Lecture Note 10

Heat Transfer Solution Methods

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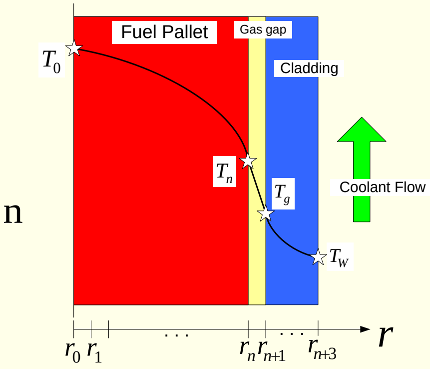
Problem to be Solved



- Single Phase Transient Heat Transfer in a Pin Level Coolant Channel with Heat Conduction in the Pellet
  - Limit to constant pressure condition to avoid solving momentum equation
    1) Continuity and energy conservation equations to be solved for the coolant along the axial direction
    2) Coolant properties (density, enthalpy, heat capacity, etc...) given as a function of temperature
    3) No mixing between flow channels
  - Neglect axial heat conduction
    1) 1-D radial heat conduction problem
    2) Temperature dependence on thermal conductivity explicitly to be modeled
  - Coupled Heat Convection and Conduction
    1) through the bulk temperature

Heat Conduction Equation

- Derivation of Heat Equation
  - Conservation of energy for a small element within the body

Rate of change in stored energy in unit volume = Volumetric heat generation rate

– Net loss rate due to conduction

$$\frac{du}{dt} = q''' - \vec{\nabla} \cdot \vec{q}''$$

$$\begin{cases} \dfrac{du}{dt} = \rho c_p \dfrac{dT}{dt} = C_p \dfrac{dT}{dt} \\ \vec{q}'' = -k \vec{\nabla} T \quad \left(\text{Fourier's Law of Heat condution}\right) \end{cases}$$

$$C_p \frac{dT}{dt} = q''' + \vec{\nabla} \cdot \left(k \vec{\nabla} T \right)$$

u : internal energy stored within the material per unit volume

ρ : density

k : thermal conductivity $\left[W / \left(m \cdot K\right)\right]$

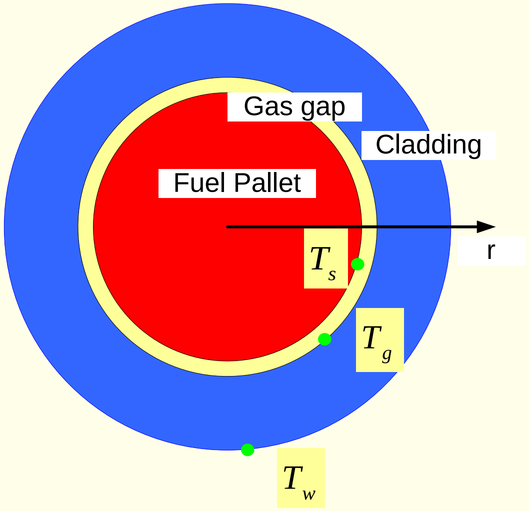
c_p : specific heat capacity $\left[J / \left(K \cdot kg\right)\right]$

C_p : volumetric heat capacity $\left[J / \left(K \cdot m^3\right)\right]$

Radial Heat Conduction in Fuel Rod Channel

- Heat Conduction Equation for Pellet and Cladding
  - Polar coordinate by neglecting axial conduction

$$C_p \frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(k(T)\, r \frac{\partial T}{\partial r} \right) + q'''$$



- Gap Conductance and Convection to Coolant
  - Newton Law of Cooling

$$q''_g = -\left(k \frac{\partial T}{\partial r} \right)_s = h_{gap}\left(T_s - T_g \right)$$

$$q''_w = -\left(k \frac{\partial T}{\partial r} \right)_w = h_w\left(T_w - T_b \right)$$

