maxima, and one more zero at 1+a leading to a total m+1 zeros.

    $\rightarrow$ (m+1)th order polynomial

    $\rightarrow$ Contradiction



- Corollary: Among the $m-th$ order polynomials passing through $(1+a, 1.0)$,

$$\tilde{T}_m(x) = \frac{T_m(x)}{T_m(1+a)}$$ has the minimax in [-1,1].

Note : For any polynomial $p_m(x)$, $\tilde{p}_m(x) = \dfrac{p_m(x)}{p_m(1+a)}$ passes through $(1+a, 1.0)$

Chebyshev Acceleration Method

- Single Parameter Method
  - Extrapolation of eigenvector using the current estimate by power method and the previous iterate

$$x^{(k)} = \omega^{(k)} x_{Pow}^{(k)} + \left(1-\omega^{(k)}\right) x^{(k-1)} = \omega^{(k)} \frac{1}{\lambda^{(k-1)}} A x^{(k-1)} + \left(1-\omega^{(k)}\right) x^{(k-1)}$$

  - The extrapolation parameter is the single parameter and it is iteration dependent

- Two Parameter Method
  - Extrapolation using two previous iterates

$$x^{(k)} = \alpha^{(k)} x_{Pow}^{(k)} + \left(1-\alpha^{(k)}+\beta^{(k)}\right) x^{(k-1)} - \beta^{(k)} x^{(k-2)}$$

Two Parameter Chebyshev Acceleration (1/4)

- Drawback's of Single Parameter Method
  - The polynomial order (K) is preset
  - The extrapolation parameter becomes very large for large K

    $\rightarrow$ too much extrapolation
- Two Parameter Method

$$x^{(k)} = \alpha^{(k)} x_{Pow}^{(k)} + (1-\alpha^{(k)} + \beta^{(k)})x^{(k-1)} - \beta^{(k)} x^{(k-2)}$$

$$= \alpha^{(k)} \frac{1}{\lambda^{(k-1)}} A x^{(k-1)} + (1-\alpha^{(k)} + \beta^{(k)})x^{(k-1)} - \beta^{(k)} x^{(k-2)}$$

For $x^{(0)} = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$ and $\beta^{(1)} = 0$

$$x^{(1)} = \alpha^{(1)} \frac{1}{\lambda^{(0)}} A \sum_{i=1}^{n} c_i u_i + (1-\alpha^{(k)}) \sum_{i=1}^{n} c_i u_i$$

$$= \sum_{i=1}^{n} c_i \left(\alpha^{(1)} \frac{\lambda_i}{\lambda^{(0)}} + (1-\alpha^{(k)}) \right) u_i = \sum_{i=1}^{n} c_i p^{(1)}(\lambda_i) u_i$$ with a first order polynomial $p^{(1)}(\gamma)$

▷ Each iteration will increase the order of polynomial of eigenvalue by 1, which leads to a k-th order polynomial at the k-th step $\rightarrow$ Let $x^{(k)} = \sum_{i=1}^{n} c_i p^{(k)}(\lambda_i) u_i$

Two Parameter Chebyshev Acceleration (2/4)

Define $\gamma_i = 2\dfrac{\lambda_i}{\lambda_2} - 1$

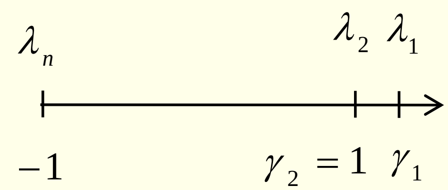
so that $\lambda_i = \dfrac{\gamma_i + 1}{2}\lambda_2$.

Normally, $\lambda_2 >> \lambda_n \sim 0$

$\lambda_i : \lambda_n \rightarrow \lambda_2$

$\gamma_i : -1 \rightarrow 1$

$$\gamma_1 = 2\frac{\lambda_1}{\lambda_2} - 1 = \frac{2}{\sigma} - 1 > 1$$



Then $x^{(k)} = \sum_{i=1}^{n} c_i p^{(k)}(\lambda_i) u_i = \sum_{i=1}^{n} c_i \eta^{(k)}(\gamma_i) u_i$

$$x^{(k)} = \eta^{(k)}(\gamma_1)\left(c_1 u_1 + \sum_{i=2}^{n} c_i u_i \frac{\eta^{(k)}(\gamma_i)}{\eta^{(k)}(\gamma_1)} \right)$$

can be minimum if $\eta^{(k)}(\gamma)$ is a Chebyshev polynomial of order k

But note that $\eta^{(k)}(\gamma_1)$ can increase indefinitely as k increases if it is an ordinary Chebyshev.

