ENGINEERING MATHEMATICS II

010.141

MODULE 2: Matrix Eigenvalue Problem

EIGENVALUES AND EIGENVECTORS

SOME DEFINITIONS

Let A be an **n × n** matrix and consider

$$A\mathbf{x} = \lambda\mathbf{x} \qquad (1)$$

λ, such that (1) has solution $x \neq 0$ is an **eigenvalue** or characteristic value of A.

x are **eigenvectors** or characteristic vectors of A.

- The **spectrum** of A is the set of eigenvalues of A;
- $\max|\lambda|$ is the **spectral radius** of A.
- The set of eigenvectors corresponding to λ (including 0) is the **eigenspace** of A for λ.

SOME DEFINITIONS (cont)

Homogeneous linear system in x_1, x_2

$\vert\rightarrow$ **Cramer's theorem**

$D(\lambda) = \det(\mathbf{A}-\lambda\mathbf{I}) = 0$

$D(\lambda)$ is the characteristic determinant, and

$D(\lambda) = 0$ is the characteristic equation

DETERMINATION OF EIGENVALUES AND EIGENVECTORS

➢ Example

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

$$Ax = \lambda x \rightarrow \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned} -5x_1 + 2x_2 &= \lambda x_1 \\ 2x_1 - 2x_2 &= \lambda x_2 \end{aligned}$$

$$\left(-5-\lambda\right)x_1 + 2x_2 = 0$$

$$2x_1 + \left(-2-\lambda\right)x_2 = 0$$

$$2x_1 + \left(-2-\lambda\right)x_2 = 0$$

$$(A - \lambda I)x = 0$$

$$\begin{aligned} \text{del}\left(A - \lambda I\right) &= \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} \\ &= \left(-5-\lambda\right)\cdot\left(-2-\lambda\right) - 4 \\ &= \lambda^2 + 7\lambda + 6 = 0 \end{aligned}$$

$$\Delta = 49 - (4)(6) = 25$$

$$\lambda = \frac{-7 \pm 5}{2} \rightarrow \lambda = -1, \; -6$$

DETERMINATION OF EIGENVALUES AND EIGENVECTORS (cont)

Eigenvector for λ = -1

$$
\begin{array}{rcrcl}
(-5+1)x_1 & + & 2x_2 & = & 0 \\
2x_1 & + & (-2+1)x_2 & = & 0 \\
\hline
-4x_1 & + & 2x_2 & = & 0 \\
2x_1 & - & x_2 & = & 0 \\
\hline
-2x_1 & + & x_2 & = & 0
\end{array}
$$

$$x_1 = \frac{x_2}{2}$$

$$\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$$

DETERMINATION OF EIGENVALUES AND EIGENVECTORS (cont)

Eigenvector for λ = -6

$$\begin{array}{rcrcl} (-5+6)x_1 & + & 2x_2 & = & 0 \\ 2x_1 & + & (-2+6)x_2 & = & 0 \\ \hline (-1)\,x_1 & + & (-1)\,2x_2 & = & (-1)0 \\ 2x_1 & + & 4x_2 & = & 0 \\ \hline x_1 & + & 2x_2 & = & 0 \end{array}$$

$$x_1 = -2x_2$$

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

EIGENVALUES

The eigenvalues of a square matrix A are the roots of the characteristic equation

$$D(\lambda) = 0$$

An **n × n matrix** has at least one **eigenvalue** and at most n different **eigenvalues**

EIGENVECTORS

If x is an eigenvector of a matrix A corresponding to an eigenvalue λ, so is kx with any $k \neq 0$.

Proof:

$$Ax = \lambda x$$

$$k(Ax) = k\lambda x$$

$$A(kx) = \lambda(kx)$$

PROBLEM

Find the eigenvalues and eigenvectors for

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

MULTIPLICITY

The algebraic multiplicity is the order M_λ of λ in the **characteristic polynomial**.

The geometric multiplicity of λ is the number of linearly independent vectors corresponding to λ. ($m\lambda$).

$$\sum M_\lambda = n$$

$$m_\lambda \underset{12}{\leq} M_\lambda$$

$$\Delta_\lambda = M_\lambda - m_\lambda \quad (\text{defect of } \lambda)$$

APPLICATIONS – Markov Process

Suppose that the 2004 state of land use in a city of 60 mi^2 of built-up area is

C: Commercially Used 25%   I: Industrially Used 20%   R: Residentially Used 55%

$$
\begin{array}{cc}
 & \begin{array}{ccc} \text{From C} & \text{From I} & \text{From R} \end{array} \\
\mathbf{A} = & \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix}
\end{array}
\begin{array}{l} \text{To C} \\ \text{To I} \\ \text{To R} \end{array}
$$

