System Control

2. The Laplace Transform

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Laplace Transformation

· Definition: $\mathcal{L}\,[f(t)] = \int_0^{\infty} f(t)e^{-st}dt = F(s)$

$\mathcal{L}$: f(t) $\Rightarrow$ F(s) , s=σ +jω (complex variable)

f(t) : a time function such that f(t)=0 for t<0

· Inverse Laplace Transformation

$$\mathcal{L}^{-1}[F(s)] = f(t)$$

Existence of Laplace Transformation

- f(t) Laplace – transformable

  if i) f(t) piecewise-continuous

  ii) f(t) of exponential order as t approaches infinity

  – $e^{\alpha t}\left|f(t)\right|$ bounded, α exist.

  – or $e^{-\sigma t}\left|f(t)\right|$ approaches zero as t approaches infinity.

- If $\lim_{t\to\infty} e^{-\sigma t}\left|f(t)\right| = \begin{cases} 0 & for\ \sigma > \sigma_c \\ \infty & for\ \sigma < \sigma_c \end{cases}$

  the σ_c : the abscissa of convergence

Existence of Laplace Transformation

example : 1) $t,\ \sin\omega t,\ t\sin\omega t\ldots$

$$\lim_{t\to\infty} e^{-\sigma t}\left|t\sin\omega t\right| = \begin{cases} 0 & if\ \sigma > 0 \\ \infty & if\ \sigma < 0 \end{cases}$$

the abscissa of convergence $\sigma_c = 0$

2) $e^{-ct},\ te^{-ct},\ e^{-ct}\sin\omega t,\quad c = const.$

$$\lim_{t\to\infty} e^{-\sigma t}\left|te^{-ct}\right| = \begin{cases} 0 & if\ \sigma > -c \\ \infty & if\ \sigma < -c \end{cases}$$

the abscissa of convergence $\sigma_c = -c$

- $e^{t^2},\quad te^{t^2}$ does not possess L. T.
- $f(t) = \begin{cases} e^{t^2} & for\quad 0 \le t \le T < \infty \\ 0 & for\quad t < 0,\ \mathrm{T} < \mathrm{t} \end{cases}$ L[f(t)] exists.

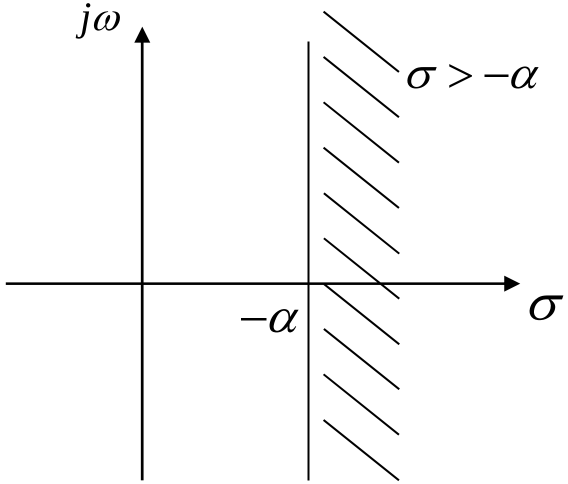
– The signals that can be physically generated always have corresponding Laplace transforms

Laplace Transformation of simple function

- Exponential function

$$f(t)=\begin{cases} 0 & for \quad t<0 \\ Ae^{-\alpha t} & for \quad t \ge 0 \end{cases}$$

A, α : constants



$$\mathcal{L}[Ae^{-\alpha t}] = \int_0^\infty Ae^{-\alpha t} \cdot e^{-st} dt$$

$$= \int_0^\infty Ae^{-(\alpha+s)t} dt$$

$$= A[-\frac{1}{s+\alpha}e^{-(s+\alpha)\cdot\infty} + \frac{1}{s+\alpha}e^{0}]$$

$$= A[-\frac{1}{s+\alpha}e^{-(\sigma+\alpha)\cdot\infty - j\omega\cdot\infty} + \frac{1}{s+\alpha}] \qquad (s=\sigma+j\omega)$$

$$= A[0+\frac{1}{s+\alpha}] = \frac{A}{s+\alpha}$$

Laplace Transformation of simple function

- step function

$$f(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\mathcal{L}[f(t)] = \int_0^\infty Ae^{-st}dt = A[-\frac{1}{s}e^{-s\cdot\infty} + \frac{1}{s}e^{-s\cdot 0}]$$

$$= A[0 + \frac{1}{s}] \qquad if \ \mathrm{Re}[s] > 0$$

- unit step input function

$$1(t - t_0) = \begin{cases} 1 & t \geq t_0 \\ 0 & t < t_0 \end{cases}$$

$$1(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\mathcal{L}[1(t)] = \frac{1}{s}$$

