Lecture Note 4

Multi-Dimensional, Multi-Group Neutron Diffusion Eigenvalue Problem

April 1, 2010

Prof. Joo Han-gyu
Department of Nuclear Engineering

2D, 1G Fixed Source Problem

- Multi-Group Neutron Balance Equation

$$\vec{\nabla} \cdot \vec{J}_g + \Sigma_{rg}\phi_g = \lambda \chi_g \left(\sum_{g'=1}^{G} \nu\Sigma_{fg'}\phi_{g'} \right) + \left(\sum_{\substack{g'=1 \\ g' \neq g}}^{G} \Sigma_{g'g}\phi_{g'} \right)$$

$$\vec{\nabla} \cdot \vec{J}_g + \Sigma_{rg}\phi_g(\vec{r}) = S_g \qquad \left(\vec{J}_g = -D_g \vec{\nabla} \phi_g \right)$$

$$\vec{\nabla} \cdot J = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} \ (2\mathrm{D})$$

$$J_u = -D \frac{\partial \phi}{\partial u}, \ \text{Fick's law} \ (u = x, y)$$

$$g = 1 \ (1 \text{ Group})$$

$$\vec{\nabla} \cdot \vec{J} + \Sigma_r \phi = S \qquad (\vec{J} = -D \vec{\nabla} \phi)$$

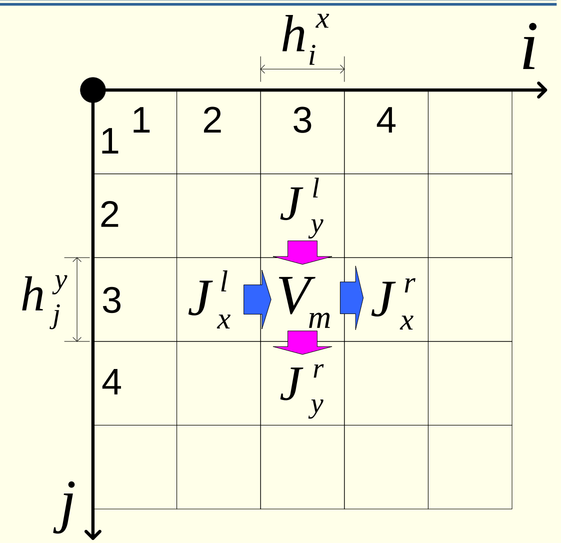
- 2D, 1G Fixed Source Problem

$$\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \Sigma_r \phi = S \qquad \phi = \phi(x, y), J_u = J_u(x, y), S = S(x, y) \qquad (u = x, y)$$

Box Scheme for 2D NDE

- Box scheme – Integration over the node (Box)

$$\underbrace{\vec{\nabla}\cdot\vec{J}}_{A} + \underbrace{\Sigma_r\phi}_{B} = \underbrace{S}_{C}$$



$$A = \int_V \vec{\nabla}\cdot\vec{J}\, dV = \int_S \vec{J}\bullet\hat{n}\,dS$$

$$= \int_0^{h_j^y}\left(J_x(h_i^x, y) - J_x(0,y)\right)dy + \int_0^{h_i^x}\left(J_y(x, h_j^y) - J_y(x,0)\right)dx$$

$$= \int_0^{h_j^y} J_x^r(y)dy - \int_0^{h_j^y} J_x^l(y)dy + \int_0^{h_i^x} J_y^r(x)dx - \int_0^{h_i^x} J_y^l(x)dx$$

$$= (\bar{J}_x^r - \bar{J}_x^l)\cdot h_j^y + (\bar{J}_y^r - \bar{J}_y^l)\cdot h_i^x$$

$$\bar{J}_u^s = \frac{1}{h^v}\int_0^{h^v} J_u^s(v)dv \quad \begin{cases} u = x, y \\ v = y, x \\ s = r, l \end{cases}$$

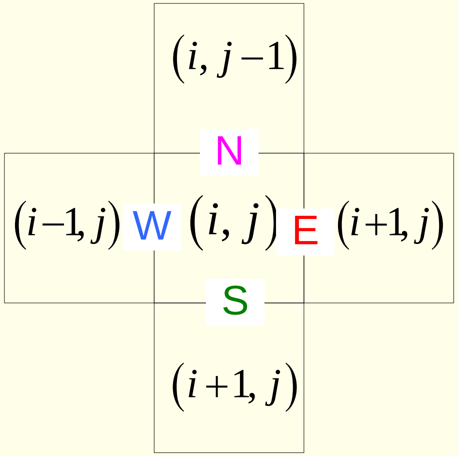
$$B = \int_V \Sigma_r\phi\, dV = \int_0^{h_j^y}\int_0^{h_i^x}\Sigma_r\phi(x,y)\,dx\,dy = \Sigma_r\bar{\phi}_m\,V_m$$

$$\otimes\ \bar{\phi}_m = \frac{1}{V_m}\int_V \phi(x,y)\,dV$$

$$C = \int_V S\,dV = \bar{S}\,V_m$$

$$\rightarrow\ \left(\bar{J}_x^r - \bar{J}_x^l\right)\cdot h_j^y + \left(\bar{J}_y^r - \bar{J}_y^l\right)\cdot h_i^x + \Sigma_r\bar{\phi}_m\,V_m = \bar{S}\,V_m$$