Thus consider a contant multiple (c_k) of Chebyshev polynomial to make $\eta^{(k)}(\gamma_1)$ constant.

$\dfrac{c_k T_k(\gamma_i)}{c_k T_k(\gamma_1)} = \tilde{T}_k(\gamma_i)$ still minimizes the error term.

$\eta^{(k)}(\gamma_1) = 1$ desired

$$\eta^{(k)}(\gamma_1) = c_k T_k(\gamma_1) = 1 \rightarrow c_k = \frac{1}{T_k(\gamma_1)}$$

$$\eta^{(k)}(\gamma) = \frac{T_k(\gamma)}{T_k(\gamma_1)} = \tilde{T}_k(\gamma)$$

Two Parameter Chebyshev Acceleration (3/4)

- Continued Derivation

$$x^{(k)} = \sum_{i=1}^{n} c_i u_i \left[\left(\alpha^{(k)} \frac{\lambda_i}{\lambda^{(k-1)}} + 1 - \alpha^{(k)} + \beta^{(k)} \right) \eta^{(k-1)}(\gamma_i) - \beta^{(k)} \eta^{(k-2)}(\gamma_i) \right] \qquad \leftarrow \lambda_i = \frac{1+\gamma_i}{2} \lambda_2$$

$$= \sum_{i=1}^{n} c_i u_i \left[\left(\alpha^{(k)} \frac{\lambda_2}{\lambda^{(k-1)}} \frac{1+\gamma_i}{2} + 1 - \alpha^{(k)} + \beta^{(k)} \right) \eta^{(k-1)}(\gamma_i) - \beta^{(k)} \eta^{(k-2)}(\gamma_i) \right] = \sum_{i=1}^{n} c_i u_i \eta^{(k)}(\gamma_i)$$

$$\eta^{(k)}(\gamma) = \left(\alpha^{(k)} \sigma \frac{1+\gamma}{2} + 1 - \alpha^{(k)} + \beta^{(k)} \right) \eta^{(k-1)}(\gamma) - \beta^{(k)} \eta^{(k-2)}(\gamma) \qquad \leftarrow \frac{\lambda_2}{\lambda^{(k-1)}} \simeq \frac{\lambda_2}{\lambda_1} = \sigma$$

– Choose $\alpha^{(k)}$ and $\beta^{(k)}$ such that the above recurrence relation be the same as $\tilde{T}_k(\gamma)$

Two Parameter Chebyshev Acceleration (4/4)

- Chebyshev Recurrence Relation for the normalized one

$$T_k(\gamma) = 2\gamma T_{k-1}(\gamma) - T_{k-2}(\gamma)$$

$$\tilde{T}_k(\gamma) = 2\gamma \frac{T_{k-1}(\gamma)}{T_k(\gamma_1)} - \frac{T_{k-2}(\gamma)}{T_k(\gamma_1)} = 2\gamma \frac{T_{k-1}(\gamma_1)}{T_k(\gamma_1)} \frac{T_{k-1}(\gamma)}{T_{k-1}(\gamma_1)} - \frac{T_{k-2}(\gamma_1)}{T_k(\gamma_1)} \frac{T_{k-2}(\gamma)}{T_{k-2}(\gamma_1)}$$

$$= 2\gamma \frac{T_{k-1}(\gamma_1)}{T_k(\gamma_1)} \tilde{T}_{k-1}(\gamma) - \frac{T_{k-2}(\gamma_1)}{T_k(\gamma_1)} \tilde{T}_{k-2}(\gamma)$$

$$\alpha^{(k)}\sigma \frac{1}{2} + 1 - \alpha^{(k)} + \beta^{(k)} = 0$$

$$\rightarrow \beta^{(k)} = (1 - \frac{\sigma}{2})\alpha^{(k)} - 1$$

Compare this with

$$\eta^{(k)}(\gamma) = \left(\alpha^{(k)}\sigma \frac{1+\gamma}{2} + 1 - \alpha^{(k)} + \beta^{(k)} \right) \eta^{(k-1)}(\gamma) - \beta^{(k)}\eta^{(k-2)}(\gamma)$$