The eigenvalue problem can be used to identify the limit state of the process, in which the state vector **x** is reproduced under the multiplication by the stochastic matrix **A** governing the process, that is, **Ax**=**x**.

$$
\mathbf{A}-\mathbf{I}=\begin{bmatrix} -0.3 & 0.1 & 0 \\ 0.2 & -0.1 & 0.2 \\ 0.1 & 0 & -0.2 \end{bmatrix} \Rightarrow
$$

the limit state of the process

$$
\mathbf{x}=[2 \quad 6 \quad 1]^T
$$

APPLICATIONS

A limit is reached if $x = A^T x$

- A^T should have eigenvalue 1

A and A^T have same eigenvalue

- A should have eigenvalue 1

We can show $V^T = [1 \quad 1 \quad 1]^T$ is an eigenvector corresponding to $\lambda = 1$

SYMMETRIC, SKEW SYMMETRIC, ORTHOGONAL MATRIES

Definitions

A real square matric n×n

- is symmetric if $A^T = A$
- is skew-symmetric if $A^T = -A$
- is orthogonal if $A^T = A^{-1}$
- Any real square matrix A

$$A = R + S$$

$$R = \frac{1}{2}\left(A + A^T\right)$$

$$S = \frac{1}{2}\left(A - A^T\right)$$

- The eigenvalues of a symmetric matrix are real
- The eigenvalues of a skew-symmetric matrix are pure imaginary or 0.

(Proof Later).

ORTHOGONAL TRANSFORMATIONS AND MATRIES

$$Y = Ax, \quad A \text{ is orthogonal}$$

Example: Plane rotation through θ.

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- An orthogonal transformation preserves the value of the inner-product of vectors

$$a \cdot b = a^T b$$

(a, b are column vectors)

ORTHOGONAL TRANSFORMATIONS AND MATRIES (cont)

Hence, it also preserves the **length** or **norm** of a vector:

$$\| a \| = \sqrt{a \cdot a} = \sqrt{a^T \cdot a}$$

Proof:

$$u = Aa \qquad A\text{, orthogonal}$$
$$v = Ab$$

We need to show $u \cdot v = a \cdot b$

$$u^T = (Aa)^T = a^T A^T$$

$$A^T A = A^{-1} A = I$$

$$u \cdot v = u^T v = a^T A^T A b = a^T A^{-1} A b$$

$$= a^T b = a \cdot b$$

ORTHONORMALITY OF COLUMN AND ROW VECTORS

A real square matrix is orthogonal iff its column vectors $a_1, \ldots, a_n$ (and also its row vectors) form an orthonormal system

$$a_j \cdot a_k = a_j^T \cdot a_k = \begin{cases} 0 & \text{if} \quad j \neq k \\ 1 & \text{if} \quad j = k \end{cases}$$

ORTHONORMALITY OF COLUMN AND ROW VECTORS

Proof: **A is orthogonal**

$$A^{-1}A = A^{T}A = I$$

$$= \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix} \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$$

$$= \begin{bmatrix} a_1^T a_1 & a_1^T a_2 & \cdots & a_1^T a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n^T a_1 & a_n^T a_2 & \cdots & a_n^T a_n \end{bmatrix}$$

$$= I$$

DETERMINANT OF AN ORTHOGONAL MATRIX

The determinant of an orthogonal matrix is +1 or -1

Proof:

$$\det(A \cdot B) = \det(B \cdot A) = \det A \cdot \det B$$

$$\det\left(A^{T}\right) = \det A$$

$$\begin{aligned} 1 &= \det\left(A \cdot A^{-1}\right) \\ &= \det\left(A \cdot A^{T}\right) \\ &= \det A \cdot \det A^{T} \\ &= (\det A)^{2} \end{aligned}$$

$$\det A = \pm 1$$

DETERMINANT OF AN ORTHOGONAL MATRIX

The eigenvalues of an orthogonal matrix A are real or complex conjugate in pairs and have absolute value = 1

$$\left(\left|\lambda\right| = 1 \quad \text{proved later}\right)$$

The orthogonal matrix in Example 1 has the characteristic equation

$$-\lambda^3 + \frac{2}{3}\lambda^2 + \frac{2}{3}\lambda - 1 = 0.$$

Now one of the eigenvalues must be real (why?), hence +1 or −1. Trying, we find −1. Division by $\lambda + 1$

gives $-(\lambda^2 - 5\lambda/3 + 1) = 0$ and the two eigenvalues $\left(5 + i\sqrt{11}\right)/6$ and $\left(5 - i\sqrt{11}\right)/6$, which have absolute value 1. Verify all of this.

