Lecture Note 5

Iterative Linear System Solution Methods (SOR, CCSI)

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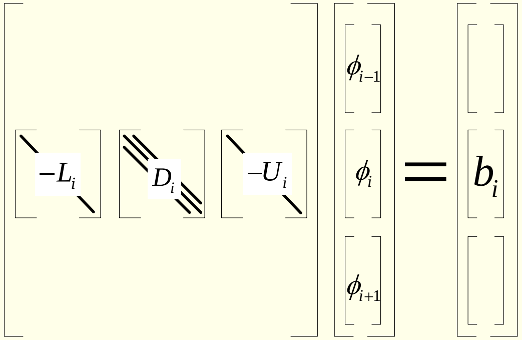
Prof. Joo Han-gyu
Department of Nuclear Engineering

Classical Iterative Solution Methods



- Block (Line) Jacobi with Natural ordering

$$-L_i\phi_{i-1} + D_i\phi_i - U_i\phi_{i+1} = b_i$$

$$D_i\phi_i^{(l)} = b_i + L_i\phi_{i-1}^{(l-1)} + U_i\phi_{i+1}^{(l-1)}$$

  - Need LU factorization, then forward and backward substitution

- Block (Line) Gauss-Seidel with Natural ordering

$$-L_i\phi_{i-1} + D_i\phi_i - U_i\phi_{i+1} = b_i$$

$$D_i\phi_i^{(l)} = b_i + L_i\phi_{i-1}^{(l)} + U_i\phi_{i+1}^{(l-1)}$$

- SOR

$$\phi_i^{(l)} = \omega\phi_{i,GS}^{(l)} + (1-\omega)\phi_i^{(l-1)}$$

Iterative Scheme and Convergence Condition

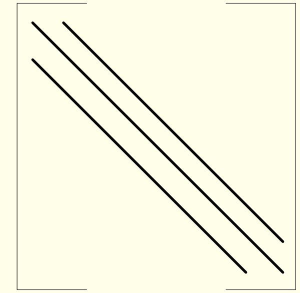
- Iterative scheme

$$A\phi = b$$

Let $A = D - L - U = M - N$

$$\begin{pmatrix} \mathrm{M} = D \text{ and } N = L + U & : \text{Jacobi} \\ \mathrm{M} = D - L \text{ and } N = U & : \text{G-S} \end{pmatrix}$$

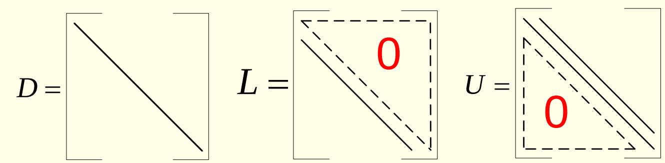
$A = \left[a_{ij}\right] =$



$$(M - N)\phi = b$$

$$M\phi = b + N\phi$$

$$M\phi^{(l)} = b + N\phi^{(l-1)} \quad : \text{Iterative scheme}$$



- Convergence criteria

$$\begin{array}{l} \;\; M\phi^{(l)} = b + N\phi^{(l-1)} \\ - \; M\phi^{*} = b + N\phi^{*} \\ \hline \end{array}$$

ϕ^{*} : Exact solution

$$Me^{(l)} = Ne^{(l-1)}$$

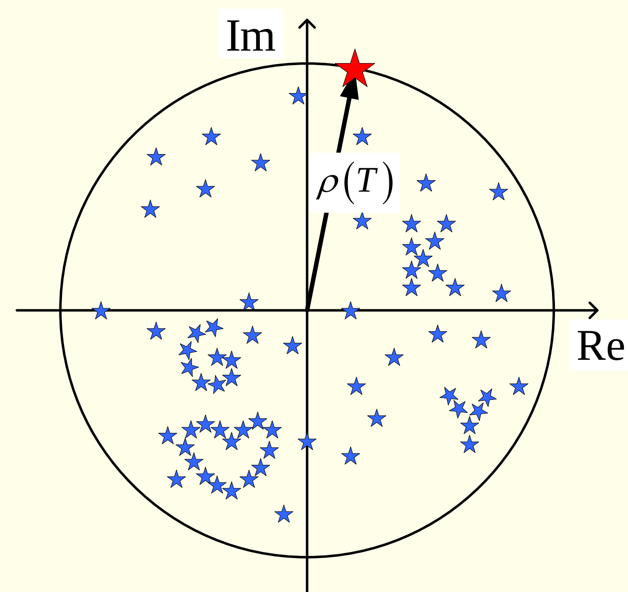
$$e^{(l)} = M^{-1}Ne^{(l-1)}$$

$\otimes\, T = M^{-1}N$: Iteration Matrix

$$e^{(l)} = Te^{(l-1)} = T^{2}e^{(l-2)} = T^{l}e^{(0)} = T^{l}\sum_{i} c_i u_i = \sum_{i} \lambda_i^{l} c_i u_i$$

Converge if $\left|\rho(T)\right| = \left|\max(\lambda_i)\right| < 1$

$\otimes\, \rho(T)$:Spectral Radius



Iterative Method in Matrix Form

- Jacobi

$$A\phi = b$$

$$\text{Let } A = D - (L + U) = M - N \qquad (\mathrm{M} = D \text{ and } N = L + U)$$

$$M\phi = b + N\phi$$

$$D\phi^{(l)} = b + (L + U)\phi^{(l-1)}$$

$$\phi^{(l)} = D^{-1}\left(b + (L + U)\phi^{(l-1)}\right)$$

$$\otimes T = M^{-1}N = D^{-1}(L + U)$$

- Gauss-Seidel

$$A\phi = b$$

$$\text{Let } A = (D - L) - U = M - N \qquad (\mathrm{M} = D - L \text{ and } N = U)$$

$$M\phi = b + N\phi$$

$$(D - L)\phi = b + U\phi$$

$$D\phi^{(l)} = b + L\phi^{(l)} + U\phi^{(l-1)}$$

$$\otimes T = M^{-1}N = (D - L)^{-1}U$$

$$\phi^{(l)} = D^{-1}\left(b + L\phi^{(l)} + U\phi^{(l-1)}\right)$$

Iterative Method in Matrix Form

- SOR

$$A\phi = b$$

$$\omega A\phi = \omega b$$

Let $\omega A = \omega\left(D - L - U\right) = \omega D - \omega L - \omega U + D - D = \underbrace{D - \omega L}_{M} - \underbrace{\left[\left(1 - \omega D\right) + \omega U\right]}_{N}$

