7. Minimum Realizations and Coprime Fractions

- ✓ What is Minimal Realization ?
- ✓ Minimal Realization and Coprime
- ✓ Computing Coprime Fractions
- ✓ Balanced Realization
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What is Minimal Realization ?

- Realization problem (IOD → SVD)
  - ∗ To apply many design techniques & computational algorithms for dynamical equations.
  - ∗ To simulate before the system is built.
  - ∗ To establish the link between SVD & IOD.
- Good realization among many realizations.

$$\left.\begin{array}{l} *\ \text{least possible dimension} \\ *\ \text{controllable \& observable} \end{array}\right\}\text{(minimal dimension)}$$

∗ easy to analysis (simple form)

$\Rightarrow$ minimal realization,

$$\begin{cases} \text{Controllable(controller) canonical form} \\ \text{Observale(observer) canonical form} \\ \text{Jordan-form} \end{cases}$$

Minimal Realization and Coprime

Definition: Degree of proper rational transfer function

For a proper rational transfer function

$$g(s) = \frac{N(s)}{D(s)},$$

If $N(s)$ and $D(s)$ is coprime,

$$\text{Degree of } g(s) := \text{Degree of } D(s).$$

Question:

What is the degree of $\hat{g}(s) = \dfrac{s+1}{s^2+2s+1}$?

Minimal Realization and Coprime

Definition:

Let SISO state equation

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

be realization of proper & coprime rational ftn $g(s)$.

Then, the state equation is said to be **irreducible** if*f*

$$\det(sI - A) = k(\text{denominatior of } g(s))$$

$$\dim A = \deg g(s)$$

where k is a nozero constant.

The irreducible state equation is called **minimal realization** of $g(s)$

Controllable Canonical Form

Realization of $g(s) = N(s) \big/ D(s)$

$$g(s) = e + \frac{\beta_1 s^{n-1} + \cdots + \beta_n}{s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n}$$

$$g(s) = \frac{\beta_1 s^{n-1} + \cdots + \beta_n}{s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n} = \frac{N(s)}{D(s)} = \frac{y(s)}{u(s)}$$

$$\Rightarrow D(s)y(s) = N(s)u(s)$$

$$\Rightarrow y(s) = N(s)D^{-1}(s)u(s)$$

Controllable canonical-form realization

Introduce new variable $v(t)$ by

$$v(s) = D^{-1}(s)u(s)$$

$$\Rightarrow \frac{D(s)v(s) = u(s)}{y(s) = N(s)v(s)} \quad \left(\frac{v(s)}{u(s)} = \frac{1}{D(s)} \right)$$

Controllable Canonical Form

Realization of Proper Rational Functions

$$\frac{v(s)}{u(s)} = \frac{1}{D(s)} = \frac{1}{s^n + \alpha_1 s^{n-1} + \cdots \alpha_n}$$

$$D(s)v(s) = u(s)$$

Define

$$\mathrm{x}(s) := \begin{bmatrix} \mathrm{x}_1(s) \\ \vdots \\ \vdots \\ \mathrm{x}_n(s) \end{bmatrix} := \begin{bmatrix} s^{n-1}v(s) \\ s^{n-2}v(s) \\ \vdots \\ v(s) \end{bmatrix} = \begin{bmatrix} s^{n-1} \\ s^{n-2} \\ \vdots \\ 1 \end{bmatrix} v(s),$$

$$s^n v(s) = -\alpha_n v(s) - \alpha_{n-1} s v(s) - \cdots - \alpha_1 s^{n-1} v(s) + u(s)$$

$$s\mathrm{x}_1(s) = -\alpha_n \mathrm{x}_n(s) - \alpha_{n-1} \mathrm{x}_{n-1}(s) - \cdots - \alpha_1 \mathrm{x}_1(s) + u(s)$$

Controllable Canonical Form

In time domain

$$\dot{x}_1(t) = -\alpha_1 x_1 - \alpha_2 x_2 - \cdots - \alpha_n x_n + u(t)$$

$$sx(s) := \begin{bmatrix} s^n \\ s^{n-1} \\ \vdots \\ s \end{bmatrix} v(s) = \begin{bmatrix} \dot{x}_1(s) \\ x_1(s) \\ \vdots \\ x_{n-1}(s) \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & \cdots & \cdots & \cdots & -\alpha_n \\ 1 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & 0 & \vdots \\ \cdots & \cdots & 0 & 1 & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 0 \end{bmatrix} u(t) \qquad (*)$$

$$v = \begin{bmatrix} 0 & 0 & \cdots & \cdots & \cdots & \cdots & 1 \end{bmatrix} x$$

Controllable Canonical Form

$$y(s) = N(s)v(s) = \sum_{i=1}^{n} \beta_i s^{n-i} v(s)$$

$$= \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \end{bmatrix} \begin{bmatrix} s^{n-1}v(s) \\ s^{n-2}v(s) \\ \vdots \\ v(s) \end{bmatrix}$$

$$= \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \end{bmatrix} \mathrm{x}(s)$$

$$\Rightarrow y(t) = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \end{bmatrix} \mathrm{x}(t)$$

Controllable Canonical Form

Controllability

$$[s\mathbf{I}-A \quad B]$$

$$=\left[\begin{bmatrix} s+\alpha_1 & \alpha_2 & \dots & & \dots & \alpha_n & \vdots & 1 \\ -1 & s & 0 & 0 & & & \vdots & 0 \\ 0 & -1 & s & 0 & 0 & & \vdots & \vdots \\ & 0 & -1 & \dots & 0 & 0 & \vdots & \vdots \\ \dots & & 0 & -1 & s & 0 & \vdots & 0 \\ 0 & 0 & \dots & 0 & -1 & s & \vdots & 0 \end{bmatrix}\right] \text{LI rows} \forall\ s$$

has rank n regardless of $\mathrm{C}=[\beta_n \quad \cdots \quad \beta_1]$ or $N(s)$.

Controllable realization from $\dfrac{N(s)}{D(s)}$ without coprimeness

$\Rightarrow$ Controllable Canonical form

Controllable Canonical Form

Theorem 7.1

Controllable canonical form is observable iff $D(s)$ and $N(s)$ are coprime.

Pf.

$$(A \Rightarrow B) \Leftrightarrow (\sim B \Rightarrow \sim A)$$

If $D(s)$ and $N(s)$ are not coprime, there exists a λ_1 such that

$$N(\lambda_1) = \beta_1 \lambda_1^{\ 3} + \beta_2 \lambda_1^{\ 2} + \beta_3 \lambda_1 + \beta_4 = 0$$

$$D(\lambda_1) = \lambda_1^{\ 4} + \alpha_1 \lambda_1^{\ 3} + \alpha_2 \lambda_1^{\ 2} + \alpha_3 \lambda_1 + \alpha_4 = 0.$$

Controllable Canonical Form

Pf. (cont)

Let us define $v' := [\lambda_1^3 \ \lambda_1^2 \ \lambda_1 \ 1] \neq 0,$

$$N(\lambda_1) = cv = 0$$

$$Av = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1^3 \\ \lambda_1^2 \\ \lambda_1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_1^4 \\ \lambda_1^3 \\ \lambda_1^2 \\ \lambda_1 \end{bmatrix} = \lambda_1 v$$

$$A^2 v = AAv = \lambda_1 Av = \lambda_1^2 v, \ \ldots$$

$$Ov = \begin{bmatrix} c \\ cA \\ cA^2 \\ cA^3 \end{bmatrix} v = \begin{bmatrix} cv \\ \lambda_1 cv \\ \lambda_1^2 cv \\ \lambda_1^3 cv \end{bmatrix} = 0$$

This implies that O does not have full rank, i.e., not observable.

Controllable Canonical Form

Pf. (cont)

$(A \Leftarrow B) \Leftrightarrow (\sim A \Rightarrow \sim B)$

If the state equation is not observable, then

by Theorem 6.O1, there exists λ_1 of A and $\mathrm{v} \neq 0$ such that

$$\begin{bmatrix} \mathrm{A} - \lambda_1 \mathrm{I} \\ \mathrm{c} \end{bmatrix} \mathrm{v} = 0.$$

or

$$(\mathrm{A} - \lambda_1 \mathrm{I})\mathrm{v}{=}0 \text{ and } \mathrm{cv} = 0.$$

$$N(\lambda_1) = \mathrm{cv} {=} \beta_1 \lambda_1{}^3 + \beta_2 \lambda_1{}^2 + \beta_3 \lambda_1 + \beta_4 = 0.$$

λ_1 is a root of $N(\lambda_1)$.

$\det(\lambda_1 \mathrm{I} - \mathrm{A}) = D(\lambda_1) = 0.$

This implies $N(s)$ and $D(s)$ are not coprime.

Observable Canonical Form

Observable canonical form realization

$$y(s) = \frac{N(s)}{D(s)} u(s)$$

$$D(s)y(s) = N(s)u(s)$$

In Time Domain

$$y^{(n)}(t) + \alpha_1 y^{(n-1)}(t) + \cdots + \alpha_n y(t)$$

$$= \beta_1 u^{(n-1)}(t) + \cdots + \beta_n u(t) \qquad (*)$$

Taking Laplace Transform with nonzero initial condition,

Observable Canonical Form

Taking Laplace Transform (non-zero initial condition)

$$
\begin{aligned}
& s^n y(s) - (s^{n-1} y(0) + s^{n-2} y^{(1)}(0) + \cdots + y^{(n-1)}(0) \\
& + \alpha_1 \left\{ s^{n-1} y(s) - \left(s^{n-2} y(0) + \cdots + y^{(n-2)}(0) \right) \right\} \\
& + \cdots + \alpha_n y(s) \\
& = \beta_1 \left\{ s^{n-1} u(s) - \left(s^{n-2} u(0) + \cdots + u^{(n-2)}(0) \right) \right\} \\
& + \beta_2 \{ \ldots \} + \ldots \\
& + \beta_n u(s)
\end{aligned}
$$

Observable Canonical Form

$$D(s)y(s) = N(s)u(s) + \{ y(0)s^{n-1} + \left(y^{(1)}(0) + \alpha_1 y(0) - \beta_1 u(0) \right) s^{n-2}$$
$$+ \cdots \left(y^{(n-1)}(0) + \alpha_1 y^{(n-2)}(0) - \beta_1 u^{(n-2)}(0) + \alpha_2 y^{(n-3)}(0) - \beta_2 u^{(n-3)}(0) + \cdots \right.$$
$$\left. + \alpha_{n-1} y(0) - \beta_{n-1} u(0) \right) \}$$

If initial state is known, output for a $u(t)$ is unique.

We choose state as;

$$\mathrm{x}_n(t) := y(t)$$
$$\mathrm{x}_{n-1}(t) := y^{(1)}(t) + \alpha_1 y(t) - \beta_1 u(t)$$
$$\vdots \qquad \qquad \vdots$$
$$\mathrm{x}_1(t) := y^{(n-1)}(t) + \alpha_1 y^{(n-2)}(t) - \beta_1 u^{(n-2)}(t) + \cdots + \alpha_{n-1} y(t) - \beta_{n-1} u(t) \quad (**)$$

Observable Canonical Form

$$\Rightarrow y = \mathrm{x}_n$$

$$\mathrm{x}_{n-1} = \dot{\mathrm{x}}_n + \alpha_1 x_n - \beta_1 u \quad \Rightarrow \dot{\mathrm{x}}_n = \mathrm{x}_{n-1} - \alpha_1 x_n + \beta_1 u$$

$$\mathrm{x}_{n-2} = \dot{\mathrm{x}}_{n-1} + \alpha_2 x_n - \beta_2 u \Rightarrow \dot{\mathrm{x}}_{n-1} = \mathrm{x}_{n-2} - \alpha_2 x_n + \beta_2 u$$

$$\vdots$$

$$\mathrm{x}_1 = \dot{x}_2 + \alpha_{n-1} x_n - \beta_{n-1} u \ \Rightarrow \ \dot{\mathrm{x}}_2 = \mathrm{x}_1 - \alpha_{n-1} x_n + \beta_{n-1} u$$

$$(*)\&(**) \Rightarrow \ \dot{\mathrm{x}}_1 = -\alpha_n x_n + \beta_n u$$

$$\Rightarrow \dot{\mathrm{x}} = \begin{bmatrix} 0 & \cdots & \cdots & 0 & -\alpha_n \\ 1 & 0 & \cdots & 0 & -\alpha_{n-1} \\ & 1 & & \vdots & \vdots \\ & & & 0 & \vdots \\ & & & 1 & -\alpha_1 \end{bmatrix} \mathrm{x} + \begin{bmatrix} \beta_n \\ \vdots \\ \vdots \\ \vdots \\ \beta_1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & \cdots & 1 \end{bmatrix} \mathrm{x}$$

Observable Canonical Form

Note:

$\{A, C\}$ does not depend on $\{\beta_i\}$, i.e. $N(s)$

$$\rho\begin{bmatrix} C \\ s\mathbf{I} - A \end{bmatrix} = n$$

$\Rightarrow \{A, C\}$ is always observable regardless of coprime between $N(s)$ & $D(s)$ (may not be controllable if not coprime)

$\Rightarrow$ Observable(or observer) canonical form.