Finite Difference Approximation of Current



- Approximation of Current

a) Relation with East

$$\bar{J}_x^r = \bar{J}_{x,e}^l$$

$$-D_m \frac{\bar{\phi}_s - \bar{\phi}_m}{\frac{h_i^x}{2}} = -D_{me} \frac{\bar{\phi}_{me} - \bar{\phi}_s}{\frac{h_{i+1}^x}{2}}$$

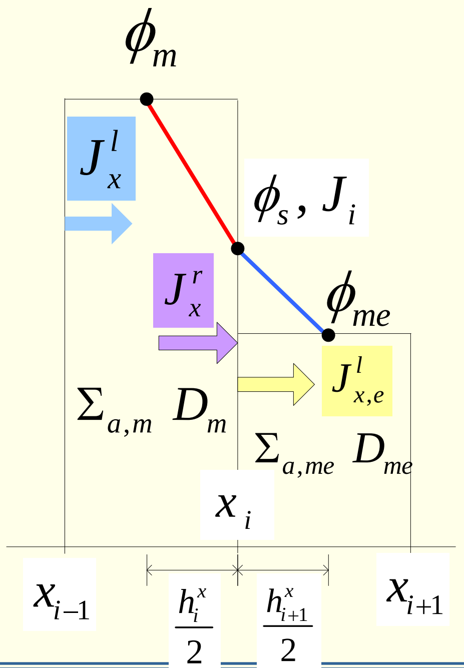
$$\phi_s = \frac{\beta_m^x}{\beta_m^x + \beta_{me}^x}\bar{\phi}_m + \frac{\beta_{me}^x}{\beta_m^x + \beta_{me}^x}\bar{\phi}_{me}$$

$$\therefore \bar{J}_x^r = -D_m \frac{\bar{\phi}_s - \bar{\phi}_m}{\frac{h_i^x}{2}} = -2\frac{\beta_m^x \beta_{me}^x}{\beta_m^x + \beta_{me}^x}\left(\bar{\phi}_{me} - \bar{\phi}_m\right) = -\tilde{D}_m^e\left(\bar{\phi}_{me} - \bar{\phi}_m\right)$$

- Relative diffusivity

$$\beta_m^x = \frac{D_m}{h_i^x}$$

$$\left(\tilde{D}_m^e = 2\frac{\beta_m \beta_{me}}{\beta_m + \beta_{me}}\right)$$



Finite Difference Approximation of Current

b) Relation for West

$$\bar{J}_x^l = -2\frac{\beta_m^x \beta_{mw}^x}{\beta_m^x + \beta_{mw}^x}\left(\bar{\phi}_m - \bar{\phi}_{mw}\right) = -\tilde{D}_m^w\left(\bar{\phi}_m - \bar{\phi}_{mw}\right) \qquad \left(\tilde{D}_m^w = 2\frac{\beta_m^x \beta_{mw}^x}{\beta_m^x + \beta_{mw}^x}\right)$$

c) Relation for South

$$\bar{J}_y^r = -2\frac{\beta_m^y \beta_{ms}^y}{\beta_m^y + \beta_{ms}^y}\left(\bar{\phi}_{ms} - \bar{\phi}_m\right) = -\tilde{D}_m^s\left(\bar{\phi}_{ms} - \bar{\phi}_m\right) \qquad \left(\tilde{D}_m^s = 2\frac{\beta_m^y \beta_{ms}^y}{\beta_m^y + \beta_{ms}^y}\right)$$

- Relative diffusivity

$$\beta_m^y = \frac{D_m}{h_i^y}$$

d) Relation for North

$$\bar{J}_y^l = -2\frac{\beta_m^y \beta_{mn}^y}{\beta_m^y + \beta_{mn}^y}\left(\bar{\phi}_m - \bar{\phi}_{mn}\right) = -\tilde{D}_m^n\left(\bar{\phi}_m - \bar{\phi}_{mn}\right) \qquad \left(\tilde{D}_m^n = 2\frac{\beta_m^y \beta_{mn}^y}{\beta_m^y + \beta_{mn}^y}\right)$$

