System Control

3. Transfer Function and State Equation

Professor Kyongsu Yi

Vehicle Dynamics and Control Laboratory

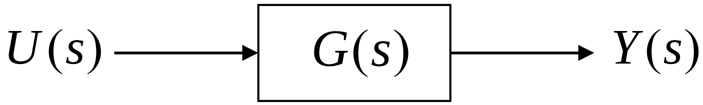
Seoul National University

Transfer Functions

• Transfer Functions (Linear Time Invariant Systems)

-The ratio of the Laplace Transform of the output (response function) to the Laplace Transform of the input (driving function) under the assumption that all initial conditions are zero.

$$\frac{Y(s)}{U(s)} = G(s)$$



• Differential Equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = b_0 x^m + b_1 x^{m-1} + \cdots + b_{m-1} x' + b_m x$$

$$\text{T. F.} \quad G(s) = \frac{\mathrm{Y(s)}}{\mathrm{X(s)}} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} \qquad n \geq m$$

• T.F.

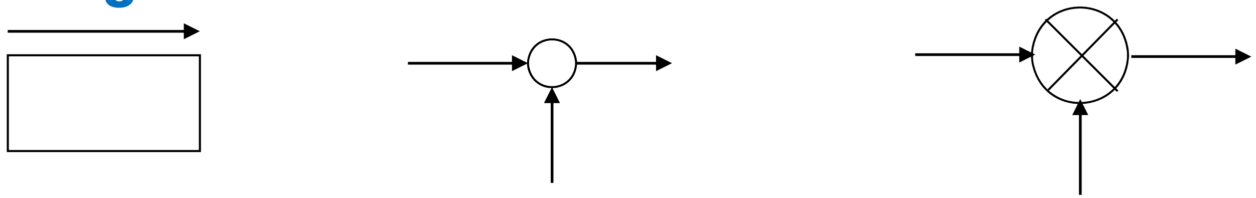
① A mathematical model

② A property of a system itself independent of the magnitude and nature of the input

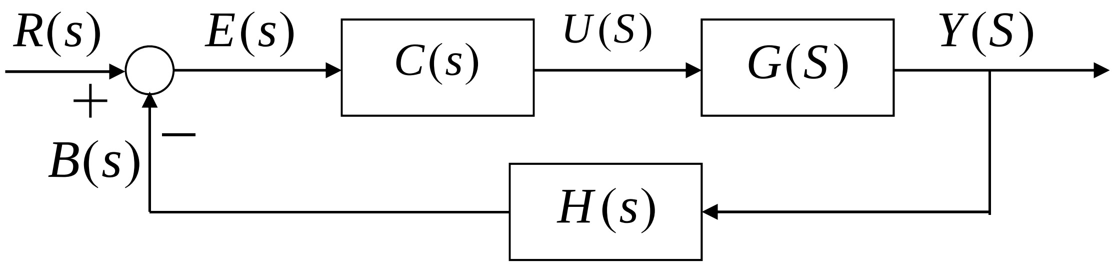
③ T. F. includes the units(input-output relations) however, does not provide any information concerning the physical structure of the system, many different systems can have identical T. F.

Feedback Control System

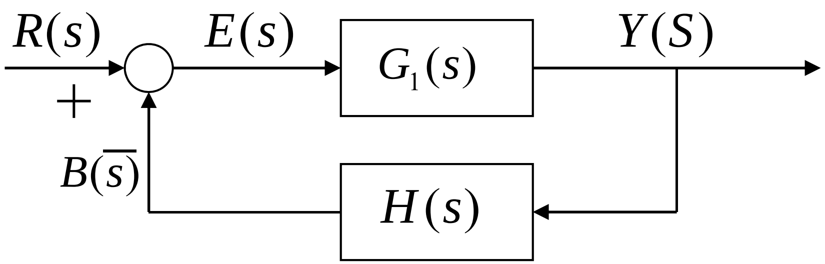
- **Block Diagram**



A block diagram (of a system) : a pictorial representation of the function performed by each component and of the flow of signals.



or



$$G_1(S) = C(s) \cdot G(s)$$

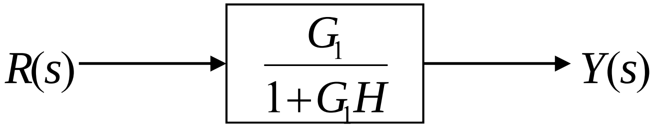
Transfer Functions

- **Open loop T.F.** $= \dfrac{B(s)}{E(s)} = \dfrac{H(s)\cdot Y(S)}{E(s)} = \dfrac{H(s)G_1(s)E(s)}{E(S)}$

  $= H(s)\cdot G_1(s)$

- **Feed forward T.F.** $= \dfrac{Y(s)}{E(s)} = G_1(s)$

- **Closed loop T.F.** $= \dfrac{Y(s)}{R(s)}$

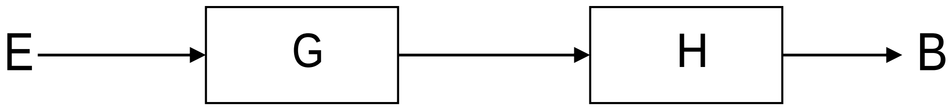


$$Y(s) = G_1(s)E(s) = G_1(s)[R(s) - B(s)]$$

$$= G_1(s)[R(s) - H(s)\cdot Y(s)]$$

$$[1 + G_1(s)H(s)]Y(s) = G_1(s)R(s)$$

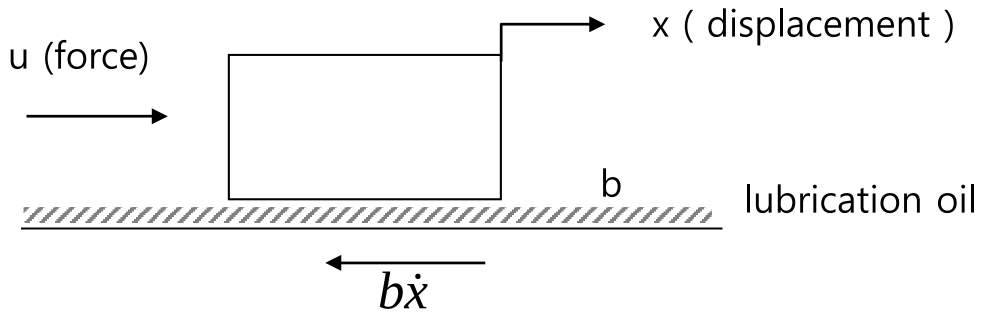
$$\frac{Y(s)}{R(s)} = \frac{G_1(s)}{1 + G_1(s)H(s)}$$



$$\frac{G_{\text{feed forward}}}{1 + G_{\text{open}}(s)}$$

Transfer Functions

Ex)