Minimal Realization

Coprime Fractions

$$g(s) = \frac{N(s)}{D(s)} = \frac{\bar{N}(s)R(s)}{\bar{D}(s)R(s)},$$

If $\bar{D}(s)$ and $\bar{N}(s)$ are coprime, controllable or observable realization of $\bar{g}(s) = \bar{N}(s)/\bar{D}(s)$ is minimal realization.

Theorem 7.2

$\{\mathbf{A},\ \mathbf{b},\ \mathbf{c},\ d\}$ is a minimal realization of $g(s)$ iff $\{\mathbf{A},\mathbf{b}\}$ is controllable and $\{\mathbf{A},\mathbf{c}\}$ is observable or iff

$$\dim \mathbf{A} = \deg g(s).$$

Minimal Realization

Pf. of Theorem 7.2

$(\Rightarrow)$

If $\{\mathbf{A},\mathbf{b}\}$ is not controllable or $\{\mathbf{A},\mathbf{c}\}$ is not observable,

the state equation can be reduced by Theorem 6.6 and 6.O6.

Thus $\{\mathbf{A},\mathbf{b},\ \mathbf{c},\ d\}$ is not minimal.

Minimal Realization

Pf. of Theorem 7.2

$(\Leftarrow)$

If $\{\mathbf{A}, \mathbf{b}, \mathbf{c}, d\}$ is controllable and observable, then $\rho(\mathrm{OC}) = n.$

However, if $\{\mathbf{A}, \mathbf{b}, \mathbf{c}, d\}$ is not minimal, there exists

a realization of $g(s)$ $\{\bar{\mathbf{A}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}, \bar{d}\}$ with $n_1 < n.$ By Theorem 4.1,

$$\mathbf{c}\mathbf{A}^m\mathbf{b} = \bar{\mathbf{c}}\bar{\mathbf{A}}^m\bar{\mathbf{b}}, \ m = 0, 1, 2, \ldots$$

$$\mathrm{OC} = \begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \\ \ldots \\ \mathbf{cA}^{n-1} \end{bmatrix} \begin{bmatrix} \mathbf{b} & \mathbf{Ab} & \ldots & \mathbf{A}^{n-1}\mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{cb} & \mathbf{cAb} & \ldots & \mathbf{cA}^{n-1}\mathbf{b} \\ \mathbf{cAb} & & & \\ \ldots & & \ldots & \ldots \\ \mathbf{cA}^{n-1}\mathbf{b} & \ldots & & \mathbf{cA}^{2(n-1)}\mathbf{b} \end{bmatrix}$$

$$= \bar{\mathrm{O}}_n \bar{\mathrm{C}}_n \ \text{ has rank } n_1 < n.$$

This is contracts to that $\{\mathbf{A}, \mathbf{b}, \mathbf{c}, d\}$ is controllable and observable.

Minimal Realization

Theorem 7.3

All minimal realization of $g(s)$ are equivalent.

Pf.

Let $\{\mathbf{A}, \mathbf{b}, \mathbf{c}, d\}$ and $\{\bar{\mathbf{A}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}, \bar{d}\}$ are minimal,

$$\mathrm{OC} = \bar{\mathrm{O}}\bar{\mathrm{C}} \quad \text{and} \quad \mathrm{OAC} = \bar{\mathrm{O}}\bar{\mathrm{A}}\bar{\mathrm{C}}$$

$$\bar{\mathrm{A}} = \bar{\mathrm{O}}^{-1}\mathrm{OAC}\bar{\mathrm{C}}^{-1} \quad = \mathrm{PAP}^{-1},$$

where $\mathrm{P} = \bar{\mathrm{O}}^{-1}\mathrm{O} = \bar{\mathrm{C}}\mathrm{C}^{-1} (\leftarrow \mathrm{OC} = \bar{\mathrm{O}}\bar{\mathrm{C}})$.

Note:

If $\{\mathbf{A}, \mathbf{b}, \mathbf{c}, d\}$ is minimal (controllable and observable),

Asymptotically stability $\Leftrightarrow$ BIBO stability

Computing Coprime Fractions

Computing Coprime Fractions

$$g(s)=\frac{N(s)}{D(s)}$$

$$\frac{N(s)}{D(s)}=\frac{\bar{N}(s)}{\bar{D}(s)}$$

If deg $\bar{D}(s) < \deg D(s)$, $D(s)$ and $N(s)$ are not coprime.

$$D(s)(-\bar{N}(s))+N(s)\bar{D}(s)=0$$

$$D(s)=D_0+D_1s+D_2s^2+D_3s^3+D_4s^4$$

$$N(s)=N_0+N_1s+N_2s^2+N_3s^3+N_4s^4$$

$$\bar{D}(s)=\bar{D}_0+\bar{D}_1s+\bar{D}_2s^2+\bar{D}_3s^3$$

$$\bar{N}(s)=\bar{N}_0+\bar{N}_1s+\bar{N}_2s^2+\bar{N}_3s^3$$

Computing Coprime Fractions

By Coefficient Comparison, := S (Sylvester resultant)

$$\begin{bmatrix} D_0 & N_0 & \vdots & 0 & 0 & \vdots & 0 & 0 & \vdots & 0 & 0 \\ D_1 & N_1 & \vdots & D_0 & N_0 & \vdots & 0 & 0 & \vdots & 0 & 0 \\ D_2 & N_2 & \vdots & D_1 & N_1 & \vdots & D_0 & N_0 & \vdots & 0 & 0 \\ D_3 & N_3 & \vdots & D_2 & N_2 & \vdots & D_1 & N_1 & \vdots & D_0 & N_0 \\ D_4 & N_4 & \vdots & D_3 & N_3 & \vdots & D_2 & N_2 & \vdots & D_1 & N_1 \\ 0 & 0 & \vdots & D_4 & N_4 & \vdots & D_3 & N_3 & \vdots & D_2 & N_2 \\ 0 & 0 & \vdots & 0 & 0 & \vdots & D_4 & N_4 & \vdots & D_3 & N_3 \\ 0 & 0 & \vdots & 0 & 0 & \vdots & 0 & 0 & \vdots & D_4 & N_4 \end{bmatrix} \begin{bmatrix} -\bar{N} \\ \bar{D}_0 \\ -\bar{N}_1 \\ \bar{D}_1 \\ -\bar{N}_2 \\ \bar{D}_2 \\ -\bar{N}_3 \\ \bar{D}_3 \end{bmatrix} = 0$$

LI vectors

Primary dependent N-column

D(s) and N(s) are coprime iff S is nonsingular

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Computing Coprime Fractions

Example 7.1

$$\frac{N(s)}{D(s)} = \frac{6s^3 + s^2 + 3s - 20}{2s^4 + 7s^3 + 15s^2 + 16s + 10}$$

Computing Coprime Fractions

Example 7.1 (cont)

$$
\begin{bmatrix}
10 & -20 & \vdots & 0 & 0 & \vdots & 0 & 0 & \vdots & 0 & 0 \\
16 & 3 & \vdots & 10 & -20 & \vdots & 0 & 0 & \vdots & 0 & 0 \\
15 & 1 & \vdots & 16 & 3 & \vdots & 10 & -20 & \vdots & 0 & 0 \\
7 & 6 & \vdots & 15 & 1 & \vdots & 16 & 3 & \vdots & 10 & -20 \\
2 & 0 & \vdots & 7 & 6 & \vdots & 15 & 1 & \vdots & 16 & 3 \\
0 & 0 & \vdots & 2 & 0 & \vdots & 7 & 6 & \vdots & 15 & 1 \\
0 & 0 & \vdots & 0 & 0 & \vdots & 2 & 0 & \vdots & 7 & 6 \\
0 & 0 & \vdots & 0 & 0 & \vdots & 0 & 0 & \vdots & 2 & 0
\end{bmatrix}
\begin{bmatrix}
-\bar{N} \\ \bar{D}_0 \\ -\bar{N}_1 \\ \bar{D}_1 \\ -\bar{N}_2 \\ \bar{D}_2 \\ -\bar{N}_3 \\ \bar{D}_3
\end{bmatrix}
= 0
$$

LI vectors

Primary dependent N-column

Computing Coprime Fractions

Example 7.1 (cont)

This monic null vector equals

$$\begin{bmatrix} -\bar{N}_0 & \bar{D}_0 & -\bar{N}_1 & \bar{D}_1 & -\bar{N}_2 & \bar{D}_2 \end{bmatrix}'$$
$$= \begin{bmatrix} 4 & 2 & -3 & 2 & 0 & 1 \end{bmatrix}.$$

Thus we have $\bar{N}(s) = -4 + 3s + 0 \cdot s^2 \quad \bar{D}(s) = 2 + 2s + s^2$

and

$$\frac{6s^3 + s^2 + 3s - 20}{2s^4 + 7s^3 + 15s^2 + 16s + 10} = \frac{3s - 4}{s^2 + 2s + 2}.$$

Computing Coprime Fractions

Theorem 7.4

$\deg g(s) = $ number of linearly independent N-columns $=: \mu$

The coefficients of a coprime fraction $g(s) = \bar{N}(s)/\bar{D}(s)$ is given by

$$\begin{bmatrix} -\bar{N}_0 & \bar{D}_0 & -\bar{N}_1 & \bar{D}_1 & \cdots & -\bar{N}_\mu & \bar{D}_\mu \end{bmatrix}'$$

QR Decomposition for column searching of S

Consider an $n \times m$ matrix M.

Then there exists an $n \times n$ orthonornal matrix $\bar{\text{Q}}$ such that

$$\bar{\mathbf{Q}}\mathbf{M} = \mathbf{R},$$

where R is an **upper** triangular matrix and

$\rho\mathbf{M} = \rho\mathbf{R}$ with LI columns in order from left to right.

$$\mathbf{M} = \mathbf{QR}, \ \bar{\mathbf{Q}}^{-1} = \bar{\mathbf{Q}}' := \mathbf{Q} \leftarrow \text{ QR decomposition.}$$

Computing Coprime Fractions

Example 7.1

$$\mathbf{R} = \begin{bmatrix} -25.1 & 3.7 & -20.6 & 10.1 & -11.6 & 11.0 & -4.1 & 5.3 \\ 0 & -20.7 & -10.3 & 4.3 & -7.2 & 2.1 & -3.6 & 6.7 \\ 0 & 0 & -10.2 & -15.6 & -20.3 & 0.8 & -16.8 & 9.6 \\ 0 & 0 & 0 & 8.9 & -3.5 & -17.9 & -11.2 & 7.3 \\ 0 & 0 & 0 & 0 & -5.0 & 0 & -12.0 & -15.0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

↑ Primary dependent N-column

Computing Coprime Fractions

Primary dependent N-column

$$\mathbf{R} = \begin{bmatrix} d & x & x & x & x & \overset{\downarrow\ \mu=2}{x} & x & x \\ 0 & n & x & x & x & x & x & x \\ 0 & 0 & d & x & x & x & x & x \\ 0 & 0 & 0 & n & x & x & x & x \\ 0 & 0 & 0 & 0 & d & 0 & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

HW 7-1

Consider

$$g(s) = \frac{\beta_1 s + \beta_2}{s^2 + \alpha_1 s + \alpha_2} =: \frac{N(s)}{D(s)}$$

and its realization

$$\dot{\mathbf{x}} = \begin{bmatrix} -\alpha_1 & -\alpha_2 \\ 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \quad y = [\beta_1 \quad \beta_2] \mathbf{x}$$

Show that the state equation is observable if and only if the Sylvester resultant of $D(s)$ and $N(s)$ is nonsingular.

