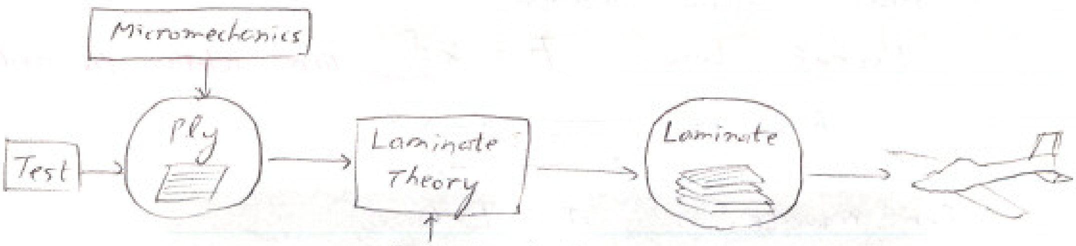
III. Ply Elasticity

Look at 3-D and 2-D Anisotropic Elasticity.

See Jones, Chap. 2 & Appendix A



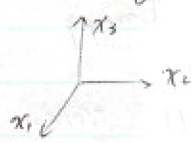
Make a brief review — 1. Jones book.

2. Bisplinghoff, Mar & Pian, "Statics of Deformable Bodies" → (tensor notation)

3. Herrmann "Applied Anisotropic Elasticity"

∘ Notation

Right hand coord. system, x_m

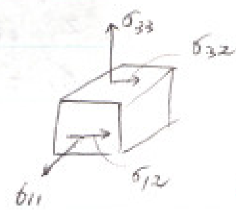


Components of stress, σ_{mn} — "Stress Tensor"

2 subscripts → 2nd order

6 independent components

Extensional	Shear
σ_{11}	$\sigma_{12} = \sigma_{21}$
σ_{22}	$\sigma_{23} = \sigma_{32}$
σ_{33}	$\sigma_{31} = \sigma_{13}$



σ_{mn} — m: face, n: direction

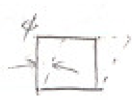
Components of Strain, ϵ_{mn} — "Strain Tensor"

2 Subscripts → 2nd order

6 independent components

Extensional	Shear
ϵ_{11}	$\epsilon_{12} = \epsilon_{21}$
ϵ_{22}	$\epsilon_{23} = \epsilon_{32}$
ϵ_{33}	$\epsilon_{31} = \epsilon_{13}$

Extension  $\epsilon_{11} \simeq \frac{u_1}{dx_1}$

Shear  $\epsilon_{12} = \frac{\phi}{2}$

Note : stress tensor symmetric : $\sigma_{mn} = \sigma_{nm}$ by equilibrium
strain tensor symmetric : $\epsilon_{mn} = \epsilon_{nm}$ by Geometrical consideration

∘ Stress - Strain Relations

Hooke's Law, $F = k\delta$ linear relation for rod

Can rewrite as $\sigma = E\epsilon$

where σ = stress = F/A

ϵ = strain = δ / l

E : modulus of Elasticity

Extending to 3-D stress, have "Generalized Hooke's Law"

$\sigma_{mn} = E_{mnpq}\,\epsilon_{pq}$

E_{mnpq} → "Elasticity tensor"

$3 \times 3 \times 3 \times 3 = 81$ components

Recall Tensor Notation Rules

Latin subscripts $(m, n, p, q, r, \cdots) \rightarrow 1, 2, 3$

Greek subscripts $(\alpha, \beta, \gamma, \cdots) \rightarrow 1, 2$

1. Subscripts that appear only once in a term are either 1, 2, or 3

$$\sigma_i = f(x_i) \longrightarrow \begin{cases} \sigma_1 = f(x_1) \\ \sigma_2 = f(x_2) \\ \sigma_3 = f(x_3) \end{cases}$$

2. Subscripts repeated in a term are "dummy" subscripts → Sum on them

$$E_{ij}\epsilon_j = E_{i1}\epsilon_1 + E_{i2}\epsilon_2 + E_{i3}\epsilon_3 = \sum_{j=1}^{3} E_{ij}\epsilon_j$$

3. No subscript can appear more than twice in a term

$f_i C_{ij} D_i$ ✗

So, in general Stress Strain $\sigma_{mn} = E_{mnpq}\cdot\epsilon_{pq}$
9 equations are represented

$$\sigma_{11} = E_{11pq}\,\epsilon_{pq}$$
$$\sigma_{12} = E_{12pq}\,\epsilon_{pq}$$
$$\sigma_{13} = \vdots$$
$$\sigma_{21} = E_{21pq}\,\epsilon_{pq}$$
$$\vdots$$
$$\sigma_{31} = E_{31pq}\,\epsilon_{pq}$$
$$\vdots$$
$$\sigma_{33} = E_{33pq}\,\epsilon_{pq}$$

Look at 1st equations and sum over p

$$\sigma_{11} = E_{11 1q}\,\epsilon_{1q} + E_{112q}\,\epsilon_{2q} + E_{113q}\,\epsilon_{3q}$$

sum over q

$$\sigma_{11} = E_{1111}\,\epsilon_{11} + E_{1112}\,\epsilon_{12} + E_{1113}\,\epsilon_{13} + E_{1121}\,\epsilon_{21} + E_{1122}\,\epsilon_{22} + E_{1123}\,\epsilon_{23} + E_{1131}\,\epsilon_{31} + E_{1132}\,\epsilon_{32} + E_{1133}\,\epsilon_{33}$$

9 Eqn.s $\longrightarrow$ 9 Terms $\Rightarrow$ 81 E_{mnpq}'s

Symmetries reduce number of independent E_{mnpq}

$$\sigma_{mn} = \sigma_{nm} \qquad\qquad \epsilon_{pq} = \epsilon_{qp}$$
$$\downarrow \qquad\qquad \downarrow$$
$$E_{mnpq} = E_{nmpq} \qquad\qquad E_{mnpq} = E_{mnqp}$$

(Equilibrium consideration)

Overall symmetry (Energy considerations)

$$E_{mnpq} = E_{pqmn}$$

So at most 21 independent constants

E_{1111}	E_{1122}	E_{1212}	E_{1213}	E_{1112}	E_{2212}	E_{3312}
E_{2222}	E_{2233}	E_{1313}	E_{1323}	E_{1113}	E_{2213}	E_{3313}
E_{3333}	E_{3311}	E_{2323}	E_{2312}	E_{1123}	E_{2223}	E_{3323}
Extension - Extension		Shear - Shear		Coupling Shear - Extension		

A Material with all 21 independent constants is Anisotropic

Have used Tensor Notation
to here Many books use a "contracted" notation
(Jones, Tsai, etc.)
Also called "Engineering" Notation
3 Major Difference

① Subscript changes

Tensor		Contracted		Physical
11	→	1	→	Extens. in 1
22	→	2	→	" 2
33	→	3	→	" 3
23	→	4	→	Rotate about 1
31	→	5	→	" 2
12	→	6	→	" 3

② Shear strain changes
Tensor shear strain is $\frac{1}{2}$ of Engineering shear strain.
We change the notation from ϵ to γ

Engineering		Tensor	Contracted
γ_{12}	=	$\epsilon_{12} + \epsilon_{21}$	ϵ_6
γ_{13}	=	$\epsilon_{13} + \epsilon_{31}$	ϵ_5
γ_{23}	=		ϵ_4

③ Elasticity Constants represented by C_{ij} instead E_{mnp}
(Still 21 components)

Tensor		Engineering
E_{mnpq}	→	C_{ij}
mn	→	i
pq	→	j

The "Generalized Hooke's Law" is

$$\sigma_{mn} = E_{mnpq}\,\epsilon_{pq} \quad \text{(Tensor notation)}$$

$$\sigma_i = C_{ij}\,\epsilon_j \quad \text{(Engineering notation)}$$

Still use summation convention

$$\sigma_i = \sum_j C_{ij}\,\epsilon_j = C_{ij}\,\epsilon_j$$

$$j = 1, 2, 3, \dots, 6$$

$C_{ij} = C_{ji}$ ← Symmetry of the Elasticity constants still applies $E_{mnpq} = E_{pqmn}$

other symm.s in $E_{mnpq} = E_{nmpq}$, etc.

all automatically included in Engineering Notation.

Be careful of 2 in shear strain.

$$2\epsilon_{mn} = \gamma_{mn}$$

- Can see usefulness of Engineering Notation by writing Tensor notation in matrix form.

$$\underbrace{\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix}}_{=\underset{\sim}{\sigma}} = \underbrace{\begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & 2E_{1123} & 2E_{1131} & 2E_{1112} \\ \vdots & & & \vdots & & \\ E_{3311} & & & & & \\ E_{2311} & & & 2E_{2323} & & \\ \vdots & & & \vdots & & \\ E_{1211} & & & 2E_{1223} & & \end{bmatrix}}_{=\underset{\sim}{E}} \underbrace{\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{31} \\ \epsilon_{12} \end{Bmatrix}}_{=\underset{\sim}{\epsilon}}$$

$\underset{\sim}{E}$ matrix not symmetric, inconvenient

Contracted (Engineering) more convenient

$$\underset{\sim}{\sigma} = \underset{\sim}{C}\,\underset{\sim}{\epsilon}$$

$$\sigma_i = C_{ij}\,\epsilon_j$$

Note $\underset{\gamma_{23}}{\underset{\parallel}{\epsilon_4}} = 2\,\epsilon_{23}$ etc.