Laplace Transformation of simple function

- Sinusoidal Functions

$$f(t) = \begin{cases} A\sin\omega t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\cos\omega t = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

$$\sin\omega t = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

$$\mathcal{L}[A\sin\omega t] = \int_0^\infty \frac{A}{2j}(e^{j\omega t} - e^{-j\omega t})e^{-st}dt$$

$$= \frac{1}{2j}(\frac{1}{s - j\omega} - \frac{1}{s + j\omega})$$

$$= \frac{A\omega}{s^2 + \omega^2}$$

$$\mathcal{L}[A\cos\omega t] = \frac{As}{s^2 + \omega^2}$$

- Ramp function

$$f(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\mathcal{L}[At] = A\int_0^\infty te^{-st}dt$$

$$= A\left(t\frac{e^{-st}}{-s}\Bigg|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}dt \right)$$

$$= A\cdot\frac{1}{s}\int_0^\infty e^{-st}dt = A\cdot\frac{1}{s^2}$$

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Laplace Transformation of simple function

- Pulse function

$$f(t) = \begin{cases} \dfrac{A}{t_0} & for \quad 0 \le t \le t_0 \\ 0 & for \quad t < 0,\ t_0 < \mathrm{t} \end{cases}$$

$$\mathcal{L}[f(t)] = \mathcal{L}\left[\frac{A}{t_0}1(t)\right] - \mathcal{L}\left[\frac{A}{t_0}1(t-t_0)\right]$$

$$= \frac{A}{t_0}\frac{1}{s}(1-e^{-t_0 s})$$

$$f(t) = \frac{A}{t_0}1(t) - \frac{A}{t_0}1(t-t_0)$$

- Impulse function

$$\mathcal{L}[f(t)] = \lim_{t_0 \to 0}\left[\frac{A}{t_0}\frac{1}{s}(1-e^{-t_0 s})\right]$$

$$f(t) = \begin{cases} \lim_{t_0 \to \infty} \dfrac{A}{t_0} & for \quad 0 \le t < t_0 \\ 0 & for \quad t < 0, \quad t_0 < t \end{cases}$$

$$= \lim_{t_0 \to 0} \frac{\dfrac{d}{dt_0}\left[A(1-e^{-t_0 s})\right]}{\dfrac{d}{dt_0}(t_0 s)} = \frac{A \cdot s}{s} = A$$

Laplace Transformation of simple function

- Unit impulse function ; impulse function of magnitude 1

$$f(t) = \begin{cases} \lim_{t_0 \to \infty} \dfrac{1}{t_0} & for \quad 0 \le t < t_0 \\ 0 \quad for & t < 0, \quad t_0 < t \end{cases}$$

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases} \qquad \mathcal{L}[\delta(t)] = \int_0^\infty \delta(t) e^{-st} dt = \int_{0-}^{0+} \delta(t) dt = 1$$

The unit-impulse function occuring at $t = t_0$

$$\delta(t) = \begin{cases} \infty & t = t_0 \\ 0 & t \neq t_0 \end{cases} \quad \mathcal{L}[\delta(t - t_0)] = \int_0^\infty \delta(t - t_0) e^{-st} dt = \int_{t_0-}^{t_0+} \delta(t - t_0) e^{-st_0} dt = e^{-t_0 s} \cdot 1$$

$$\delta(t - t_0) = \frac{d}{dt} 1(t - t_0)$$

Useful Theorems

Theorem 1. Linearity

$$\mathcal{L}[af(t)] = aF(s)$$

Theorem 2. Superposition

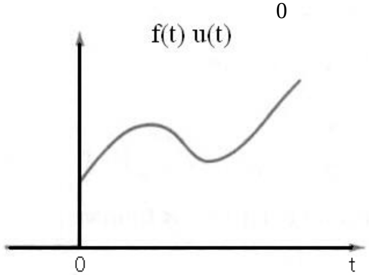
$$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$$