$$\left(M - N\right)\phi = \omega b$$

$$M\phi = \omega b + N\phi$$

$$M\phi^{(l)} = \omega b + N\phi^{(l-1)}$$

$$\left(D - \omega L\right)\phi^{(l)} = \omega b + \left(\left(1 - \omega\right)D + \omega U\right)\phi^{(l-1)}$$

$$D\phi^{(l)} = \omega\left(b + L\phi^{(l)} + U\phi^{(l-1)}\right) + \left(1 - \omega\right)D\phi^{(l-1)}$$

$$\phi^{(l)} = \omega\phi_{G.S}^{(l)} + \left(1 - \omega\right)\phi^{(l-1)}$$

$$\therefore T = M^{-1}N$$

$$= \left(D - \omega L\right)^{-1}\left[\left(1 - \omega D\right) + \omega U\right]$$

– How to find a proper ω for min $\rho\left(T\right)$?

$\rightarrow$ can be answered by knowing the property of Jacobi iteration Matrix

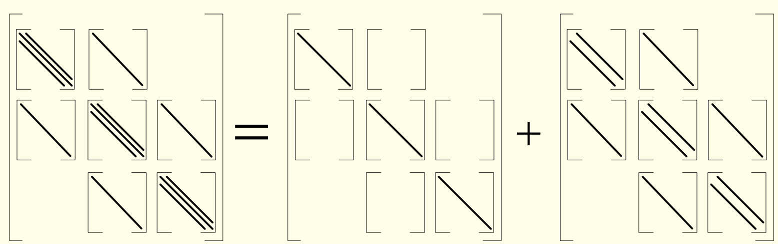
Properties of Jacobi Iteration Matrix

- Properties of Jacobi Iteration Matrix

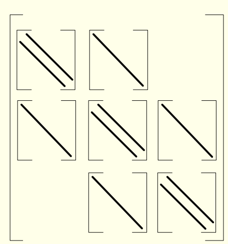
  1) Jacobi Iteration Matrix with various Iteration Matrix Structure

  ㉠ Natural ordering $(2D)$

$$\text{Let } A = D - (L + U) = M - N \qquad (\mathrm{M} = D \text{ and } N = L + U)$$



$$T = M^{-1}N = D^{-1}(L + U) =$$

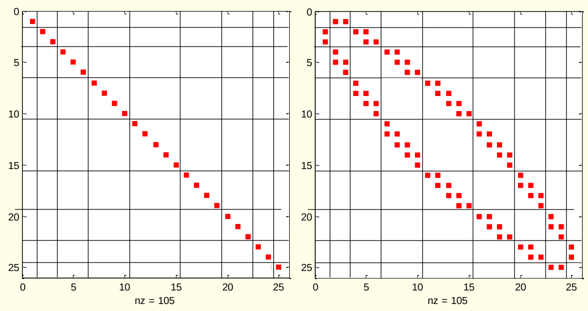


Properties of Jacobi Iteration Matrix

㉡ Point Jacobi Iteration Matrix with C-M

$$\text{Let } A = D - (L+U) = M - N \qquad (\mathrm{M} = D \text{ and } N = L+U)$$

$$A =$$



$$T = D^{-1}(L+U) = \begin{bmatrix} D_1^{-1} & 0 & & & 0 \\ 0 & D_2^{-1} & 0 & & \\ & 0 & D_3^{-1} & 0 & \\ & & 0 & D_4^{-1} & 0 \\ 0 & & & 0 & D_5^{-1} \end{bmatrix} \begin{bmatrix} 0 & U_1 & & & 0 \\ L_2 & 0 & U_2 & & \\ & L_3 & 0 & U_3 & \\ & & L_4 & 0 & U_4 \\ 0 & & & L_5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & D_1^{-1}U_1 & & & 0 \\ D_2^{-1}L_2 & 0 & D_2^{-1}U_2 & & \\ & D_3^{-1}L_3 & 0 & D_3^{-1}U_3 & \\ & & D_4^{-1}L_4 & 0 & D_4^{-1}U_4 \\ 0 & & & D_5^{-1}L_5 & 0 \end{bmatrix}$$

– Diagonal blocks are all zero : consistently ordered

Properties of Jacobi Iteration Matrix

ㄷ Red-Black

Let $A = D - (L+U) = M - N \qquad (\mathrm{M} = D \text{ and } N = L + U)$

$$A = \begin{bmatrix} D_1 & U_1 \\ L_2 & D_2 \end{bmatrix} = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} + \begin{bmatrix} 0 & U_1 \\ L_2 & 0 \end{bmatrix}$$

$$T = D^{-1}(L+U) = \begin{bmatrix} D_1^{-1} & 0 \\ 0 & D_2^{-1} \end{bmatrix}\begin{bmatrix} 0 & U_1 \\ L_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & D_1^{-1}U_1 \\ D_2^{-1}L_2 & 0 \end{bmatrix}$$

♠ Discussion

– Depending on the ordering scheme, the diagonal blocks of the Jacobi iteration matrix can be 0 or not.

– Consistent ordering scheme = ordering with which the diagonal blocks are all 0 (C-M, R-B)

– The natural ordering can be a consistent ordering scheme for the Line (Block) Jacobi and is NOT a consistent scheme for the pointwise Jacobi except for 1D problems

Properties of Jacobi Iteration Matrix

2) Diagonal Dominance

$$-\tilde{D}_m^e h_j^y \overline{\phi}_{me} - \tilde{D}_m^w h_j^y \overline{\phi}_{mw} - \tilde{D}_m^s h_i^x \overline{\phi}_{ms} - \tilde{D}_m^n h_i^x \overline{\phi}_{mn} + (\tilde{D}_m^e h_j^y + \tilde{D}_m^w h_j^y + \tilde{D}_m^s h_i^x + \tilde{D}_m^n h_i^x + \Sigma_r V_m)\, \overline{\phi}_m = \overline{S}\, V_m$$

$$s_i^{OD} = \sum_{j \neq i} \left| a_{ij} \right| < a_{ii} \quad \forall i$$ ← Digonal Dominance

– Jacobi Iteration Matrix T

$$T = \left[t_{ij} \right] = D^{-1} (L + U) = \left[-\frac{a_{ij}}{a_{ii}} \right] \qquad \otimes t_{kk} = 0$$

$$\sum \left| t_{ij} \right| < 1$$

– Let u be the fundamental eigenvector of T with λ being the corresponding eigenvalue

Namely, $Tu = \lambda u$

Properties of Jacobi Iteration Matrix

2) Diagonal Dominance - conti.