COMPLEX MATRICES: HERMITIAN, SKEW-HERMITIAN, UNITARY

Definitions:

A square matrix $A = [a_{jk}]$ is

- Hermitian if $\overline{A}^T = A$ — Symmetric
- $\text{Skew}-\text{Hermitian if}$ $\overline{A}^T = -A$ ➡ Skew-symmetric
- Unitary if $\overline{A}^T = A^{-1}$ — Orthogonal

COMPLEX MATRICES: HERMITIAN, SKEW-HERMITIAN, UNITARY (cont)

- If A is hermitian $a_{jj} = \bar{a}_{jj} \rightarrow$ diagonal elements are real
- If A is skew-hermitian $a_{jj} = -\bar{a}_{jj} \rightarrow$ diagonal elements are pure imaginary
- If a hermitian matrix is real $\overline{A}^T = A^T = A \rightarrow$ symmetric
- If a skew-hermitian matrix is real $\overline{A}^T = A^T = -A \rightarrow$ skew-symmetric
- If a matrix is real and unitary, $\overline{A}^T = A^T = A^{-1} \rightarrow$ orthogonal

EIGENVALUES

Theorem:

- The eigenvalues of a Hermitian matrix are **real**.
- The eigenvalues of a skew-Hermitian matrix are **pure imaginary or 0**.
- The eigenvalues of a unitary matrix have **absolute value of "1"**.

EIGENVALUES (cont)

Proof: Let λ be an eigenvalue of **A**, x be a corresponding eigenvector.

$$\mathbf{A}x = \lambda x$$

(a) Assume A is Hermitian

$$x^T A x = \bar{x}^T \lambda x = \lambda \bar{x}^T x$$

$$\bar{x}^T x = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \bar{x}_1 x_1 + \cdots + \bar{x}_n x_n$$

$$= |x_1|^2 + \cdots + |x_n|^2$$

$$\neq 0 \qquad \text{since } x \neq 0$$

EIGENVALUES (cont)

$$\lambda = \frac{\bar{x}^T A x}{\bar{x}^T x}$$

λ is real if $\bar{x}^T A x$ is real

$$\underbrace{\bar{x}^T A x}_{\text{number}} = \left(\bar{x}^T A x\right)^T$$

$$= x^T A^T \bar{x}$$

$$= x^T \bar{A} \bar{x} = \left(\overline{\bar{x}^T A x}\right)$$

Hermitian: $\bar{A}^T = A$ or $\bar{A} = A^T$

EIGENVALUES (cont)

(b) If A is skew-Hermitian

$$\lambda = \frac{\bar{x}^T A x}{\bar{x}^T x} \quad \text{since we made no use of propertry.}$$

$$\bar{x}^T A x = \left(\bar{x}^T A x\right)^T$$

$$= x^T A^T \bar{x}$$

$$= -x^T \bar{A}\,\bar{x} = -\left(\overline{\bar{x}^T A x}\right)$$

$$\bar{A}^T = -A$$

(c) If A is unitary

$$Ax = \lambda x \quad \text{and} \quad \left(\bar{A}\,\bar{x}\right)^T = \left(\overline{\lambda x}\right)^T = \bar{\lambda}\,\bar{x}^T$$

EIGENVALUES (cont)

Multiplying:

$$\left(\overline{\mathrm{A}}\,\overline{\mathrm{x}}\right)^{\mathrm{T}}\mathrm{Ax} = \overline{\lambda}\,\overline{\mathrm{x}}^{\mathrm{T}}\lambda\mathrm{x}$$
$$= \overline{\lambda}\,\lambda\,\overline{\mathrm{x}}^{\mathrm{T}}\mathrm{x}$$
$$= |\lambda|^2\,\overline{\mathrm{x}}^{\mathrm{T}}\mathrm{x}$$

$$\left(\overline{\mathrm{A}}\,\overline{\mathrm{x}}\right)^{\mathrm{T}}\mathrm{Ax} = \overline{\mathrm{x}}^{\mathrm{T}}\,\overline{\mathrm{A}}^{\mathrm{T}}\mathrm{Ax}$$
$$= \overline{\mathrm{x}}^{\mathrm{T}}\mathrm{A}^{-1}\mathrm{Ax} = \overline{\mathrm{x}}^{\mathrm{T}}\mathrm{x}$$

$$|\lambda|^2 = 1$$

FORMS

Forms:

$\bar{x}^T A^T x$ is called a <u>form</u> in $x_1, \ldots, x_n$ and A is its coefficient matrix.