Balanced Realization

Balanced Realization

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$
$$y = \mathbf{c}\mathbf{x}$$

If the system is controllable, observable, and asymptotically stable, there exist $\mathbf{W}_c > 0, \mathbf{W}_o > 0$ such that

$$\mathbf{A}\mathbf{W}_c + \mathbf{W}_c\mathbf{A}' = -\mathbf{b}\mathbf{b}'$$

and

$$\mathbf{A}\mathbf{W}_o + \mathbf{W}_o\mathbf{A}' = -\mathbf{c}\mathbf{c}'.$$

Balanced Realization

Different Minimal Realization has different $\mathbf{W}_c$ and $\mathbf{W}_o$.

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & -4/\alpha \\ 4\alpha & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 2\alpha \end{bmatrix} u \leftarrow g(s) = \frac{3s+18}{s^2+3s+18}$$

$$y = \begin{bmatrix} -1 & -2/\alpha \end{bmatrix}\mathbf{x}$$

$$\mathbf{W}_c = \begin{bmatrix} 0.5 & 0 \\ 0 & \alpha^2 \end{bmatrix} \text{ and } \mathbf{W}_o = \begin{bmatrix} 0.5 & 0 \\ 0 & 1/\alpha^2 \end{bmatrix}$$

$$\mathbf{W}_c\mathbf{W}_o = \begin{bmatrix} 0.25 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{W}_c = \mathbf{W}_o = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix} \text{ for } \alpha = 1 \leftarrow \text{Balanced Realization}$$

Balanced Realization

Theorem 7.5

Let $\{\mathbf{A},\ \mathbf{b},\ \mathbf{c}\}$ and $\{\bar{\mathbf{A}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}\}$ be minimal and equivalent, let $\mathbf{W_c W_o}$ and $\bar{\mathbf{W}}_\mathbf{c}\bar{\mathbf{W}}_\mathbf{o}$ be the product of their controllability and observability Grammians.

$\mathbf{W_c W_o}$ and $\bar{\mathbf{W}}_\mathbf{c}\bar{\mathbf{W}}_\mathbf{o}$ are similar and their eigenvalues are all real and positive.

Balanced Realization

Pf. of Theorem 7.5

$$\bar{\mathbf{A}} = \mathbf{PAP}^{-1} \qquad \bar{\mathbf{b}} = \mathbf{Pb} \qquad \bar{\mathbf{c}} = \mathbf{cP}^{-1}$$

$$\bar{\mathbf{A}}\bar{\mathbf{W}}_c + \bar{\mathbf{W}}_c\bar{\mathbf{A}}' = -\bar{\mathbf{b}}\bar{\mathbf{b}}'$$

and

$$\bar{\mathbf{A}}'\bar{\mathbf{W}}_o + \bar{\mathbf{W}}_o\bar{\mathbf{A}} = -\bar{\mathbf{c}}'\bar{\mathbf{c}}$$

$$\mathbf{PAP}^{-1}\bar{\mathbf{W}}_c + \bar{\mathbf{W}}_c(\mathbf{P}')^{-1}\mathbf{A}'\mathbf{P}' = -\mathbf{Pbb}'\mathbf{P}'$$

which implies

$$\mathbf{AP}^{-1}\bar{\mathbf{W}}_c(\mathbf{P}')^{-1} + \mathbf{P}^{-1}\bar{\mathbf{W}}_c(\mathbf{P}')^{-1}\mathbf{A}' = -\mathbf{bb}'$$

$$\mathbf{W}_c = \mathbf{P}^{-1}\bar{\mathbf{W}}_c(\mathbf{P}')^{-1} \text{ or } \bar{\mathbf{W}}_c = \mathbf{PW}_c\mathbf{P}'$$

$$\mathbf{W}_o = \mathbf{P}'\bar{\mathbf{W}}_o\mathbf{P} \text{ or } \bar{\mathbf{W}}_o = (\mathbf{P}')^{-1}\mathbf{W}_o\mathbf{P}^{-1}$$

$$\mathbf{W}_c\mathbf{W}_o = \mathbf{P}^{-1}\bar{\mathbf{W}}_c(\mathbf{P}')^{-1}\mathbf{P}'\bar{\mathbf{W}}_o\mathbf{P} = \mathbf{P}^{-1}\bar{\mathbf{W}}_c\bar{\mathbf{W}}_o\mathbf{P} \rightarrow \textit{similar}$$

Balanced Realization

Pf. of Theorem 7.5 (cont)

By Theorem 3.6, since $\mathbf{W}_c$ is symmetric and positive definite,

$$\mathbf{W}_c = \mathbf{Q}'\mathbf{D}\mathbf{Q} = \mathbf{Q}'\mathbf{D}^{1/2}\mathbf{D}^{1/2}\mathbf{Q} := \mathbf{R}'\mathbf{R},$$

where $\mathbf{R}$ is not orthogonal but nonsingular.

$$\det(\sigma^2\mathbf{I} - \mathbf{W}_c\mathbf{W}_o) = \det(\sigma^2\mathbf{I} - \mathbf{R}'\mathbf{R}\mathbf{W}_o) = \det(\sigma^2\mathbf{I} - \mathbf{R}\mathbf{W}_o\mathbf{R}')$$

This implies that $\mathbf{W}_c\mathbf{W}_o$ and $\mathbf{R}\mathbf{W}_o\mathbf{R}'$ have the same eigenvalues.
Since $\mathbf{R}\mathbf{W}_o\mathbf{R}'$ is symmetric and positive definite,
all eigenvalues are real and positive. (Q.E.D.)

Note:

$\mathbf{W_c W_o}$ of any minimal realization is similar Σ^2,
where $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n)$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$

Balanced Realization

Theorem 7.6

For any $n-$dimensional minimal equation {A, b, c},
there exists an equivalence transformation $\bar{x} = Px$ such that

$$\bar{\mathbf{W}}_c = \bar{\mathbf{W}}_o = \mathbf{\Sigma}.$$

This is called a *balanced realization.*

Pf. Theorem 7.6

$$\mathbf{W}_c = \mathbf{R}'\mathbf{R}$$

$\mathbf{R}'\mathbf{W}_o\mathbf{R}$:real and symmetric

$$\rightarrow \mathbf{R}'\mathbf{W}_o\mathbf{R} = \mathbf{U}\Sigma^2\mathbf{U}' \leftarrow \mathbf{U}: \textit{orthonormal}$$

$$\mathbf{P}^{-1} = \mathbf{R}'\mathbf{U}\mathbf{\Sigma}^{-1/2} \text{ or } \mathbf{P} = \mathbf{\Sigma}^{1/2}\mathbf{U}'(\mathbf{R}')^{-1}$$

$$\bar{\mathbf{W}}_c = \mathbf{P}\mathbf{W}_c\mathbf{P}' = \mathbf{\Sigma}^{1/2}\mathbf{U}'(\mathbf{R}')^{-1}\mathbf{R}'\mathbf{R}\mathbf{R}^{-1}\mathbf{U}\mathbf{\Sigma}^{1/2} = \mathbf{\Sigma} \quad (\leftarrow \mathbf{W}_c = \mathbf{R}'\mathbf{R})$$

$$\bar{\mathbf{W}}_o = (\mathbf{P}')^{-1}\mathbf{W}_o\mathbf{P}^{-1} = \mathbf{\Sigma}^{-1/2}\mathbf{U}'\mathbf{R}\mathbf{W}_o\mathbf{R}'\mathbf{U}\mathbf{\Sigma}^{-1/2} = \mathbf{\Sigma} \quad (\leftarrow \mathbf{U}\mathbf{\Sigma}^2\mathbf{U}' = \mathbf{R}\mathbf{W}_o\mathbf{R}')$$

Balanced Realization

Note:

If $\bar{\mathbf{W}}_c = \mathbf{I}, \ \bar{\mathbf{W}}_o = \boldsymbol{\Sigma}^2$, it is called input-normal realization.

If $\bar{\mathbf{W}}_c = \boldsymbol{\Sigma}^2, \ \bar{\mathbf{W}}_o = \mathbf{I}$, it is called output-normal realization.

Balanced realization can be used in *system reduction.*

$$\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix} \mathbf{x}$$

$$\mathbf{W}_c = \mathbf{W}_o = \boldsymbol{\Sigma} = \mathrm{diag}(\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2)$$

$$\dot{\mathbf{x}}_1 = \mathbf{A}_{11}\mathbf{x}_1 + \mathbf{b}_1 u$$

$$y = \mathbf{c}_1 \mathbf{x}.$$

If $\boldsymbol{\Sigma}_2$ is much smaller than $\boldsymbol{\Sigma}_1$,

the reduced one is close to the original one.

Realization using Hankel Matrix

Realization from the Hankel matrix

$$g(s) = \frac{\beta_0 s^n + \beta_1 s^{n-1} + \cdots + \beta_n}{s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n}$$

$$= h(0) + h(1)s^{-1} + \cdots + h(n)s^{-n} + \cdots\cdots$$

(infinite series)

$\{h(i), i = 0, 1, \cdots\}$: Markov Parameters

$$\left.\begin{aligned}
&h(0) = \beta_0 \\
&h(1) = -\alpha_1 h(0) + \beta_1 \\
&h(2) = -\alpha_1 h(1) - \alpha_2 h(0) + \beta_2 \\
&\vdots \\
&h(n) = -\alpha_1 h(n-1) - \alpha_2 h(n-2) \cdots - \alpha_n h(0) + \beta_n
\end{aligned}\right\} \ldots\ldots(*)$$

$$h(n+i) = -\alpha_1 h(n+i-1) - \alpha_2 h(n+i-2) \cdots - \alpha_n h(i), i = 1, 2, \cdots$$

Realization using Hankel Matrix

Hankel matrix (m × n order)

$$\mathbf{T}(\mathrm{m,n}) := \begin{bmatrix} h(1) & h(2) & \cdots & h(n) \\ h(2) & h(3) & & \\ \vdots & & & \\ h(m) & h(m+1) & \cdots & h(m+n-1) \end{bmatrix}$$

(!! $h(0)$ is not involved)

Realization using Hankel Matrix

Theorem

Proper transfer function $g(s)$ has degree n iff

$$\rho\mathbf{T}(n,n)=\rho\mathbf{T}(n+k,n+l)=n\ \forall\ k,l=1,2,3$$

Pf)

$(\Rightarrow)$ If $\deg g(s)=n$

$$h(n+i)=\sum_{j=1}^{n}-\alpha_j h(n+i-j)\ (\beta_k \text{ are not involved})$$

for $i=1,2,\cdots$

$\rightarrow (n+1)$th row of $\mathbf{T}(n+1,\infty)$ can be written as

a L.C. of n rows of $\mathbf{T}(n,\infty)$

$\rightarrow \rho\mathbf{T}(n,\infty)=\rho\mathbf{T}(n+1,\infty)$

$\rightarrow \rho\mathbf{T}(n+i,\infty)=\rho\mathbf{T}(n+i+1,\infty),\ i=1,2,\cdots$

$\rightarrow \rho\mathbf{T}(n,\infty)=\rho\mathbf{T}(\infty,\infty)$

Realization using Hankel Matrix

$$\rightarrow \rho \mathbf{T}(n,\infty) = n(\because o/w \; \exists \; \overline{n} < n \text{ satisfying})$$

$$\rightarrow \rho \mathbf{T}(n,n) = \rho \mathbf{T}(n+k, n+l) = n \;\; \forall k,l$$

$$(\Leftarrow) \; \rho \mathbf{T}(n,n) = \rho \mathbf{T}(n+k, n+l) = n$$

implies $\exists \; \{\alpha_j\} \;\; \ni$

$$h(n+i) = \sum_{j=1}^{n} -\alpha_j \cdot h(n+i-j)$$

If we find $\{\beta_i\}$ using $(*)$

$$g(s) = \sum_{i=o}^{\infty} h(i) s^{-i} = \frac{\beta_0 s^n + \cdots + \beta_n}{s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n}$$

$$\rightarrow \deg g(s) = n$$

Realization using Hankel Matrix

Row Searching Algorithm, Appendix A in 2nd Ed.