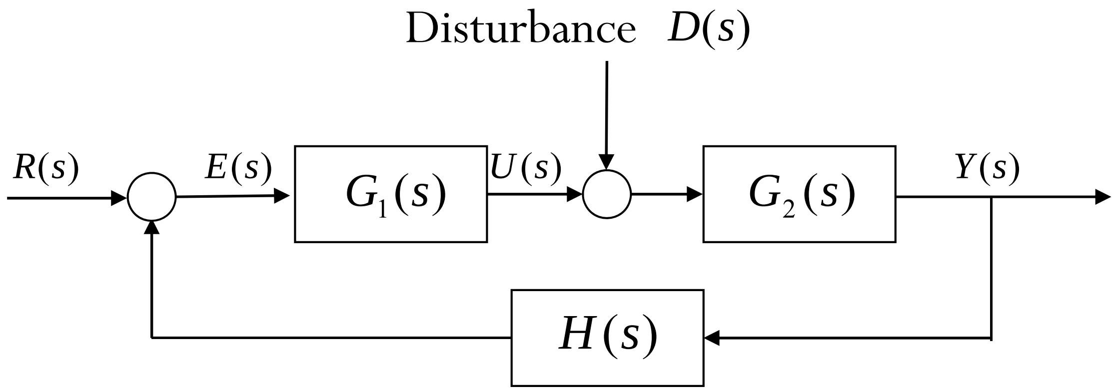
$$m\ddot{x} = u - b\dot{x} \qquad\qquad m\ddot{x} + b\dot{x} = u$$

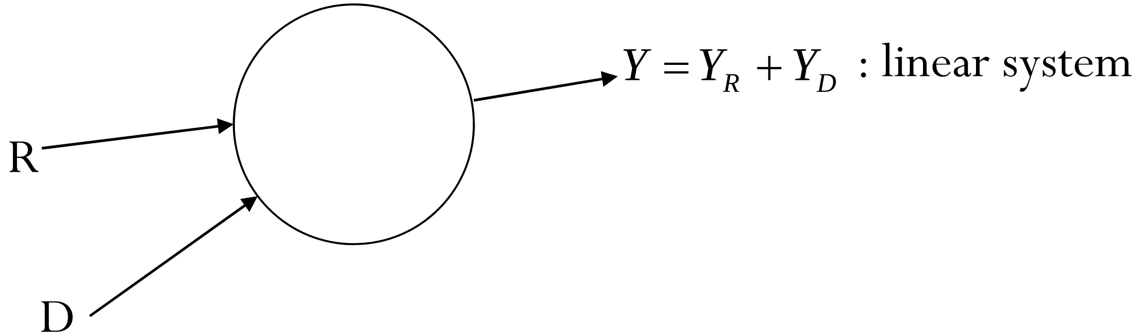
$$x(0) = 0$$
$$\dot{x}(0) = 0$$

$$ms^2 X(s) + bsX(s) = U(s)$$
$$(ms^2 + bs)X(s) = U(s)$$

$$\frac{X(s)}{U(s)} = \frac{1}{s(ms+b)}$$: Transfer Function

Closed-loop system subjected to a disturbance





$$Y_D(s) = G_2(s)[D(s) + U(s)$$

$$= G_2(s)D(s) + G_2(s)G_1(s)E(s)$$

$$= G_2(s)D(s) + G_2(s)G_1(s)[-H(s)Y_D(s)]$$

Closed-loop system subjected to a disturbance

$$R=0: \quad \frac{Y_D(s)}{D(s)} = \frac{G_2}{1+G_1G_2H} = G_D(s)$$

$$D=0: \quad \frac{Y_R(s)}{R(s)} = \frac{G_1G_2}{1+G_1G_2H} = G_R(s)$$

$$Y(s) = \frac{G_1G_2}{1+G_1G_2H}R + \frac{G_2}{1+G_1G_2H}D$$

$$= \frac{G_2}{1+G_1G_2H}[G_1R+D]$$

- $G_1G_2H >> 1$

$$G_D(s) \cong \frac{G_2}{G_1G_2H} = \frac{1}{G_1H}$$

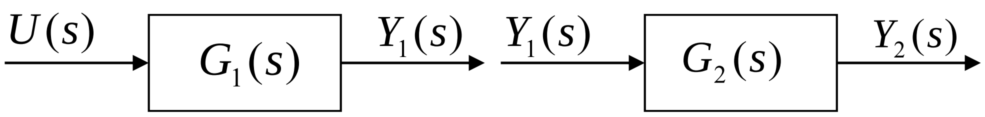
$$G_1H >> 1 \qquad G_D = \varepsilon << 1$$

The effect of the disturbance is reduced → Advantage of the closed-loop system

$$G_R(s) \approx \frac{G_1G_2}{G_1G_2H} = \frac{1}{H}$$

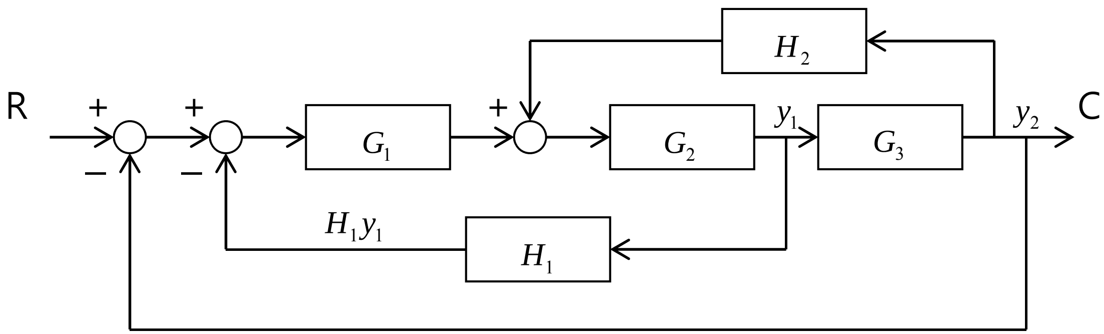
Block Diagram Reduction

ex1)