↑ Tensor

$$\sigma_{11} = \cdots\ 2E_{1123}\,\epsilon_{23} + \cdots$$

$$\sigma_1 = \cdots\ C_{14}\,\epsilon_4 + \cdots \qquad \epsilon_4 = 2\epsilon_{23}, \text{ etc.}$$

C is symmetric.

$$C_{14} = E_{1123}$$

- COMPLIANCE

Just as we have $\sigma_{mn} = E_{mnpq}\,\epsilon_{pq}$

Also have inverse $\epsilon_{mn} = S_{mnpq}\,\sigma_{pq}$

S_{mnpq} → Compliance tensor

$$\underset{\sim}{\sigma} = \underset{\sim}{E}\,\underset{\sim}{\epsilon}$$

$$\underset{\sim}{\epsilon} = \underbrace{\underset{\sim}{E}^{-1}}_{=\underset{\sim}{S}}\,\underset{\sim}{\sigma}$$

Same symmetries as E_{mnpq}

Writing out Tensor Relations as matrices,

$$\underbrace{\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \vdots \\ \sigma_{12} \end{Bmatrix}}_{\sigma} = \underbrace{\begin{bmatrix} E_{1111} & E_{1122} & \cdots & 2E_{1112} \\ \vdots & & & \vdots \\ E_{12} & & & 2E_{1212} \end{bmatrix}}_{E} \underbrace{\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \vdots \\ \epsilon_{12} \end{Bmatrix}}_{\epsilon}$$

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \vdots \\ \epsilon_{12} \end{Bmatrix}_{6\times 1} = \begin{bmatrix} S_{1111} & & 2S_{1123} & \\ & & & \\ & & 2S_{1223} & 2S_{1212} \end{bmatrix}_{6\times 6} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \vdots \\ \sigma_{12} \end{Bmatrix}_{6\times 1}$$

Look @ Problem Set. 3

Wish to write in Engineering Notation

$$\begin{Bmatrix} \epsilon_{1} \\ \epsilon_{2} \\ \vdots \\ \epsilon_{6} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & & & & & \\ \vdots & & & & & \\ S_{61} & & & & & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{1} \\ \sigma_{2} \\ \vdots \\ \sigma_{6} \end{Bmatrix}$$

Symmetric relate S_{ij} to S_{mnpq}

For Elasticity Matrix, use

$$\begin{Bmatrix} \sigma_{1} \\ \sigma_{2} \\ \vdots \\ \sigma_{6} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{16} \\ & & & \\ C_{61} & & & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{1} \\ \epsilon_{2} \\ \vdots \\ \epsilon_{6} \end{Bmatrix}$$

Symm.

$$C_{ij}^{-1} = S_{ij}$$

All symmetric matrices in Engineering notation.

☐ Problem Set #1

$$\text{Weight Figure of Merit} = \frac{\sigma_{ULT}}{(\text{Specific Gravity})}$$

$$\text{Cost} \quad '' \quad = \frac{\sigma_{ULT}}{(S\cdot G)\,(\$/lb)}$$

Stress - Strain Relations (Engineering Notation)

$$\underset{\sim}{\sigma} = \underset{\sim}{C}\,\underset{\sim}{\epsilon}\,, \qquad \underset{\sim}{\epsilon} = \underset{\sim}{S}\,\underset{\sim}{\sigma}$$

where

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_6 \end{bmatrix} \qquad \underset{\sim}{\epsilon} = \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_6 \end{Bmatrix}$$

$\sigma_6 = \sigma_{12}$, etc. $\qquad \epsilon_6 = \gamma_{12} = 2\epsilon_{12}$

$\underset{\sim}{C}$: elasticity $\}$ 6×6 Symm. matrices

$\underset{\sim}{S}$: compliance $\}$ 21 Independent constants

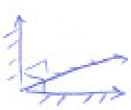
$$\underset{\sim}{S} = \underset{\sim}{C}^{-1}$$

- Types of Materials

- Fully Anisotropic → 21 constants

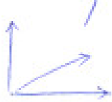
Grain along non-orthogonal axes.
Different stiffness along each direction
(Some crystals)

- Monoclinic Material → 13 constants

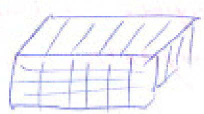


1 axis ⊥ other two
Different stiffness along each direction
(some crystals, some composites)

- Orthotropic Material → 9 constants



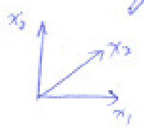
3 axis ⊥ to each other
Different stiffness along each direction

Important Practical Case →  Crystals. Composites.

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & & & \\ & S_{22} & S_{23} & & 0 & \\ & & S_{33} & & & \\ & & & S_{44} & & 0 \\ & \text{Symm.} & & & S_{55} & \\ & & & & & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}$$

 but no shear 

- Transversely Isotropic Material → 5 constants



3 perpendicular axes

x_1 is stiffer than $x_2 = x_3$

same stiffness any direction in x_2, x_3 plane

Wood or composite

Like ortho-tropic but additionally

$S_{33} = S_{22}$

$S_{13} = S_{12}$

$S_{55} = S_{66}$

$S_{44} = 2(S_{22} - S_{23}) \longleftarrow G = \frac{E}{2(1+\nu)}$

$$\underline{S} = \left[\begin{array}{ccc|ccc} S_{11} & S_{12} & S_{12} & & & \\ & S_{22} & S_{23} & & 0 & \\ & & S_{22} & & & \\ \hline & & & 2(S_{22}-S_{23}) & & \\ & \text{Symm.} & & & S_{55} & \\ & & & & & S_{55} \end{array}\right]$$

5 constants

- Isotropic Material → 2 constants

Same properties in all directions

Most metals, Resin

Many crystals randomly oriented
Polycrystalone material

Same as Transversely Isotropic, but additionally

$S_{22} = S_{11}$

$S_{13} = S_{12}$

$S_{55} = S_{44} = 2(S_{11} - S_{12})$

$S_{66} = S_{44} = 2(S_{11} - S_{12})$

Only 2 constants S_{11} and S_{12}

S_{mn}'s traditionally expressed in terms of Modulus of Elasticity E and Poisson's Ratio ν

$$S_{11} = \frac{1}{E} \; ; \qquad S_{12} = -\frac{\nu}{E}$$

with these

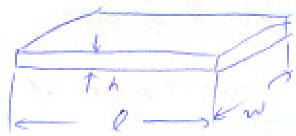
$$\epsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu\sigma_2 - \nu\sigma_3\right]$$

$$\vdots$$

$$\epsilon_4 = \frac{2(1+\nu)}{E}\sigma_4, \quad \text{etc.}$$

• 2-Dim. Plane Stress Approximations

Many structures are thin. (plate)



$h \ll l, w$

Also, not heavily loaded through thickness

$$\underset{\sigma_{33}}{\sigma_3},\ \underset{\sigma_{23}}{\sigma_4},\ \underset{\sigma_{31}}{\sigma_5} \ll \sigma_1,\ \sigma_2,\ \underset{\sigma_{12}}{\sigma_6}$$

Assume $\sigma_3 = \sigma_4 = \sigma_5 = 0$ in stress-strain

Only deal with σ_1, σ_2, σ_6

in 3-D $\underset{\sim}{\sigma} = \underset{\sim}{C}\,\underset{\sim}{\epsilon}$ → 6×6 matrix

in 2-D $\underset{\sim}{\sigma} = \underset{\sim}{Q}\,\underset{\sim}{\epsilon}$ → 3×3 matrix

For Transversely 3×1 isotropic mat'l

$$\sigma_1 = C_{11}\epsilon_1 + C_{12}\epsilon_2 + C_{13}\epsilon_3 + 0 + 0 + 0$$

$$\sigma_2 = \text{etc.}$$

$$0 = \sigma_3 = C_{12}\epsilon_1 + C_{12}\epsilon_2 + C_{22}\epsilon_3 + 0 + 0 + 0$$

Solve for ϵ_3 and put into others

$$\sigma_6 = C_{66}\epsilon_6$$

$$\Rightarrow \left\{ \begin{aligned} \sigma_1 &= Q_{11}\epsilon_1 + Q_{12}\epsilon_2 \\ \sigma_2 &= Q_{21}\epsilon_1 + Q_{22}\epsilon_2 \\ \sigma_6 &= Q_{66}\epsilon_6 \end{aligned} \right\} \text{ Transversely isotropic}$$

In general, for fibers not along looks

$$\sigma_1 = Q_{11}\epsilon_1 + Q_{12}\epsilon_2 + Q\ \epsilon_6$$
$$\sigma_2 = \text{--}$$
$$\sigma_6 = Q_{61}\epsilon_1 + Q_{62}\epsilon_2 + Q_{66}\epsilon_6$$

$[Q]_{3\times3}$ matrix symm.