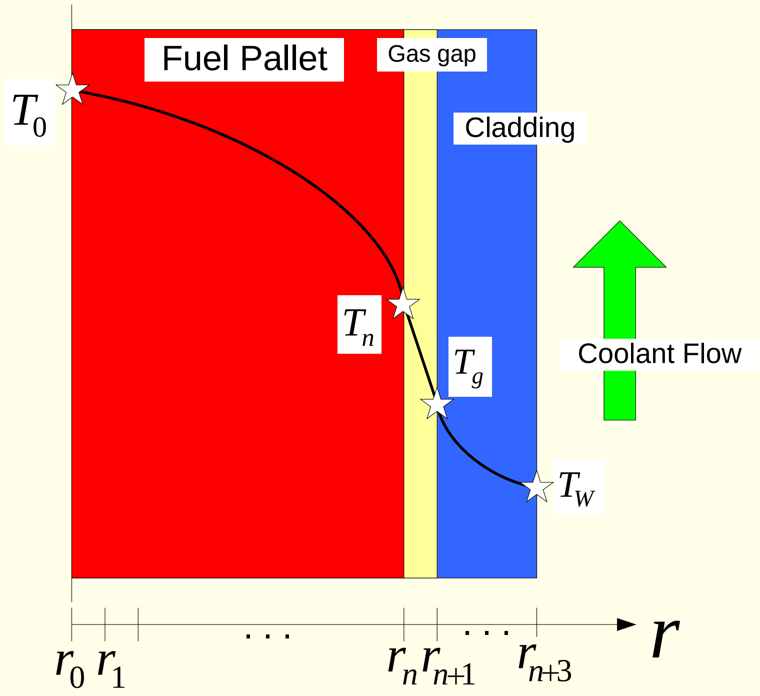
h: Heat Transfer Coefficient $(\mathrm{w/m^2} - K)$

Discretization For numerical Solution of Radial Heat Conduction

- Equidistance Meshing for Easier Finite Difference

n intervals in the pellet $\rightarrow$ r_0 to r_n mesh points with point scheme

2 intervals in the cladding $\rightarrow$ $(n+4)$ total mesh points



Spatial Discretization

- Finite Difference Approximation with Equidistance Meshing for Interior Points

$$\frac{1}{r}\frac{\partial T}{\partial r}\left(k(T)\,r\frac{\partial T}{\partial r}\right)_i = \frac{1}{r_i\Delta r}\left(\left[k(T)\,r\frac{\partial T}{\partial r}\right]_{i+\frac{1}{2}} - \left[k(T)\,r\frac{\partial T}{\partial r}\right]_{i-\frac{1}{2}}\right)$$

$$= \frac{1}{r_i\Delta r}\left(k_{i+\frac{1}{2}}\left(r_i+\frac{\Delta r}{2}\right)\frac{T_{i+1}-T_i}{\Delta r} - k_{i-\frac{1}{2}}\left(r_i-\frac{\Delta r}{2}\right)\frac{T_i-T_{i-1}}{\Delta r}\right) \qquad \text{where } k_{i+\frac{1}{2}} = \frac{k_{i+1}+k_i}{2}$$

$$= \frac{1}{\Delta r^2}\left(k_{i+\frac{1}{2}}\left(1+\frac{\Delta r}{2r_i}\right)(T_{i+1}-T_i) - k_{i-\frac{1}{2}}\left(1-\frac{\Delta r}{2r_i}\right)(T_i-T_{i-1})\right)$$

$$= \frac{1}{\Delta r^2}\left[k_{i-\frac{1}{2}}\left(1-\frac{\Delta r}{2r_i}\right)T_{i-1} - \left(k_{i-\frac{1}{2}}+k_{i+\frac{1}{2}}+\frac{\Delta r}{2r_i}(k_{i+\frac{1}{2}}-k_{i-\frac{1}{2}})\right)T_i + k_{i+\frac{1}{2}}\left(1+\frac{\Delta r}{2r_i}\right)T_{i+1}\right]$$

- Temporal Differencing by Theta Method (Crank-Nicholson if theta=0.5, 2nd order accurate)

$$C_p\frac{T^{(l+1)}-T^{(l)}}{\Delta t} = \theta f^{(l+1)} + \bar{\theta} f^{(l)} \qquad \text{where } \bar{\theta} = 1-\theta$$

- Resulting Tridiagonal Linear System

$$b_i T_{i-1}^{(l+1)} + a_i T_i^{(l+1)} + c_i T_{i+1}^{(l+1)} = \theta q_i^{(l+1)} + \bar{\theta} q_i^{(l)} + g\left(T_{i-1}^{(l)}, T_i^{(l)}, T_{i+1}^{(l)}\right)$$