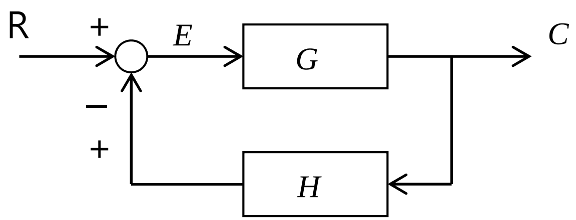


$$Y_2(s) = G_2(s)Y_1(s) = G_2(s)G_1(s)U(s)$$

ex2)

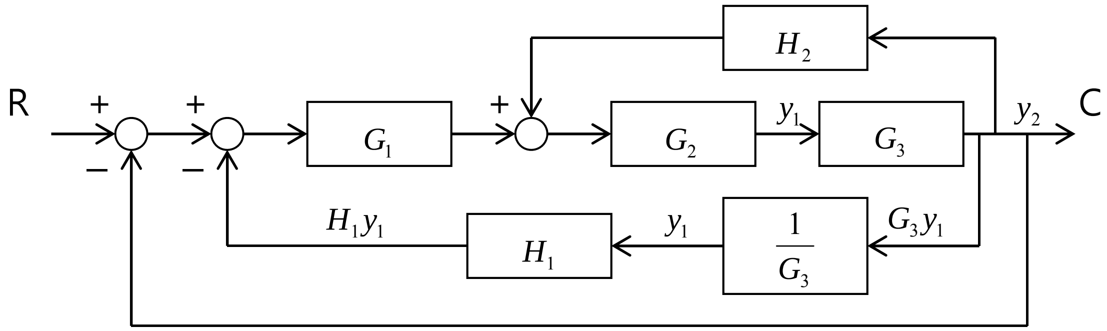


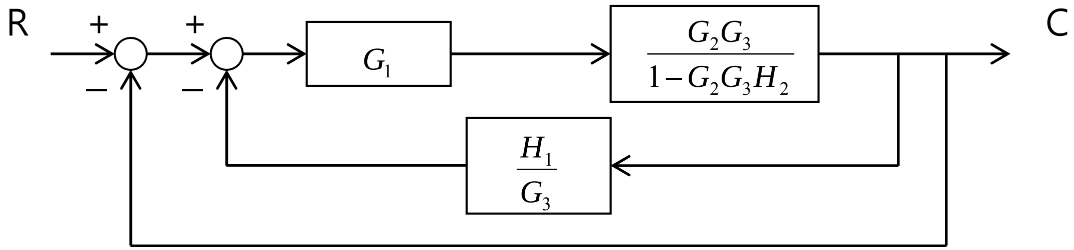
*

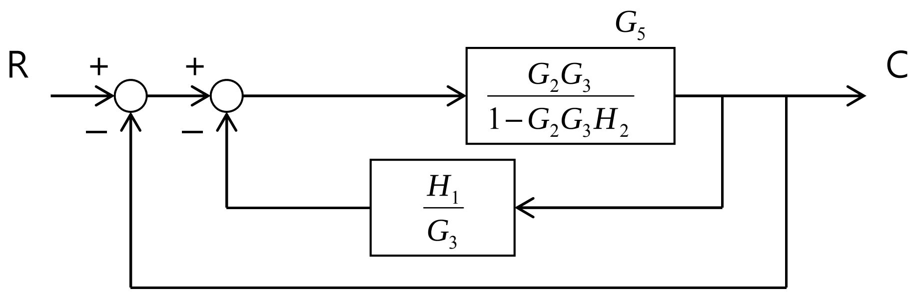


$$\frac{C}{R} = \frac{G}{1 \pm GH}$$

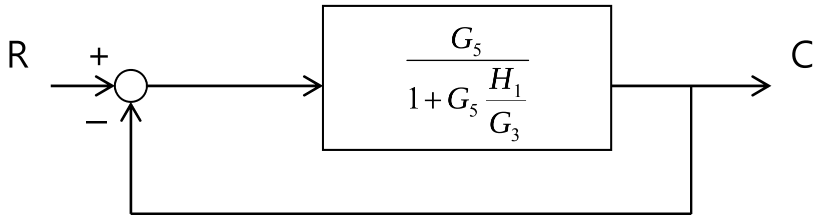
Block Diagram Reduction



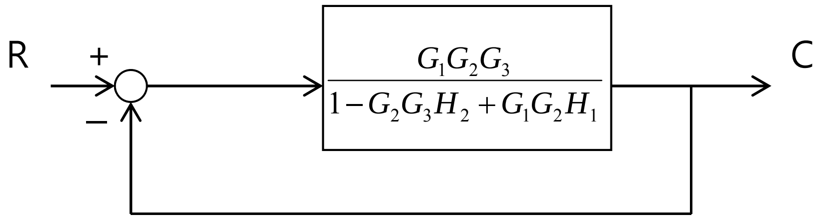




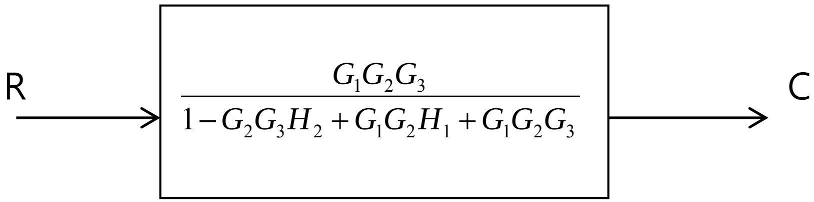
Block Diagram Reduction



$$\frac{\frac{G_1G_2G_3}{1-G_2G_3H_2}}{1+\frac{G_1G_2G_3}{1-G_2G_3H_2}\frac{H_1}{G_3}}=\frac{G_1G_2G_3}{1-G_2G_3H_2+G_1G_2H_1}$$



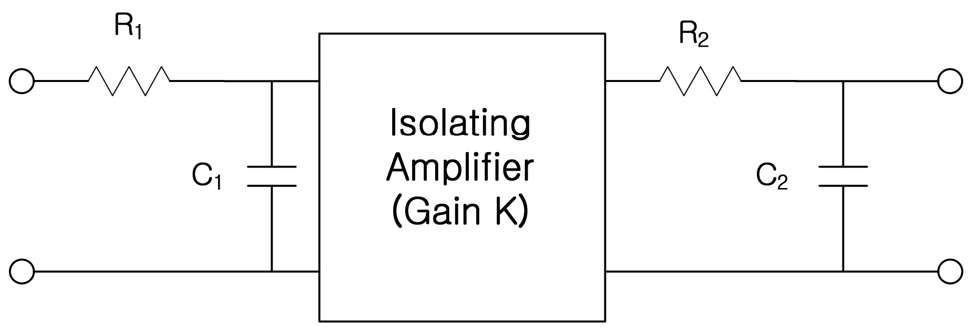
$$\frac{\frac{G_1G_2G_3}{1-G_2G_3H_2+G_1G_2H_1}}{1+\frac{G_1G_2G_3}{1-G_2G_3H_2+G_1G_2H_1}}=\frac{G_1G_2G_3}{1-G_2G_3H_2+G_1G_2H_1+G_1G_2G_3}$$



No Loading Effect

- **No Loading Effect**

Block can be connected in series only if the output of one block is not affected by the next following block.