– NBE with Box scheme

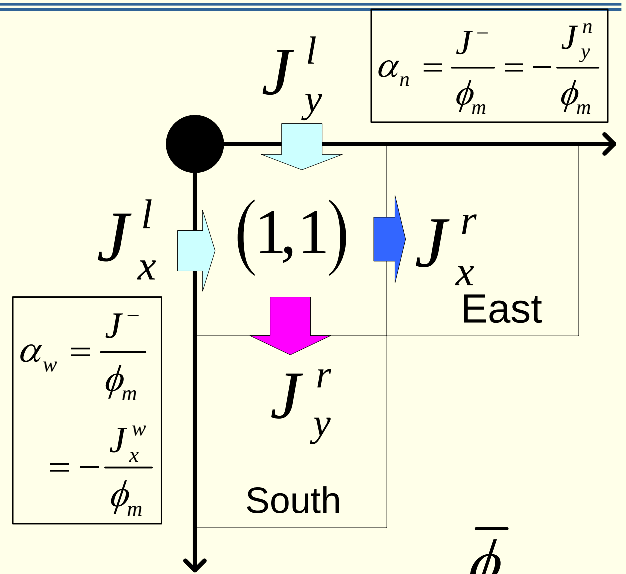
$$\left(\bar{J}_x^r - \bar{J}_x^l\right)\cdot h_j^y + \left(\bar{J}_y^r - \bar{J}_y^l\right)\cdot h_i^x + \Sigma_r \bar{\phi}_m \; V_m = \bar{S} \; V_m$$

$$\left[-\tilde{D}_m^e\left(\bar{\phi}_{me} - \bar{\phi}_m\right) - \left(-\tilde{D}_m^w\left(\bar{\phi}_m - \bar{\phi}_{mw}\right)\right)\right]\cdot h_j^y + \left[-\tilde{D}_m^s\left(\bar{\phi}_{ms} - \bar{\phi}_m\right) - \left(-\tilde{D}_m^n\left(\bar{\phi}_m - \bar{\phi}_{mn}\right)\right)\right]\cdot h_i^x + \Sigma_r \bar{\phi}_m \; V_m = \bar{S} \; V_m$$

$$-\tilde{D}_m^e h_j^y \bar{\phi}_{me} - \tilde{D}_m^w h_j^y \bar{\phi}_{mw} - \tilde{D}_m^s h_i^x \bar{\phi}_{ms} - \tilde{D}_m^n h_i^x \bar{\phi}_{mn} + \left(\tilde{D}_m^e h_j^y + \tilde{D}_m^w h_j^y + \tilde{D}_m^s h_i^x + \tilde{D}_m^n h_i^x + \Sigma_r V_m\right)\bar{\phi}_m = \bar{S} \; V_m$$

Finite Difference Approximation of Current

- Boundary Condition (At First Node – West and North Boundary)



a) West Boundary

$$\bar{J}_x^w = \bar{J}_x^l$$

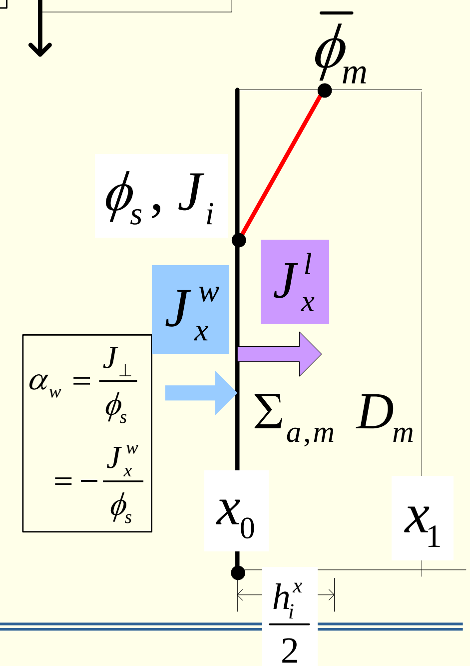
$$-\alpha_w \phi_s = -D_m \frac{\bar{\phi}_m - \phi_s}{\frac{h_i^x}{2}}$$

$$\phi_s = \frac{\frac{D_m}{h_i^x}}{\frac{\alpha_w}{2} + \frac{D_m}{h_i^x}} \bar{\phi}_m = \frac{\beta_m^x}{\beta_w + \beta_m^x} \bar{\phi}_m$$

$$\left(\beta_m^x = \frac{D_m}{h_i^x}, \beta_w = \frac{\alpha_w}{2} \right)$$

$$\bar{J}_x^l = -\frac{\alpha_w \beta_m^x}{\beta_w + \beta_m^x} \bar{\phi}_m = -2 \frac{\beta_w \beta_m^x}{\beta_w + \beta_m^x} \bar{\phi}_m = -\tilde{D}_m^{\alpha w} \bar{\phi}_m$$

$$\left(\tilde{D}_m^{\alpha w} = 2 \frac{\beta_w \beta_m^x}{\beta_w + \beta_m^x} \right)$$



As $\alpha_w \to \infty$, $\tilde{D}_m^{\alpha w} \to 2\beta_m^x$ (for zero flux)