Boundary Conditions

- At the Center, symmetric

$$T(r) = ar^2 + T_0$$

$$T_1 = a\Delta r^2 + T_0 \rightarrow \quad a = \frac{T_1 - T_0}{\Delta r^2}$$

$$\left. \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r} \right) \right|_0 = 4ak = \frac{4k}{\Delta r^2}\left(T_1 - T_0 \right)$$

  – Theta Method

$$\frac{C_p}{\Delta t}\left(T_0^{(l+1)} - T_0^{(l)} \right) = \theta\left(\frac{4k}{\Delta r^2}T_1^{(l+1)} + q^{(l+1)} \right) + \bar{\theta}\left(\frac{4k}{\Delta r^2}T_1^{(l)} + q^{(l)} \right)$$

$$\left(\frac{C_p}{\Delta t} + \frac{4\theta k}{(\Delta r)^2} \right)T_0^{l+1} - \frac{4\theta k}{\Delta r^2}T_1^{(l+1)} = \frac{C_p}{\Delta t}T_0^{(l)} + \bar{\theta}\frac{4k}{\Delta r^2}\left(T_1^{(l)} - T_0^{(l)} \right) + \theta q^{(l+1)} + \bar{\theta} q^{(l)}$$

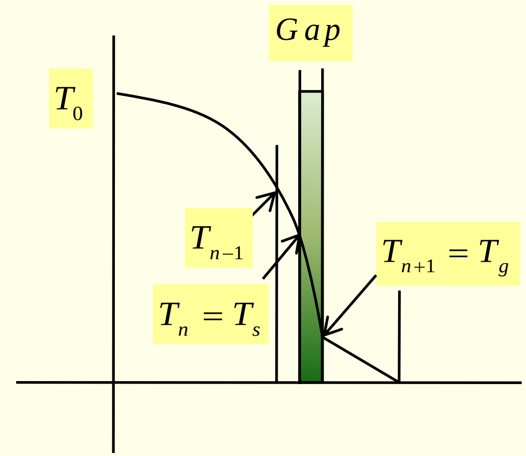
Boundary Conditions

- At Pellet Side of the Gap
  - Suppose a quadratic shape within pellet

$$T(\xi) = a\xi^2 + b\xi + T_n, \quad \xi = r - r_n, \quad T_n = T_s$$

  - Heat Flux Continuity $\quad -k_n b = h_g (T_n - T_g) \rightarrow b = -\frac{h_g}{k_n}(T_n - T_g)$
  - at the (n-1) th point $\quad a\Delta r^2 - b\Delta r + T_n = T_{n-1}$

$$a = \frac{1}{\Delta r}\left[\frac{T_{n-1} - T_n}{\Delta r} - \frac{h_g}{k_n}(T_n - T_g)\right]$$



  - Laplacian at $r = r_s$

· Pellet Side $\quad T'(\xi) = 2a\xi + b$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)\bigg|_{r_n} = \frac{1}{r_n + \xi}\cdot\frac{\partial}{\partial \xi}\left(k(r_n + \xi)\frac{\partial T}{\partial \xi}\right)\bigg|_{\xi=0} = \frac{1}{r_n + \xi}\cdot\frac{\partial}{\partial \xi}\left(k(r_n + \xi)(2a\xi + b)\right)\big|_{\xi=0} = 2ak_n + \left(\frac{k_n}{r_n} + \frac{\partial k}{\partial \xi}\bigg|_s\right)b$$

$$= \frac{1}{\Delta r^2}\left[2k_n T_{n-1} - \left(2k_n + h_g\Delta r\left(3 + \frac{1}{n} - \frac{k_{n-1}}{k_n}\right)\right)T_n + h_g\Delta r\left(3 + \frac{1}{n} - \frac{k_{n-1}}{k_n}\right)T_g\right]$$

$$r_n = n\Delta r$$

$$\left(\frac{\partial k}{\partial \xi}\bigg|_s = \frac{k_n - k_{n-1}}{\Delta r}\right)$$

Boundary Conditions

- At Cladding Side of the Gap
  - Suppose a quadratic shape within cladding

$$T(\xi) = a\xi^2 + b\xi + T_g, \quad \xi = r - r_g, \quad T_{n+1} = T_g$$

  - Heat Flux Continuity $\quad q_g'' \cdot 2\pi r_g = h_g \left(T_n - T_g\right) \cdot 2\pi r_s$