$$T = \begin{bmatrix} -1 & -1 & 2 & -2 & 1 \\ 1 & -1 & 4 & -2 & 4 \\ -1 & -3 & 8 & -6 & 6 \\ 5 & 1 & -4 & 10 & 1 \\ 7 & 1 & -2 & 10 & 4 \end{bmatrix}$$

$$k_1 T = \begin{bmatrix} 1 & & & & \\ -2 & 1 & & & \\ -4 & & 1 & & \\ 2 & & & 1 & \\ 1 & & & & 1 \end{bmatrix} T = \begin{bmatrix} -1 & -1 & 2 & -2 & 1 \\ 3 & 1 & 0 & 2 & 2 \\ 3 & -1 & 0 & 2 & 2 \\ 3 & 1 & 0 & 6 & 3 \\ 6 & 0 & 0 & 8 & 5 \end{bmatrix} := T_1$$

Realization using Hankel Matrix

$$k_2 T_1 = \begin{bmatrix} 1 & 0 & & & \\ 0 & 1 & & & \\ & -1 & 1 & & \\ & -1 & & 1 & \\ & -2 & & & 1 \end{bmatrix} T_1 = \begin{bmatrix} -1 & -1 & 2 & -2 & 1 \\ 3 & 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 4 & 1 \\ 0 & -2 & 0 & 4 & 1 \end{bmatrix} := T_2$$

$$k_3 = I$$

$$k_4 T_2 = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & -1 & 1 \end{bmatrix} T_2 = \begin{bmatrix} -1 & -1 & 2 & -2 & 1 \\ 3 & 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Realization using Hankel Matrix

$$k := k_4 k_3 k_2 k_1$$

$$kT = \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \vdots \\ \vdots \\ \bar{a}_n \end{bmatrix} \Rightarrow \begin{array}{c} \text{If } i\text{-th row is} \\ \text{dependent row, } \vec{a}_i = 0 \end{array}$$

$$k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ .. & 1 & 0 & 0 \\ .. & k_{ij} & \ddots & 0 \\ .. & .. & .. & 1 \end{bmatrix} \Rightarrow \begin{array}{c} \vec{a}_i = \sum_{j}^{j=i-1} k_{ij} a_j + a_i = 0 \\ a_i = -\sum_{j}^{j=i-1} k_{ij} a_j \end{array}$$

$$\begin{bmatrix} k_{i1} & k_{i2} & \cdots & k_{i(i-1)} & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = 0$$

Realization using Hankel Matrix

Consider s.e. $\{A, B, C, D\}$

$$g(s) = D + C(sI - A)^{-1}B = D + s^{-1}C(I - s^{-1}A)^{-1}B$$

$$= D + CBs^{-1} + CABs^{-2} + CA^2Bs^{-3} \cdots$$

$\Rightarrow \{A, B, C, D\}$ is a realization of $g(s)$ iff

$$D = h(0) \,\&\, h(i) = CA^{i-1}B \quad i = 1, 2, \cdots$$

Realization:

$$\{A, B, C, D\} \leftarrow h(i) = CA^{i-1}B \leftarrow g(s) = \sum h(i)s^{-i} \leftarrow g(s)$$

Realization using Hankel Matrix

Here, $g(s) \xrightarrow{H(n,n)} \{A, B, C, D\}$ realization

Let $g(s) = \dfrac{N(s)}{D(s)}$, $\deg D(s) = n$

(may not be coprime) $\quad (\deg g(s) \le n)$

Determine $\deg g(s)$ (Hankel matrix Rank check)

$$\mathbf{T}(n+1, n) = \begin{bmatrix} h(1) & & h(n) \\ \vdots & & \vdots \\ h(n+1) & & h(2n) \end{bmatrix} \begin{matrix} \left.\right\} \quad \sigma \; LI \text{ rows} \\ n+1-\sigma \; LD \text{ rows} \end{matrix}$$

where σ can be determined by row searching algorithm,

$\begin{bmatrix} h(\sigma+1) & \ldots & h(2\sigma) \end{bmatrix}$ is primary dependent row.

Realization using Hankel Matrix

$$Note: \text{ If } D(s) \,\&\, N(s) \text{ are coprime, } \sigma = n$$

$$\text{otherwise } \sigma < n.$$

$$\rightarrow \begin{bmatrix} a_1 & a_2 & \cdots & a_\sigma & 1 & 0 & \cdots & 0 \end{bmatrix} \mathbf{T}(n+1, n) = 0$$

$$\text{If } \sigma = n,\ h(n+1) = -\alpha_1 h(n) - \alpha_2 h(n-1) - \ldots - \alpha_n h(1)$$

$$\rightarrow a_i = \alpha_{n-i}, i = 1, \cdots, n$$

$$\text{If } \sigma < n,\ h(\sigma+1) = -\sum_{i=1}^{\sigma} a_i h(i)$$

$$\rightarrow a_i \neq \alpha_{n-i}, i = 1, \cdots, n$$

Realization using Hankel Matrix

Claim:

$$A = \begin{bmatrix} 0 & 1 & & & & 0 \\ \vdots & 0 & 1 & & & \\ \vdots & \vdots & 0 & & & \\ \vdots & \vdots & & \ddots & & \\ 0 & 0 & & & 0 & 1 \\ -a_1 & -a_2 & & & & -a_\sigma \end{bmatrix}, \; B = \begin{bmatrix} h(1) \\ h(2) \\ \vdots \\ \vdots \\ \vdots \\ h(\sigma) \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}, \qquad D = h(0)$$

is controllable & observable.

Realization using Hankel Matrix

$$\text{Since } h(\sigma+i) = -a_1 h(\sigma+i-1) - a_2 h(\sigma+i-2) \cdots - a_\sigma h(i)$$

$$i = 1, 2, 3 \cdots$$

$$AB = \begin{bmatrix} h(2) \\ \vdots \\ \vdots \\ h(\sigma+i) \end{bmatrix}, A^2 B = \begin{bmatrix} h(3) \\ h(4) \\ \vdots \\ h(\sigma+2) \end{bmatrix} \cdots A^k B = \begin{bmatrix} h(k+1) \\ h(k+2) \\ \vdots \\ h(k+\sigma) \end{bmatrix}$$

$$\left(C = \begin{bmatrix} 1 & 0 & \cdots \end{bmatrix} \right)$$

$$\Rightarrow CB = h(1), CAB = h(2), \cdots CA^2 B = h(3) \cdots$$

$$\Rightarrow \{A, B, C, D\} \text{ is realization of } g(s)$$

Realization using Hankel Matrix

Controllability matrix

$$\begin{bmatrix} B & AB & \cdots & A^{\sigma-1}B \end{bmatrix} = \mathbf{T}(\sigma,\sigma) \Rightarrow \text{controllable}$$

Observability matrix

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\sigma-1} \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \Rightarrow \text{observable}$$

If A is realized by $n > \sigma$, {A,B} is not controllable,

but {A,C} is observable. $\rightarrow$ Observability Realization

$\mathbf{T}(\sigma,\sigma) = \mathbf{OC}.$

Realization using Hankel Matrix

On the other hand,

$$A = \begin{bmatrix} 0 & & & & & -a_1 \\ 1 & 0 & & & & \vdots \\ & 1 & \ddots & & & \vdots \\ & & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \vdots \\ 0 & & & & 1 & -a_\sigma \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} h(1) & \cdots & \cdots & \cdots & \cdots & h(\sigma) \end{bmatrix}$$

Realization using Hankel Matrix

$$\Rightarrow C = \begin{bmatrix} B & AB & \cdots & \cdots & \cdots & A^{\sigma-1}B \end{bmatrix} = \begin{bmatrix} 1 & 0 & & & \\ 0 & 1 & & & \\ 0 & 0 & \ddots & & \\ 0 & \vdots & & \ddots & \\ 0 & 0 & & & 1 \end{bmatrix}$$

$$O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\sigma-1} \end{bmatrix} = \mathbf{T}(\sigma, \sigma)$$

$\rightarrow$ Always controllable $\rightarrow$ Controllability Realization

$\mathbf{T}(\sigma, \sigma) = \mathbf{OC}.$

Realization using Hankel Matrix

$$\tilde{\mathbf{T}}(\sigma,\sigma) := \mathbf{OAC} = \begin{bmatrix} & 1 & \\ & & \dots \\ -a_\sigma & -a_{\sigma-1} & -a_{\sigma-2} \end{bmatrix} \begin{bmatrix} h(1) & \dots & h(\sigma) \\ \dots & \dots & \dots \\ h(\sigma) & \dots & h(2\sigma) \end{bmatrix}$$

$$= \begin{bmatrix} h(2) & \dots & h(\sigma+1) \\ \dots & \dots & \dots \\ h(\sigma+1) & \dots & h(2\sigma+1) \end{bmatrix}$$

$$\mathbf{AT}(\sigma,\sigma) = \tilde{\mathbf{T}}(\sigma,\sigma)$$

$$\mathbf{A} = \tilde{\mathbf{T}}(\sigma,\sigma)^{-1}\mathbf{T}(\sigma,\sigma)$$

If we know σ, we can determine $\mathbf{A}$.

Realization using Hankel Matrix

Example 7.2

$$g(s)=\frac{4s^2-2s-6}{2s^4+2s^3+2s^2+3s+1}$$
$$=0s^{-1}+2s^{-2}-3s^{-3}-2s^{-4}+2s^{-5}+3.5s^{-6}+\dots$$

$$\mathbf{T}(4,4)=\begin{bmatrix}0 & 2 & -3 & -2\\ 2 & -3 & -2 & 2\\ -3 & -2 & 2 & 3.5\\ -2 & 2 & 3.5 & \dots\end{bmatrix},\quad \rho\mathbf{T}(4,4)=3=\sigma=\deg g(s)$$

$$\mathbf{A}=\tilde{\mathbf{T}}(3,3)\mathbf{T}^{-1}(3,3)=\begin{bmatrix}2 & -3 & -2\\ -3 & -2 & 2\\ -2 & 2 & 3.5\end{bmatrix}\begin{bmatrix}0 & 2 & -3\\ 2 & -3 & -2\\ -3 & -2 & 2\end{bmatrix}^{-1}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -0.5 & -1 & 0\end{bmatrix}$$

$$b=[0\quad 2\quad -3]',\ c=[1\quad 0\quad 0]$$

Realization using Hankel Matrix

Example 7.2 (cont)

Without calculating $\tilde{\mathbf{T}}(3,3)\mathbf{T}^{-1}(3,3),$ by row searching algorothm

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \dots & 1 & 0 & 0 \\ \dots & \dots & 1 & 0 \\ a_3 & a_2 & a_1 & 1 \end{bmatrix} \mathbf{T}(4,3) = \begin{bmatrix} c_1 \neq 0 \\ c_2 \neq 0 \\ c_3 \neq 0 \\ 0 \end{bmatrix}$$

$[a_3 \quad a_2 \quad a_1 \quad 1]\mathbf{T}(4,3) = 0,$ by transpose,

$$\mathbf{T}(3,4)\mathbf{a} = \begin{bmatrix} 0 & 2 & -3 & -2 \\ 2 & -3 & -2 & 2 \\ -3 & -2 & 2 & 3.5 \end{bmatrix} \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ 1 \end{bmatrix} = 0$$

$\mathbf{a}$ is null vector of $\mathbf{T}(3,4)$.