Homework Prob. → Relation between 3-D and 2-D

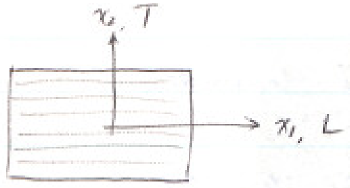
- Properties of Single Ply

Ply → flat → plane stress

2-D Stress - Strain Eqn.s are

$$\underset{\sim}{\sigma} = \underset{\sim}{Q}\ \underset{\sim}{\epsilon}$$

↳ 3 X 3 matrix (matrix of 6 constants)



L : longitudinal

T : transverse

On this set of axes — orthotropic

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{Bmatrix} \qquad 4 \text{ constants}$$

Also cloth (0/90) weave works this way, some funny products may not

From Strength of Materials, we are familiar with Engineering Constants

$$E_L \qquad E_T \qquad \nu_{LT} \qquad G_{LT}$$

These come from experimental tests.

Formal definitions from

$$\epsilon_1 = \frac{1}{E_L}(\sigma_1 - \nu_{LT}\sigma_2)$$
$$\epsilon_2 = \frac{1}{E_T}(\sigma_2 - \nu_{TL}\sigma_1)$$
$$\epsilon_6 = \frac{1}{G_{LT}}\sigma_6$$

Questions :

a) How to find Engineering Constants ?

b) How to relate them to Elastic Constants Q_{ij}

a) Tests for Engineering Constants

① Longitudinal Tests

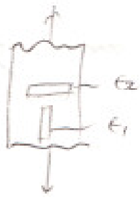
Apply known P (dead weight, calibrated machine)

long, narrow specimen

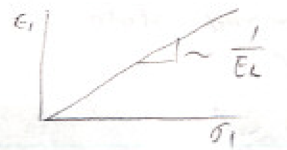
know $\sigma_1 = P/A$ (except near ends, reinforce there)

$\sigma_2 = 0, \quad \sigma_6 = 0$

Measure ϵ_1, ϵ_2 (ϵ_6 ?) with strain gages



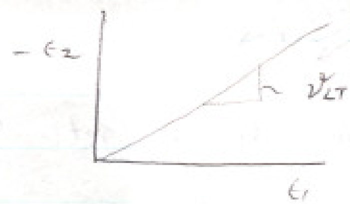
Plot



$$\epsilon_1 = \frac{\sigma_1}{E_L}$$

$$\epsilon_2 = -\nu_{LT}\epsilon_1$$

$$\epsilon_6 = 0$$



From this test $\rightarrow$ 2 constants

② Transverse Tension

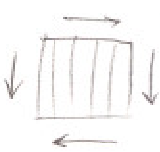
Same deal, apply known P

From this test, got E_T, ν_{TL}

③ Shear Tests

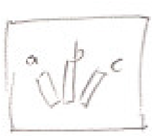
Apply know shear σ_6

(not too easy)

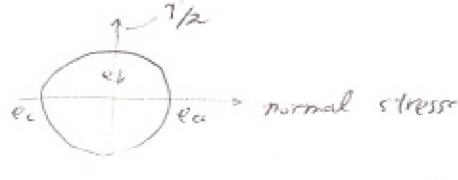


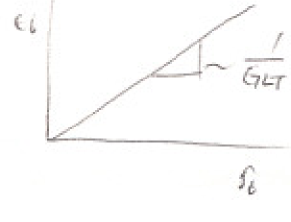
Measure shear strain with a rosette

$\epsilon_6 = e_c - e_a$



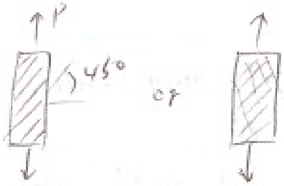
Mohr's circle





Shear Tests difficult to perform

Easier to test 45° Ply in tension



$[\pm 45]_s$ laminate

This gives a mixed state of stress in axis system of the material.

But can untangle to get G

So, we have,

$$\epsilon_1 = \frac{1}{E_L}\sigma_1 - \frac{\nu_{TL}}{E_T}\sigma_2$$

$$\epsilon_2 = -\frac{\nu_{LT}}{E_L}\sigma_1 + \frac{1}{E_T}\sigma_2$$

$$\epsilon_6 = \frac{1}{G_{LT}}\sigma_6$$

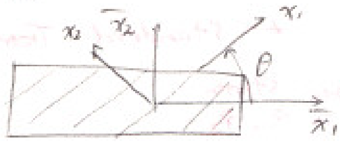
Because of symmetry

$$\frac{\nu_{TL}}{E_T} = \frac{\nu_{LT}}{E_L}$$

ν_{LT} : Major Poisson's Ratio ~ 0.3

$\nu_{TL} = E_T/E_L \cdot \nu_{LT} \approx 0.02$

- Rotation of Plies



$x_1, x_2 \to$ ply axes

$\bar{x}_1, \bar{x}_2 \to$ laminate axes (also x, y)

Ply at angle θ from lamina axis $\bar{x}_1$

($+\theta \to x_1$ going towards $\bar{x}_2$)

In $x_1 x_2$ (ply axes) $\hookrightarrow$ 2-D orthotropic Material

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{Bmatrix} \quad \text{or} \quad \underset{\sim}{Q} = \underset{\sim}{Q}\,\underset{\sim}{\epsilon}$$

Q's from E_L, E_T, ν_{LT}, G_{LT}

To find Stress - Strain in $\bar{x}_1, \bar{x}_2$ (laminate axes), first relate Stresses in 2 axis systems.

$$\underset{\substack{\uparrow \\ \text{stress tensor} \\ \text{in } x_1, x_2}}{\sigma_{mn}} = \underbrace{l_{m\bar{p}}\ l_{n\bar{q}}}_{\substack{\downarrow \\ \text{Direction Cosine}}}\ \bar{\sigma}_{pq} \quad \leftarrow \text{Standard Transform Law}$$

$\bar{\sigma}_{pq}$: stress in $\bar{x}_1, \bar{x}_2$

Direction Cosine $= \cos(\text{angle } x_m \text{ and } \bar{x}_p)$

Table of Cosines

	X_1	X_2
$\bar{X}_1$	$\cos\theta$	$\cos(90+\theta) = -\sin\theta$
$\bar{X}_2$	$\cos(90-\theta) = \sin\theta$	$\cos\theta$

$$\sigma_{11} = l_{1\bar{1}}\, l_{1\bar{1}}\, \bar{\sigma}_{11} + l_{1\bar{2}}\, l_{1\bar{2}}\, \bar{\sigma}_{22} + l_{1\bar{1}}\, l_{1\bar{2}}\, \bar{\sigma}_{12} + l_{1\bar{2}}\, l_{1\bar{1}}\, \bar{\sigma}_{21}$$

$$= \cos^2\theta\, \bar{\sigma}_{11} + \sin^2\theta\, \bar{\sigma}_{22} + \cos\theta \sin\theta\, \bar{\sigma}_{12} + \sin\theta\cos\theta\, \bar{\sigma}_{21}$$

σ_{22} = etc.

σ_{12} = etc.

So obtain

$$\underset{\text{Ply}}{\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}} = \underbrace{\begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & -cs & (c^2-s^2) \end{bmatrix}}_{T_\sigma} \underset{\text{Laminate}}{\begin{Bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_6 \end{Bmatrix}}$$

or $\underset{\sim}{\sigma} = \underset{\sim}{T_\sigma}\, \underset{\sim}{\bar{Q}}$

where $c = \cos\theta$, $s = \sin\theta$

Also for strain

$$\underset{\substack{\text{Tensor Strain}\\ \text{in } x_1, x_2}}{\epsilon_{mn}} = l_{m\bar{p}}\, l_{n\bar{q}}\, \underset{\substack{\text{Tensor Strain}\\ \text{in } \bar{x}_1, \bar{x}_2}}{\bar{\epsilon}_{pq}} \quad \leftarrow \text{Standard Transform Law}$$

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \frac{1}{2}\epsilon_6 \end{Bmatrix} = T_\sigma \begin{Bmatrix} \bar{\epsilon}_1 \\ \bar{\epsilon}_2 \\ \frac{1}{2}\bar{\epsilon}_6 \end{Bmatrix}$$

Recall
$\epsilon_{12} = \frac{1}{2}\gamma_{12} = \frac{1}{2}\epsilon_6$

Absorb the $\frac{1}{2}$ into T_σ gives

$$\underset{\text{Ply}}{\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{Bmatrix}} = \underbrace{\begin{bmatrix} c^2 & s^2 & cs \\ s^2 & c^2 & -cs \\ -2cs & 2cs & (c^2-s^2) \end{bmatrix}}_{T_\epsilon} \underset{\text{Laminate}}{\begin{Bmatrix} \bar{\epsilon}_1 \\ \bar{\epsilon}_2 \\ \bar{\epsilon}_6 \end{Bmatrix}}$$

or $\underset{\sim}{\epsilon} = \underset{\sim}{T_\epsilon}\, \underset{\sim}{\bar{\epsilon}}$