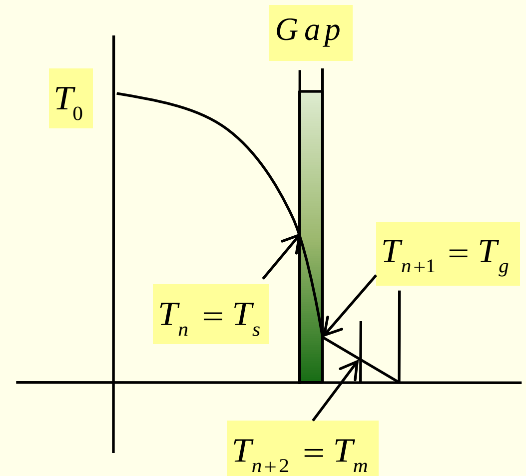
$$k_{n+1} b = -h_g \left(T_n - T_g\right) \frac{r_s}{r_g}$$

  - At the center of clad $\quad T(\xi)\Big|_{\xi = \frac{d}{2}} \equiv T_m = \frac{ad^2}{4} + \frac{bd}{2} + T_w$

$$\rightarrow a = \frac{4}{d^2}\left[T_m - T_g + \frac{dh_g}{2k_g}\frac{r_s}{r_g}\left(T_n - T_g\right)\right]$$

  - Laplacian at $r = r_g$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)_{r_g} = \frac{1}{d^2}\left[8k_g\left(T_m - T_g\right) - 4dh_g\frac{r_s}{r_g}\left(T_n - T_g\right)\right] - h_g\frac{r_s}{r_g}\left[\frac{1}{r_g} + \frac{2}{d}\left(\frac{k_m}{k_g} - 1\right)\right]\left(T_n - T_g\right)$$



Discretization for Cladding

- For the wall side of the cladding of thickness d

  – Suppose a quadratic shape $T(\xi) = a\xi^2 + b\xi + T_w \qquad (\xi = r - r_w)$

  – Heat Flux Continuity at Wall $-k\nabla T = h(T_w - T_b) \qquad (\nabla T = 2\xi + b)$

  $$-k_w b = h_w (T_w - T_b) \quad \rightarrow b = -\frac{h_w}{k_w}(T_w - T_b)$$

  – At the center of clad $T(\xi)\Big|_{\xi=-\frac{d}{2}} \equiv T_m = \frac{ad^2}{4} - \frac{bd}{2} + T_w \quad \rightarrow a = \frac{4}{d^2}\left[T_m - T_w - \frac{dh_w}{2k_w}(T_w - T_b)\right]$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)\Bigg|_{r_w} = \frac{1}{r_w+\xi}\cdot\frac{\partial}{\partial \xi}\left(k(r_w+\xi)\frac{\partial T}{\partial \xi}\right)\Bigg|_{\xi=0} = \frac{1}{r_w+\xi}\cdot\frac{\partial}{\partial \xi}\left(k(r_w+\xi)(2a\xi+b)\right)\Big|_{\xi=0} = 2ak_w + \left(\frac{k_w}{r_w} + \frac{\partial k}{\partial \xi}\right)b$$

– Laplacian at $r = r_w$ $\left(\frac{\partial k}{\partial \xi}\Big|_w = \frac{k_w - k_m}{d/2}\right)$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)\Bigg|_{r_w} = \frac{1}{d^2}\left[8k_w(T_m - T_w) - 4dh_w(T_w - T_b)\right] - h_w\left[\frac{1}{r_w} + \frac{2}{d}\left(1 - \frac{k_m}{k_w}\right)\right](T_w - T_b)$$

Final Equation to Solve

- Tridiagonal Linear System

$$\begin{bmatrix} a_0 & c_0 & & & & & & \\ b_1 & a_1 & c_1 & & & & & \\ & \ddots & \ddots & \ddots & & & & \\ & & b_i & a_i & c_i & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & b_{n+2} & a_{n+2} & c_{n+2} \\ & & & & & b_{n+3} & a_{n+3} \end{bmatrix} \begin{bmatrix} T_0^{(l+1)} \\ T_1^{(l+1)} \\ \vdots \\ T_i^{(l+1)} \\ \vdots \\ T_{n+2}^{(l+1)} \\ T_{n+3}^{(l+1)} \end{bmatrix} = \begin{bmatrix} S_0 \\ S_1 \\ \vdots \\ S_i \\ \vdots \\ S_{n-1} \\ S_{n+3} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ f_c\left(T_b\right) \end{bmatrix}$$

$$S_i = \theta q_i^{(l+1)} + \bar{\theta} q_i^{(l)} + g\left(T_{i-1}^{(l)}, T_i^{(l)}, T_{i+1}^{(l)}\right)$$