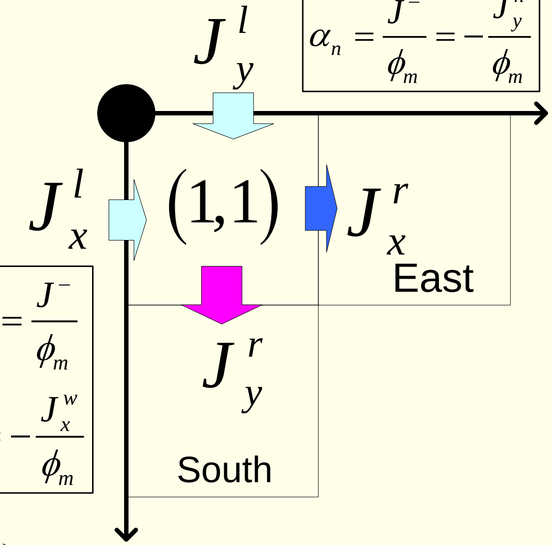
As $\alpha_w \to 0$, $\tilde{D}_m^{\alpha w} \to 0$ (for zero current)

SNURPL

Finite Difference Approximation of Current

b) North Boundary

$$\alpha_n = \frac{J^-}{\phi_m} = -\frac{J_y^n}{\phi_m}$$



$$\alpha_w = \frac{J^-}{\phi_m} = -\frac{J_x^w}{\phi_m}$$

$$\bar{J}_y^n = \bar{J}_y^l$$

$$-\alpha_n \phi_s = -D_m \frac{\bar{\phi}_m - \phi_s}{\frac{h_j^y}{2}}$$

$$\phi_s = \frac{\frac{D_m}{h_j^y}}{\frac{\alpha_n}{2} + \frac{D_m}{h_j^y}} \bar{\phi}_m = \frac{\beta_m^y}{\beta_n + \beta_m^y} \bar{\phi}_m \qquad \left(\beta_m^y = \frac{D_m}{h_j^y}, \beta_n = \frac{\alpha_n}{2} \right)$$

$$\bar{J}_y^l = -\frac{\alpha_n \beta_m^y}{\beta_n + \beta_m^y} \bar{\phi}_m = -2\frac{\beta_n \beta_m^y}{\beta_n + \beta_m^y} \bar{\phi}_m = -\tilde{D}_m^{\alpha n} \bar{\phi}_m$$

$$\left(\tilde{D}_m^{\alpha n} = 2\frac{\beta_n \beta_m^y}{\beta_n + \beta_m^y} \right)$$

– NBE with Box scheme at First Node

$$\begin{cases} \bar{J}_x^r = -\tilde{D}_m^e \left(\bar{\phi}_{me} - \bar{\phi}_m \right) \\ \bar{J}_y^r = -\tilde{D}_m^s \left(\bar{\phi}_{ms} - \bar{\phi}_m \right) \end{cases}$$

$$\left(\bar{J}_x^r - \bar{J}_x^l \right) \cdot h_j^y + \left(\bar{J}_y^r - \bar{J}_y^l \right) \cdot h_i^x + \Sigma_r \bar{\phi}_m \; V_m = \bar{S} \; V_m$$

$$\left[-\tilde{D}_m^e \left(\bar{\phi}_{me} - \bar{\phi}_m \right) - \left(-\tilde{D}_m^{\alpha w} \bar{\phi}_m \right) \right] \cdot h_j^y + \left[-\tilde{D}_m^s \left(\bar{\phi}_{ms} - \bar{\phi}_m \right) - \left(-\tilde{D}_m^{\alpha n} \bar{\phi}_m \right) \right] \cdot h_i^x + \Sigma_r \bar{\phi}_m \; V_m = \bar{S} \; V_m$$

$$-\tilde{D}_m^e h_j^y \bar{\phi}_{me} - \tilde{D}_m^s h_i^x \bar{\phi}_{ms} + \left(\tilde{D}_m^e h_j^y + \tilde{D}_m^{\alpha w} h_j^y + \tilde{D}_m^s h_i^x + \tilde{D}_m^{\alpha n} h_i^x + \Sigma_r V_m \right) \bar{\phi}_m = \bar{S} \; V_m$$

Finite Difference Approximation of Current

- Boundary Condition (Eastern side – East Boundary)