– k-th row $\quad \sum_j t_{kj} u_j = \lambda u_k$ $\quad u = [u_1, \cdots, u_k, \cdots, u_n]$

Let u_k be the maximum element (Absolute)

$$\sum_{j \neq k} t_{kj} u_j = \lambda u_k \qquad \because t_{kk} = 0$$

$$|\lambda| |u_k| \leq \sum_{j \neq k} |t_{kj}| |u_j| \leq \sum_{j \neq k} |t_{kj}| |u_k| \qquad \because |u_k| \text{ is Max}$$

$$\lambda \leq \sum_{j \neq k} |t_{kj}| < 1$$

♠ Discussion

– If A is diagonally dominant, then $\rho(T) < 1$ where $T = D^{-1}(L+U)$

– As absorption increases, $\rho(T)$ approaches to 0. So converges faster.

– As the diffusion coefficient increases, $\rho(T)$ approaches to 1. So converges slower.

Properties of Jacobi Iteration Matrix

3) Eigenvalues of T appear in $\pm$ pairs for the consistent ordering scheme

$$T = D^{-1}(L+U) = \begin{bmatrix} 0 & D_1^{-1}U_1 & & & 0 \\ D_2^{-1}L_2 & 0 & D_2^{-1}U_2 & & \\ & D_3^{-1}L_3 & 0 & D_3^{-1}U_3 & \\ & & D_4^{-1}L_4 & 0 & D_4^{-1}U_4 \\ 0 & & & D_5^{-1}L_5 & 0 \end{bmatrix} \quad \text{(Block Tridiagonal Structure)}$$

Let $T\psi = \lambda\psi$, then $-\lambda$ is an eigenvalue of T as well

$$\phi = \left[(-1)^i \psi_i\right] = \begin{bmatrix} -\psi_1 \\ \psi_2 \\ (-1)^i \psi_i \\ \vdots \\ (-1)^N \psi_N \end{bmatrix}$$

Proof) – Let $\phi_i = (-1)^i \psi_i$

$$T\psi = \lambda\psi$$

$$D^{-1}(L+U)\psi = \lambda\psi$$

$$D_i^{-1}L_i\psi_{i-1} + D_i^{-1}U_i\psi_{i+1} = \lambda\psi_i \quad \rightarrow L_i\psi_{i-1} + U_i\psi_{i+1} = \lambda D_i\psi_i$$

– Apply T to ϕ $(T\phi)$

$$D_i^{-1}L_i\phi_{i-1} + D_i^{-1}U_i\phi_{i+1} = D_i^{-1}L_i(-1)^{i-1}\psi_{i-1} + D_i^{-1}U_i(-1)^{i+1}\psi_{i+1}$$

$$= -(-1)^i\left[D_i^{-1}L_i\psi_{i-1} + D_i^{-1}U_i\psi_{i+1}\right] = -(-1)^i\lambda\psi_i = -\lambda\phi_i$$

$\therefore -\lambda$ is another eigenValue corresponding $\phi = \left[(-1)^i \psi_i\right]$

Properties of SOR Iteration Matrix (1/5)

$$T_{SOR} = M^{-1}N = \left(D - \omega L\right)^{-1}\left[\left(1-\omega\right)D + \omega U\right]$$

For which ω, $\rho(\mathrm{T})$ gets minimized?

T : SOR Iteration Matrix

T_J : Jacobi Iteration Matrix

- Relation eigenvalues of Jacobi and SOR iteration materix
  - Let $T\phi = \gamma\phi$ and $T_J\psi = \lambda\psi$

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_i \\ \vdots \\ \phi_N \end{bmatrix}, \psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_i \\ \vdots \\ \psi_N \end{bmatrix}$$

$$T\phi = \gamma\phi$$

$$\left(D - \omega L\right)^{-1}\left[\left(1-\omega\right)D + \omega U\right]\phi = \gamma\phi$$

$$\left(1-\omega\right)D\phi + \omega U\phi = \gamma\left(D - \omega L\right)\phi$$

$$\left(1-\omega\right)D_i\phi_i + \omega U_i\phi_{i+1} = \gamma\left(D_i\phi_i - \omega L_i\phi_{i-1}\right) \rightarrow \quad \text{Assume } \phi_i = \alpha^i\psi_i$$

$$\left(1-\omega\right)D_i\alpha^i\psi_i + \omega U_i\alpha^{i+1}\psi_{i+1} = \gamma\left(D_i\alpha^i\psi_i - \omega L_i\alpha^{i-1}\psi_{i-1}\right)$$

$$\left(1-\omega\right)D_i\psi_i + \omega U_i\alpha\psi_{i+1} = \gamma D_i\psi_i - \omega\frac{\gamma}{\alpha}L_i\psi_{i-1}$$

$$\omega\left(\frac{\gamma}{\alpha}L_i\psi_{i-1} + U_i\alpha\psi_{i+1}\right) = (\gamma + \omega - 1)D_i\psi_i$$

Properties of SOR Iteration Matrix (1/5)

$T_{SOR} = M^{-1}N = (D - \omega L)^{-1}\left[(1-\omega)D + \omega U\right]$ For which ω, ρ(T) gets minimized?