  – Bulk coolant temperature of the previous step is used to break the coupling between the heat conduction equation and the convection equation

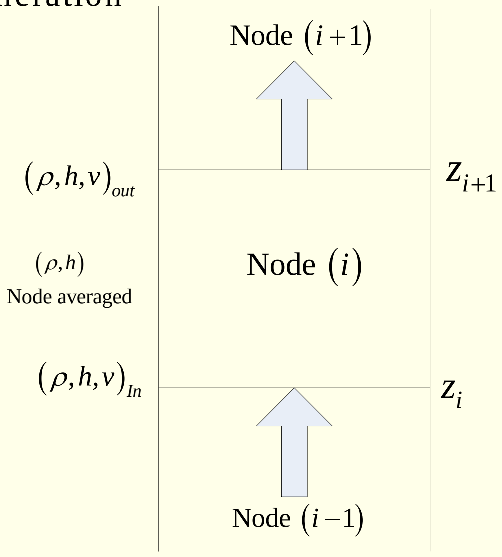
Solution of Heat Convection

- Governing Equations

$$\frac{\partial \rho}{\partial t} = \Gamma - \nabla \bullet \rho v$$

Mass conservation : $\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial z} = 0$ with $\Gamma = 0$ indicating no mass generation

Energy conservation : $\frac{\partial \rho h}{\partial t} + \frac{\partial \rho h v}{\partial z} = q_c + \frac{\zeta}{A_c} q_w \equiv q$



- Integration over a Node (Box scheme)

Mass conservation : $\frac{d\overline{\rho}}{dt} + \frac{1}{\Delta z}\left([\rho v]_{out} - [\rho v]_{in}\right) = 0$

Energy conservation : $\frac{d\overline{\rho h}}{dt} + \frac{1}{\Delta z}\left([\rho h v]_{out} - [\rho h v]_{in}\right) = \overline{q}$

ζ : Heated Perimeter

A_c :Channel Area

Temporal Differencing by Theta Method

$$\frac{\rho^{(l+1)}-\rho^{(l)}}{\Delta t}+\frac{\theta}{\Delta z}\left(\left[\rho v\right]_{out}^{(l+1)}-\left[\rho v\right]_{in}^{(l+1)}\right)+\frac{\bar{\theta}}{\Delta z}\left(\left[\rho v\right]_{out}^{(l)}-\left[\rho v\right]_{in}^{(l)}\right)=0$$

$$\frac{\left[\rho h\right]^{(l+1)}-\left[\rho h\right]^{(l)}}{\Delta t}+\frac{\theta}{\Delta z}\left(\left[\rho h v\right]_{out}^{(l+1)}-\left[\rho h v\right]_{in}^{(l+1)}\right)+\frac{\bar{\theta}}{\Delta z}\left(\left[\rho h v\right]_{out}^{(l)}-\left[\rho h v\right]_{in}^{(l)}\right)=\theta q^{(l+1)}+\bar{\theta} q^{(l)}$$

$-$Divide by $\frac{\theta}{\Delta z}$ and move known terms to RHS except terms in time deirvative

$$\frac{\Delta z}{\theta \Delta t}\left(\rho^{(l+1)}-\rho^{(l)}\right)+\left[\rho v\right]_{out}^{(l+1)}=\left[\rho v\right]_{in}^{(l+1)}-\frac{\bar{\theta}}{\theta}\left(\left[\rho v\right]_{out}^{(l)}-\left[\rho v\right]_{in}^{(l)}\right)\cdots (1)$$

$$\frac{\Delta z}{\theta \Delta t}\left(\left[\rho h\right]^{(l+1)}-\left[\rho h\right]^{(l)}\right)+\left[\rho h v\right]_{out}^{(l+1)}=\left[\rho h v\right]_{in}^{(l+1)}-\frac{\bar{\theta}}{\theta}\left(\left[\rho h v\right]_{out}^{(l)}-\left[\rho h v\right]_{in}^{(l)}\right)+\Delta z\left(q^{(l+1)}+\frac{\bar{\theta}}{\theta} q^{(l)}\right)\cdots (2)$$