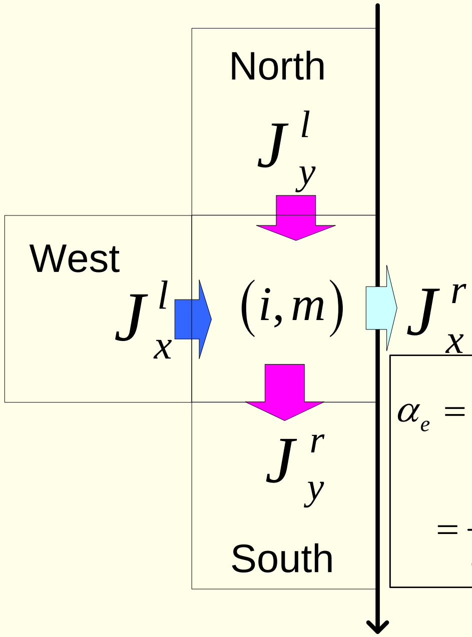
  a) East Boundary

$$\bar{J}_x^{e} = \bar{J}_x^{r}$$

$$\alpha_e \phi_s = -D_m \frac{\bar{\phi}_m - \phi_s}{\frac{h_i^x}{2}}$$

$$\phi_s = \frac{\frac{D_m}{h_i^x}}{\frac{\alpha_e}{2} + \frac{D_m}{h_i^x}}\bar{\phi}_m = \frac{\beta_m^x}{\beta_e + \beta_m^x}\bar{\phi}_m \qquad \left(\beta_m^x = \frac{D_m}{h_i^x}, \beta_e = \frac{\alpha_e}{2}\right)$$

$$\bar{J}_x^{r} = \frac{\alpha_e \beta_m^x}{\beta_e + \beta_m^x}\bar{\phi}_m = 2\frac{\beta_e \beta_m^x}{\beta_e + \beta_m^x}\bar{\phi}_m = \tilde{D}_m^{\alpha e}\bar{\phi}_m \qquad \left(\tilde{D}_m^{\alpha e} = 2\frac{\beta_e \beta_m^x}{\beta_e + \beta_m^x}\right)$$



$$\alpha_e = \frac{J^+}{\phi_m} = \frac{J_x^e}{\phi_m}$$

$$\begin{cases} \bar{J}_x^{l} = -\tilde{D}_m^{w}\left(\bar{\phi}_m - \bar{\phi}_{mw}\right) \\ \bar{J}_y^{r} = -\tilde{D}_m^{s}\left(\bar{\phi}_{ms} - \bar{\phi}_m\right) \\ \bar{J}_y^{l} = -\tilde{D}_m^{n}\left(\bar{\phi}_m - \bar{\phi}_{mn}\right) \end{cases}$$

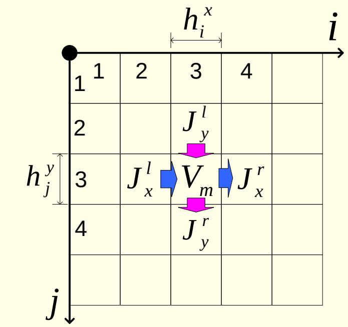
– NBE with Box scheme at Eastern side

$$\left(\bar{J}_x^{r} - \bar{J}_x^{l}\right)\cdot h_j^y + \left(\bar{J}_y^{r} - \bar{J}_y^{l}\right)\cdot h_i^x + \Sigma_r \bar{\phi}_m V_m = \bar{S} V_m$$

$$\left[\left(\tilde{D}_m^{\alpha e}\bar{\phi}_m\right) - \left(-\tilde{D}_m^{w}\left(\bar{\phi}_m - \bar{\phi}_{mw}\right)\right)\right]\cdot h_j^y + \left[\left(-\tilde{D}_m^{s}\left(\bar{\phi}_{ms} - \bar{\phi}_m\right) - \left(-\tilde{D}_m^{n}\left(\bar{\phi}_m - \bar{\phi}_{mn}\right)\right)\right)\right]\cdot h_i^x + \Sigma_r \bar{\phi}_m V_m = \bar{S} V_m$$

$$-\tilde{D}_m^{w} h_j^y \bar{\phi}_{mw} - \tilde{D}_m^{s} h_i^x \bar{\phi}_{ms} - \tilde{D}_m^{n} h_i^x \bar{\phi}_{mn} + \left(\tilde{D}_m^{\alpha e} h_j^y + \tilde{D}_m^{w} h_j^y + \tilde{D}_m^{s} h_i^x + \tilde{D}_m^{n} h_i^x + \Sigma_r V_m\right)\bar{\phi}_m = \bar{S} V_m$$

Finite Difference Approximation of Current



- Problem to Solve
  - At First Node – West and North Boundary

$$-\tilde{D}^{e}_{m} h^{y}_{j}\bar{\phi}_{me} - \tilde{D}^{s}_{m} h^{x}_{i}\bar{\phi}_{ms} + (\tilde{D}^{e}_{m} h^{y}_{j} + \tilde{D}^{\alpha w}_{m} h^{y}_{j} + \tilde{D}^{s}_{m} h^{x}_{i} + \tilde{D}^{\alpha n}_{m} h^{x}_{i} + \Sigma_{r} V_{m})\,\bar{\phi}_{m} = \bar{S}\,V_{m}$$