$$\longrightarrow \begin{cases} \alpha^{(k)}\sigma \dfrac{1+\gamma}{2} + 1 - \alpha^{(k)} + \beta^{(k)} = 2\gamma \dfrac{T_{k-1}(\gamma_1)}{T_k(\gamma_1)} \\ \beta^{(k)} = \dfrac{T_{k-2}(\gamma_1)}{T_k(\gamma_1)} \end{cases}$$

$$\text{Coef. of } \gamma \rightarrow \alpha^{(k)} = \frac{4}{\sigma} \frac{T_{k-1}(\gamma_1)}{T_k(\gamma_1)}$$

$$\begin{cases} \gamma_1 = \dfrac{2}{\sigma} - 1 \\ T_k(\gamma_1) = 2\gamma_1 T_{k-1}(\gamma_1) - T_{k-2}(\gamma_1) \end{cases}$$

Equality of the contant term holds

Practical Estimation of σ

- Dominance Ratio σ
  - σ is important because σ is used for calculating extrapolation parameters

$$\sigma = \frac{\lambda_2}{\lambda_1} = \frac{\left\| x^{(l)} - x^{*} \right\|}{\left\| x^{(l-1)} - x^{*} \right\|} \quad \leftrightarrow \quad e^{(l)} = \sigma e^{(l-1)}$$

$$\begin{array}{rl} & x^{(l)} - x^{*} = \sigma \left(x^{(l-1)} - x^{*} \right) \\ - & x^{(l-1)} - x^{*} = \sigma \left(x^{(l-2)} - x^{*} \right) \\ \hline & x^{(l)} - x^{(l-1)} = \sigma \left(x^{(l-1)} - x^{(l-2)} \right) \end{array}$$

$\tilde{e}^{(l)} \equiv x^{(l)} - x^{(l-1)}$: pseudo error

$$\sigma = \frac{\left\| e^{(l)} \right\|}{\left\| e^{(l-1)} \right\|} = \frac{\left\| \tilde{e}^{(l)} \right\|}{\left\| \tilde{e}^{(l-1)} \right\|}$$

Not valid

if extrapolation performed

Error Reduction at l-th step of chebyshev (1/4)

- Theoretical Error Reduction Factor
  - let $x^{(0)} = \sum_{i=1}^{N} c_i u_i$

$\eta^{(l)}(\gamma)$: l-th order normalized chebyshev polynomial

x^* : Exact solution

$$x^{(l)} = \sum_{i=1}^{N} \left[\eta^{(l)}(\gamma_i) c_i u_i \right]$$

$$\begin{aligned}
e^{(l)} &= x^{(l)} - x^* \\
&= \sum_{i=1}^{N} \left[\eta^{(l)}(\gamma_i) c_i u_i \right] - x^* \\
&= \left[c_1 u_1 + \eta^{(l)}(\gamma_2) c_2 u_2 + O(\gamma_3) \right] - x^* \quad \leftarrow \eta^{(l)}(\gamma_1) = 1 \\
&= \eta^{(l)}(1) c_2 u_2 + O(\gamma_3) \quad \leftarrow \text{Let } x^* = c_1 u_1
\end{aligned}$$

$$\zeta_l^* = \frac{\left\| e^{(l)} \right\|}{\left\| e^{(0)} \right\|} \approx \frac{\eta^{(l)}(1) \left\| c_2 u_2 \right\|}{\left\| c_2 u_2 \right\|} = \eta^{(l)}(1) = \frac{1}{T_l(\gamma_1)}$$

$$\gamma_1 = \frac{2}{\sigma} - 1 \text{ with exact } \sigma$$

Error Reduction at l-th step of chebyshev (2/4)

- Error Reduction in Practice

$$x^{(l-1)} = \sum_{i=1}^{N} \left[\eta^{(l-1)}(\gamma_i) c_i u_i \right] = c_1 u_1 + \eta^{(l-1)}(1) c_2 u_2 + \cdots$$

$$\left(\because \eta^{(l-1)}(\gamma_1) = 1,\ \gamma_2 = 1,\ \frac{\lambda_2}{\lambda_1} = \sigma \right)$$

$$\begin{aligned} x_{Pow}^{(l)} &= A \frac{x^{(l-1)}}{\lambda_1} \\ &= \sum_{i=1}^{N} \left[\frac{\lambda_i}{\lambda_1} \eta^{(l-1)}(\gamma_i) c_i u_i \right] \\ &= \frac{\lambda_1}{\lambda_1} \eta^{(l-1)}(\gamma_1) c_1 u_1 + \frac{\lambda_2}{\lambda_1} \eta^{(l-1)}(\gamma_2) c_2 u_2 + \cdots \\ &= c_1 u_1 + \sigma \eta^{(l-1)}(1) c_2 u_2 + \cdots \end{aligned}$$