Formulation in terms of Volume Enthalpy Change

- Unknowns

  2 average values(ρ,h) + 3 outlet values ($\rho_{out}, h_{out}, v_{out}$)

- Auxiliary State Equations

$$\frac{\Delta z}{\theta \Delta t}\left(\rho^{(l+1)} - \rho^{(l)}\right) + [\rho v]_{out}^{(l+1)} = [\rho v]_{in}^{(l+1)} - \frac{\bar{\theta}}{\theta}\left([\rho v]_{out}^{(l)} - [\rho v]_{in}^{(l)}\right) \cdots (1)$$

Assumption : $h^{(l+1)} = \frac{1}{2}\left(h_{out}^{(l+1)} + h_{in}^{(l+1)}\right)$

$$\begin{cases} \rho^{(l+1)} = \rho^{(l)} + \left.\frac{d\rho}{dh}\right|^{(l)} \left(h^{(l+1)} - h^{(l)}\right) \Rightarrow \rho^{(l+1)} - \rho^{(l)} = \left.\frac{d\rho}{dh}\right|^{(l)} \left(h^{(l+1)} - h^{(l)}\right) \\ [\rho h]^{(l+1)} = [\rho h]^{(l)} + \left.\frac{d[\rho h]}{dh}\right|^{(l)} \left(h^{(l+1)} - h^{(l)}\right) \Rightarrow [\rho h]^{(l+1)} - [\rho h]^{(l)} = \left.\frac{d[\rho h]}{dh}\right|^{(l)} \left(h^{(l+1)} - h^{(l)}\right) \end{cases}$$

- Conversion in terms of Volume Enthalpy Change $\left(X \equiv h^{(l+1)} - h^{(l)}\right)$

$$(1) \rightarrow [\rho v]_{out}^{(l+1)} = -\frac{\Delta z}{\theta \Delta t} \times \left.\frac{d\rho}{dh}\right|^{(l)} \left(h^{(l+1)} - h^{(l)}\right) + [\rho v]_{in}^{(l+1)} - \frac{\bar{\theta}}{\theta}\left([\rho v]_{out}^{(l)} - [\rho v]_{in}^{(l)}\right) \equiv \alpha X + \beta$$

$$\begin{aligned} [\rho h v]_{out}^{(l+1)} &= [\rho v]_{out}^{(l+1)} \cdot h_{out}^{(l+1)} \\ &= (\alpha X + \beta)\left(2X + 2h^{(l)} - h_{in}^{(l+1)}\right) \\ &= 2\alpha X^2 + \gamma X + \delta \quad \text{where } \gamma = \alpha\left(2h^{(l)} - h_{in}^{(l+1)}\right) + 2\beta,\ \delta = \beta(2h^{(l)} - h_{in}^{(l+1)}) \end{aligned}$$

Solution Sequence

- Quadratic Equation

$$(2) \rightarrow \frac{\Delta z}{\theta \Delta t} \cdot \frac{d\overline{\rho h}}{dh}\bigg|^{l} X + \left(2\alpha X^2 + \gamma X + \delta\right) = [\rho hv]_{in}^{(l+1)} - \frac{\bar{\theta}}{\theta}\left([\rho hv]_{out}^{(l)} - [\rho hv]_{in}^{(l)}\right) + \Delta z\left(q^{(l+1)} + \frac{\bar{\theta}}{\theta} q^{(l)}\right)$$

$$2\alpha X^2 + \left(\frac{\Delta z}{\theta \Delta t} \cdot \frac{d\overline{\rho h}}{dh}\bigg|^{(l)} + \gamma\right) X + \delta - [\rho hv]_{in}^{(l+1)} + \frac{\bar{\theta}}{\theta}\left([\rho hv]_{out}^{(l)} - [\rho hv]_{in}^{(l)}\right) - \Delta z\left(q^{(l+1)} + \frac{\bar{\theta}}{\theta} q^{(l)}\right) = 0$$

$$\Leftrightarrow aX^2 + bX + c = 0$$

- Two Roots

$$X_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$X_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{\Delta z}{\theta \Delta t}\left([\rho h]^{(l+1)} - [\rho h]^{(l)}\right) + [\rho hv]_{out}^{(l+1)} = [\rho hv]_{in}^{(l+1)} - \frac{\bar{\theta}}{\theta}\left([\rho hv]_{out}^{(l)} - [\rho hv]_{in}^{(l)}\right) + \Delta z\left(q^{(l+1)} + \frac{\bar{\theta}}{\theta} q^{(l)}\right) \cdots (2)$$