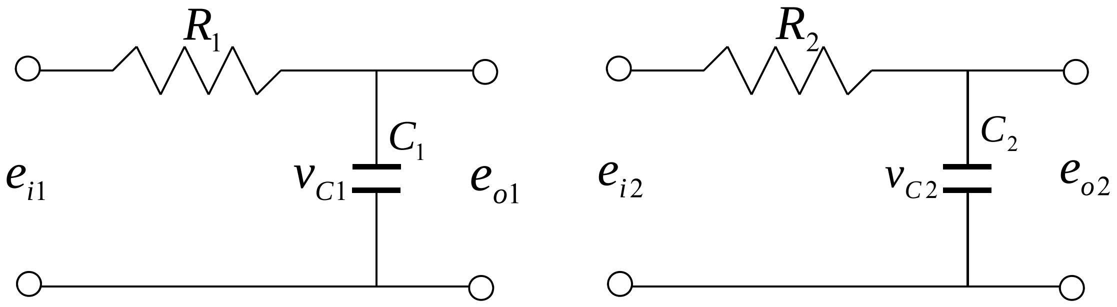
$$V(s) = I(s)Z(s)$$

$Z(s)$: complex impedance

If the input impedance of the second element is infinite, the output of the first element is not affected by connecting it to the second element.

No Loading Effect

ex1)



$$e_{i1} = R_1 i_1 + e_{o1}$$

$$\frac{dv_{C1}}{dt} = \frac{1}{C_1} i_1, \qquad v_{C1} = e_{o1}$$

$$\Rightarrow \frac{de_{o1}}{dt} = \frac{1}{C_1}\left[\frac{1}{R_1}(e_{i1} - e_{o1})\right] = -\frac{1}{C_1 R_1} e_{o1} + \frac{1}{C_1 R_1} e_{i1}$$

$$\frac{E_{o1}(s)}{E_{i1}(s)} = \frac{\frac{1}{C_1 R_1}}{S + \frac{1}{C_1 R_1}} \quad \left(\frac{1}{C_1 R_1} \rightarrow a_1\right)$$

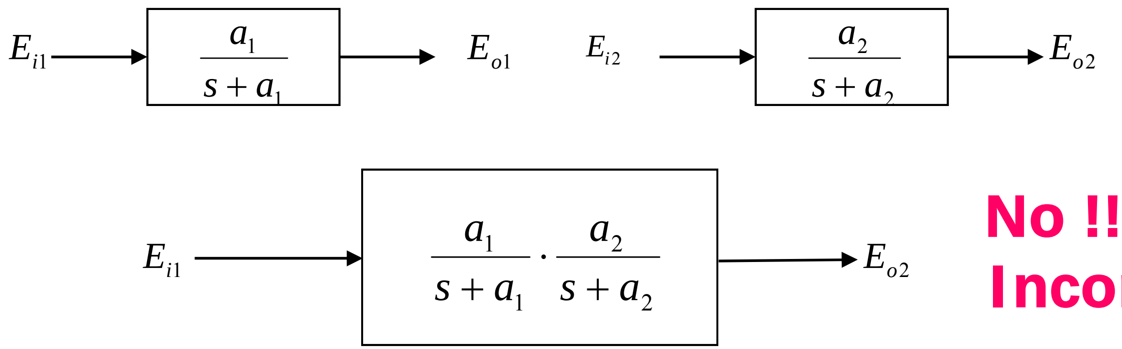
$$\frac{E_{o2}(s)}{E_{i2}(s)} = \frac{\frac{1}{C_2 R_2}}{S + \frac{1}{C_2 R_2}} \quad \left(\frac{1}{C_2 R_2} \rightarrow a_2\right)$$

No Loading Effect

ex1)

$$\frac{E_{o1}(s)}{E_{i1}(s)} = \frac{\frac{1}{C_1R_1}}{S + \frac{1}{C_1R_1}} \quad \frac{1}{C_1R_1} \rightarrow a_1$$

$$\frac{E_{o2}(s)}{E_{i2}(s)} = \frac{\frac{1}{C_2R_2}}{S + \frac{1}{C_2R_2}} \quad \frac{1}{C_2R_2} \rightarrow a_2$$



No !!

Incorrect

P.90.Ogata

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{R_1C_1R_2C_2s^2 + (R_1C_1 + R_2C_2 + R_1C_2)s + 1}$$

Mason's Gain Formula

The overall gain

$$P = \frac{1}{\Delta}\sum_k P_k \Delta_k$$

P_k = path gain of k-th forward path

Δ = determinant

= $1 - \sum_a L_a - \sum_{b,c} L_b L_c - \sum_{d,e,f} L_d L_e L_f + \cdots$

$\sum_a L_a$ = sum of all individual loop gains

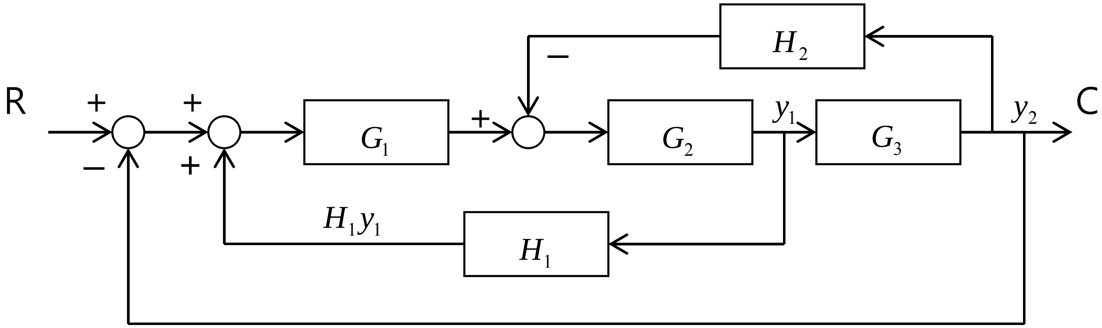
$\sum_{b,c} L_b L_c$ = sum of gain products of all possible combination of two non touching loops