  - At Inner Nodes

$$-\tilde{D}^{e}_{m} h^{y}_{j}\bar{\phi}_{me} - \tilde{D}^{w}_{m} h^{y}_{j}\bar{\phi}_{mw} - \tilde{D}^{s}_{m} h^{x}_{i}\bar{\phi}_{ms} - \tilde{D}^{n}_{m} h^{x}_{i}\bar{\phi}_{mn} + (\tilde{D}^{e}_{m} h^{y}_{j} + \tilde{D}^{w}_{m} h^{y}_{j} + \tilde{D}^{s}_{m} h^{x}_{i} + \tilde{D}^{n}_{m} h^{x}_{i} + \Sigma_{r} V_{m})\,\bar{\phi}_{m} = \bar{S}\,V_{m}$$

  - At Eastern Node

$$-\tilde{D}^{w}_{m} h^{y}_{j}\bar{\phi}_{mw} - \tilde{D}^{s}_{m} h^{x}_{i}\bar{\phi}_{ms} - \tilde{D}^{n}_{m} h^{x}_{i}\bar{\phi}_{mn} + (\tilde{D}^{\alpha e}_{m} h^{y}_{j} + \tilde{D}^{w}_{m} h^{y}_{j} + \tilde{D}^{s}_{m} h^{x}_{i} + \tilde{D}^{\alpha n}_{m} h^{x}_{i} + \Sigma_{r} V_{m})\,\bar{\phi}_{m} = \bar{S}\,V_{m}$$

  - At last Node – East and South Boundary

$$-\tilde{D}^{w}_{m} h^{y}_{j}\bar{\phi}_{mw} - \tilde{D}^{n}_{m} h^{x}_{i}\bar{\phi}_{mn} + (\tilde{D}^{\alpha e}_{m} h^{y}_{j} + \tilde{D}^{w}_{m} h^{y}_{j} + \tilde{D}^{\alpha s}_{m} h^{x}_{i} + \tilde{D}^{n}_{m} h^{x}_{i} + \Sigma_{r} V_{m})\,\bar{\phi}_{m} = \bar{S}\,V_{m}$$

- Matrix Form

$$M\phi = \frac{1}{k}\chi\,\nu\Sigma_{f}\phi \mathrm{V_m} + \Sigma_{s}\,\phi \mathrm{V_m} + \mathrm{S}\mathrm{V_m}$$

Diagonal Entry

- Diagonal Entry

  – If $h_x^i = h_y^j = h$ and $D_{me} = D_{mw} = D_{ms} = D_{mn} = D$

  – for inner nodes

$$\tilde{D}_m = 2\frac{\beta_m \beta_{me}}{\beta_m + \beta_{me}} = \frac{D}{h} \qquad \left(\beta_m^x = \frac{D}{h}, \beta_{me}^x = \frac{D}{h}\right)$$

  – for reflective boundary Condition $(J = 0, \alpha = 0)$

$$\tilde{D}_m^\alpha = 2\frac{\beta_n \beta_m^y}{\beta_n + \beta_m^y} = 0 \qquad \left(\beta_m^y = \frac{D}{h}, \beta_n = \frac{\alpha_n}{2} = 0\right)$$

  – for zero flux boundary Condition $(\phi = 0, \alpha = \infty)$

$$\tilde{D}_m^\alpha = 2\frac{\beta_s \beta_m^y}{\beta_s + \beta_m^y} = 2\frac{D}{h} \qquad \left(\beta_m^y = \frac{D}{h}, \beta_s = \frac{\alpha_s}{2} = \infty\right)$$

  – Diagonal Entry is $\left(\tilde{D}_m^e h + \tilde{D}_m^w h + \tilde{D}_m^s h + \tilde{D}_m^n h + \Sigma_r V_m\right)$

  – Let $\gamma = \tilde{D} h$, Default Diagonal Entry is $\left(\tilde{D}_m^e h + \tilde{D}_m^w h + \tilde{D}_m^s h + \tilde{D}_m^n h + \Sigma_r V_m\right) = 4\gamma + \Sigma_r V_m$

$$\begin{cases} \text{Reflective BC} \rightarrow & -\gamma \\ \text{Zero Flux BC} \rightarrow & +\gamma \end{cases}$$