Realization using Hankel Matrix

Balanced Form

$$\mathbf{T}(\sigma,\sigma)=\mathbf{OC}.$$

$$\mathbf{T}(\sigma,\sigma)=\mathbf{K}\Lambda\mathbf{L}'=\mathbf{K}\Lambda^{1/2}\Lambda^{1/2}\mathbf{L}'$$

$$\mathbf{O}=\mathbf{K}\Lambda^{1/2}\ \text{and}\ \mathbf{C}=\Lambda^{1/2}\mathbf{L}'.$$

$$\tilde{\mathbf{T}}(\sigma,\sigma)=\mathbf{OAC}\rightarrow\mathbf{A}=\mathbf{O}^{-1}\tilde{\mathbf{T}}(\sigma,\sigma)\mathbf{C}^{-1}$$

$$\mathbf{A}=\Lambda^{-1/2}\mathbf{K}'\tilde{\mathbf{T}}(\sigma,\sigma)\mathbf{L}\Lambda^{-1/2}.$$

$$\mathbf{CC}'=\Lambda^{1/2}\mathbf{L}'\mathbf{L}\Lambda^{1/2}=\Lambda$$

$$\mathbf{O}'\mathbf{O}=\Lambda^{1/2}\mathbf{K}'\mathbf{K}\Lambda^{1/2}=\Lambda$$

$$\rightarrow\text{Balanced Realization}$$

Summary

Realizations of $g(s) = \dfrac{\beta_1 s^2 + \beta_2 s + \beta_3}{s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3}$ (assume coprime)

Controllable form

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & \\ & 0 & 1 \\ -\alpha_3 & -\alpha_2 & -\alpha_1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{C} = [\beta_3 \quad \beta_2 \quad \beta_1], \qquad D = h(0)$$

Observable form

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & -\alpha_3 \\ 1 & 0 & -\alpha_2 \\ 0 & 1 & -\alpha_1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \beta_3 \\ \beta_2 \\ \beta_1 \end{bmatrix}$$

$$\mathbf{C} = [0 \quad 0 \quad 1], \qquad D = h(0)$$

Controllability form

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & -\alpha_3 \\ 1 & 0 & -\alpha_2 \\ 0 & 1 & -\alpha_1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{C} = [h(1) \quad h(2) \quad h(3)], \quad D = h(0)$$

Observability form

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & \\ & 0 & 1 \\ -\alpha_3 & -\alpha_2 & -\alpha_1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} h(1) \\ h(2) \\ h(3) \end{bmatrix}$$

$$\mathbf{C} = [1 \quad 0 \quad 0], \qquad D = h(0)$$

HW 7-2

Show that the two state equations

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u \quad y = \begin{bmatrix} 2 & 2 \end{bmatrix}\mathbf{x}$$

and

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}u \quad y = \begin{bmatrix} 2 & 0 \end{bmatrix}\mathbf{x}$$

are realizations of $(2s+2)/(s^2 - s - 2)$.

Are they minimal realization?

Are they algebraically equavalent?

Degree of Transfer Matrices

Definition: MIMO case

Degree of a proper rational matrix $\hat{G}(s)$

is defined as the degree of Least Common Denominator (LCD)

of all coprime minors of $\hat{G}(s)$.

Degree of Transfer Matrices

Example

$$\hat{G}_1(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

$$\text{The minors of order 1 : } \frac{1}{s+1}, \frac{1}{s+1}, \ldots$$

$$\text{The minors of order 2 : } 0$$

$$\text{LCM of denominators } = s+1 = \Delta(s)$$

$$\Rightarrow \delta(\hat{G}_1) = 1$$

Degree of Transfer Matrices

Example

$$\hat{G}_2(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{1}{s+1} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

$$\text{The minors of order 1 : } \frac{1}{(s+1)}$$

$$\text{The minors of order 2 : } \frac{1}{(s+1)^2}$$

$$\Rightarrow \quad \text{LCM of denominators } = (s+1)^2$$

$$\Rightarrow \quad \delta(\hat{G}_2) = 2$$

Degree of Transfer Matrices

Example 7.5

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \frac{s}{s+1} & \frac{1}{(s+1)(s+2)} & \frac{1}{s+3} \\ \frac{-1}{s+1} & \frac{1}{(s+1)(s+2)} & \frac{1}{s} \end{bmatrix}$$

1×1 minors : entries

$$2 \times 2 \text{ minors : } \left. \begin{matrix} \frac{\cancel{(s+1)}}{(s+1)\cancel{(s+1)}(s+2)} \\ \frac{\cancel{s}(s+4)}{\cancel{s}(s+1)(s+3)} \\ \frac{3}{s(s+1)(s+2)(s+3)} \end{matrix} \right\} \Leftarrow \text{(all should be coprime)}$$

$$\text{LCD of all minors } = s(s+1)(s+2)(s+3)$$

$$\Rightarrow \delta(\hat{\mathbf{G}}) = 4$$

Minimal Realizations-Matrix Case

Minimal Realizations-Matrix Case

Theorem 7.M2

$\{\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{D}\}$ is a minimal realization of $\mathbf{G}(s)$ iff

$\{\mathbf{A},\mathbf{B}\}$ is controllable and $\{\mathbf{A},\mathbf{C}\}$ is observable or iff

$$\dim \mathbf{A} = \deg \mathbf{G}(s).$$

Minimal Realizations-Matrix Case

Pf. of Theorem 7.M2

($\Rightarrow$) If not controllable or observable, There exists a zero state equivalent equation with lesser dimension which is not minimal.

($\Leftarrow$) If not minimal, $\exists$ $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ with $\bar{n} < n$, Theorem 4.1 implies

$$\mathbf{CA}^m\mathbf{B} = \bar{\mathbf{C}}\bar{\mathbf{A}}^m\bar{\mathbf{B}} \quad \text{for } m = 0, 1, 2, \ldots$$

$$OC = \bar{O}_n\bar{C}_n \qquad (*)$$

where $O,\ C,\ \bar{O}_n,\ \bar{C}_n$ are, respectly, $nq \times n, n \times np, nq \times \bar{n},$ and $\bar{n} \times np$.

Using Sylvester inequility

$$\rho(\bar{O}_n) + \rho(\bar{C}_n) - \bar{n} \le \rho(\bar{O}_n\bar{C}_n) \le \min(\rho(\bar{O}_n), \rho(\bar{C}_n))$$

which is proved in [6], and $\rho(\bar{O}_n) = \rho(\bar{C}_n) = \bar{n}$, we have $\rho(\bar{O}_n\bar{C}_n) = \bar{n}$.

From (*), $\rho(OC) = \rho(\bar{O}_n\bar{C}_n) = \bar{n} < n$.

This implies {A, B, C, D} is not controllable or observable.

The remaining part will be given in the remainder of this chapter.

Minimal Realizations-Matrix Case

Theorem 7.M3

All minimal realizations of G(s) are equivalent.

Pf.

Consider two minimal realizations {A, B, C, D} and $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$.

$$OC = \bar{O}\bar{C}$$

$$\bar{O}'OC\bar{C}' = \bar{O}'\bar{O}\bar{C}\bar{C}'$$

$$(\bar{O}'\bar{O})^{-1}\bar{O}'O = \bar{C}\bar{C}'(C\bar{C}')^{-1} := \mathrm{P}$$

$$O\mathbf{A}C = \bar{O}\bar{\mathbf{A}}\bar{C}$$

$$\bar{O}'O\mathbf{A}C\bar{C}' = \bar{O}'\bar{O}\bar{\mathbf{A}}\bar{C}\bar{C}'$$

$$\bar{\mathbf{A}} = (\bar{O}'\bar{O})^{-1}\bar{O}'O\mathbf{A}C\bar{C}'(\bar{C}\bar{C}')^{-1} = \mathrm{PAP}^{-1}$$

This shows {A, B, C, D} and $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ are equivalent.

Minimal Realizations-Matrix Case

Exmaple 7.6

$$\mathbf{G}(s)=\begin{bmatrix}\dfrac{4s-10}{2s+1} & \dfrac{3}{s+2}\\ \dfrac{1}{(2s+1)(s+2)} & \dfrac{1}{(s+2)^2}\end{bmatrix},\ \Delta(s)=(2s+1)(s+2)^2$$

Minimal realization has 3-dimension.

6-dim. in (4.39) and 4-dim. in (4.44) are not minimal.

By Matlab, [am,bm,cm,dm]=minreal(a,b,c,d);

$$\dot{\mathbf{x}}=\begin{bmatrix}-0.8625 & -4.0897 & 3.2544\\ 0.2921 & -3.0508 & 1.2709\\ -0.0944 & 0.3377 & -0.5867\end{bmatrix}\mathbf{x}+\begin{bmatrix}0.3218 & -0.5305\\ 0.0459 & -0.4983\\ -0.1688 & 0.0840\end{bmatrix}\mathbf{u}$$

$$\mathbf{y}=\begin{bmatrix}0 & -0.0339 & 35.5281\\ 0 & -2.1031 & -0.5720\end{bmatrix}\mathbf{x}+\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}\mathbf{u}$$

Matrix Polynomial Fractions

Matrix Polynomial Fractions

$$\mathbf{G}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)$$

Example 7.5 can be expressed as a *right fraction*

$$\mathbf{G}(s) = \begin{bmatrix} s & 1 & s \\ -1 & 1 & s+3 \end{bmatrix} \begin{bmatrix} s+1 & 0 & 0 \\ 0 & (s+1)(s+2) & 0 \\ 0 & 0 & s(s+3) \end{bmatrix}^{-1}$$

$\mathbf{G}(s) = \bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)$ is called a *left fraction.*

$$\mathbf{G}(s) = [\mathbf{N}(s)\mathbf{R}(s)][\mathbf{N}(s)\mathbf{R}(s)]^{-1}$$

$$= \mathbf{N}(s)\mathbf{R}(s)\mathbf{R}^{-1}(s)\mathbf{D}^{-1}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)$$

$\rightarrow$ Right (left) fraction is not unique.

$\rightarrow$ Right (left) coprime fraction is needed.

If $\mathbf{D}(s) = \hat{\hat{\mathbf{D}}}(s)\mathbf{R}(s)$ and $\mathbf{N}(s) = \mathbf{N}(s)\mathbf{R}(s)$,

$\mathbf{R}(s)$ is called *common right devider*.

$\mathbf{A}(s) = \mathbf{B}(s)\mathbf{C}(s)$

$\mathbf{B}$: *left devider* of $\mathbf{A}$

$\mathbf{C}$: *right devider* of $\mathbf{A}$

$\mathbf{A}$: *right multiple* of $\mathbf{B}(s)$

$\mathbf{A}$: *left multiple* of $\mathbf{C}(s)$

Matrix Polynomial Fractions

Definition 7.2 A square polynomial matrix M(s) is called a unimodular matrix if its determinant is nonzero and independent of s

Examples of unimodular matrix

$$\begin{bmatrix} 2s & s^2+s+1 \\ 2 & s+1 \end{bmatrix}, \begin{bmatrix} -2 & s^{10}+s+1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} s & s+1 \\ s-1 & s \end{bmatrix}$$

Products of unimodular matrices are clearly unimodular.

$$\det \mathbf{M}_1(s) \det \mathbf{M}_2(s) = \det\left[\mathbf{M}_1(s)\mathbf{M}_2(s)\right] = c \neq 0$$

Inverse of unimodular matrix is unimodular.

$$\det \mathbf{M}(s) \det \mathbf{M}^{-1}(s) = \det\left[\mathbf{M}(s)\mathbf{M}^{-1}(s)\right] = \det \mathbf{I} = 1$$

Matrix Polynomial Fractions

Definition 7.3 A square polynomial matrix R(s) is a *greatest common right divider (gcrd)* of D(s) and N(s) if

1) R(s) is *common right divider (crd)* of D(s) and N(s)

2) R(s) is left multiple of every crd of D(s) and N(s)

If a gcrd is a unimodular, D(s) and N(s) are right coprime.

Left coprime can be defined in a similar manner.

Greatest common right(left) devider $\mathrm{M}(s)$ is unimodular in

$$\mathbf{N}_r(s) = \bar{\mathbf{N}}_r(s)\mathrm{M}(s),\ \ \mathbf{D}_r(s) = \bar{\mathbf{D}}_r(s)\mathrm{M}(s) \text{ or}$$

$$\mathbf{N}_l(s) = \mathrm{M}(s)\bar{\mathbf{N}}_l(s),\ \ \mathbf{D}_l(s) = \mathrm{M}(s)\bar{\mathbf{D}}_l(s),$$

where $\det \mathrm{M}(s)$ is independent of s.