$\sum_{d,e,f} L_d L_e L_f$ = sum of gain products of all possible combination of three non touching loops

Δ_k = cofactors of the k-th forward path determinant of the graph with the loops touching the k-th forward path removed, that is, the cofactor Δ_k is obtained from Δ by removing the loops that touch path P_k

Mason's Gain Formula

Ex 3-13) Ogata



① **One Forward path :** $P_1 = G_1 G_2 G_3$

② **Three Individual Loops :** $L_1 = G_1 G_2 H_1$

$$L_2 = -G_2 G_3 H_2$$

$$L_3 = -G_1 G_2 G_3$$

③ **No Non-touching Loops :** $\Delta = 1 - \left(L_1 + L_2 + L_3\right)$

$$= 1 - G_1 G_2 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3$$

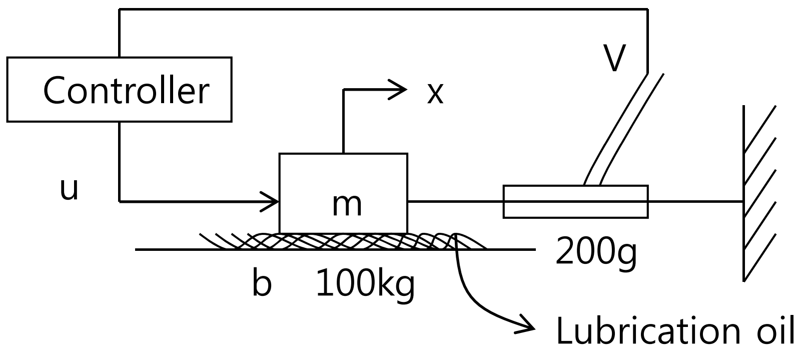
④ Δ_1 **:** P_1 **touches all loops**

$$\Delta_1 = 1$$

⑤ $$\frac{C(s)}{R(s)} = \frac{1}{\Delta}\left(P_1 \Delta_1\right)$$

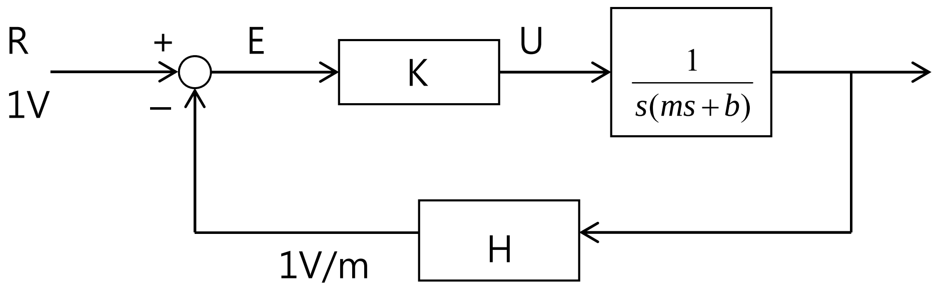
$$= \frac{G_1 G_2 G_3}{1 - G_1 G_2 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3}$$

State Equation



$$m\ddot{x} = u - b\dot{x}$$

$$(ms^2 + bs)X(s) = U(s)$$



State Equation

- $m\ddot{x} + b\dot{x} = u$

$$x_1 = x \qquad \dot{x}_1 = \dot{x} = x_2$$

$$x_2 = \dot{x} \qquad \dot{x}_2 = \ddot{x} = -\frac{b}{m}\dot{x} + \frac{u}{m} = -\frac{b}{m}x_2 + \frac{1}{m}u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

$$y = x_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

First order matrix differential Eq.
➔ State Equation

State Equation

- **State** x

The smallest set of variables such that knowledge of these variables at $t = t_0$, together with the knowledge of the input for $t \geq t_0$, completely determines the behavior of the system at any time $t \geq t_0$

- **State Variables**

The variables making up the smallest set of variables that determine the state of the dynamic system

ex) x_1 : displacement x_2 : velocity

- **State Vector**

x_1 , x_2 , x_3 ... state variables

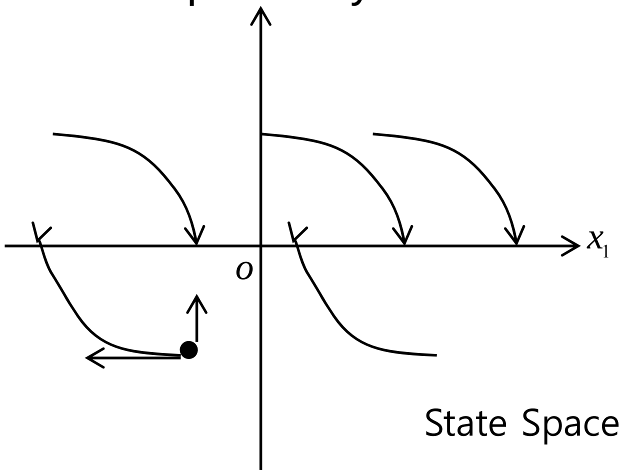
$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

A vector that determines uniquely the system state x(t) for any time once the state at $t = t_0$ is given and the input u(t) for $t \geq t_0$ is specified

State Equation

- **State Space**

The n-dimensional space, whose coordinate axes consist of the x_1 axis, x_2 axis, ..., x_n axis is called a state space. Any state can be represented by a point in the state space



$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, \cdots, x_n; u_1, u_2, \cdots, u_r; t) \\ \dot{x}_2 = f_2(x_1, x_2, \cdots, x_n; u_1, u_2, \cdots, u_r; t) \\ \qquad \vdots \\ \dot{x}_n = f_n(x_n, x_2, \cdots, x_n; u_1, u_2, \cdots, u_r; t) \end{cases}$$

$$\begin{cases} y_1 = g_1(x_1, x_2, \cdots, x_n; u_1, u_2, \cdots, u_r; t) \\ \qquad \vdots \\ y_m = g_m(x_1, x_2, \cdots, x_n; u_1, u_2, \cdots, u_r; t) \end{cases}$$

$$\dot{x} = f(x,u,t)$$
$$y = g(x,u,t)$$

$$f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \qquad g = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_m \end{bmatrix}$$