Example: (n = 2)

$$\bar{x}^T A x = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} \bar{x}_1 & \bar{x}_2 \end{bmatrix} \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}$$

$$= a_{11}\bar{x}_1 x_1 + a_{12}\bar{x}_1 x_2 + a_{21}\bar{x}_2 x_1 + a_{22}\bar{x}_2 x_2$$

FORMS (cont)

In general

$$\bar{x}^T A^T x = \sum_j \bar{x}^T a_{ij} x_j = \sum_j \sum_i \left[\bar{x}^T\right]_i a_{ij} x_j = \sum_j \sum_i \bar{x}_i a_{ij} x_j$$

If x and A are real

$$\bar{x}^T A x = \sum_j \sum_i a_{ij} x_i x_j$$

Quadratic form.

We can assume A symmetric.

QUADRATIC FORM

$$a_{ij}x_i x_j + a_{ji}x_j x_i = \frac{(a_{ij}+a_{ji})}{2}x_i x_j + \frac{(a_{ij}+a_{ji})}{2}x_j x_i$$

Example:

$$\begin{aligned}
X^T A x &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3x_1 + 4x_2 \\ 6x_1 + 2x_2 \end{bmatrix} \\
&= x_1(3x_1 + 4x_2) + x_2(6x_1 + 2x_2) \\
&= 3x_1^2 + 4x_1x_2 + 6x_1x_2 + 2x_2^2 \\
&= 3x_1^2 + 10x_1x_2 + 2x_2^2 \\
&= 3x_1^2 + 5x_1x_2 + 5x_1x_2 + 2x_2^2 \\
&= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned}$$

QUADRATIC FORM (cont)

If A is Hermitian or skew-Hermitian; the form is called **Hermitian** or **skew-Hermitian form**.

Theorem:

For every choice of x, the value of an **Hermitian form** is **real**, and the value of a **skew-Hermitian** form is **pure imaginary** or **0**.

Proof: **Homework 1 (week 7)**

Properties of Unitary Matrices – Complex Vector Space C^n.

Inner Product:

$$a \cdot b = \bar{a}^T b$$

QUADRATIC FORM (cont)

Length or **Norm**

$$\| a \| = \sqrt{a \cdot a} = \sqrt{\bar{a}^T \cdot a} = \sqrt{\bar{a}_1 a_1 + \bar{a}_2 a_2 + \cdots + \bar{a}_n a_n}$$

$$= \sqrt{|a_1|^2 + \cdots + |a_n|^2}$$

Theorem:

A unitary transformation, y = Ax, A unitary, preserves the value of the inner product and the norm.

Proof:

$$u \cdot v = \bar{u}^T v = \left(\overline{A a}\right)^T (Ab) = \left(\overline{A a}\right)^T (Ab) = \bar{a}^T \bar{A}^T Ab$$

$$= \bar{a}^T A^{-1} Ab = \bar{a}^T b = a \cdot b$$

QUADRATIC FORM (cont)

Theorem:

A square matrix is unitary iff its column vectors (row vectors) form a unitary system, i.e.,

Proof:

$$a_j a_k = a_j^T a_k = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

Theorem:

The determinant of a unitary matrix has absolute value 1.

Proof:

$$1 = \det\left(A \cdot A^{-1}\right) = \det\left(A \cdot \overline{A}^T\right) = \det A \cdot \det\left(\overline{A}^T\right) = \det A \cdot \det\left(\overline{A}\right)$$

$$= \det A \cdot \overline{\det A} = \left|\det A\right|^2$$

SIMULARITY OF MATRICES-BASIS OF EIGENVECTORS-DIAGONALIATION

Similarity of Matrices:

An $n \times n$ matrix $\hat{A}$ is called similar to an n x n matrix A if

$$\hat{A} = P^{-1}AP$$

for some $\left(\text{non singular}\right)$ n x n matrix P.

EIGENVALUES AND EIGENVECTORS OF SIMILAR MATRICES

$$A \underset{\substack{\text{Similarity}\\ \text{Transfornation}}}{\longrightarrow} \hat{A}$$

If $\hat{A}$ is similar to A, then $\hat{A}$ has <u>same</u> eigenvalues as A. Furthermore, if x is an eigenvector of A, $y = P^{-1}x$ is an eigenvector of $\hat{A}$ corresponding to the same eigenvalue.