Placing into Ply axes Stress Strain

$$\sigma = Q\,\epsilon$$
$$T_\sigma \bar{\sigma} = Q\, T_\epsilon\, \bar{\epsilon}$$
$$\bar{\sigma} = \underbrace{T_\sigma^{-1} Q\, T_\epsilon}_{=\bar{Q}}\, \bar{\epsilon}$$

or $\bar{\sigma} = \bar{Q}\,\bar{\epsilon}$ $\leftarrow$ Stress - Strain Relation in laminate $\bar{x}_1, \bar{x}_2$ axes

Now Note Inverses

$$\begin{cases} T_\sigma^{-1} = T_\sigma(-\theta) = T_\epsilon^T \\ T_\epsilon^{-1} = T_\epsilon(-\theta) = T_\sigma^T \end{cases}$$

So Rotated Q matrix is

$$\boxed{\bar{Q} = T_\epsilon^T\, Q\, T_\epsilon}$$

Q fully populated now

Also in Jones Notation,

Laminated Axes $\bar{x}_1, \bar{x}_2 \rightarrow x, y$

Laminated Stress $\bar{\sigma}_i \rightarrow \sigma_x, \sigma_y, \tau_{xy}$

Laminated Strain $\bar{\epsilon}_i \rightarrow \epsilon_x, \epsilon_y, \gamma_{xy}$

Final Laminated Stress – Strain Eqn.s

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Multiplyis, out matrices

$$\bar{Q}_{11} = Q_{11}\cos^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}(\sin^4\theta + \cos^4\theta)$$

$\bar{Q}_{22}$ = etc.

$\bar{Q}_{66}$ = etc.

Similarly can transform compliances.

from $\epsilon = S\sigma$

obtain $\bar{\epsilon} = \bar{S}\,\bar{\sigma}$

where $\bar{S} = T_\sigma^T S\, T_\sigma$

- Alternate ways of Rotating

$$\bar{E}_{mnpq} = l_{mr}\, l_{ns}\, l_{pt}\, l_{qu}\, E_{rstu}$$

Also, can mathematically reduce $\bar{Q}_{ij}$ by $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, etc.

can then express

$$\bar{Q}_{ij} = A_{ij} + B_{ij}\cos 2\theta + C_{ij}\cos 4\theta + D\sin 2\theta + E\sin 4\theta$$

A, B, C, D, E → depend only on 4 invariants

4. Laminate Theory

Can now manipulate orthotropic plies in plane stress.

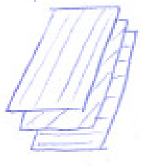
$$\underset{\sim}{\sigma} = \underset{\sim}{Q}\,\underset{\sim}{\epsilon}, \quad \bar{\sigma} = \bar{\underset{\sim}{Q}}\,\bar{\underset{\sim}{\epsilon}}$$

where $\bar{\underset{\sim}{Q}} = f(\underset{\sim}{Q}, \theta)$

Similarly, have

$$\underset{\sim}{\epsilon} = \underset{\sim}{S}\,\underset{\sim}{\sigma}, \text{ etc.}$$

But, composites are actually used as laminates



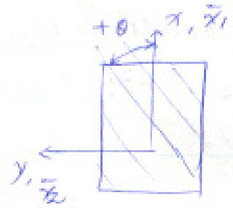
- Many plies (lamina) are arranged at many θ
- carry load, provide stiffness, strength, etc.

Note other laminates → electronics – circuit boards,

capacitors, active materials (piezoelectrics),
thermal barrier – coats for engine combustors, etc.

- Laminate Notation

Need keep track of ply orientation
Use a compact notation



θ : ply angle

Note: Usually 0° direction corresponds to principal loading direction.

Laminates specified as

$[\pm 30 / 0_2]_s$ → Top +30, −30, 0, 0 | 0, 0, −30, +30 Bottom

± ; repeat ; symm.

$[\pm 30 / 0_3]_T$ → +30, −30, 0, 0, 0

total

$[0/30]_{2s}$ → 0, 30, 0, 30 | 30, 0, 30, 0

repeat group

etc.

Typical Laminates may bear

Cross Ply – $[0_2/90_2]_T$ → 0, 0, 90, 90

Angle Ply – $[\pm\theta]_s$

Quasi-Isotropic – $[0/\pm 45/90]_s$, $[0/\pm 60]_s$

etc.

- In-plane Stress, Strain & Stiffness

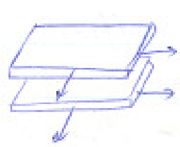
(Symmetric laminates – no bending)

Basic Assumptions:

1. Plies are all glued together.
2. Plies are in plane-stress.

$$\sigma_{33} = \sigma_{23} = \sigma_{13} = 0$$

- Strain



because of gluing, ϵ_{ij}'s all same

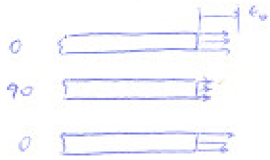
$$\underset{\sim}{\epsilon}^{\circ} = \underset{\sim}{\epsilon} = \begin{Bmatrix} \epsilon_x^{\circ} \\ \epsilon_y^{\circ} \\ \gamma_{xy}^{\circ} \end{Bmatrix}$$

laminate ; each lamina

- Stress

Is stress in 0° and 90° same?

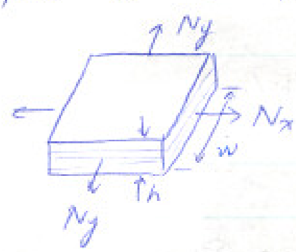
Not same.



Q_{ij}' different

To find stresses, look @ average forces in plies.

Define N ← force / unit width or laminate



$$N_x = \frac{\text{load in } x}{w}$$

Total load − P (lb)

$$N = \frac{P}{w} \quad (\text{lb/in.})$$

Average Stress $(\sigma_x)_A = N_x / h$, h : laminate thickness

$$N_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x \, dz$$

In discrete plies, $N_x = \sum_{k=1}^{n} \sigma_x^{(k)} t_k$

n: number of plies, $k = 1, 2, 3, \cdots$ top to down

Similarly
$$N_y = \sum_{k=1}^{n} \sigma_y^{(k)} t_k$$
$$N_{xy} = \sum_{k=1}^{n} \tau_{xy}^{(k)} t_k$$

will then have

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{Bmatrix} \epsilon_x^\circ \\ \epsilon_y^\circ \\ \gamma_{xy}^\circ \end{Bmatrix}$$

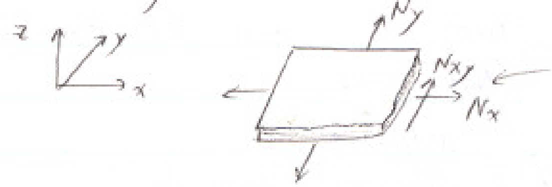
$$N_x = \sum_{k=1}^{n} \left[\bar{Q}_{11}^{(k)} \epsilon_x + \bar{Q}_{12}^{(k)} \epsilon_y + \bar{Q}_{16}^{(k)} \gamma_{xy} \right] t_k$$
$$= \underbrace{\left[\sum_{k=1}^{n} \bar{Q}_{11}^{(k)} t_k \right]}_{=A_{11}} \epsilon_x^\circ + \underbrace{\left[\sum_{k=1}^{n} \bar{Q}_{12}^{(k)} t_k \right]}_{=A_{12}} \epsilon_y^\circ + \underbrace{\left[\sum \bar{Q}_{16}^{(k)} t_k \right]}_{=A_{16}} \gamma_{xy}^\circ$$

Similarly for N_y & N_{xy}

So finally

$$\underbrace{\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix}}_{=\underset{\sim}{N}} = \underbrace{\begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}}_{=\underset{\sim}{A}} \underbrace{\begin{Bmatrix} \epsilon_x^\circ \\ \epsilon_y^\circ \\ \gamma_{xy}^\circ \end{Bmatrix}}_{=\underset{\sim}{\epsilon}^\circ}$$

Given a symmetric laminate



Have form Relation

$$\underset{\sim}{N} = \underset{\sim}{A} \; \underset{\sim}{\epsilon}^{0}$$

where,

$$\underset{\sim}{N} = \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \text{Force (lbs/in.)}$$

$$\underset{\sim}{\epsilon}^{0} = \begin{Bmatrix} \epsilon_x^{0} \\ \epsilon_y^{0} \\ \gamma_{xy}^{0} \end{Bmatrix} = \text{Midplane Strains (in./in.) (laminate axes)}$$

$$\underset{\sim}{A} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} = \text{Extensional Stiffness (lb/in.)}$$

and $A_{ij} = \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} t_k$

Could also write as equivalent moduli

$$\underset{\sim}{\bar{\sigma}} = \underset{\sim}{E}^{eq} \underset{\sim}{\epsilon}^{0}$$

Average stress $= N/h$

So $\underbrace{\frac{1}{h}\underset{\sim}{N}}_{=\bar{\sigma}} = \underbrace{\frac{1}{h}\underset{\sim}{A}}_{=E^{eq}} \underset{\sim}{\epsilon}^{0}$ ← lbs/in² = modulus

$$\underset{\sim}{E}^{eq} = \begin{bmatrix} E_{11}^{eq} & E_{12}^{eq} & E_{16}^{eq} \\ E_{12}^{eq} & - & - \\ E_{16}^{eq} & - & - \end{bmatrix}$$ ← like $\underset{\sim}{Q}$ matrix for the laminate

These are <u>not</u> the Engineering constants for the Laminate.