$\bar{J}_x^n = 0$

$\bar{J}_x^w = 0$ $\phi_y^e = 0$

2γ	3γ	3γ	3γ	4γ
3γ	4γ	4γ	4γ	5γ
3γ	4γ	4γ	4γ	5γ
3γ	4γ	4γ	4γ	5γ
5γ	5γ	5γ	5γ	6γ

$\phi_y^s = 0$

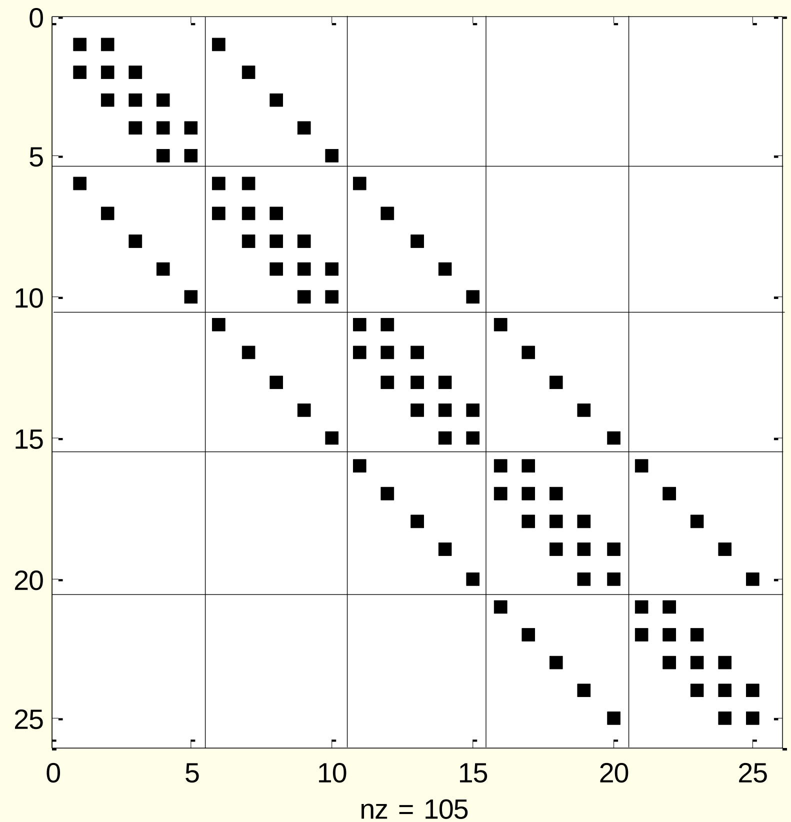
Node Ordering Scheme (1/3)

- Natural Ordering

m →

n ↓

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

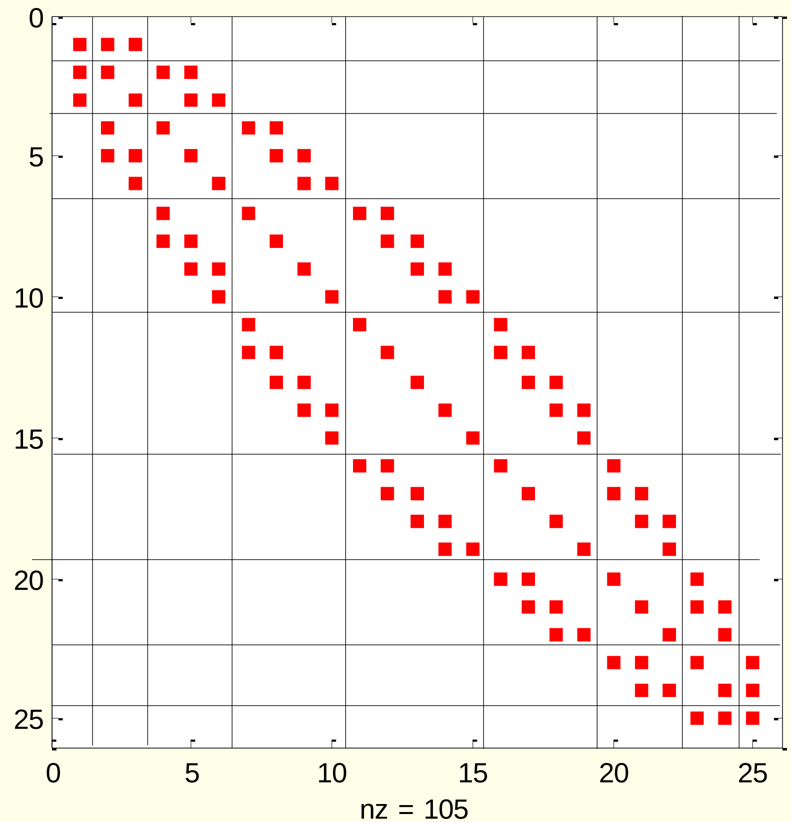


☆ Block Tridiagonal Matrix

- Size of the Block = m x m (representing each row of meshes) → Bandwidth=*m*
- Number of the Diagonal Blocks = n
- Off-diagonal Blocks represents coupling between rows of meshes
- Penta-diagonal Matrix

Node Ordering Scheme (2/3)