T : SOR Iteration Matrix

T_J : Jacobi Iteration Matrix

- Relation eigenvalues of Jacobi and SOR iteration materix
  - Let $T\phi = \gamma\phi$ and $T_J\psi = \lambda\psi$

$$T\phi = \gamma\phi$$

$$(D - \omega L)^{-1}\left[(1-\omega)D + \omega U\right]\phi = \gamma\phi$$

$$(1-\omega)D\phi + \omega U\phi = \gamma(D - \omega L)\phi$$

$$(1-\omega)D_i\phi_i + \omega U_i\phi_{i+1} = \gamma\left(D_i\phi_i - \omega L_i\phi_{i-1}\right)$$

$$\omega L_i\phi_{i-1} + \omega U_i\phi_{i+1} = (\gamma + \omega - 1)D_i\phi_i$$

Assume $\phi_i = \alpha^i\psi_i$

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_i \\ \vdots \\ \phi_N \end{bmatrix}, \psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_i \\ \vdots \\ \psi_N \end{bmatrix}$$

$$\omega\left(\frac{\gamma}{\alpha}L_i\psi_{i-1} + U_i\alpha\psi_{i+1}\right) = (\gamma + \omega - 1)D_i\psi_i$$

Properties of SOR Iteration Matrix (2/5)

$$\omega\left(\frac{\gamma}{\alpha}L_i\psi_{i-1}+U_i\alpha\psi_{i+1}\right)=(\gamma+\omega-1)D_i\psi_i$$

$$T_J\psi=\lambda\psi$$

$$L_i\psi_{i-1}+U_i\psi_{i+1}=\lambda D_i\psi_i$$

$$\omega\left(\frac{\gamma}{\alpha}L_i\psi_{i-1}+U_i\alpha\psi_{i+1}\right)=(\gamma+\omega-1)\frac{1}{\lambda}\left(L_i\psi_{i-1}+U_i\psi_{i+1}\right)$$

$$\omega\frac{\gamma}{\alpha}=(\gamma+\omega-1)\frac{1}{\lambda}$$

$$\rightarrow\frac{\gamma}{\alpha^2}=1\rightarrow\alpha=\sqrt{\gamma}$$

$$\omega\alpha=(\gamma+\omega-1)\frac{1}{\lambda}$$

$$\sqrt{\gamma}\omega\lambda=(\gamma+\omega-1)$$

$$\phi_i=\sqrt{\gamma}^{\,i}\psi_i$$

$$\phi=\begin{bmatrix}\sqrt{\gamma}\psi_1\\ \gamma\psi_2\\ \sqrt{\gamma}^{\,i}\psi_i\\ \vdots\\ \sqrt{\gamma}^{\,N}\psi_N\end{bmatrix}$$

If $\omega=1$, T is iteration matrix of Gauss seidel

$$\sqrt{\gamma}\lambda=\gamma\quad\rightarrow\quad\gamma=\lambda^2$$

$\rightarrow$ Gauss Seidel is two times faster than Jacobi.

Properties of SOR Iteration Matrix (3/5)

$$\sqrt{\gamma}\,\omega\lambda = (\gamma+\omega-1) \;\rightarrow\; \gamma\omega^2\lambda^2 = (\gamma+\omega-1)^2 \;\rightarrow$$ Find root γ which is dependent on ω.

$$\gamma^2 + \left[2(\omega-1)-\omega^2\lambda^2\right]\gamma + (\omega-1)^2 = 0$$

$$\gamma^2 - 2\left[(1-\omega)+\frac{\omega^2\lambda^2}{2}\right]\gamma + (\omega-1)^2 = 0$$

$$\gamma = (1-\omega)+\frac{\omega^2\lambda^2}{2} \pm \sqrt{\omega^2\lambda^2(1-\omega)+\left(\frac{\omega^2\lambda^2}{2}\right)^2} = (1-\omega)+\frac{\omega^2\lambda^2}{2} \pm \omega\left|\lambda\right|\sqrt{(1-\omega)+\frac{\omega^2\lambda^2}{4}}$$

$$\begin{cases} \gamma_+ = (1-\omega)+\dfrac{\omega^2\lambda^2}{2}+\omega\lambda\sqrt{\delta} \\ \gamma_- = (1-\omega)+\dfrac{\omega^2\lambda^2}{2}-\omega\lambda\sqrt{\delta} \end{cases}$$

where $\delta = (1-\omega)+\dfrac{\omega^2\lambda^2}{4}$ when will $|\gamma|$ be minimum?

Properties of SOR Iteration Matrix (4/5)

- Proof of Property-conti.
  - If $\delta<0$, then γ is complex.

$$\delta = 1-\omega+\frac{\omega^2\lambda^2}{4}<0$$

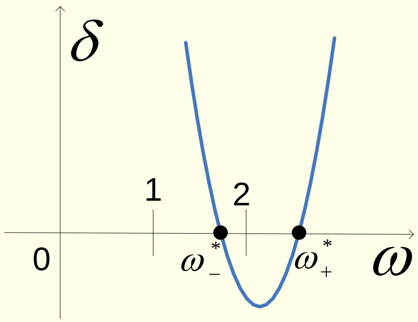
$$\left|\gamma\right|^2=\left(1-\omega+\frac{\omega^2\lambda^2}{2}\right)^2-\omega^2\lambda^2\left(-\delta\right)\rightarrow\gamma=\omega-1$$

  - If $\delta=0$

$$\delta = 1-\omega+\frac{\omega^2\lambda^2}{4}=0$$

$$\omega^2\lambda^2-4\omega+4=0$$

$$\omega=\frac{1}{\lambda^2}\left(2\pm\sqrt{4-4\lambda^2}\right)=\frac{2}{\lambda^2}\left(1\pm\sqrt{1-\lambda^2}\right)$$



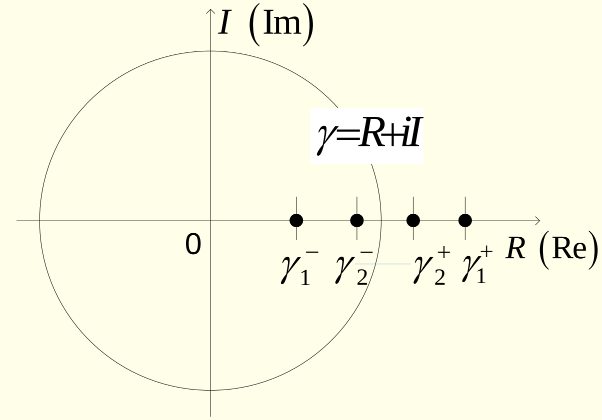
$$\begin{cases}\omega_+^*=\dfrac{2}{\lambda^2}\left(1+\sqrt{1-\lambda^2}\right)=\dfrac{2}{1-\sqrt{1-\lambda^2}}>2\\[2ex]\omega_-^*=\dfrac{2}{\lambda^2}\left(1-\sqrt{1-\lambda^2}\right)=\dfrac{2}{1+\sqrt{1-\lambda^2}}>1\end{cases}$$

$$\left(\lambda=\rho\left(T_J\right)<1\right)$$

Properties of SOR Iteration Matrix (5/5)