Also, have Inverse Relations,

$$\underset{\sim}{\epsilon}^{0} = \underset{\sim}{a} \; \underset{\sim}{N} \quad \text{where } \underset{\sim}{a} = \underset{\sim}{A}^{-1}$$

This only applies for symmetric laminates (no bending)

Deal later with unsymm. laminates

- Properties of $\underset{\sim}{A}$ matrix

$$\underset{\sim}{A} = \sum_{k=1}^{N} \bar{Q}^{(k)} t_k$$

$$\left.\begin{aligned} \bar{Q}_{11} &= c^4 Q_{11} + s^4 Q_{22} + \cdots \\ \bar{Q}_{12} &= c^2 s^2 (Q_{11} + \cdots \\ \bar{Q}_{22} &= \\ &\vdots \end{aligned}\right\} \text{see Handout, also Jones, p. 51}$$

- Remarks on $\underset{\sim}{A}$
  1. Thickness (area) weighted stiffness $\bar{Q}_{ij}$
  2. Independent of stacking order
  3. Balanced Laminates

     "a $-\theta$ for every $+\theta$"

     $\bar{Q}_{11}$, $\bar{Q}_{12}$, $\bar{Q}_{22}$, $\bar{Q}_{66}$ not sensitive to sign
     $(c^4, s^4, c^2, s^2, \cdots)$

     $\bar{Q}_{16}$, $\bar{Q}_{26}$ are affected $(c^3 s, cs^3, \cdots)$

     $$A_{16} = A_{26} = 0$$

     $\therefore$ Balanced laminates are orthotropic.
  4. Quasi-isotropic laminate

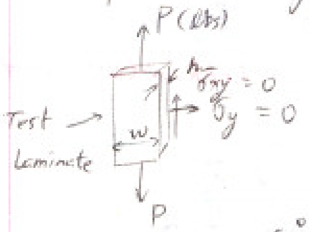
     $[0/\pm 60]_s$, $[0/\pm 45/90]_s$ ← primary

     $[0/\pm 30/\pm 60/90]_s$ ← Built up from $[0/\pm 60]_s$

     $A_{22} = A_{11}$, $A_{66} = f(A_{11}, A_{12})$

     $\therefore$ Quasi-isotropic have "isotropic" stiffness

- Example of using $\underset{\sim}{A}$ matrix —



$$N_x = P/w, \quad \bar{\sigma}_x = \frac{N_x}{h} = \frac{P}{hw}$$

($\bar{\sigma}_x$ ← Average stress)

$$N_y = 0$$
$$N_{xy} = 0$$

$$\underset{\sim}{\epsilon}^{\circ} = \underset{\sim}{A}^{-1} \underset{\sim}{N} = \underset{\sim}{a} \underset{\sim}{N}$$

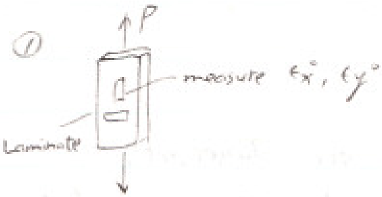
$$\epsilon_x^{\circ} = a_{11} N_x + a_{12} N_y + \cancel{a_{16} N_{xy}} = a_{11} \frac{P}{w} = a_{11} h \bar{\sigma}_x$$

$$\epsilon_y^{\circ} = a_{12} \frac{P}{w} = a_{12} h \bar{\sigma}_x$$

$$\gamma_{xy}^{\circ} = a_{16} h \bar{\sigma}_x = 0 \quad \leftarrow \text{If balanced}$$

- Laminate Engineering Constants

  Constants we get from mechanical tests on laminates (as for plies)

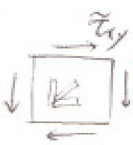
① 

$\bar{\sigma}_x = \frac{P}{wh}$, $\quad \bar{\sigma}_x = \bar{E}_L \epsilon_x^\circ$ ← Engineering Stiffness or Modulus $\bar{E}_L$

$\bar{\nu}_{LT} = -\epsilon_y^\circ / \epsilon_x^\circ$ ← Engineering Poisson's Ratio

② 

$\bar{\sigma}_y = \frac{P}{wh}$, $\quad \bar{\sigma}_y = \bar{E}_T \epsilon_y^\circ$ ← Engineering Transverse Stiffness $\bar{E}_T$

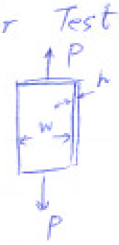
③ 

measure γ_{xy}°

$\bar{\tau}_{xy} = \bar{G}_{LT} \gamma_{xy}^\circ$ ← Engineering shear Stiffness $\bar{G}_{LT}$

· obtaining Laminate Engineering Constants

For Test ①



$$N_x = \frac{P}{w} = h\,\bar{\sigma}_x$$
$$N_y = 0$$
$$N_{xy} = 0$$
$$\underset{\sim}{\epsilon}^\circ = \underset{\sim}{A}^{-1} \underset{\sim}{N} = \underset{\sim}{a} \underset{\sim}{N}$$

$$\epsilon_x^\circ = a_{11} N_x = a_{11} h\,\bar{\sigma}_x$$
$$\epsilon_y^\circ = a_{12} N_x = a_{12} h\,\bar{\sigma}_x = \frac{a_{12}}{a_{11}} \epsilon_x^\circ$$
$$\epsilon_{xy}^\circ = 0 \text{ for balanced laminate}$$

(if it weren't, $\epsilon_{xy}^\circ = a_{16} N_x = \frac{a_{16}}{a_{11}} \epsilon_x^\circ$)

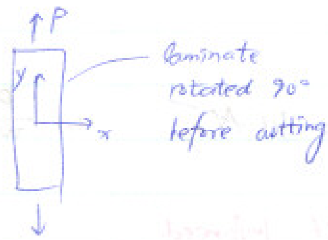
Note : $\bar{E}_x = \frac{\bar{\sigma}_x}{\epsilon_x^\circ} = \frac{1}{a_{11} h}$

$\bar{\nu}_{xy} = -\frac{\epsilon_y^\circ}{\epsilon_x^\circ} = -\frac{a_{12}}{a_{11}}$

If not balanced, would also have $\eta_{xy,x}$

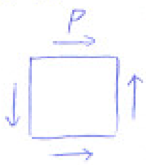
Lekhnitski coefficient

For test ②



$$N_y = \frac{P}{w} = h\,\bar{\sigma}_y$$
$$N_x = 0, \quad N_{xy} = 0$$
$$\epsilon_x^\circ = a_{12} N_y = a_{12} h\,\bar{\sigma}_y$$
$$\epsilon_y^\circ = a_{22} N_y = a_{22} h\,\bar{\sigma}_y$$
$$\epsilon_{xy}^\circ = 0 \text{ if balanced}$$
$$\bar{E}_y = \frac{\bar{\sigma}_y}{\epsilon_y^\circ} = \frac{1}{a_{22} h}$$

Test ②



$$N_{xy} = \frac{P}{w} = h\,\bar{\sigma}_{xy}$$
$$N_x = 0, \quad N_y = 0$$
$$\left.\begin{aligned} \epsilon_x^\circ &= a_{16} N_{xy} = 0 \\ \epsilon_y^\circ &= a_{26} N_{xy} = 0 \end{aligned}\right\} \text{Balanced Laminate}$$
$$\epsilon_{xy}^\circ = a_{66} N_{xy} = a_{66} h \bar{\sigma}_{xy}$$
$$\bar{G}_{xy} = \bar{\sigma}_{xy} / \epsilon_{xy}^\circ = 1/a_{66} h$$

For balanced laminate,

$$\underset{\sim}{A} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}, \quad \underset{\sim}{a} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & a_{66} \end{bmatrix}$$

$$a_{66} = \frac{1}{A_{66}}$$
$$\bar{G}_{xy} = \frac{1}{a_{66} h} = \frac{A_{66}}{h} = E_{66}^{eq}$$

But $a_{11} \neq \frac{1}{A_{11}}$

So $\bar{E}_x \neq E_{11}^{eq}$

Engineering Equivalent

When does difference come up?