Solution Sequence

- Linear Formulation

$$[\rho h v]_{out}^{(l+1)} = [\rho v]_{out}^{(l+1)} \cdot h_{out}^{(l+1)} \cong [\rho v]_{out}^{(l+1)} \cdot h_{out}^{(l)} = (\alpha X + \beta) h_{out}^{(l)}$$

$$\beta = [\rho v]_{in}^{(l+1)} - \frac{\bar{\theta}}{\theta}\left([\rho v]_{out}^{(l)} - [\rho v]_{in}^{(l)}\right)$$

$$(2) \rightarrow \frac{\Delta z}{\theta \Delta t} \cdot \frac{d\overline{\rho h}}{dh}\bigg|^{l} X + (\alpha X + \beta) h_{out}^{(l)} = [\rho h v]_{in}^{(l+1)} - \frac{\bar{\theta}}{\theta}\left([\rho h v]_{out}^{(l)} - [\rho h v]_{in}^{(l)}\right) + \Delta z\left(q^{(l+1)} + \frac{\bar{\theta}}{\theta} q^{(l)}\right)$$

$$\left(\frac{\Delta z}{\theta \Delta t} \cdot \frac{d\overline{\rho h}}{dh}\bigg|^{(l)} + \alpha h_{out}^{(l)}\right) X = [\rho h v]_{in}^{(l+1)} - \frac{\bar{\theta}}{\theta}\left([\rho h v]_{out}^{(l)} - [\rho h v]_{in}^{(l)}\right) + \Delta z\left(q^{(l+1)} + \frac{\bar{\theta}}{\theta} q^{(l)}\right) - h_{out}^{(l)} \beta$$

$$= \Delta z\left(q^{(l+1)} + \frac{\bar{\theta}}{\theta} q^{(l)}\right) + [\rho h v]_{in}^{(l)} - h_{out}^{(l)} [\rho v]_{in}^{(l+1)} + \frac{\bar{\theta}}{\theta}\left[h_{out}^{(l)}\left([\rho v]_{out}^{(l)} - [\rho v]_{in}^{(l)}\right) - \left([\rho h v]_{out}^{(l)} - [\rho h v]_{in}^{(l)}\right)\right]$$

$$AX = B \rightarrow X = \frac{B}{A}$$

- Choice of the two solutions (X_1, X_2) of quadratic formulation
  - Choose one closer to X

Final Solution

$$h^{(l+1)} = h^{(l)} + X$$

$$\rho^{(l+1)} = \rho^{(l)}\left(h^{(l+1)}\right), \quad \left[\rho h\right]^{(l+1)} \cong \rho^{(l+1)} \cdot h^{(l+1)}$$

$$(1) \rightarrow \left[\rho v\right]_{out}^{(l+1)} = \left[\rho v\right]_{in}^{(l+1)} - \frac{\Delta z}{\theta \Delta t}\left(\rho^{(l+1)} - \rho^{(l)}\right) - \frac{\bar{\theta}}{\theta}\left(\left[\rho v\right]_{out}^{(l)} - \left[\rho v\right]_{in}^{(l)}\right)$$

$$(2) \rightarrow \left[\rho h v\right]_{out}^{(l+1)} = \left[\rho h v\right]_{in}^{(l+1)} - \frac{\Delta z}{\theta \Delta t}\left(\left[\rho h\right]^{(l+1)} - \left[\rho h\right]^{(l)}\right) - \frac{\bar{\theta}}{\theta}\left(\left[\rho h v\right]_{out}^{(l)} - \left[\rho h v\right]_{in}^{(l)}\right) + \Delta z\left(q^{(l+1)} + \frac{\bar{\theta}}{\theta} q^{(l)}\right)$$