Linear Systems

- **Linear Systems**

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

- **Linearization (of nonlinear system)**

$$\dot{x} = f(x,u,t) = f(x,u)$$

$$x = x_0, u = u_0, \dot{x} = \dot{x}_0 = f_0 = 0$$

$$x = x_0 + \Delta x, u = u_0 + \Delta u$$

$$\dot{x} = f(x_0, u_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0 u_0} \Delta x + \left.\frac{\partial f}{\partial u}\right|_{x_0 u_0} \Delta u + \left.\frac{\partial^2 f}{\partial x^2}\right|_{x_0 u_0} \Delta x^2 + \left.\frac{\partial^2 f}{\partial u^2}\right|_{x_0 u_0} \Delta u^2 + \cdots$$

$$\cong f_0 + K_1 \Delta x + K_2 \Delta u$$

$$\dot{x} - \dot{x}_0 = K_1 \Delta x + K_2 \Delta u \qquad \Delta x, \Delta u : small$$

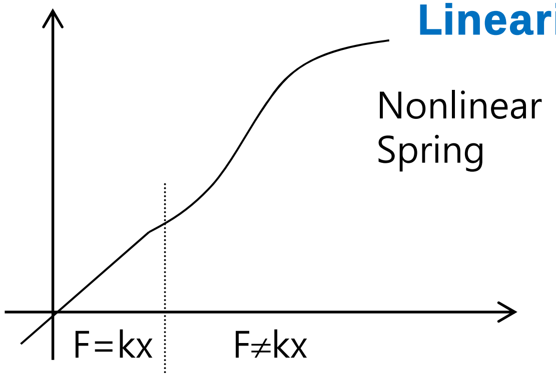
$$\Delta x = K_1 \Delta x + K_2 \Delta u \qquad Approximation$$

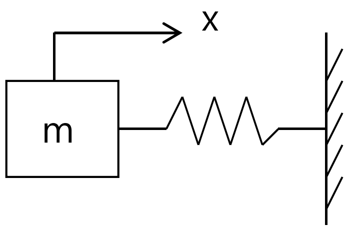
VDCL Vehicle Dynamics & Control Laboratory

Linear Systems

- **Nonlinear Systems → Linear systems**

Linearization





- **State : mathematical concept, not physical meaning**

$$\begin{cases} x_1 = x \\ x_2 = \dot{x} \end{cases}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{b}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

$$\hat{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} x + \dot{x} \\ \dot{x} \end{bmatrix} = Tx$$

$$x = T^{-1}\hat{x}$$

$$T^{-1}\dot{\hat{x}} = AT^{-1}\hat{x} + BU$$

$$\begin{cases} \dot{\hat{x}} = TAT^{-1}\hat{x} + TBU \\ y = CT^{-1}\hat{x} \end{cases}$$

$$\begin{cases} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}U \\ y = \hat{C}\hat{x} \end{cases}$$

Linear Systems

- **Model :**

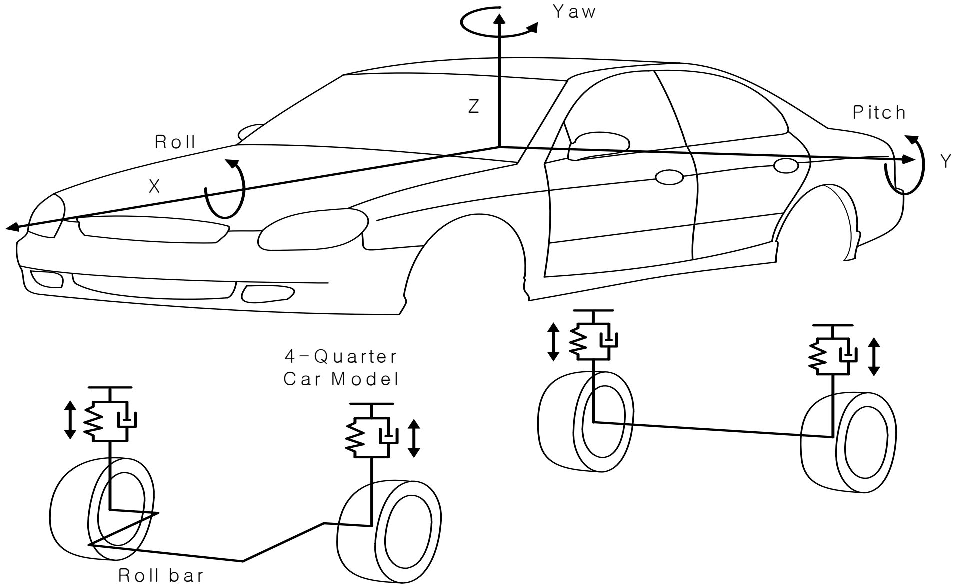
  Differential eq.
  Transfer Functions
  State eq.

- **State Eq.** ⇄ **T.F.** (→ **Unique**; ← **Not unique**)

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

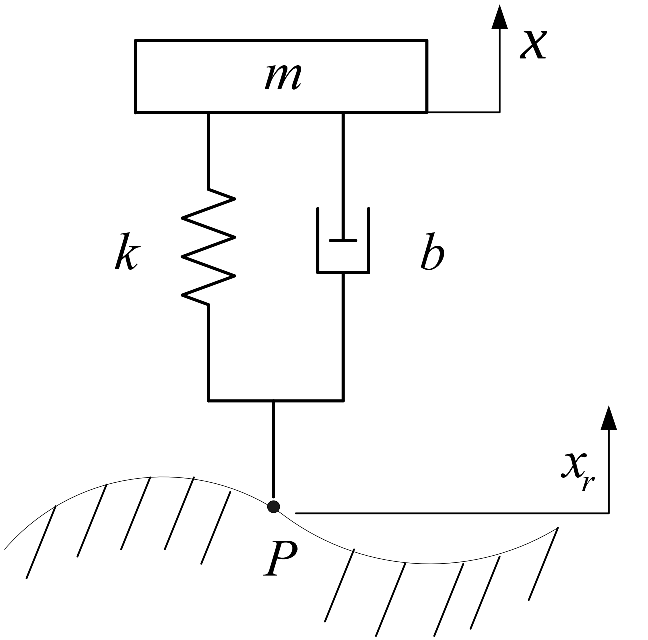
$$sX(s) = AX(s) + BU(s)$$
$$X(s) = (sI - A)^{-1} BU(s)$$
$$Y(s) = CX(s) + DU(s)$$
$$= \left[C(sI - A)^{-1} B + D \right] U(s)$$
$$= G(s)U(S)$$