Matrix Polynomial Fractions

Definition 7.4:

$$\mathbf{G}(s) = \underbrace{\mathbf{N}_r(s)\mathbf{D}_r^{-1}(s)}_{\text{right coprime}} = \underbrace{\mathbf{D}_l^{-1}(s)\mathbf{N}_l(s)}_{\text{left coprime}}$$

$$\Rightarrow \text{Characteristic polynomial} = \det \mathbf{D}_r(s) = \det \mathbf{D}_l(s)$$

$$\Rightarrow \deg \mathbf{G}(s) = \deg\ \det \mathbf{D}_r(s) = \deg\ \det \mathbf{D}_l(s)$$

$$\mathbf{G}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s) = [\mathbf{N}(s)\mathbf{R}(s)][\mathbf{D}(s)\mathbf{R}(s)]^{-1}$$

Define $\mathbf{D}_1(s) = \mathbf{D}(s)\mathbf{R}(s)$, $\mathbf{N}_1(s) = \mathbf{N}(s)\mathbf{R}(s)$

$$\det \mathbf{D}_1(s) = \det[\mathbf{D}(s)\mathbf{R}(s)] = \det \mathbf{D}(s) \det \mathbf{R}(s)$$

$$\deg \det \mathbf{D}_1(s) = \deg \det \mathbf{D}(s) + \deg \det \mathbf{R}(s)$$

If $\mathbf{R}(s)$ is unimodular, $\deg \det \mathbf{D}_1(s) = \deg \det \mathbf{D}(s)$

Then $\mathbf{D}_1(s)$ and $\mathbf{N}_1(s)$ are right coprime.

Matrix Polynomial Fractions

Column and Row Reducedness

Degree of polynimial vector: the highest power in all entries.

$\delta_{ci}\mathbf{M}(s) =$ degree of ith column of $\mathbf{M}(s)$: *column degree*

$\delta_{ri}\mathbf{M}(s) =$ degree of ith row of $\mathbf{M}(s)$: *row degree*

$$\mathbf{M}(s) = \begin{bmatrix} s+1 & s^3 - 2s + 5 & -1 \\ s-1 & s^2 & 0 \end{bmatrix}$$

$$\rightarrow \delta_{c1} = 1,\ \delta_{c2} = 3,\ \delta_{c3} = 0,\ \delta_{r1} = 3,\ \delta_{r2} = 2$$

Matrix Polynomial Fractions

Definition 7.5

A nonsingular polynomial matrix $\mathbf{M}(s)$degree is column reduced if

$$\deg \det \mathbf{M}(s) = \textit{sum of all column degrees.}$$

It is row reduced if

$$\deg \det \mathbf{M}(s) = \textit{sum of all row degrees.}$$

Example :

$$\mathbf{M}(s) = \begin{bmatrix} 3s^2 + 2s & 2s+1 \\ s^2 + s - 3 & s \end{bmatrix}$$

$$\Delta(\mathrm{s}) = s^3 - s^2 + 5s + 3 \Rightarrow \deg \Delta(\mathrm{s}) = \delta_{c1} + \delta_{c2} = 2+1$$

$$\rightarrow \textit{column reduced}$$

$$\Rightarrow \deg \Delta(\mathrm{s}) \neq \delta_{r1} + \delta_{r2} = 2+2$$

$$\rightarrow \textit{not row reduced}$$

Matrix Polynomial Fractions

$\mathbf{M}(s)$ can be expressed as

$$\mathbf{M}(s) = \mathbf{M}_{hc}\mathbf{H}_c(s) + \mathbf{M}_{lc}(s)$$

Example:

$$\mathbf{M}(s) = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} 2s & 1 \\ s-3 & 0 \end{bmatrix}$$

$\searrow$ nonsingular $\leftrightarrow$ column reduced

or can be expressed as

$$\mathbf{M}(s) = \mathbf{H}_r(s)\mathbf{M}_{hr} + \mathbf{M}_{lr}(s)$$

Example:

$$\mathbf{M}(s) = \begin{bmatrix} s^2 & 0 \\ 0 & s^2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 2s & 2s+1 \\ s-3 & s \end{bmatrix}$$

$\searrow$ singular $\leftrightarrow$ not row reduced

Matrix Polynomial Fractions

$$\mathbf{G}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s) = \bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)$$

right coprime left coprime

$\mathbf{D}(s)$: column reduced, $\bar{\mathbf{D}}(s)$: row reduced

$\Rightarrow$

$$\deg \mathbf{G}(s) = \text{ sum of column degrees of } \mathbf{D}(s)$$
$$= \text{ sum of row degrees of } \bar{\mathbf{D}}(s)$$

If G(s) is strictly proper, then

$$\delta_{ci}\mathbf{N}(s) < \delta_{ci}\mathbf{D}(s), \; i = 1, 2, \ldots$$

The converse is not necessarily true, ex,

$$\mathbf{N}(s)\mathbf{D}^{-1}(s) = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} s^2 & s-1 \\ s+1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{-2s-1}{1} & \dfrac{2s^2-s+1}{1} \end{bmatrix}$$

The reason is that $\mathbf{D}(s)$ is not column reduced.

Matrix Polynomial Fractions

Theorem 7.8

If $\mathbf{D}(s)$ is column reduced, then

$\mathbf{N}(s)\mathbf{D}^{-1}(s)$ is proper (strictly proper) iff

$\delta_{ci}\mathbf{N}(s) \leq \delta_{ci}\mathbf{D}(s) \quad \left[\delta_{ci}\mathbf{N}(s) < \delta_{ci}\mathbf{D}(s)\right]$ for $i = 1, 2, 3, \ldots$

Pf.

Necessity part follows from the preceding examples.

To show sufficiency,

$$\mathbf{D}(s) = \mathbf{D}_{hc}\mathbf{H}_c(s) + \mathbf{D}_{lc}(s) = \left[\mathbf{D}_{hc} + \mathbf{D}_{lc}(s)\mathbf{H}_c^{-1}(s)\right]\mathbf{H}_c(s)$$

$$\mathbf{N}(s) = \mathbf{N}_{hc}\mathbf{H}_c(s) + \mathbf{N}_{lc}(s) = \left[\mathbf{N}_{hc} + \mathbf{N}_{lc}(s)\mathbf{H}_c^{-1}(s)\right]\mathbf{H}_c(s)$$

$$\mathbf{G}(s) := \mathbf{N}(s)\mathbf{D}^{-1}(s) = \left[\mathbf{N}_{hc} + \mathbf{N}_{lc}(s)\mathbf{H}_c^{-1}(s)\right]\left[\mathbf{D}_{hc} + \mathbf{D}_{lc}(s)\mathbf{H}_c^{-1}(s)\right]^{-1}$$

$$\lim_{x\to\infty}\mathbf{G}(s) = \mathbf{N}_{hc}\mathbf{D}_{hc}^{-1}$$

$\mathbf{D}_{hc}^{-1}$ is nonsingular since column reduced

$\Rightarrow$ *proper*

$\mathbf{N}_{hc} = 0$ for $\delta_{ci}\mathbf{N}(s) < \delta_{ci}\mathbf{D}(s)$

$\Rightarrow$ *strictly proper*

Matrix Polynomial Fractions

Corollary 7.8

If $\mathbf{D}(s)$ is row reduced, then

$\bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)$ is proper (strictly proper) iff

$$\delta_{ri}\bar{\mathbf{N}}(s) \le \delta_{ri}\bar{\mathbf{D}}(s) \quad \left[\delta_{ri}\bar{\mathbf{N}}(s) < \delta_{ri}\bar{\mathbf{D}}(s)\right] \text{ for } i = 1, 2, 3, \dots$$

HW 7-3

Find the characteristic polynomials and degrees of the following proper rational matrix of

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{1}{(s+1)^2} & \dfrac{s+3}{s+2} & \dfrac{1}{s+5} \\ \dfrac{1}{(s+3)^2} & \dfrac{s+1}{s+4} & \dfrac{1}{s} \end{bmatrix}.$$

Use two methods: minors and column degrees.

You may use Matlab for coprime fraction.

Matrix Polynomial Fractions

Computing Matrix Coprime Fractions

For the given left fraction $\bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)$,

not necessarily left coprime,

we can find the right coprime fraction $\mathbf{N}(s)\mathbf{D}^{-1}(s)$

$$\mathbf{G}(s) = \bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)$$

$$\bar{\mathbf{N}}(s)\mathbf{D}(s) = \bar{\mathbf{D}}(s)\mathbf{N}(s)$$

$$\bar{\mathbf{D}}(s)(-\mathbf{N}(s)) + \bar{\mathbf{N}}(s)\mathbf{D}(s) = \mathbf{0}$$

where

$$\bar{\mathbf{D}}(s) = \bar{\mathbf{D}}_0 + \bar{\mathbf{D}}_1 s + \bar{\mathbf{D}}_2 s^2 + \bar{\mathbf{D}}_3 s^3 + \bar{\mathbf{D}}_4 s^4$$

$$\bar{\mathbf{N}}(s) = \bar{\mathbf{N}}_0 + \bar{\mathbf{N}}_1 s + \bar{\mathbf{N}}_2 s^2 + \bar{\mathbf{N}}_3 s^3 + \bar{\mathbf{N}}_4 s^4$$

$$\mathbf{D}(s) = \mathbf{D}_0 + \mathbf{D}_1 s + \mathbf{D}_2 s^2 + \mathbf{D}_3 s^3$$

$$\mathbf{N}(s) = \mathbf{N}_0 + \mathbf{N}_1 s + \mathbf{N}_2 s^2 + \mathbf{N}_3 s^3$$

Matrix Polynomial Fractions

$$
\mathbf{SM} := \begin{bmatrix}
\bar{\mathbf{D}}_0 & \bar{\mathbf{N}}_0 & \vdots & \mathbf{0} & \mathbf{0} & \vdots & \mathbf{0} & \mathbf{0} & \vdots & \mathbf{0} & \mathbf{0} \\
\bar{\mathbf{D}}_1 & \bar{\mathbf{N}}_1 & \vdots & \bar{\mathbf{D}}_0 & \bar{\mathbf{N}}_0 & \vdots & \mathbf{0} & \mathbf{0} & \vdots & \mathbf{0} & \mathbf{0} \\
\bar{\mathbf{D}}_2 & \bar{\mathbf{N}}_2 & \vdots & \bar{\mathbf{D}}_1 & \bar{\mathbf{N}}_1 & \vdots & \bar{\mathbf{D}}_0 & \bar{\mathbf{N}}_0 & \vdots & \mathbf{0} & \mathbf{0} \\
\bar{\mathbf{D}}_3 & \bar{\mathbf{N}}_3 & \vdots & \bar{\mathbf{D}}_2 & \bar{\mathbf{N}}_2 & \vdots & \bar{\mathbf{D}}_1 & \bar{\mathbf{N}}_1 & \vdots & \bar{\mathbf{N}}_0 & \bar{\mathbf{N}}_0 \\
\bar{\mathbf{D}}_4 & \bar{\mathbf{N}}_4 & \vdots & \bar{\mathbf{D}}_3 & \bar{\mathbf{N}}_3 & \vdots & \bar{\mathbf{D}}_2 & \bar{\mathbf{N}}_2 & \vdots & \bar{\mathbf{N}}_1 & \bar{\mathbf{N}}_1 \\
\mathbf{0} & \mathbf{0} & \vdots & \bar{\mathbf{D}}_4 & \bar{\mathbf{N}}_4 & \vdots & \bar{\mathbf{D}}_3 & \bar{\mathbf{N}}_3 & \vdots & \bar{\mathbf{N}}_2 & \bar{\mathbf{N}}_2 \\
\mathbf{0} & \mathbf{0} & \vdots & \mathbf{0} & \mathbf{0} & \vdots & \bar{\mathbf{D}}_4 & \bar{\mathbf{N}}_4 & \vdots & \bar{\mathbf{N}}_3 & \bar{\mathbf{N}}_3 \\
\mathbf{0} & \mathbf{0} & \vdots & \mathbf{0} & \mathbf{0} & \vdots & \mathbf{0} & \mathbf{0} & \vdots & \bar{\mathbf{N}}_4 & \bar{\mathbf{N}}_4
\end{bmatrix}
\begin{bmatrix} -\mathbf{N}_0 \\ \mathbf{D}_0 \\ -\mathbf{N}_1 \\ \mathbf{D}_1 \\ -\mathbf{N}_2 \\ \mathbf{D}_2 \\ -\mathbf{N}_3 \\ \mathbf{D}_3 \end{bmatrix} = \mathbf{0}
$$