For $w \ll l$ case

Unconstrained

$$N_y = N_{xy} = 0$$
$$\epsilon_x^\circ = \bar{\sigma}_x / \bar{E}_x$$

Engineering Stiffness

For $w \gg l$ case

Can't say $N_y = N_{xy} = 0$

$\epsilon_y^\circ = \epsilon_{xy}^\circ = 0$ is a better approximation

then would have

$$\underset{\sim}{N} = \underset{\sim}{A}\,\underset{\sim}{\epsilon}^\circ$$
$$N_x = A_{11}\epsilon_x^\circ + 0 + 0$$
$$N_y = A_{12}\epsilon_x^\circ$$
$$N_{xy} = A_{16}\epsilon_x^\circ$$
$$N_x = \bar{\sigma}_x h = A_{11}\epsilon_x^\circ$$

$$\frac{\bar{\sigma}_x}{\epsilon_x^\circ} = \frac{A_{11}}{h} = E_{11}^{eq}$$

Equivalent stiffness

- Effect of Boundaries

Note also this effect in isotropic materials.

Correct stiffness there $\rightarrow$ $E^{eq} = E/(1-\nu^2)$

Because plies are constrained by neighbors, usually more convenient to work with $\underset{\sim}{A}$ (or $\underset{\sim}{E}^{*}$) rather than $\bar{E}_x$, $\bar{E}_y$, etc.

- Ply Stresses

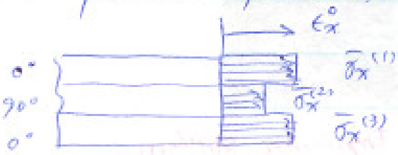
Given a laminate description, can find laminate engineering constants for input to Plate and shell problems.

Given loads P_x, P_y, P_{xy}, can find N_x, N_y, N_{xy} and then one gets laminate strains from

$$\underset{\sim}{\epsilon}^\circ = \underset{\sim}{a}\,\underset{\sim}{N}$$

Average laminate stresses $\bar{\underset{\sim}{\sigma}} = \underset{\sim}{N}/h$

Now want to look at individual stresses in k^{th} ply. (to predict failure)



Ply strains $\bar{\epsilon}_x^{(k)}$ all the same

Ply stresses $\bar{\sigma}_x^{(k)}$ all different

(Bar indicates laminate coordinate system)

How do we calculate $\underset{\sim}{\sigma}^{(k)}$

Note: no bar — want stresses in ply coordinate system

Two Paths for getting $\underset{\sim}{\sigma}^{(k)}$

Path # 1

Know $\bar{\underset{\sim}{\epsilon}}^{(k)} = \underset{\sim}{\epsilon}^\circ$ laminate strain

ply strain, ply k, laminate coords.

Also know

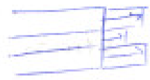
$$\bar{\underset{\sim}{\sigma}}^{(k)} = \bar{\underset{\sim}{Q}}^{(k)}\,\bar{\underset{\sim}{\epsilon}}^{(k)}$$

ply strain, laminate coordinate

ply stress, ply k, laminate coords.

Therefore, can calculate stresses in ply

$\bar{\sigma}_x^{(k)}$, etc.

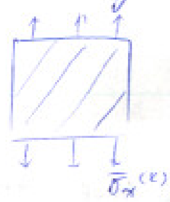


$\longrightarrow N_x$

$$N_x = \sum_{k=1}^{N} \bar{\sigma}_x^{(k)} t_k$$

(can check out N_x)

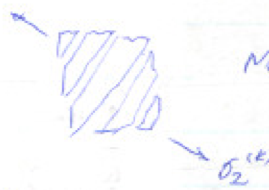
Unfortunately, $\bar{\underset{\sim}{\sigma}}^{(k)}$ isn't very useful.



$\theta = 45^\circ$

Type of Failure

Fiber failure

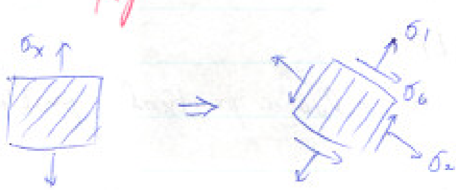


Matrix failure

Given $\bar{\underset{\sim}{\sigma}}^{(k)}$ get $\underset{\sim}{\sigma}^{(k)}$ by transformation

$$\underset{\sim}{\sigma}^{(k)} = \underset{\sim}{T_\sigma}^{(k)} \bar{\underset{\sim}{\sigma}}^{(k)}$$

ply laminate



Also, get ply strain in ply coordinates.

$$\underset{\sim}{\varepsilon}^{(k)} = \underset{\sim}{S}^{(k)} \underset{\sim}{\sigma}^{(k)}$$

So system we have

$$
\begin{array}{ccccccc}
P & \rightarrow & \underset{\sim}{N} & \xrightarrow{a} & \underset{\sim}{\varepsilon}^{\circ} & \xrightarrow{\text{same}} & \bar{\underset{\sim}{\varepsilon}}^{(k)} \\
 & & & & & & \downarrow \bar{Q}^{k} \\
 & & & & & & \bar{\underset{\sim}{\sigma}}^{(k)} \\
 & & & & & & \downarrow T_\sigma \\
 & & & & \underset{\sim}{\varepsilon}^{(k)} & \xleftarrow{S} & \underset{\sim}{\sigma}^{k}
\end{array}
$$

Path #2

Given $\bar{\underset{\sim}{\varepsilon}}^{(k)}$, go directly to $\underset{\sim}{\varepsilon}^{(k)}$ by transformation

$$\underset{\sim}{\epsilon}^{(k)} = T_\epsilon^{(k)} \, \bar{\underset{\sim}{\epsilon}}^{(k)}$$

ply coords — laminate coordinates

Then from $\underset{\sim}{\epsilon}^{(k)}$, get $\sigma^{(k)}$ from ply stress-strain equations.

$$\underset{\sim}{\sigma}^{(k)} = \underset{\sim}{Q}^{(k)} \, \underset{\sim}{\epsilon}^{(k)}$$

So, have another system.

$$\begin{tikzcd} P \arrow[r] & \underset{\sim}{N} \arrow[r, "\underset{\sim}{a}"] & \underset{\sim}{\epsilon}^{\circ} \arrow[r, "\text{same}"] & \bar{\underset{\sim}{\epsilon}}^{(k)} \arrow[d, "T_\epsilon^{(k)}"] \\ & & \underset{\sim}{\sigma}^{(k)} & \underset{\sim}{\epsilon}^{(k)} \arrow[l, "\underset{\sim}{Q}^{(k)}"'] \end{tikzcd}$$

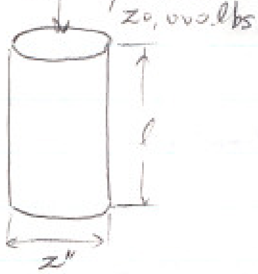
So, summarizing what we know so far, we have arrived at,

- In-Plane Classical Laminate Plate Theory (CLPT)

$$\begin{tikzcd} \text{Loads} \arrow[d] & & \text{Constraints} \arrow[d] & \\ N \arrow[r] & A \arrow[r] & \epsilon^{\circ} \arrow[d] & \cdots \text{ Laminate} \\ \bar{\sigma} \arrow[d, "\text{transform}"] & \bar{Q} \arrow[u, "\text{sum}"] \arrow[l] & \bar{\epsilon} \arrow[l, "\#1"'] \arrow[d] & \cdots \text{ Ply in laminate coordinate system} \\ \sigma \arrow[r, shift left] & Q \arrow[u, "\text{transform}"] \arrow[l, shift left] \arrow[r, shift left] & \epsilon \arrow[l, shift left, "\#2"'] & \cdots \text{ Ply (Ply coords syst)} \\ & \text{Laminate Properties} \arrow[u] & & \end{tikzcd}$$

Example of In-Plane CLPT

Use previous system to solve a practical problem.