- Pseudo Error of the l-th Power Estimate

$$\begin{aligned} \tilde{e}_{Pow}^{(l)} &= x_{Pow}^{(l)} - x^{(l-1)} \\ &= \left[c_1 u_1 + \sigma \eta^{(l-1)}(1) c_2 u_2 + \cdots \right] - \left[c_1 u_1 + \eta^{(l-1)}(1) c_2 u_2 + \cdots \right] \\ &\approx (\sigma - 1) \eta^{(l-1)}(1) c_2 u_2 \end{aligned}$$

Error Reduction at l-th step of chebyshev (3/4)

- Error Reduction factor $\left(\text{practically} - \text{Conti.}\right)$

$$\tilde{e}^{(l)}_{Pow} = x^{(l)}_{Pow} - x^{(l-1)} \approx \left(\sigma - 1\right)\eta^{(l-1)}(1)c_2 u_2$$

$$\tilde{e}^{(1)}_{Pow} = x^{(1)}_{Pow} - x^{(0)} \approx \left(\sigma - 1\right)\eta^{(0)}(1)c_2 u_2$$

$$\eta^{(0)}(1) = 1$$

$$\zeta_l \equiv \frac{\left\|\tilde{e}^{(l)}_{Pow}\right\|}{\left\|\tilde{e}^{(1)}_{Pow}\right\|} = \frac{\left\|x^{(l)}_{Pow} - x^{(l-1)}\right\|}{\left\|x^{(1)}_{Pow} - x^{(0)}\right\|} = \frac{\left(\sigma - 1\right)\eta^{(l-1)}(1)\ \left\|c_2 u_2\right\|}{\left(\sigma - 1\right)\eta^{(0)}(1)\left\|c_2 u_2\right\|} = \eta^{(l-1)}(1) = \tilde{T}_{l-1}(1) = \frac{T_{l-1}(1)}{T_{l-1}(\gamma_1)} = \frac{1}{T_{l-1}(\gamma_1)}$$

$$= \zeta^*_{l-1} \quad \rightarrow \text{can be obtained during iteration without knowing } \sigma$$

* Error reduction in practice obtained after power iteration at the l-th step

can be the theoretical error reduction for (l-1) th step if all the parameters used (σ) are correct.

But in reality $\zeta_l < \zeta^*_{l-1}$ because faster decay of higher modes in early stage

where higher modes are present

Error Reduction at l-th step of chebyshev (4/4)

- New dominance ratio (σ') from chebyshev acceleration

Let $\zeta_l = \tilde{T}_{l-1}(\gamma_2') = \dfrac{T_{l-1}(\gamma_2')}{T_{l-1}(\gamma_1)}$ ← Error reduction factor practically obtained is a value corresponding to a $\gamma > 1$

$$T_{l-1}(\gamma_2') = \zeta_l T_{l-1}(\gamma_1) = \frac{\zeta_l}{\zeta_{l-1}^*} = \xi$$

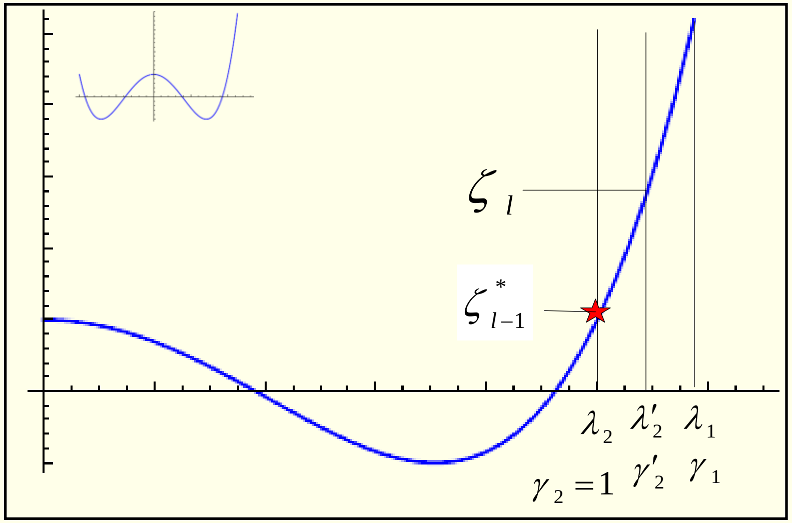
$$\cosh\left[(l-1)\cosh\gamma_2'\right] = \xi$$

$$\gamma_2' = \cosh\left(\frac{\cosh^{-1}\xi}{l-1}\right)$$

$$2\frac{\lambda_2'}{\lambda_2} - 1 = \gamma_2'$$

$$\lambda_2' = \lambda_2 \frac{1+\gamma_2'}{2}$$

$$\sigma' = \sigma \frac{1+\gamma_2'}{2} \qquad (\text{New dominance ratio})$$



smaller σ → larger γ_1 → smaller ζ_{l-1}^*

⊗means that previously used σ was too low.