Vehicle Suspension



Vehicle Suspension

Ex 1) Simplified Quarter Car Model



$$m\ddot{x} = -b(\dot{x} - \dot{x}_r) - k(x - x_r)$$

$$m\ddot{x} + b(\dot{x} - \dot{x}_r) + k(x - x_r) = 0$$

$$m\ddot{x} + b\dot{x} + kx = b\dot{x}_r + kx_r$$

Laplace Transform

$$(ms^2 + bs + k)X(s) = (bs + k)X_r(s)$$

The transfer function

$$\frac{X(s)}{X_r(s)} = \frac{bs + k}{ms^2 + bs + k}$$

State Eq : $\ddot{x} = -\frac{b}{m}\dot{x} - \frac{k}{m}x + \frac{b}{m}\dot{x}_r + \frac{k}{m}x_r$

let $x_1 = x \qquad u = \dot{x}_r$

$x_2 = \dot{x}$

$x_3 = x_r$

then $\dot{x}_1 = \dot{x} = x_2$

$$\dot{x}_2 = \ddot{x} = -\frac{b}{m}x_2 - \frac{k}{m}x_1 + \frac{k}{m}x_3 + \frac{b}{m}u$$

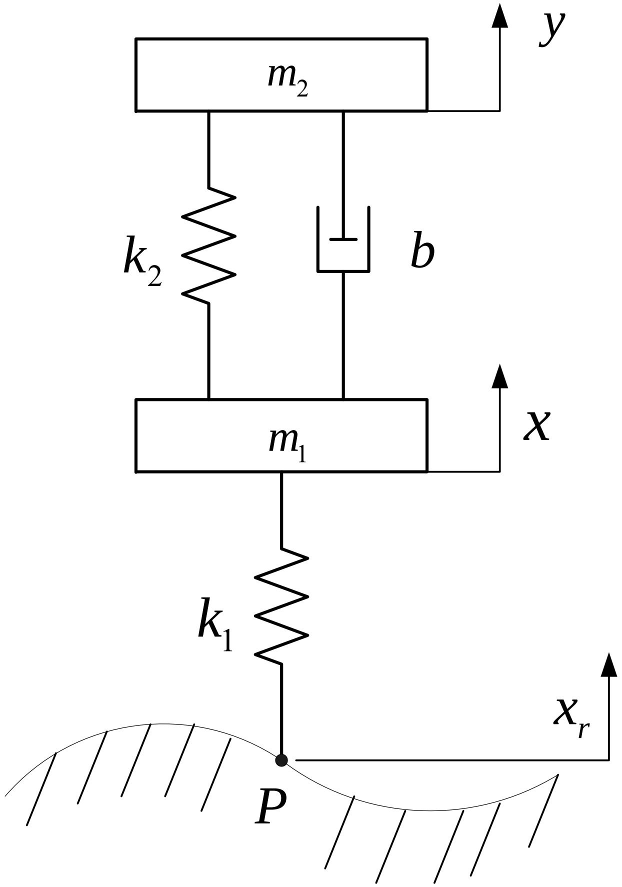
$$\dot{x}_3 = \dot{x}_r = u$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{k}{m} & -\frac{b}{m} & \frac{k}{m} \\ 0 & 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ \frac{b}{m} \\ 1 \end{bmatrix}u$$

VDCL Vehicle Dynamics & Control Laboratory

Vehicle Suspension

Ex 2) Another Quarter Car Model (2 DOF ¼ Car model)



Applying the Newton's second law to the system, we obtain

$$m_1\ddot{x} = -k_2(x-y) - b(\dot{x}-\dot{y}) - k_1(x-x_r)$$
$$m_2\ddot{y} = k_2(x-y) + b(\dot{x}-\dot{y})$$

Hence we have

$$m_1\ddot{x} + b\dot{x} + (k_1+k_2)x = b\dot{y} + k_2 y + k_1 x_r$$
$$m_2\ddot{y} + b\dot{y} + k_2 y = b\dot{x} + k_2 x$$

Taking Laplace Transform

$$(m_1 s^2 + bs + k_1 + k_2)X(s) = (bs+k_2)Y(s) + k_1 X_r(s)$$
$$(m_2 s^2 + bs + k_2)Y(s) = (bs+k_2)X(s)$$

Eliminating X(s) from the last two equations, we have

$$\frac{Y(s)}{X_r(s)} = \frac{k_1(bs+k_2)}{m_1 m_2 s^4 + (m_1+m_2)bs^3 + \left[k_1 m_2 + (m_1+m_2)k_2\right]s^2 + k_1 bs + k_1 k_2}$$

Ex 2) Another Quarter Car Model (2 DOF ¼ Car model)

State Equation :

let

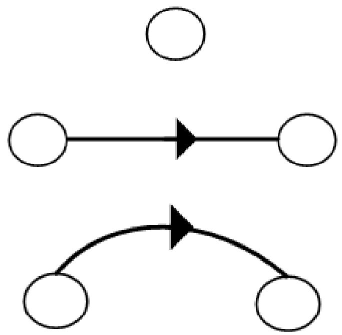
$$
\begin{aligned}
x_1 &= y - x \\
x_2 &= \dot{y} \\
x_3 &= x - x_r \\
x_4 &= \dot{x}
\end{aligned}
$$

$$
\begin{aligned}
\dot{x}_1 &= \dot{y} - \dot{x} = x_2 - x_4 \\
x_2 &= \ddot{y} = -\frac{k_2}{m_2} x_1 - \frac{b}{m_2}\left(x_2 - x_4\right) \\
\dot{x}_3 &= \dot{x} - \dot{x}_r = x_4 - \dot{x}_r \\
\dot{x}_4 &= \ddot{x} = \frac{k_2}{m_1} x_1 + \frac{b}{m_1}\left(x_2 - x_4\right) - \frac{k_1}{m_1} x_3
\end{aligned}
$$

$$
\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -\dfrac{k_2}{m_2} & -\dfrac{b}{m_2} & 0 & -\dfrac{b}{m_2} \\ 0 & 0 & 0 & 1 \\ \dfrac{k_2}{m_1} & \dfrac{b}{m_1} & -\dfrac{k_1}{m_1} & -\dfrac{b}{m_1} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} u \qquad u = \dot{x}_r
$$