Tubular compression member

Assume l short

(no buckling)

Material T300/934 Gr/Ep

Ply Engineering Properties

$E_L = 20$ Msi, $E_T = 1.4$ Msi, $\nu_{LT} = .29$, $G_{LT} = .7$ Msi

Ply thicknesses : .005" (5 mils)

Lay up $[\,0/\pm 45/90\,]_s$

Referring to grand scheme, already have Ply Eng'g Consts (get by micromechanics and Test)

Step #1 : Find $\underset{\sim}{Q}$

$$\nu_{TL} = \frac{E_T}{E_L}\nu_{LT} = .020$$

$$Q_{11} = \frac{E_L}{1-\nu_{LT}\nu_{TL}} = \frac{20}{1-.29(.02)} = 20.12 \text{ Msi}$$

↑ not much different from E_L

$$Q_{12} = \frac{\nu_{LT}E_T}{1-\nu_{LT}\nu_{TL}} = .408 \text{ Msi}$$

$$Q_{22} = \frac{E_T}{1-\nu_{LT}\nu_{TL}} = 1.41 \text{ Msi} \simeq E_T$$

$$Q_{16} = 0, \quad Q_{26} = 0$$

$$Q_{66} = G_{LT} = .7 \text{ Msi}$$

$$\underset{\sim}{Q} = \begin{bmatrix} 20.12 & .408 & 0 \\ .408 & 1.41 & 0 \\ 0 & 0 & .70 \end{bmatrix} (\text{Msi})$$

Step #2 : Compute $\bar{Q}$ for each ply

0° Plies : Trivial $\rightarrow$ $\underset{\sim}{\bar{Q}_0} = \underset{\sim}{Q}$

90° Plies : Easy $\rightarrow$ Reverse 1 & 2

$$\underset{\sim}{\bar{Q}_{90}} = \begin{bmatrix} 1.41 & .408 & 0 \\ .408 & 20.12 & 0 \\ 0 & 0 & .70 \end{bmatrix} \text{Msi}$$

45° Plies : Harder $\rightarrow$ Use Transform formulas.

$$\bar{Q}_{11} = c^4 Q_{11} + s^4 Q_{22} + 2c^2s^2(Q_{12} + 2Q_{66})$$

Note : $\begin{cases} \cos\theta = \sin\theta = .707 \\ \cos^2\theta = \sin^2\theta = .500 \end{cases}$

$$\bar{Q}_{11} = .25(20.12) + .25(1.41) + .500(.408 + 2[.7])$$
$$= 6.29 \text{ Msi}$$

Similarly

$$\bar{Q}_{12} = 4.89 \text{ Msi}$$

$\bar{Q}_{16}$ = etc. , $\bar{Q}_{26}$ = , $\bar{Q}_{66}$ =

$$\bar{Q}_{45} = \begin{bmatrix} 6.29 & 4.89 & 4.68 \\ 4.89 & 6.29 & 4.68 \\ 4.68 & 4.68 & 5.18 \end{bmatrix} \text{Msi}$$

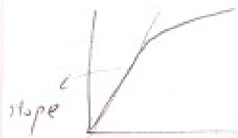
−45° Plies : Easy. Same as +45 except $\bar{Q}_{16}$ and $\bar{Q}_{26}$ change signs

Note : $\sin(-\theta) = -\sin\theta \longrightarrow$ only s, s^3

$$\cos(-\theta) = \cos\theta$$

$$\underset{\sim}{\bar{Q}}_{-45} = \begin{bmatrix} 6.29 & 4.89 & -4.68 \\ 4.89 & 6.29 & -4.68 \\ -4.68 & -4.68 & 5.46 \end{bmatrix} \text{ Msi}$$

☐ Problem Set. 4



$\sigma_{xy} \neq 0$
$\epsilon_{xy} = 0$

$$\begin{Bmatrix} \sigma_x \\ 0 = \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \cancel{\bar{Q}_{16}} \\ \bar{Q}_{12} & \bar{Q}_{22} & \cancel{} \\ \bar{Q}_{16} & \bar{Q}_{26} & \cancel{} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$\sigma_{xy} = 0$
$\epsilon_{xy} \neq 0$

$$\bar{Q}_{11} = 39 + Q_{66}$$
$$\bar{Q}_{22} = 39 + Q_{66}$$
$$\bar{Q}_{12} = 39 - Q_{66}$$

$$\epsilon_y = -\frac{\bar{Q}_{12}}{\bar{Q}_{22}} \epsilon_x$$

$$\epsilon_y = -.73\,\epsilon_x, \quad \tau_{xy} = .42\,\sigma_x$$

$$E_L = 140, \quad \bar{Q}_{11} = 40, \quad E_x = 22$$

2. $Q_{22} = Q_{11}(90-\theta)$

$Q_{26} = Q_{16}($ " $)$

Step 3 Assemble $\underset{\sim}{A}$ matrix

$$\underset{\sim}{A} = \sum_{k=1}^{N} \bar{Q}^{(k)} t_k$$

Note :
- Thickness all the same
- Order doesn't matter here
- Symmetric $\bar{Q}$ $\rightarrow$ symmetric $\underset{\sim}{A}$
- Also, $\bar{Q}_{11}^{-45°} = \bar{Q}_{11}^{45°}$ (etc. for 12, 22, 66)
  but $\bar{Q}_{16}^{-45°} = -\bar{Q}_{16}^{45}$ (and for 26),
  so 16 and 26 terms cancel

So summing,

$$A_{11} = t\,(2\,\bar{Q}_{11}^{0} + 2\bar{Q}_{11}^{90} + 4\,\bar{Q}_{11}^{45})$$
$$= 0.005\,(2(20.12\times10^6) + 2(1.41\times10^6)$$

$+4(6.29\times10^6)) = 0.341\times10^6\ \text{lb/in.}$

$$A_{12} = t(2Q_{12}^{0} + 2Q_{12}^{90} + 4Q_{12}^{45})$$
$$= .005\,(2(.408\times10^6) + 2(.408\times10^6) + 4(4.09\times10^6))$$
$$= .106\times10^6\ \text{lb/in.}$$

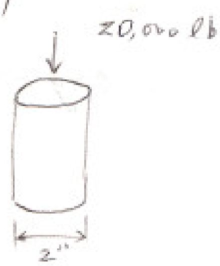
$A_{22} = .341\times10^6$ lb/in.

$A_{66} = .118\times10^6$ lb/in.

$A_{16} = 0, \quad A_{26} = 0$

$$\underset{\sim}{A} = \begin{bmatrix} .341 & .106 & 0 \\ .106 & .341 & 0 \\ 0 & 0 & .118 \end{bmatrix} \times 10^6\ \text{lb/in.}$$

Step. 4 : Establish Loading



Loading Assume thin,
load distributes evenly

$$N_x = \frac{P}{\text{circumference}} = \frac{P}{2\pi r} = \frac{20{,}000}{2\pi(1)} = -3{,}183\ \text{lb/in.}$$

compress

Assume unrestrained, $N_y = 0$, $N_{xy} = 0$

Step. 5 : Calculate Laminate Strain

$$\underset{\sim}{\epsilon}^{\circ} = \underset{\sim}{a}\,\underset{\sim}{N}, \quad \underset{\sim}{a} = \underset{\sim}{A}^{-1}$$

can invert 3x3 matrix, or else,

$$\underset{\sim}{a} = \begin{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix}^{-1} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \frac{1}{A_{66}} \end{bmatrix} = \begin{bmatrix} \frac{A_{22}}{A_{11}A_{22}-A_{12}^2} & \frac{-A_{12}}{A_{11}A_{22}-A_{12}^2} & 0 \\ \frac{-A_{12}}{A_{11}A_{22}-A_{12}^2} & \frac{A_{11}}{A_{11}A_{22}-A_{12}^2} & 0 \\ 0 & 0 & \frac{1}{A_{66}} \end{bmatrix}$$

$$\underset{\sim}{a} = \begin{bmatrix} 3.25 & -1.01 & 0 \\ -1.01 & 3.25 & 0 \\ 0 & 0 & 8.47 \end{bmatrix} \times 10^{-6}\ \text{in/lb}$$

$$\epsilon_x^{\circ} = a_{11} N_x = 3.25\times10^{-6}(-3183) = -0.0103$$
$$\epsilon_y^{\circ} = a_{12} N_x = -1.01\times10^{-6}(-3183) = +.0032$$
$$\epsilon_{xy}^{\circ} = a_{16} N_x = 0$$

Step. 6 Calculate Ply strains in laminate coordinates

Jones Notation $\bar{\epsilon}^{(K)} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$

All $\bar{\epsilon}^{(k)}$ equal to ϵ^{0}

step #7: Calculate Ply stresses in laminate coordinates system

Jones Notation $\bar{\sigma}^{(k)} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$

$$\bar{\sigma}^{(k)} = \bar{Q}^{(k)} \underbrace{\bar{\epsilon}^{(k)}}_{=\epsilon^{0}}$$

$$\begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\sigma}_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^{0} \\ \epsilon_y^{0} \\ \gamma_{xy}^{0} \end{Bmatrix}$$

$0°$ Ply: $\sigma_x = \bar{Q}_{11}\epsilon_x^{0} + \bar{Q}_{12}\epsilon_y^{0}$

$= 20.12 \times 10^{6}(-.0103) + .408 \times 10^{6}(+.0032)$

$= -206$ (Ksi) (high)

$\sigma_y = \bar{Q}_{12}\epsilon_x^{0} + \bar{Q}_{22}\epsilon_y^{0}$, $\sigma_{xy} = 0$

$= .306$ (Ksi) (low)

$90°$ Ply: $\sigma_x = \bar{Q}_{11}^{90}\epsilon_x^{0} + \bar{Q}_{12}^{90}\epsilon_y^{0}$

$= -13$ Ksi

$\sigma_y = 60$ Ksi

$\sigma_{xy} = 0$

$+45°$ Ply: $\sigma_x = \bar{Q}_{11}\epsilon_x^{0} + \bar{Q}_{12}\epsilon_y^{0} + \cancel{\bar{Q}_{16}\gamma_{xy}^{0}}$ 0

$= -49$ Ksi

$\sigma_y = \bar{Q}_{12}\epsilon_x^{0} + \bar{Q}_{22}\epsilon_y^{0} + \cancel{\bar{Q}_{26}\gamma_{xy}^{0}}$ 0