- Proof of Property-conti.

$$As\ \omega \uparrow \rightarrow \gamma^{+} = (1-\omega) + \frac{\omega^2\lambda^2}{2} \uparrow \quad \Rightarrow \text{ Choose smaller } \omega\left(\omega_{-}^{*}\right)$$



$$\omega_{opt} = \omega_{-}^{*} = \frac{2}{1+\sqrt{1-\lambda^2}} = \frac{2}{1+\sqrt{1-\rho_{GS}}}$$

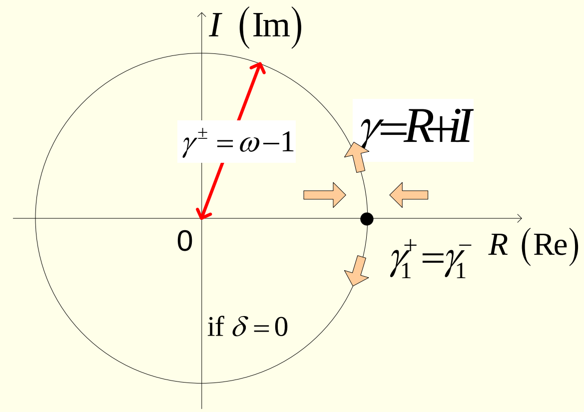
– When $\omega = \omega_{opt}$

$$\because \gamma = \omega - 1 > 1 \text{ for } \omega_{+}^{*}$$

$$\delta = 1-\omega+\frac{\omega^2\lambda^2}{4} = 0$$

$$\omega^2\lambda^2 = 4(\omega-1)$$

$$\begin{aligned}\gamma_{\pm} &= (1-\omega)+\frac{\omega^2\lambda^2}{2} \pm \omega\lambda\sqrt{\delta} \\ &= (1-\omega)+\frac{\omega^2\lambda^2}{2} \\ &= (1-\omega)+2(\omega-1) \\ &= \omega-1\end{aligned}$$



Estimation of ρ_{GS}

- Eigenvalue Problem for Iteration Matrix

$$M^{-1}N\phi = \lambda\phi \qquad Inverse\ Power\ Method : M\phi^{(l)} = \frac{1}{\lambda}N\phi^{(l-1)}\ ? \qquad M = D - L$$

- Solve an arbitrary linear system iteratively

$$M\phi^{(l)} = \tilde{b} + N\phi^{(l-1)}$$
$$M\phi^{(l-1)} = \tilde{b} + N\phi^{(l-2)}$$

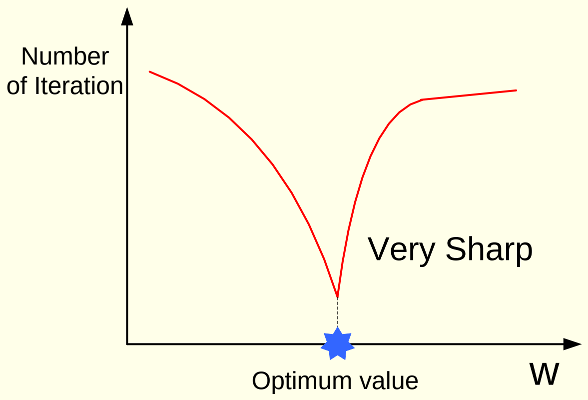
$$M(\phi^{(l)} - \phi^{(l-1)}) = N(\phi^{(l-1)} - \phi^{(l-2)}) \rightarrow \tilde{e}^{(l)} = M^{-1}N\tilde{e}^{(l-1)} = T\tilde{e}^{(l-1)} = \rho\tilde{e}^{(l-1)}$$

$$\rho(M^{-1}N) = \frac{\left\|\tilde{e}^{(l)}\right\|}{\left\|\tilde{e}^{(l-1)}\right\|}$$

- Practical way to obtain ρ_{GS}
  - perform a few G-S iteration with b=0
  - monitor pseudo error reduction, then determine ρ_{GS}

Cyclic-Chebyshev Iterative Method

- Disadvantage of SOR
  - Good performance with ω_{opt}
  - Bad performance when ω is out of an ω_{opt}, even a little.
  - Time and cost are needed for obtaining ω_{opt}



- Normal Iterative Scheme

$$A\phi = b$$

$$(M - N)\phi = b$$

$$M\phi^{(l)} = N\phi^{(l-1)} + b$$

$$\phi^{(l)} = M^{-1}N\phi^{(l-1)} + M^{-1}b$$

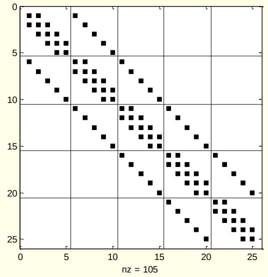
$$= T\phi^{(l-1)} + g \qquad \textit{for Jacobi}$$

$$= D^{-1}(L + U)\phi^{(l-1)} + g$$

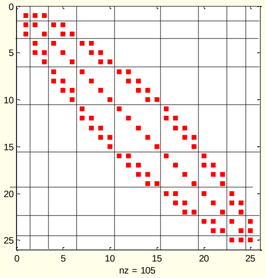
$$= D^{-1}L\phi^{(l-1)} + D^{-1}U\phi^{(l-1)} + g$$

Cyclic-Chebyshev Iterative Method

- Consistently ordered scheme for Jacobi



[Natural ordering for Line Jacobi]



[Cuthill-Mckee ordering]