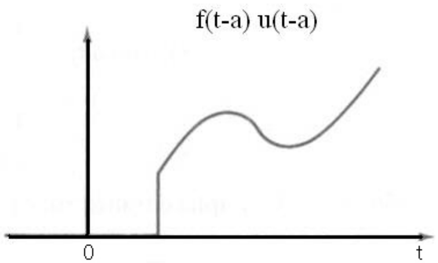
Theorem 3. Translation in time.

$$\mathcal{L}[f(t-a)1(t-a)] = \int_0^\infty f(t-a)1(t-a)e^{-st}dt \qquad (a>0)$$

$$= \int_{-a}^\infty f(\tau)1(\tau)e^{-s(\tau+a)}d\tau \qquad (let.\, t-a=\tau)$$

$$= \int_0^\infty f(\tau)1(\tau)e^{-s\tau}e^{-sa}d\tau = e^{-sa}F(s) \qquad (\because f(\tau)1(\tau)=0 \quad for\ \tau<0)$$





Useful Theorems

Theorem 4. Complex differentiations

$$\mathcal{L}[tf(t)] = -\frac{d}{ds}F(s)$$

$$\mathcal{L}[1] = \frac{1}{s} \qquad \mathcal{L}[t \cdot 1] = -\frac{d}{ds}F(s) = \frac{1}{s^2}$$

$$\mathcal{L}[t^2] = \mathcal{L}[t \cdot t] = -\frac{d}{ds}\left(\frac{1}{s^2}\right) = \frac{2}{s^3} \qquad \mathcal{L}[tf(t)] = -\frac{d}{ds}F(s)$$

$$proof) \quad let. \ F(s) = \int_0^\infty f(t)e^{-st}dt$$

$$\frac{d}{ds}F(s) = \int_0^\infty \frac{\partial}{\partial s}\left[f(t)e^{-st}\right]dt = -\int_0^\infty tf(t)e^{-st}dt = -\mathcal{L}[tf(t)]$$

$$similarly, \quad \mathcal{L}[t^2 f(t)] = (-1)^2\frac{d^2}{ds^2}F(s) \qquad \mathcal{L}[t^n f(t)] = (-1)^n\frac{d^n}{ds^n}F(s)$$

Theorem 5. Translation in the s-domain

$$\mathcal{L}[e^{at} f(t)] = F(s-a) \qquad \mathcal{L}[e^{at}\cos\omega t] = \frac{(s-a)}{(s-a)^2+\omega^2}$$

Useful Theorems

Theorem 6. Real Differentiation

$$Df(t) = \frac{d}{dt} f(t)$$

$$proof)\quad let.\quad \mathscr{L}[\frac{d}{dt} f(t)] = \int_0^\infty \frac{d}{dt} f(t) e^{-st} dt$$

$$= f(t)e^{-st}\Big|_0^\infty - \int_0^\infty f(t)e^{-st} dt(-s)$$

$$= -f(0) + s \cdot F(s)$$

$$= s \cdot F(s) - f(0)$$

$$similarly,\quad \mathscr{L}[D^2 f(t)] = \mathscr{L}[D \cdot Df(t)] = \mathscr{L}[Df'(t)]$$

$$= s\{s \cdot F(s) - f(0)\} = s^2 \cdot F(s) - s \cdot f(0) - f'(0)$$

$$\mathscr{L}[\frac{d^n}{dt^n} f(t)] = s^n \cdot F(s) - s^{n-1} \cdot f(0) - s^{n-2} \cdot f'(0)$$

$$\ldots - f^{(n-1)}(0)$$

Useful Theorems

Theorem 7. Real Integration

$$\int_0^t f(t)dt = D^{-1}f(t) - D^{-1}f(0)$$

$$\mathcal{L}\left[\int_0^t f(t)dt\right] = \int_0^\infty \int_0^t f(\tau)d\tau \cdot e^{-st}dt$$

$$= -\frac{1}{s}e^{-st}\int_0^t f(\tau)d\tau\Big|_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st}f(t)dt$$

$$= \frac{1}{s}\int_0^t f(\tau)d\tau\Big|_{t=0} + \frac{1}{s}\int_0^\infty e^{-st}f(t)dt = \frac{F(s)}{s} + \frac{f^{-1}(0)}{s}$$