$\mathbf{S}$: Generalized resultant: $8q \times 4(q+p)$

$\bar{\mathbf{D}}_i : q \times q,\ \bar{\mathbf{N}}_i : q \times p,\ \mathbf{D}_i : p \times p,\ \mathbf{N}_i : q \times p$

Matrix Polynomial Fractions

Example 7.7

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{4s-10}{2s+1} & \dfrac{3}{s+2} \\ \dfrac{1}{(2s+1)(s+2)} & \dfrac{s+1}{(s+2)^2} \end{bmatrix}$$

$$\mathbf{G}(s) = \begin{bmatrix} (2s+1)(s+2) & 0 \\ 0 & (2s+1)(s+2)^2 \end{bmatrix}^{-1}$$

$$\times \begin{bmatrix} (4s-10)(s+2) & 3(2s+1) \\ s+2 & (s+1)(2s+1) \end{bmatrix} =: \bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)$$

Matrix Polynomial Fractions

$$\bar{\mathbf{D}}(s) = \begin{bmatrix} 2s^2+5s+2 & 0 \\ 0 & 2s^3+9s^2+12s+4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 12 \end{bmatrix} s + \begin{bmatrix} 2 & 0 \\ 0 & 9 \end{bmatrix} s^2 + \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} s^3$$

$$\bar{\mathbf{N}}(s) = \begin{bmatrix} 4s^2-2s-20 & 6s+3 \\ s+2 & 2s^2+3s+1 \end{bmatrix}$$

$$= \begin{bmatrix} -20 & 3 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 6 \\ 1 & 3 \end{bmatrix} s + \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} s^2 + \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} s^3$$

By Matlab,

$$[\mathrm{q},\ \mathrm{r}] = \mathrm{qr}(\mathrm{S}) \rightarrow \mathrm{S} = \mathrm{qr}$$

Matrix Polynomial Fractions

↓Primary dependent *n*2 vector

s1 ←→ r1

$$r = \begin{bmatrix} d1 & 0 & x & x & x & x & x & x & x & 0 & x & x \\ 0 & d2 & x & x & x & x & x & x & 0 & x & x & x \\ 0 & 0 & n1 & x & x & x & x & x & x & x & x & x \\ 0 & 0 & 0 & n2 & x & x & x & x & x & x & x & x \\ 0 & 0 & 0 & 0 & d1 & x & x & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & d2 & x & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & n1 & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & x & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d1 & x & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d2 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

z1=null(s1), z1b=z1/z1(8) for making monic vector

Matrix Polynomial Fractions

Primary dependent *n*1 vector

s2 ↔ r2

$$
r = \begin{bmatrix}
d1 & 0 & x & x & x & x & x & x & x & 0 & x & x \\
0 & d2 & x & x & x & x & x & x & 0 & x & x & x \\
0 & 0 & n1 & x & x & x & x & x & x & x & x & x \\
0 & 0 & 0 & n2 & x & x & x & x & x & x & x & x \\
0 & 0 & 0 & 0 & d1 & x & x & x & x & x & x & x \\
0 & 0 & 0 & 0 & 0 & d2 & x & x & x & x & x & x \\
0 & 0 & 0 & 0 & 0 & 0 & n1 & x & x & x & x & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & x & x & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d1 & x & x & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d2 & x & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

z2=null(s2), z2b=z2/z2(10) for making monic vector

Matrix Polynomial Fractions

$$
\begin{bmatrix} -\mathbf{N}_0 \\ \cdots \\ \mathbf{D}_0 \\ \cdots \\ -\mathbf{N}_1 \\ \cdots \\ \mathbf{D}_1 \\ \cdots \\ -\mathbf{N}_2 \\ \cdots \\ \mathbf{D}_2 \end{bmatrix}
=
\begin{bmatrix}
-n_0^{11} & -n_0^{12} \\
-n_0^{21} & -n_0^{22} \\
d_0^{11} & d_0^{12} \\
d_0^{21} & d_0^{22} \\
-n_1^{11} & -n_1^{12} \\
-n_1^{21} & -n_1^{22} \\
d_1^{11} & d_1^{12} \\
d_2^{21} & d_2^{22} \\
-n_2^{11} & -n_2^{12} \\
-n_2^{21} & -n_2^{22} \\
d_2^{11} & d_2^{12} \\
d_2^{21} & d_2^{22}
\end{bmatrix}
=
\begin{bmatrix}
10 & 7 \\
-0.5 & -1 \\
1 & 1 \\
0 & 2 \\
1 & -4 \\
0 & 0 \\
2.5 & 2 \\
0 & 1 \\
-2 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 0
\end{bmatrix}
$$

z1b=z1/z1(8)

z2b=z2/z2(10)

Matrix Polynomial Fractions

$$\mathbf{D}(s) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 2.5 & 2 \\ 0 & 1 \end{bmatrix} s + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} s^2$$

$$= \begin{bmatrix} s^2 + 2.5s + 1 & 2s + 1 \\ 0 & s + 2 \end{bmatrix}$$

$$\mathbf{N}(s) = \begin{bmatrix} -10 & -7 \\ 0.5 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix} s + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} s^2$$

$$= \begin{bmatrix} 2s^2 - s - 10 & 4s - 7 \\ 0.5 & 1 \end{bmatrix}$$

$$\mathbf{G}(s) = \begin{bmatrix} (2s-5)(s+2) & 4s-7 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} (s+2)(s+0.5) & 2s+1 \\ 0 & s+2 \end{bmatrix}^{-1}$$

column degrees $\mu_1 = 2,\ \ \mu_2 = 1,\ \ \deg\det \mathbf{D}(s) = 2 + 1 = 3$

Matrix Polynomial Fractions

Note:

$$\deg \mathbf{G}(s) = \deg \det \mathbf{D}(s) = \sum \mu_i$$

$$= \text{total number of linearly independent } \bar{N}\text{-columns in S}$$

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{G}(s) = \hat{\hat{\mathbf{N}}}(s)\mathbf{D}^{-1}(s) = [\mathbf{N}(s)\mathbf{P}][\mathbf{D}(s)\mathbf{P}]^{-1} = \mathbf{N}(s)\mathbf{D}^{-1}(s)$$

The columns of $\mathbf{N}(s)\mathbf{D}(s)$ can be arbitrarily permutated.

Matrix Polynomial Fractions

Theorem 7.M4

Let $\mathrm{G(s)}=\bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)$ be a left fraction,not necessarily left coprime

Let $\mu_i, i=1,2,\ldots,p,$ be the number of linearly independent $\bar{N}_i - columns.$

$$\deg \mathbf{G}(s) = \mu_1 + \mu_2 + \cdots + \mu_p$$

A right coprime fraction $\mathbf{N}(\mathrm{s})\mathbf{D}^{-1}(s)$ can be obtained by computing

p monic null vectors using p *matrices* formed from

each primary dependent $\bar{N}_i - column$ and its LHS LI columns.

Note:

The column-degree coefficient matrix $\mathbf{D}_{hc}$

can be a unit upper trangular matrix.

$$\mathbf{D}_{hc} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} : column\ echelon\ \ form$$

$\rightarrow$ Realization will be nicer.

Matrix Polynomial Fractions

Dual: Compute a left coprime fraction $\bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)$ from a right fraction $\mathbf{N}(s)\mathbf{D}^{-1}(s)$

$$\left[-\bar{\mathbf{N}}_0\ \bar{\mathbf{D}}_0 \vdots -\bar{\mathbf{N}}_1\ \bar{\mathbf{D}}_1 \vdots -\bar{\mathbf{N}}_2\ \bar{\mathbf{D}}_2 \vdots -\bar{\mathbf{N}}_3\ \bar{\mathbf{D}}_3\right]\mathbf{T} = \mathbf{0}$$

$$\mathbf{T} := \begin{bmatrix} \mathbf{D}_0 & \mathbf{D}_1 & \mathbf{D}_2 & \mathbf{D}_3 & \mathbf{D}_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{N}_0 & \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_3 & \mathbf{N}_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{D}_0 & \mathbf{D}_1 & \mathbf{D}_2 & \mathbf{D}_3 & \mathbf{D}_4 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_0 & \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_3 & \mathbf{N}_4 & \mathbf{0} & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_0 & \mathbf{D}_1 & \mathbf{D}_2 & \mathbf{D}_3 & \mathbf{D}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{N}_0 & \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_3 & \mathbf{N}_4 & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}_0 & \mathbf{D}_1 & \mathbf{D}_2 & \mathbf{D}_3 & \mathbf{D}_4 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{N}_0 & \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_3 & \mathbf{N}_4 \end{bmatrix}$$

Matrix Polynomial Fractions

Corollary 7.M4

Let G(s)=$\mathbf{N}(s)\mathbf{D}^{-1}(s)$ be a right fraction,not necessarily right coprime

Let $v_i, i=1,2,...,q,$ be the number of linearly independent $\bar{N}_i - rows$ in T.

$$\deg \mathbf{G}(s) = v_1 + v_2 + \cdots + v_q$$

A left coprime fraction $\bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)$ can be obtained by computing q monic null vectors using q *matrices* formed from each primary dependent $\bar{N}_i - rows$ and its preceeding LI rows.

Note:

The *row echelon form* can be also defined.

Realizations from Coprime Fractions

Realizations from Coprime Fractions

$$\mathbf{G}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)$$

$$\mathbf{H}(s) := \begin{bmatrix} s^{\mu_1} & 0 \\ 0 & s^{\mu_2} \end{bmatrix} = \begin{bmatrix} s^4 & 0 \\ 0 & s^2 \end{bmatrix}$$

$$\mathbf{L}(s) := \begin{bmatrix} s^{\mu_1 - 1} & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & s^{\mu_2 - 1} \\ \vdots & \vdots \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s^3 & 0 \\ s^2 & 0 \\ s & 0 \\ 1 & 0 \\ 0 & s \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{y}(s) = \mathbf{N}(s)\mathbf{D}^{-1}(s)\mathbf{u}(s)$$

$$\mathbf{v}(s) = \mathbf{D}^{-1}(s)\mathbf{u}(s) \rightarrow \mathbf{D}^{-1}(s)\mathbf{v}(s) = \mathbf{u}(s)$$

$$\mathbf{y}(s) = \mathbf{N}(s)\mathbf{v}(s)$$

Realizations from Coprime Fractions

Define state variables

$$\mathbf{x}(s) = \mathbf{L}(s)\mathbf{v}(s) = \begin{bmatrix} s^{\mu_1 - 1} & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & s^{\mu_2 - 1} \\ \vdots & \vdots \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1(s) \\ v_2(s) \end{bmatrix}$$

$$= \begin{bmatrix} s^3 v_1(s) \\ s^2 v_1(s) \\ s v_1(s) \\ v_1(s) \\ s v_2(s) \\ v_2(s) \end{bmatrix} =: \begin{bmatrix} x_1(s) \\ x_2(s) \\ x_3(s) \\ x_4(s) \\ x_5(s) \\ x_6(s) \end{bmatrix} \rightarrow \mathbf{x}(t) = \begin{bmatrix} \dddot{v}_1 \\ \ddot{v}_1 \\ \dot{v}_1 \\ v_1 \\ \dot{v}_2 \\ v_2 \end{bmatrix}$$

Realizations from Coprime Fractions

$$x_1(t) = v_1^{(3)}(t) \quad x_2(t) = \ddot{v}_1(t) \quad x_3(t) = \dot{v}_1(t) \quad x_4(t) = v_1(t)$$

$$x_5(t) = \dot{v}_2(t) \quad x_6(t) = v_2(t)$$

$$\dot{x}_2 = x_1 \quad \dot{x}_3 = x_2 \quad \dot{x}_4 = x_3 \quad \dot{x}_6 = x_5$$