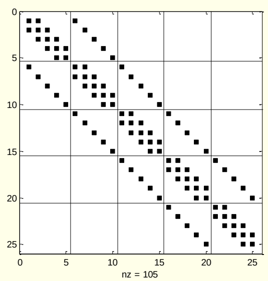
$$T = D^{-1}(L+U) =$$

$$\begin{bmatrix} D_1^{-1} & & & & & & \\ & D_2^{-1} & & & & & \\ & & D_3^{-1} & & & & \\ & & & D_4^{-1} & & & \\ & & & & D_5^{-1} & & \\ & & & & & D_6^{-1} & \\ & & & & & & D_7^{-1} \end{bmatrix} \begin{bmatrix} 0 & U_1 & & & & & \\ L_2 & 0 & U_2 & & & & \\ & L_3 & 0 & U_3 & & & \\ & & L_4 & 0 & U_4 & & \\ & & & L_5 & 0 & U_5 & \\ & & & & L_6 & 0 & U_6 \\ & & & & & L_7 & 0 \end{bmatrix} = \begin{bmatrix} 0 & D_1^{-1}U_1 & & & & & \\ D_2^{-1}L_2 & 0 & D_2^{-1}U_2 & & & & \\ & D_3^{-1}L_3 & 0 & D_3^{-1}U_3 & & & \\ & & D_4^{-1}L_4 & 0 & D_4^{-1}U_4 & & \\ & & & D_5^{-1}L_5 & 0 & D_5^{-1}U_5 & \\ & & & & D_6^{-1}L_6 & 0 & D_6^{-1}U_6 \\ & & & & & D_7^{-1}L_7 & 0 \end{bmatrix}$$

Cyclic-Chebyshev Iterative Method

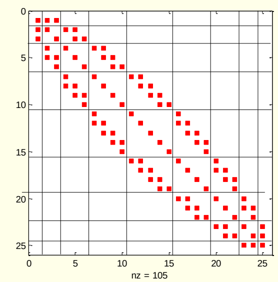
- Solution vector of consistently ordered scheme for Jacobi

[Natural ordering for Line Jacobi]



$$\phi = \begin{bmatrix} \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_m \end{bmatrix} \\ \begin{bmatrix} \phi_{m+1} \\ \vdots \\ \phi_{2m} \end{bmatrix} \\ \begin{bmatrix} \phi_{2m+1} \\ \vdots \\ \phi_{3m} \end{bmatrix} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} \vec{\phi}_1 \\ \vec{\phi}_2 \\ \vec{\phi}_3 \\ \vec{\phi}_4 \\ \vdots \end{bmatrix}$$

[Cuthill-Mckee ordering]



$$\phi = \begin{bmatrix} [\phi_1] \\ \begin{bmatrix} \phi_2 \\ \phi_3 \end{bmatrix} \\ \begin{bmatrix} \phi_4 \\ \phi_5 \\ \phi_6 \end{bmatrix} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} \vec{\phi}_1 \\ \vec{\phi}_2 \\ \vec{\phi}_3 \\ \vec{\phi}_4 \\ \vec{\phi}_5 \\ \vdots \end{bmatrix}$$

Cyclic-Chebyshev Iterative Method

- Change the sequence of variables

$$
\begin{bmatrix} \vec{\phi}_1 \\ \vec{\phi}_2 \\ \vec{\phi}_3 \\ \vec{\phi}_4 \\ \vec{\phi}_5 \\ \vec{\phi}_6 \\ \vec{\phi}_7 \end{bmatrix}
=
\begin{bmatrix}
0 & D_1^{-1}U_1 & & & & & \\
D_2^{-1}L_2 & 0 & D_2^{-1}U_2 & & & & \\
 & D_3^{-1}L_3 & 0 & D_3^{-1}U_3 & & & \\
 & & D_4^{-1}L_4 & 0 & D_4^{-1}U_4 & & \\
 & & & D_5^{-1}L_5 & 0 & D_5^{-1}U_5 & \\
 & & & & D_6^{-1}L_6 & 0 & D_6^{-1}U_6 \\
 & & & & & D_7^{-1}L_7 & 0
\end{bmatrix}
\begin{bmatrix} \vec{\phi}_1 \\ \vec{\phi}_2 \\ \vec{\phi}_3 \\ \vec{\phi}_4 \\ \vec{\phi}_5 \\ \vec{\phi}_6 \\ \vec{\phi}_7 \end{bmatrix}
+ \vec{g}
$$

$$
\left(
\begin{array}{l}
\vec{\phi}_1 = D_1^{-1}U_1\vec{\phi}_2 + \vec{g}_1 \\
\vec{\phi}_3 = D_3^{-1}L_3\vec{\phi}_2 + D_3^{-1}U_3\vec{\phi}_4 + \vec{g}_3 \\
\vec{\phi}_5 = D_5^{-1}L_5\vec{\phi}_4 + D_5^{-1}U_5\vec{\phi}_6 + \vec{g}_5 \\
\vdots
\end{array}
\right)
$$

$$
\phi^{(l)} = T\phi^{(l-1)} + \vec{g}
$$

$$
\vec{\phi}_i^{(l)} = D_i^{-1}L_i\vec{\phi}_{i-1}^{(l-1)} + D_i^{-1}U_i\vec{\phi}_{i+1}^{(l-1)} + \vec{g}
$$

$$
\begin{bmatrix}
\begin{bmatrix} \vec{\phi}_1 \\ \vec{\phi}_3 \\ \vec{\phi}_5 \\ \vec{\phi}_7 \end{bmatrix} \\
\begin{bmatrix} \vec{\phi}_2 \\ \vec{\phi}_4 \\ \vec{\phi}_6 \end{bmatrix}
\end{bmatrix}
=
\left[
\begin{array}{ccc|cccc}
D_1^{-1}U_1 & & & & & & \\
D_3^{-1}L_3 & D_3^{-1}U_3 & & & & & \\
 & D_5^{-1}L_5 & D_5^{-1}U_5 & & & & \\
 & & D_7^{-1}L_7 & 0 & & & \\
\hline
 & & & D_2^{-1}L_2 & D_2^{-1}U_2 & & \\
 & & & & D_4^{-1}L_4 & D_4^{-1}U_4 & \\
 & & & & & D_6^{-1}L_6 & D_6^{-1}U_6
\end{array}
\right]
\begin{bmatrix}
\begin{bmatrix} \vec{\phi}_2 \\ \vec{\phi}_4 \\ \vec{\phi}_6 \end{bmatrix} \\
\begin{bmatrix} \vec{\phi}_1 \\ \vec{\phi}_3 \\ \vec{\phi}_5 \\ \vec{\phi}_7 \end{bmatrix}
\end{bmatrix}
+ \vec{g}'
$$

$$
\begin{bmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{bmatrix}
=
\begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}
\begin{bmatrix} \tilde{\phi}_2 \\ \tilde{\phi}_1 \end{bmatrix}
+
\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}
$$