Chebyshev Acceleration Logic (1/3)

- Chebyshev Acceleration Logic

1. Perform power iteration for first n_0 iterations ($n_0 \geq 3$) and estimate σ

2. Start chebyshev acceleration at (n_0+1)th step

① Determine $x_{pow}^{(n_0+1)} = A\hat{x}^{(n_0+1)} = \dfrac{1}{\lambda^{(n_0)}} A x^{(n_0)}$

② Determine $\lambda^{(n_0+1)} = \lambda^{(n_0)} \dfrac{\left\langle x_{pow}^{(n_0+1)}, x_{pow}^{(n_0+1)} \right\rangle}{\left\langle x_{pow}^{(n_0+1)}, x^{(n_0)} \right\rangle}$

③ Determine pseudo error $\left\| \tilde{e}^{(n_0+1)} \right\| = \left\| x_{pow}^{(n_0+1)} - x^{(n_0)} \right\|$

④ Determine extrapolation parameter $\alpha = \dfrac{2}{2-\sigma}, \; \beta = 0 \;\leftarrow\; \alpha\sigma \dfrac{\gamma+1}{2} + 1 - \alpha = \dfrac{\gamma}{\gamma_1}$

⑤ Perform extrapolation $x^{(n_0+1)} = \alpha x_{pow}^{(n_0+1)} + (1-\alpha) x^{(n_0)}$

Chebyshev Acceleration Logic (2/3)

3. Continue chebyshev acceleration at least two more steps, $l = n_0 + 3,\ m = l - n_0 \quad (\text{chebyshev order})$

① Determine $x_{pow}^{(l)} = A\hat{x}^{(l)} = \dfrac{1}{\lambda^{(l-1)}} A x^{(l-1)}$

② Determine $\lambda^{(l)} = \lambda^{(l-1)} \dfrac{\left\langle x_{pow}^{(l)}, x_{pow}^{(l)} \right\rangle}{\left\langle x_{pow}^{(l)}, x^{(l-1))} \right\rangle}$

③ Determine pseudo error $\left\| \tilde{e}^{(m)} \right\| = \left\| x_{pow}^{(l)} - x^{(l-1)} \right\|$

④ Determine extrapolation parameter $\alpha = \dfrac{4}{\sigma} \dfrac{T_{m-1}(\gamma_1)}{T_m(\gamma_1)} \qquad \beta = \left(1 - \dfrac{\sigma}{2}\right)\alpha - 1 \qquad \left(\gamma_1 = \dfrac{2}{\sigma} - 1\right)$

⑤ Perform extrapolation $x^{(l)} = \alpha x_{pow}^{(l)} + (1 - \alpha + \beta) x^{(l-1)} - \beta x^{(l-2)}$

⑥ Monitor Relative Error Reduction factor

$$\zeta_m = \frac{\left\| \tilde{e}^{(m)} \right\|}{\left\| \tilde{e}^{(1)} \right\|} \qquad \xi = \frac{\zeta_m}{\zeta_{m-1}^*} = \zeta_m T_{m-1}(\gamma_1)$$

Chebyshev Acceleration Logic (3/3)

⑦ Estimate new dominance ratio (from cheby order 2)

$$\gamma_2' = \cosh\left(\frac{\cosh^{-1}\xi}{m-1}\right) \quad \text{if } \xi > 1, \left(\xi = \frac{\zeta_m}{\zeta_{m-1}^*}\right) \qquad \gamma_2' = \cos\left(\frac{\cos^{-1}\xi}{m-1}\right) \quad \text{if } \xi \le 1$$

$$\sigma' = \sigma\frac{1+\gamma'}{2}$$

4. Reset cycle if neccessary

① If $\zeta_m > \zeta_{m-1}^*$: Error reduction with the current σ is not effective.