- Cuthill-Mckee Ordering

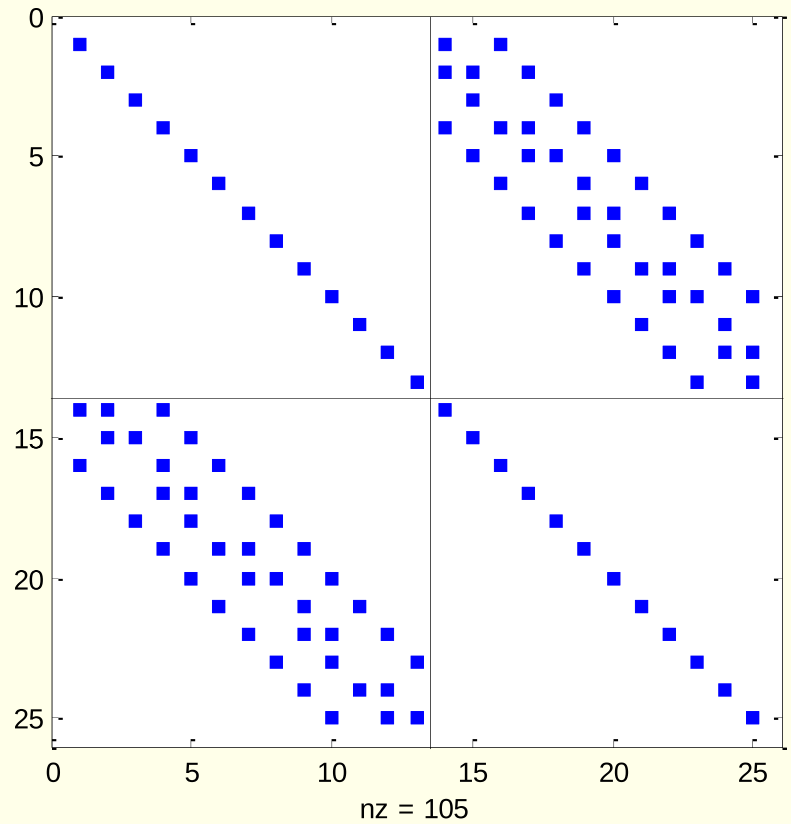


Max. bandwidth =m, but much less fill-in when elimination

Node Ordering Scheme (3/3)

- Red-Black Ordering

1	14	2	15	3
16	4	17	5	18
6	19	7	20	8
21	9	22	10	23
11	24	12	25	13



Classical Iterative Solution Methods

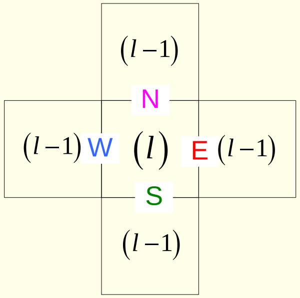
- Classical Iterative Solution Methods
  - Jacobi
  - Gauss-Seidel
  - Successive Over-Relaxtion (SOR)
  - Cyclic Chebyshev Semi Iterative Method

$$M\phi = b$$

$$M_m = (\cdots, -\gamma_m^n, \cdots - \gamma_m^w, d_m, \cdots, -\gamma_m^e, \cdots, -\gamma_m^s, \cdots)$$

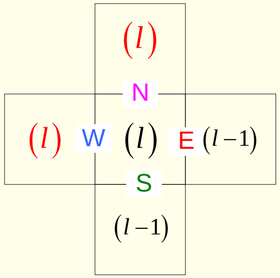
- Point Jacobi

$$\phi_m^{(l)} = \frac{1}{d_m}\left(b_m + \sum_{d=e,w,s,n} \gamma_m^d \phi_{md}^{(l-1)} \right)$$



- Point Gauss-Seidel

$$\phi_m^{(l)} = \frac{1}{d_m}\left(b_m + \sum_{d=w,n} \gamma_m^d \phi_{md}^{(l)} + \sum_{d=e,s} \gamma_m^d \phi_{md}^{(l-1)} \right)$$

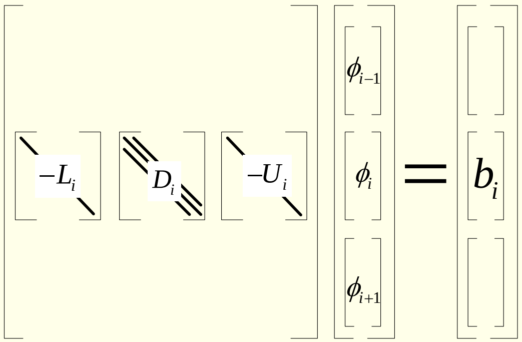


Classical Iterative Solution Methods

- Block (Line) Jacobi with Natural ordering

$$-L_i\phi_{i-1} + D_i\phi_i - U_i\phi_{i+1} = b_i$$

$$D_i\phi_i^{(l)} = b_i + L_i\phi_{i-1}^{(l-1)} + U_i\phi_{i+1}^{(l-1)}$$



  – Need LU factorization, then forward and backward substitution

- Block (Line) Gauss-Seidel with Natural ordering

$$-L_i\phi_{i-1} + D_i\phi_i - U_i\phi_{i+1} = b_i$$

$$D_i\phi_i^{(l)} = b_i + L_i\phi_{i-1}^{(l)} + U_i\phi_{i+1}^{(l-1)}$$

- SOR

$$\phi_i^{(l)} = \omega\phi_{i,GS}^{(l)} + (1-\omega)\phi_i^{(l-1)}$$