EIGENVALUES AND EIGENVECTORS OF SIMILAR MATRICES (cont)

Proof: $\mathrm{Ax} = \lambda \mathrm{x} \quad \lambda$ an eigenvalue $\mathrm{x} \neq 0$

$$\begin{aligned} P^{-1}Ax &= P^{-1}APP^{-1}x \\ &= \hat{A}P^{-1}x \\ &= \lambda P^{-1}x \end{aligned}$$

$$\hat{A}\,\underbrace{P^{-1}x}_{y} = \lambda\, P^{-1}x$$

$$\hat{A}y = \lambda y$$

PROPERTIES OF EIGENVECTORS

Linear Independence

Let λ_1, λ_2, …, λ_n be distinct eigenvalues of an $n \times n$ matrix. The corresponding eigenvectors are linearly independent.

Proof

Suppose it is not the case. Let r be such that $\{x_1, \ldots, x_r\}$ linearly independent.

$$r < n$$

PROPERTIES OF EIGENVECTORS

$\{x_1, x_2, \ldots, x_r, x_{r+1}\}$ is linearly dependent.

$$\exists \ c_1, c_2, \cdots, c_{r+1} \ \text{ not all zero such that :}$$

$$c_1 x_1 + c_2 x_2 + \cdots + c_r x_r + c_{r+1} x_{r+1} = 0$$

$$A(c_1 x_1 + c_2 x_2 + \cdots + c_r x_r + c_{r+1} x_{r+1}) = A(0)$$

$$c_1 Ax_1 + c_2 Ax_2 + \cdots + c_r Ax_r + c_{r+1} Ax_{r+1} = 0$$

$$c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + \cdots + c_r \lambda_r x_r + c_{r+1} \lambda_{r+1} x_{r+1} = 0$$

$$\lambda_{r+1}(c_1 x_1 + c_2 x_2 + \cdots + c_r x_r + c_{r+1} x_{r+1} = 0) -$$

$$(c_1 \lambda_1 x_1 + \cdots + c_{r+1} \lambda_{r+1} x_{r+1} = 0)$$

$$c_1(\lambda_{r+1} - \lambda_1)x_1 + \cdots + c_r(\lambda_{r+1} - \lambda_1)x_r = 0$$

$$c_1 = 0 = c_2 = \cdots = c_r$$

$$c_{r+1} x_{r+1} = 0 \quad \rightarrow \quad x_{r+1} \neq 0 \ \text{ since it is an eigenvector. So, } c_{r+1} = 0$$

$$x_{r+1} = 0 \qquad \text{which is a contradiction}$$

BASIS OF EIGENVECTORS

If an $n \times n$ matrix A has **n distinct eigenvalues**, then A has a basis of eigenvectors.

A Hermitian, skew-Hermitian or unitary matrix has a basis of eigenvectors that is a unitary system. A symmetric matrix has an orthonormnal basis of eigenvectors.

(not proved here)

DIAGONALIZATION

If an n × n matrix A has a basis of eigenvectors then

$$D = X^{-1}AX$$

is diagonal with the eigenvalues of A or the diagonal. X is the matrix of eigenvectors as column vectors.

$$D^m = X^{-1}A^mX$$

For example,

$$D^2 = D \cdot D = X^{-1}AX \cdot X^{-1}AX = X^{-1}A^2X$$

DIAGONALIZATION

Example: Diagonalize

$$\mathbf{A} = \begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1.0 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix}$$

$$\lambda_1 = 3, \lambda_2 = -4, \lambda_3 = 0$$

$$\begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix} \begin{bmatrix} -3 & -4 & 0 \\ 9 & 4 & 0 \\ -3 & -12 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

TRANSFORMATION OF FORMS TO PRINCIPAL AXES

Explained for **quadratic forms**

$$Q = x^T A x$$

We assume: A is symmetric

$\rightarrow$ A has an orthonormal basis of n eigenvectors.

X is **orthogonal**

$$\rightarrow \quad X^{-1} = X^T$$

$$D = X^{-1}AX \quad \rightarrow \quad XD = AX$$

$$A = XDX^{-1} = XDX^T$$

TRANSFORMATION OF FORMS TO PRINCIPAL AXES (cont)

$$Q = x^T A x = x^T X D X^{-1} x = x^T X D X^T x$$

Set $\quad y = X^T x, \quad y = X^{-1} x \quad \Rightarrow \quad Xy = x$

And $\quad x^T X = \left(X^T x\right)^T = y^T \quad \Rightarrow \quad X^T x = y$

$$Q = y^T D y = [y_1 \cdots y_n] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ \vdots \\ y_n \end{bmatrix}$$

$$Q = [y_1 \quad y_2 \quad \cdots \quad y_n] \begin{bmatrix} \lambda_1 y_1 & & & \\ & \lambda_2 y_2 & & \\ & & \ddots & \\ & & & \lambda_n y_n \end{bmatrix}$$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$