$= -30$ Ksi

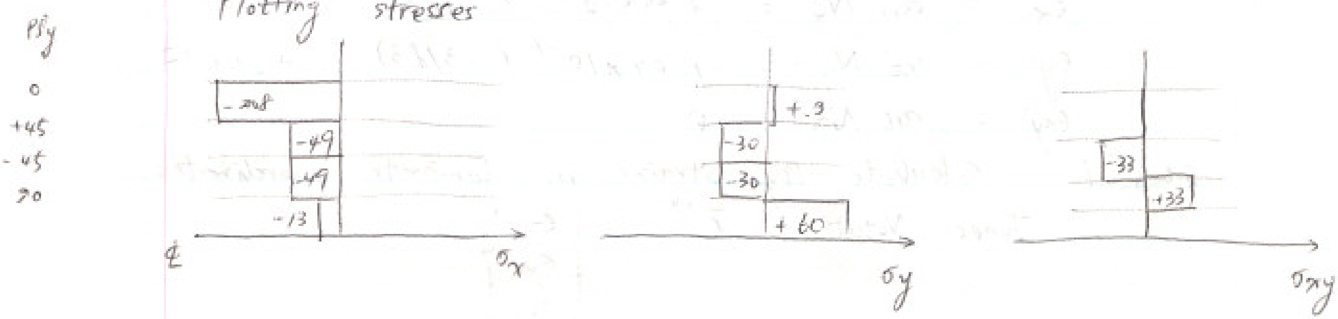
$\sigma_{xy} = \bar{Q}_{16}\epsilon_x^{0} + \bar{Q}_{26}\epsilon_y^{0} + \cancel{\bar{Q}_{66}\gamma_{xy}^{0}}$ 0

$= -33$ Ksi

$-45°$ Ply: same as $+45°$, but

$\sigma_{xy}(-45°) = -\sigma_{xy}(+45°)$

Plotting stresses



step #8 : Calculate ply stress $\underset{\sim}{\sigma}$ in ply coordinates

$$\underset{\sim}{\sigma} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} \leftarrow \text{Jones Notation}$$

$$\underset{\sim}{\sigma} = T_\sigma \, \bar{\underset{\sim}{\sigma}} \rightarrow \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -cs & cs & (c^2-s^2) \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}$$

0° ply
$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} -206 \\ 30 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -206 \\ .30 \\ 0 \end{Bmatrix} \text{Ksi}$$

90° ply
$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} -13 \\ 60 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 60 \\ -13 \\ 0 \end{Bmatrix} \text{Ksi}$$

+45° ply
$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 1 \\ .5 & .5 & -1 \\ -.5 & .5 & 0 \end{bmatrix} \begin{Bmatrix} -49 \\ -30 \\ -33 \end{Bmatrix} = \begin{Bmatrix} -73 \\ -6.5 \\ 9.5 \end{Bmatrix} \text{Ksi}$$

−45° ply − same sign as +45° by σ_6 change sign

Summary of stress (Ksi)

Ply θ	σ_1	σ_2	σ_6
0°	−206	.3	0
+45°	−73	−6.5	9.5
−45°	−73	−6.5	−9.5
90°	60	−13	0

Note: Compare to strength of unidirectional material

Compress Ultimate (1-dir.) = 160 Ksi

Compress Ult. (2-dir.) = 25 Ksi

Shear = 10 Ksi

Fiber failure in 0° ply, Reinforce strut

Look also at ply axis strains

step #9 Calculate ply strains $\underset{\sim}{\epsilon}$ in ply coords.

$$\underset{\sim}{\epsilon} = \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{Bmatrix} \leftarrow \text{Jones notation}$$

$$\underset{\sim}{\epsilon} = \underset{\sim}{S}\,\underset{\sim}{\sigma} \rightarrow \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}$$

$$S_{11} = \frac{1}{E_L} = \frac{1}{20\times10^6} = .050\times10^{-6}$$

$$S_{12} = -\frac{\nu_{LT}}{E_L} = -\frac{.29}{20\times10^6} = -.0145\times10^{-6}$$

$$S_{22} = \frac{1}{E_T} = \frac{1}{1.4\times10^6} = .7143\times10^{-6}$$

$$S_{66} = \frac{1}{G_{LT}} = \frac{1}{.7\times10^6} = 1.429\times10^{-6}$$

$$S_{16} = S_{26} = 0$$

0° ply

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{Bmatrix} = \overset{\times10^{-6}}{\begin{bmatrix} .0500 & -.0145 & 0 \\ -.0145 & .7143 & 0 \\ 0 & 0 & 1.429 \end{bmatrix}} \overset{\times10^{3}}{\begin{Bmatrix} -206 \\ .30 \\ 0 \end{Bmatrix}} = \begin{Bmatrix} -.0103 \\ .0032 \\ 0 \end{Bmatrix}$$

90° ply

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{Bmatrix} = \begin{bmatrix} .0500 & -.0145 & 0 \\ -.0145 & .7143 & 0 \\ 0 & 0 & 1.429 \end{bmatrix} \begin{Bmatrix} 60 \\ -13 \\ 0 \end{Bmatrix} = \begin{Bmatrix} .0032 \\ -.0103 \\ 0 \end{Bmatrix}$$

+45° ply

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{Bmatrix} = \begin{bmatrix} .0500 & -.0145 & 0 \\ & \text{etc.} & \\ & & \end{bmatrix} \begin{Bmatrix} -73 \\ -6.5 \\ 9.5 \end{Bmatrix} = \begin{Bmatrix} -.0036 \\ -.0036 \\ .0136 \end{Bmatrix}$$

−45° ply — same as +45° by ϵ_6 changes sign

Summary of Strains

Ply θ	ϵ_1	ϵ_2	ϵ_6
0	−.0103	.0032	0
+45	−.0036	−.0036	.0136
−45	−.0036	−.0036	−.0136
90	.0032	−.0103	0

Sometimes use a max. strain criteria instead of max. stress
($\epsilon_1 \simeq 7000\ \mu\epsilon$)

Also can do Ply Stress Analysis by Path #2

Steps 1. ~ 6 same as before.

Step #7A Calculate ply strain $\underset{\sim}{\epsilon}$ in ply coords

$$\underset{\sim}{\epsilon} = \underset{\sim}{T_\epsilon}\,\underset{\sim}{\bar{\epsilon}} \rightarrow \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & sc \\ s^2 & c^2 & -sc \\ -2cs & 2cs & (c^2-s^2) \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_{xy} \end{Bmatrix}$$

0° Ply $-$ $\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} -.0103 \\ .0032 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -.0103 \\ .0032 \\ 0 \end{Bmatrix}$

90° Ply $\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} -.0103 \\ .0032 \\ 0 \end{Bmatrix} = \begin{Bmatrix} .0032 \\ -.0103 \\ 0 \end{Bmatrix}$

etc. 45° & −45° plus

Step # 8A : Calculate ply stress $\underset{\sim}{\sigma}$ in ply coords.

$\underset{\sim}{\sigma} = \underset{\sim}{Q}\,\underset{\sim}{\epsilon} \rightarrow \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{Bmatrix}$

0° ply $\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} 20.12 & .408 & 0 \\ .408 & 1.41 & 0 \\ 0 & 0 & .7 \end{bmatrix} \begin{Bmatrix} -.0103 \\ .0032 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -206 \\ 3 \\ 0 \end{Bmatrix}$ Ksi

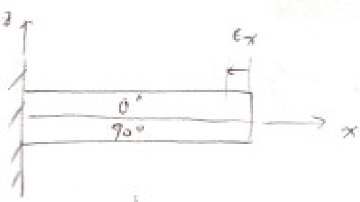
$\times 10^6$

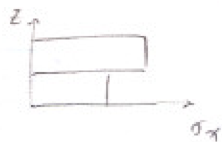
90° ply $\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} & & \\ & \text{same} & \\ & & \end{bmatrix} \begin{Bmatrix} .0032 \\ -.0103 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 60 \\ -13 \\ 0 \end{Bmatrix}$ Ksi

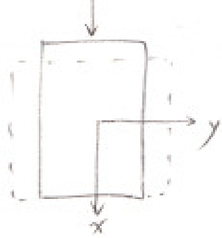
Same for +45° & −45°

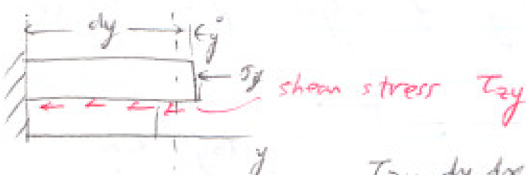
Same Results from Path #1 and Path #2

↳ easier (but no $\bar{\underset{\sim}{Q}}$)









$$\tau_{zy}\,dy\,dx = \sigma_y h\,(\epsilon_y^F - \epsilon_y^0)\,dy\,dx$$
$$\tau_{zy} = \sigma_y h\,(\epsilon_y^F - \epsilon_y^0)$$