Theorem 8. Final value Theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Theorem 9. Initial value Theorem

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

Theorem 10. Complex Integration

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)ds$$

Inverse Laplace Transformation

$$\mathcal{L} : f(t) \rightarrow F(s)$$

$$\mathcal{L}^{-1} : F(s) \rightarrow f(t)$$

$$f(t) = \mathcal{L}^{-1}\left[F(s)\right] = \frac{1}{2\pi j}\int_{c-j\omega}^{c+j\omega} F(s)e^{st}ds$$ (c : real constant)

*** Inverse Laplace Transformation by Partial Fraction Method**

$$F(s) = \frac{P(s)}{Q(s)} = \frac{a_m s^m + a_{m-1}s^{m-1} + \cdots + a_0}{s^n + b_{n-1}s^{n-1} + \cdots b_1 s + b_0} \qquad (n \geq m)$$

$$s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0 \qquad \Rightarrow \text{real complex conjugate,} \quad \text{a , b : real num.}$$

$$= (s - c_1)(s - c_2)\cdots(s^2 + d_1 s + d_2)$$

$$\Rightarrow F(s) = \frac{\alpha_1}{s - c_1} + \frac{\alpha_2}{s - c_2} + \cdots + \frac{\beta_1 s + \beta_2}{s^2 + d_1 s + d_2} + \cdots$$

Examples of Inverse Laplace Transformation

ex) $$F(s)=\frac{1}{(s+2)^2(s+3)}=\frac{a}{(s+2)}+\frac{b}{(s+2)^2}+\frac{c}{(s+3)}$$

$$a(s+2)(s+3)+b(s+3)+c(s+2)^2=1$$

$$let\ s=-2,\ than\ b=1,\quad let\ s=-3,\ than\ c=1$$

$$a(s+2)+b+c\cdot\frac{s+2}{s+3}=\frac{1}{s+3}\ \xrightarrow{\frac{d}{ds}}\ a+c\cdot\frac{(s+3)-(s+2)}{s+3}=\frac{-1}{(s+3)^2}$$

$$\therefore\ a=-1,\ b=1,\ c=1,\qquad F(s)=\frac{-1}{(s+2)}+\frac{1}{(s+2)^2}+\frac{1}{(s+3)}$$

=> Inverse Laplace Transformation

$$f(t)=-e^{-2t}+te^{-2t}+e^{-3t}$$ (partial fraction method)

ex) $$F(s)=\frac{10}{s^2+6s+25}=\frac{10\times\frac{1}{4}\times 4}{(s+3)^2+4^2}=\frac{10}{4}\frac{4}{(s+3)^2+4^2}$$

$$\therefore\ f(t)=\frac{10}{4}\sin 4te^{-3t}$$

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Solution of Differential Equation by Laplace Transformation

$$y'' + 2y' + 4y = 1 \quad y(0) = 0 \ , \ y'(0) = 2$$

$$\text{L.T: } s^2Y(s) - sy(0) - y'(0) + 2\{sY(s) - y(0)\} + 4Y(s) = \frac{1}{s}$$

$$(s^2 + 2s + 4)Y(s) = \frac{1}{s} + 2 = \frac{2s+1}{s}$$

$$Y(s) = \frac{2s+1}{s(s^2+2s+4)} = \frac{1}{4s} - \frac{1}{4}\frac{s+1-1}{(s+1)^2 + \left(\sqrt{3}\right)^2}$$

$$\therefore y(t) = \frac{1}{4} - \frac{1}{4}\cos\sqrt{3}te^{-t} + \frac{1}{4\sqrt{3}}\sin\sqrt{3}te^{-t}$$

Dynamic System → Mathematical Model (Differential eq.) —L. T.→ S-domain

Solution in Time Domain ←Inverse L.T.— S-domain