To develope x_1 and x_5,

$$\mathbf{D}(s) = \mathbf{D}_{hc}\mathbf{H}(s) + \mathbf{D}_{lc}\mathbf{L}(s)$$

$$\left[\mathbf{D}_{hc}\mathbf{H}(s) + \mathbf{D}_{lc}\mathbf{L}(s)\right]\mathbf{v}(s) = \mathbf{u}(s)$$

$$\mathbf{H}(s)\mathbf{v}(s) + \mathbf{D}_{hc}^{-1}\mathbf{D}_{lc}\mathbf{L}(s)\mathbf{v}(s) = \mathbf{D}_{hc}^{-1}\mathbf{u}(s)$$

$$\mathbf{H}(s)\mathbf{v}(s) = -\mathbf{D}_{hc}^{-1}\mathbf{D}_{lc}\mathbf{x}(s) + \mathbf{D}_{hc}^{-1}\mathbf{u}(s)$$

$$\mathbf{D}_{hc}^{-1}\mathbf{D}_{lc} =: \begin{bmatrix} \alpha_{111} & \alpha_{112} & \alpha_{113} & \alpha_{114} & \alpha_{121} & \alpha_{122} \\ \alpha_{211} & \alpha_{212} & \alpha_{213} & \alpha_{214} & \alpha_{221} & \alpha_{222} \end{bmatrix}$$

$$\mathbf{D}_{hc}^{-1} =: \begin{bmatrix} 1 & b_{12} \\ 0 & 1 \end{bmatrix}$$

Realizations from Coprime Fractions

$$\begin{bmatrix} sx_1(s) \\ sx_5(s) \end{bmatrix} = -\begin{bmatrix} \alpha_{111} & \alpha_{112} & \alpha_{113} & \alpha_{114} & \alpha_{121} & \alpha_{122} \\ \alpha_{211} & \alpha_{212} & \alpha_{213} & \alpha_{214} & \alpha_{221} & \alpha_{222} \end{bmatrix} \mathbf{x}(s)$$

$$+\begin{bmatrix} 1 & b_{12} \\ 0 & 1 \end{bmatrix} \mathbf{u}(s)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_5 \end{bmatrix} = -\begin{bmatrix} \alpha_{111} & \alpha_{112} & \alpha_{113} & \alpha_{114} & \alpha_{121} & \alpha_{122} \\ \alpha_{211} & \alpha_{212} & \alpha_{213} & \alpha_{214} & \alpha_{221} & \alpha_{222} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 & b_{12} \\ 0 & 1 \end{bmatrix} \mathbf{u}$$

$$\mathbf{N}(s) = \begin{bmatrix} \beta_{111} & \beta_{112} & \beta_{113} & \beta_{114} & \beta_{121} & \beta_{122} \\ \beta_{211} & \beta_{212} & \beta_{213} & \beta_{214} & \beta_{221} & \beta_{222} \end{bmatrix} \mathbf{L}(s)$$

$$\mathbf{y}(s) = \begin{bmatrix} \beta_{111} & \beta_{112} & \beta_{113} & \beta_{114} & \beta_{121} & \beta_{122} \\ \beta_{211} & \beta_{212} & \beta_{213} & \beta_{214} & \beta_{221} & \beta_{222} \end{bmatrix} \mathbf{x}(s)$$

Realizations from Coprime Fractions

$$\dot{\mathbf{x}} = \begin{bmatrix} -\alpha_{111} & -\alpha_{112} & -\alpha_{113} & -\alpha_{114} & \vdots & -\alpha_{121} & -\alpha_{122} \\ 1 & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\alpha_{211} & -\alpha_{212} & -\alpha_{213} & -\alpha_{214} & \vdots & -\alpha_{221} & -\alpha_{222} \\ 0 & 0 & 0 & 0 & \vdots & 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 & b_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \cdots & \cdots \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} \beta_{111} & \beta_{112} & \beta_{113} & \beta_{114} & \vdots & \beta_{121} & \beta_{122} \\ \beta_{211} & \beta_{212} & \beta_{213} & \beta_{214} & \vdots & \beta_{221} & \beta_{222} \end{bmatrix} \mathbf{x}$$

Controllable canonical form

Realizations from Coprime Fractions

Example 7.8

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{4s-10}{2s+1} & \dfrac{3}{s+2} \\ \dfrac{1}{(2s+1)(s+2)} & \dfrac{s+1}{(s+2)^2} \end{bmatrix} =: \mathbf{G}(\infty) + \mathbf{G}_{sp}(s)$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -\dfrac{12}{2s+1} & \dfrac{3}{s+2} \\ \dfrac{1}{(2s+1)(s+2)} & \dfrac{s+1}{(s+2)^2} \end{bmatrix}$$

$$\mathbf{G}_{sp}(s) = \begin{bmatrix} -6s-12 & -9 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} s^2+2.5s+1 & 2s+1 \\ 0 & s+2 \end{bmatrix}^{-1}$$

Realizations from Coprime Fractions

$$\mathbf{H}(s) = \begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix} \qquad \mathbf{L}(s) = \begin{bmatrix} s & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{D}(s) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \mathbf{H}(s) + \begin{bmatrix} 2.5 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{L}(s)$$

$$\mathbf{N}(s) = \begin{bmatrix} -6 & -12 & -9 \\ 0 & 0.5 & 1 \end{bmatrix} \mathbf{L}(s)$$

$$\mathbf{D}_{hc}^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{D}_{hc}^{-1}\mathbf{D}_{lc} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2.5 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2.5 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

Realizations from Coprime Fractions

$$\dot{\mathbf{x}} = \begin{bmatrix} -2.5 & -1 & \vdots & 3 \\ 1 & 0 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 & -2 \\ 0 & 0 \\ \cdots & \cdots \\ 0 & 1 \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} -6 & -12 & \vdots & -9 \\ 0 & 0.5 & \vdots & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{u}$$

Observable canonical form can be obtained by using Left coprime fraction.

Realizations from Coprime Fractions

Note:

Let $\{\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}\}$ be a minimalrealization

$$\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}=\mathbf{N}(s)\mathbf{D}^{-1}(s)=\bar{\mathbf{D}}^{-1}(s)\bar{\mathbf{N}}(s)$$

which implies

$$\frac{1}{\det(s\mathbf{I}-\mathbf{A})}\mathbf{C}\left[\mathrm{Adj}(s\mathbf{I}-\mathbf{A})\right]\mathbf{B}+\mathbf{D}=\frac{1}{\det\mathbf{D}(s)}\mathbf{N}(s)\left[\mathrm{Adj}(\mathbf{D}(s))\right]$$

$$=\frac{1}{\det\bar{\mathbf{D}}(s)}\left[\mathrm{Adj}(\bar{\mathbf{D}}(s))\right]\bar{\mathbf{N}}(s)$$

$*\deg\mathbf{G}(s)=\deg\det\mathbf{D}(s)=\deg\det\bar{\mathbf{D}}(s)=\dim\mathbf{A}\leftarrow$ Proof of Theorem 7.M2

$*$characteristic polynomial of $\mathbf{G}(s)=k_1\det\mathbf{D}(s)=k_2\det\bar{\mathbf{D}}(s)=k_3\det(s\mathbf{I}-\mathbf{A})$

$*$the set of column degrees of $\mathbf{D}(s)=$ the set of controllable indices of $(\mathbf{A},\mathbf{B})$

$*$the set of row degrees of $\bar{\mathbf{D}}(s)=$ the set of observability indices of $(\mathbf{A},\mathbf{C})$

HW 7-4

Find a right coprime fraction of

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{s^2+1}{s^3} & \dfrac{2s+1}{s^2} \\ \dfrac{s+2}{s^2} & \dfrac{2}{s} \end{bmatrix}$$

and then a minimal realization.

Realizations from Matrix Markov Parameters

Realizations from Matrix Markov Parameters

$$\mathbf{G}(s) = \mathbf{H}(1)s^{-1} + \mathbf{H}(2)s^{-2} + \mathbf{H}(3)s^{-3} + \cdots$$

$$\mathbf{T} = \begin{bmatrix} \mathbf{H}(1) & \mathbf{H}(2) & \mathbf{H}(3) & \cdots & \mathbf{H}(r) \\ \mathbf{H}(2) & \mathbf{H}(3) & \mathbf{H}(4) & \cdots & \mathbf{H}(r+1) \\ \mathbf{H}(3) & \mathbf{H}(4) & \mathbf{H}(5) & \cdots & \mathbf{H}(r+2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{H}(r) & \mathbf{H}(r+1) & \mathbf{H}(r+2) & \cdots & \mathbf{H}(2r-1) \end{bmatrix}$$

$$\tilde{\mathbf{T}} = \begin{bmatrix} \mathbf{H}(2) & \mathbf{H}(3) & \mathbf{H}(4) & \cdots & \mathbf{H}(r+1) \\ \mathbf{H}(3) & \mathbf{H}(4) & \mathbf{H}(5) & \cdots & \mathbf{H}(r+2) \\ \mathbf{H}(4) & \mathbf{H}(5) & \mathbf{H}(6) & \cdots & \mathbf{H}(r+3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{H}(r+1) & \mathbf{H}(r+2) & \mathbf{H}(r+3) & \cdots & \mathbf{H}(2r) \end{bmatrix}$$

$$\mathbf{T} = OC \quad \text{and} \quad \tilde{\mathbf{T}} = O\mathbf{A}C \rightarrow O'\tilde{\mathbf{T}}C' = O'O\mathbf{A}CC'$$

$$\rightarrow \mathbf{A} = (O'O)^{-1}O'\tilde{\mathbf{T}}C'(CC')^{-1}$$

$$= O^{+}\tilde{\mathbf{T}}C^{+}$$

Realizations from Matrix Markov Parameters

Theorem 7.M7

A strictly proper rational matrix $\mathbf{G}(s)$ has degree n iff the matrix $\mathbf{T}$ has rank n.

Realizations from Matrix Markov Parameters

By singular value decomposition,

$$\mathbf{T} = \mathbf{K}\begin{bmatrix} \mathbf{\Lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{L}'$$

Choosing nonsingular part $(n-\text{dim})$,

$$\mathbf{T} = \bar{\mathbf{K}}\mathbf{\Lambda}^{1/2}\mathbf{\Lambda}^{1/2}\bar{\mathbf{L}}' =: OC$$

$$O = \bar{\mathbf{K}}\mathbf{\Lambda}^{1/2} \text{ and } C = \mathbf{\Lambda}^{1/2}\bar{\mathbf{L}}'$$

$$O^+ = \left[(\bar{\mathbf{K}}\mathbf{\Lambda}^{1/2})'\bar{\mathbf{K}}\mathbf{\Lambda}^{1/2}\right]^{-1}(\bar{\mathbf{K}}\mathbf{\Lambda}^{1/2})'$$

$$O^+ = \mathbf{\Lambda}^{-1/2}\bar{\mathbf{K}}'$$

$$C^+ = \bar{\mathbf{L}}\mathbf{\Lambda}^{-1/2}$$

$$\mathbf{A} = O^+\tilde{\mathbf{T}}C^+$$

$\mathbf{B}$ = first p columns of C

$\mathbf{C}$ = first q rows of O

$$O'O = \mathbf{\Lambda}^{1/2}\bar{\mathbf{K}}'\bar{\mathbf{K}}\mathbf{\Lambda}^{1/2} =$$

$$CC' = \mathbf{\Lambda}^{1/2}\bar{\mathbf{L}}'\bar{\mathbf{L}}\mathbf{\Lambda}^{1/2} =$$

$\leftarrow$ Balanced realization

Summary

Degree of transfer function

Coprimeness and minimal realization

Computing Coprime fraction (Sylvester matrix)

Controllable form, Observable form

Controllability form, Observability form (from Henkel Matrix)

Balanced realization

Degree of transfer function matrix, Unimodular

Greatest common right divisor, Left(right) multiple

column(row) degree, column(row) reduced,

Coprimeness of transfer function matrix,

Computing Right(Left) coprime fraction

Minimum realizations(controllable/Observable/Balanced-Henkel)