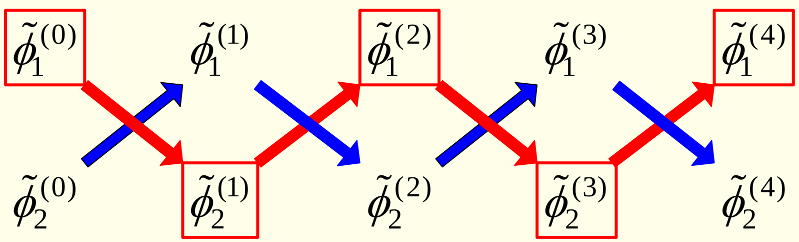
Cyclic-Chebyshev Iterative Method

- Change the order of variables

$$\begin{bmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{bmatrix} = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} \tilde{\phi}_2 \\ \tilde{\phi}_1 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \tilde{\phi}_1^{(l)} = T_1 \tilde{\phi}_2^{(l-1)} + g_1 \\ \tilde{\phi}_2^{(l)} = T_2 \tilde{\phi}_1^{(l-1)} + g_2 \end{cases}$$

  – Cyclic iterative method (advangtage of consistently ordering)



  – One path is enough.

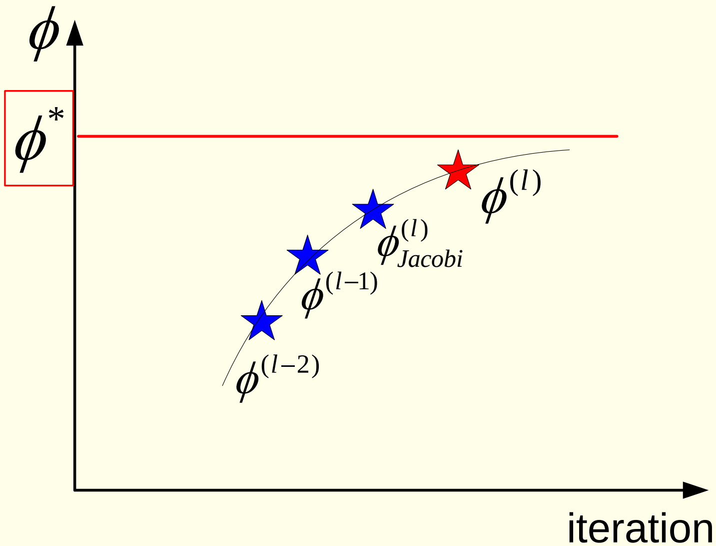
Cyclic-Chebyshev Iterative Method

- Extrapolation
  - Jacobi

$$\phi^{(l)}_{Jacobi} = T\phi^{(l-1)} + g$$

  - Extrapolation

$$\phi^{(l)} = \alpha^{(l)}\phi^{(l)}_{Jacobi} + \left(1-\alpha^{(l)}+\beta^{(l)}\right)\phi^{(l-1)} - \beta^{(l)}\phi^{(l-2)}$$



  - l-th error

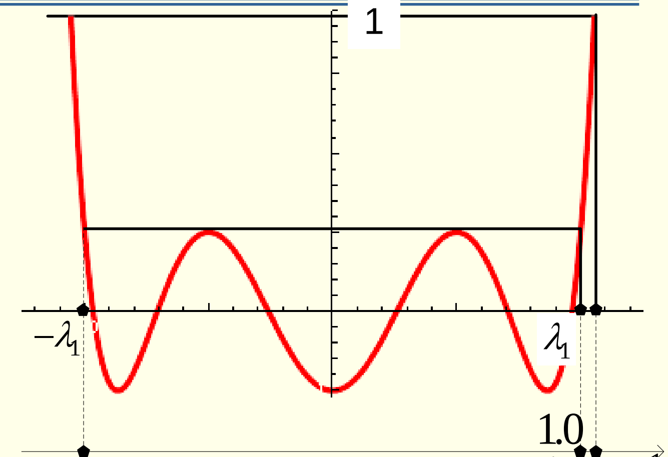
$$\begin{array}{rl} & \phi^{(l)} = \alpha^{(l)}\left(T\phi^{(l-1)}+g\right) + \left(1-\alpha^{(l)}+\beta^{(l)}\right)\phi^{(l-1)} - \beta^{(l)}\phi^{(l-2)} \\ - & \phi^{*} = \alpha^{(l)}\left(T\phi^{*}+g\right) + \left(1-\alpha^{(l)}+\beta^{(l)}\right)\phi^{*} - \beta^{(l)}\phi^{*} \\ \hline & e^{(l)} = \alpha^{(l)}Te^{(l-1)} + \left(1-\alpha^{(l)}+\beta^{(l)}\right)e^{(l-1)} - \beta^{(l)}e^{(l-2)} \end{array}$$

$\left(\phi^{*}\ :\ \text{Exact solution}\right)$

$$e^{(l)} = \sum_{i=1}^{n} p^{(l)}(\lambda_i)c_i u_i \qquad with\ e^{(0)} = \sum_{i=1}^{n} c_i u_i$$

  - Should minimize the l-th error $\rightarrow$ Chebyshev polynomial

Cyclic-Chebyshev Iterative Method



– Change of variable

$$\left(\gamma_i = \frac{\lambda_i}{\lambda_1} \quad , \quad \gamma_0 = \frac{1}{\lambda_1} \right)$$

$$e^{(l)} = \sum_{i=1}^{n} p^{(l)}(\lambda_i) c_i u_i = \sum_{i=1}^{n} \eta^{(l)}(\gamma_i) c_i u_i$$

$$= \eta^{(l)}(\gamma_0) \sum_{i=1}^{n} \frac{\eta^{(l)}(\gamma_i)}{\eta^{(l)}(\gamma_0)} c_i u_i \quad \rightarrow \eta^{(l)}(\gamma) = \frac{T_l(\gamma)}{T_l(\gamma_1)} = \tilde{T}_l(\gamma)$$ for minimization