② $m \ge m_{MAX}$

③ Replace $\sigma = \sigma'$

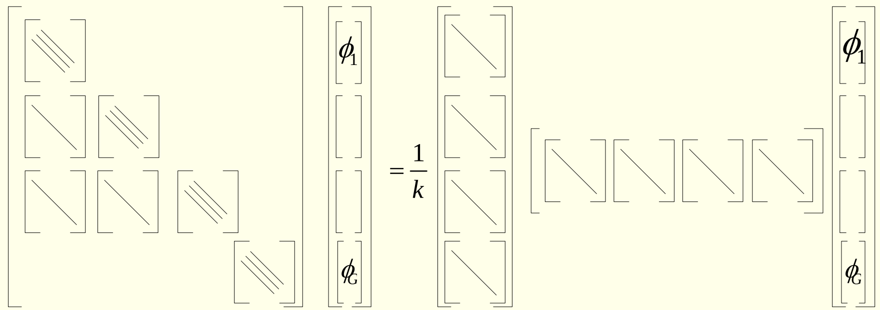
④ Reset $m = 0$

5. continue 3,4 until convergence

$$\frac{\left\|\tilde{e}^{(m)}\right\|}{\left\|x^{(l)}\right\|} < \varepsilon \quad \text{and } m \ge 3$$

Chebyshev for Multi Group Problem

- Group Major ordering



$$M\phi = \frac{1}{k}\chi\nu\Sigma_f\phi$$

$$M\phi = \frac{1}{k}F\phi \qquad \left(F = \chi\nu\Sigma_f\right)$$

$$M\phi = \frac{1}{k}\chi\underbrace{\nu\Sigma_f\cdot\phi}_{\psi}$$

$$M\phi = \frac{1}{k}\chi\psi \qquad \rightarrow \text{Apply Chebyshev to } \psi$$

Eigenvalue Update with Fission Source

$$M\phi = \frac{1}{k}F\phi$$

$$\underbrace{M^{-1}F}_{A}\phi = k\phi$$

Power Method with Scaling: $\phi^{(l)} = M^{-1}F\dfrac{\phi^{(l-1)}}{k^{(l-1)}}$

Inverse Power Method: $M\phi^{(l)} = F\dfrac{\phi^{(l-1)}}{k^{(l-1)}} = \dfrac{1}{k^{(l-1)}}\chi\psi^{(l-1)}$, $\chi \in R^{N,K}$, $N = G \times K$

Let $\Gamma = \left[I_K \cdots I_K \right] \in R^{K,N}$ with I_K being identity matrix of Rank K ($N = K \times G$)

$$\rightarrow \Gamma\chi = I_K \quad \because \sum_{g=1}^{G}\chi_g^k \equiv 1 \text{ irrespective of } k$$

$\Gamma M\phi^{(l)} = \dfrac{1}{k^{(l-1)}}\psi^{(l-1)}$ $\rightarrow$ $K \times K$ linear system: Equation for entire group

For large l, $M\phi^{(l)} = \dfrac{1}{k}\chi\psi^{(l)} \rightarrow \Gamma M\phi^{(l)} = \dfrac{1}{k}\psi^{(l)}$

$$\rightarrow k = \frac{<\psi^{(l)},\psi^{(l)}>}{<\psi^{(l)},\Gamma M\phi^{(l)}>} = \frac{<\psi^{(l)},\psi^{(l)}>}{<\psi^{(l)},\dfrac{1}{k^{(l-1)}}\psi^{(l-1)}>} = k^{(l-1)}\frac{<\psi^{(l)},\psi^{(l)}>}{<\psi^{(l)},\psi^{(l-1)}>}$$

Fission Source Iteration

1. Determine fission source at each node and fission source adjustment parameter λ

$$\psi_i = \sum_{g'=1}^{G} \nu\Sigma_{fg'}^{i}\phi_{g'}^{i} \qquad \lambda = \frac{1}{k}$$

Loop over groups

2. Determine source at each node for the current group

$$s_{i,g} = \lambda\chi_g\psi_i + \sum_{g'=G\min(g)}^{G\max(g)} \Sigma_{g'g}^{i}\phi_{g'}^{i}$$

3. Solve for flux for the group

4. Sweep over groups (one sweep)

5. Upscattering sweep if necessary

6. Determine new fission source

7. Estimate new k

8. Estimate pseudo error

9. Estimate dominance ratio

10. Determine extrapolation parameters

11. Do extrapolation

12. Monitor effectiveness of the current Cheby cycle