• Recurrence relation of Chebyshev polynomial

$$T_l(\gamma) = 2\gamma T_{l-1}(\gamma) - T_{l-2}(\gamma)$$

$$\frac{T_l(\gamma)}{T_l(\gamma_0)} = 2\gamma \frac{T_{l-1}(\gamma)}{T_l(\gamma_0)} - \frac{T_{l-2}(\gamma)}{T_l(\gamma_0)} \quad = 2\gamma \frac{T_{l-1}(\gamma_0)}{T_l(\gamma_0)} \frac{T_{l-1}(\gamma)}{T_{l-1}(\gamma_0)} - \frac{T_{l-2}(\gamma_0)}{T_l(\gamma_0)} \frac{T_{l-2}(\gamma)}{T_{l-2}(\gamma_0)}$$

$$\tilde{T}_l(\gamma) = 2\gamma \frac{T_{l-1}(\gamma_0)}{T_l(\gamma_0)} \tilde{T}_{l-1}(\gamma) - \frac{T_{l-2}(\gamma_0)}{T_l(\gamma_0)} \tilde{T}_{l-2}(\gamma)$$

Cyclic-Chebyshev Iterative Method

- Extrapolation parameters
  - Compare two formula

$$\begin{cases} \tilde{T}_l(\gamma) = 2\gamma \dfrac{T_{l-1}(\gamma_0)}{T_l(\gamma_0)} \tilde{T}_{l-1}(\gamma) - \dfrac{T_{l-2}(\gamma_0)}{T_l(\gamma_0)} \tilde{T}_{l-2}(\gamma) \\ \eta^{(l)}(\gamma) = \alpha^{(l)} \lambda_1 \gamma \eta^{(l-1)}(\gamma) + \left(1 - \alpha^{(l)} + \beta^{(l)}\right) \eta^{(l-1)}(\gamma) - \beta^{(l)} \eta^{(l-2)}(\gamma) \end{cases}$$

$$\Rightarrow \quad \boxed{\alpha^{(l)} \lambda_1 = 2 \frac{T_{l-1}(\gamma_0)}{T_l(\gamma_0)}} \qquad \alpha^{(l)} = \frac{2}{\lambda_1} \frac{T_{l-1}(\gamma_0)}{T_l(\gamma_0)} = \frac{2}{\lambda_1} \frac{T_{l-1}\left(\dfrac{1}{\lambda_1}\right)}{T_l\left(\dfrac{1}{\lambda_1}\right)} = 2\gamma_0 \frac{T_{l-1}(\gamma_0)}{T_l(\gamma_0)}$$

$$\beta^{(l)} = \frac{T_{l-2}(\gamma_0)}{T_l(\gamma_0)} = \frac{T_{l-2}\left(\dfrac{1}{\lambda_1}\right)}{T_l\left(\dfrac{1}{\lambda_1}\right)}$$

-Aftermath

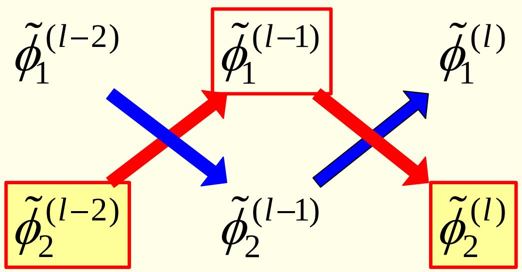
$$1 - \alpha^{(l)} + \beta^{(l)} = 1 - 2\gamma_0 \frac{T_{l-1}(\gamma_0)}{T_l(\gamma_0)} + \frac{T_{l-2}(\gamma_0)}{T_l(\gamma_0)} = 1 - \frac{T_l(\gamma_0) + T_{l-2}(\gamma_0)}{T_l(\gamma_0)} + \frac{T_{l-2}(\gamma_0)}{T_l(\gamma_0)} = 0$$

Cyclic-Chebyshev Iterative Method

- Advantage of cyclic iterative method for applying Chebyshev

$$\phi^{(l)} = \alpha^{(l)}\phi^{(l)}_{Jacobi} + \left(1 - \alpha^{(l)} + \beta^{(l)}\right)\phi^{(l-1)} - \beta^{(l)}\phi^{(l-2)}$$

$$\Rightarrow \quad \phi^{(l)} = \alpha^{(l)}\phi^{(l)}_{Jacobi} - \beta^{(l)}\phi^{(l-2)} \qquad \left(\because\ 1 - \alpha^{(l)} + \beta^{(l)} = 0\right)$$



- Do it only for half.
- Convergence ratio $\quad \rho_{CCSi} = \frac{1}{2}\left(\omega_{opt} - 1\right)$
- Total calculating quantity

$$\left[\text{calculate half (each step)}\right] \times \left[\text{calculate double (iteration)}\right] = 1$$

- Good for not exact ω_{opt}

Consistently Ordered Matrix

Let the $N \times N$ matrix A be partitioned into the form

$$A = \begin{bmatrix} A_{1,1} & \cdots & A_{1,q} \\ \vdots & & \vdots \\ A_{q,1} & \cdots & A_{q,q} \end{bmatrix}, \qquad (9\text{-}2.1)$$

where $A_{i,j}$ is an $n_i \times n_j$ submatrix and $n_1 + \cdots + n_q = N$.

Definition 9-2.1 The $q \times q$ block matrix A of (9-2.1) is said to have *Property* $\mathscr{A}$ if there exists two disjoint nonempty subsets S_R and S_B of $\{1, 2, \ldots, q\}$ such that $S_R \cup S_B = \{1, 2, \ldots, q\}$ and such that if† $A_{i,j} \neq 0$ and $i \neq j$, then $i \in S_R$ and $j \in S_B$ or $i \in S_B$ and $j \in S_R$.

Definition 9-2.2. The $q \times q$ block matrix A of (9-2.1) is said to be *consistently ordered* if for some t there exist disjoint nonempty subsets $S_1, \ldots, S_t$ of $\{1, 2, \ldots, q\}$ such that $\bigcup_{i=1}^{t} S_i = \{1, \ldots, q\}$ and such that if $A_{i,j} \neq 0$ with $i \neq j$ and S_k is the subset containing i, then $j \in S_{k+1}$ if $j > i$ and $j \in S_{k-1}$ if $j < i$.