Signal Flow

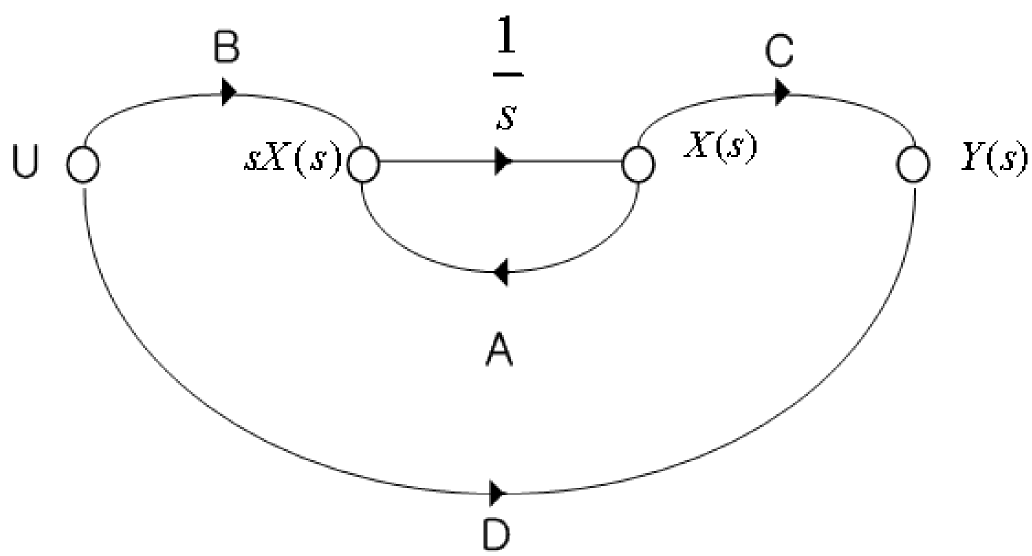
$$\dot{x}(t) = Ax + Bu$$
$$y = Cx + Du$$



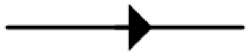
: signal

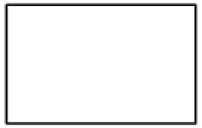
: relation dynamic $\left(\frac{1}{s}, z^{-1}\right)$

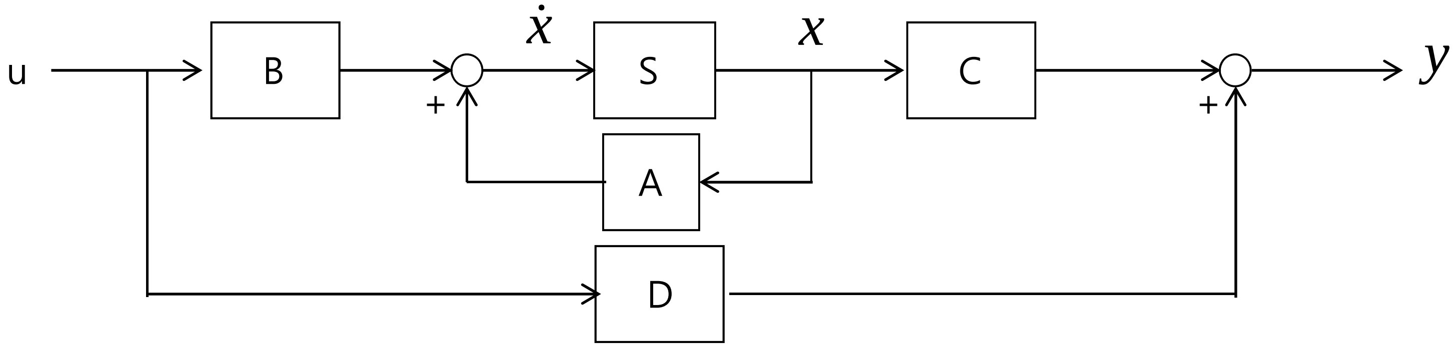
: static relation

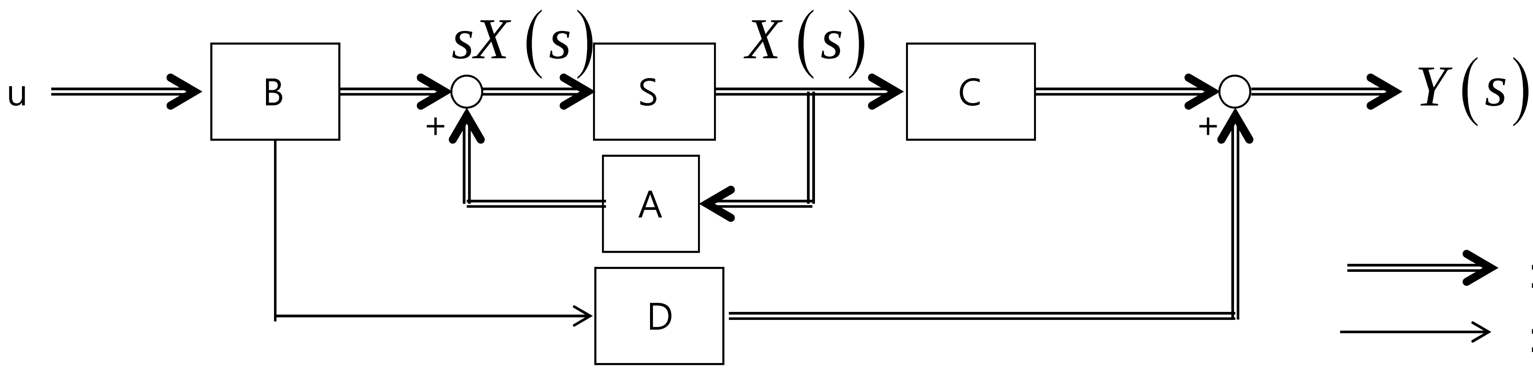


Block Diagram


: Signals on Line


: Transfer function inside the box (block)





: vector

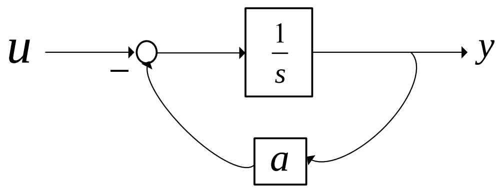
: scalar

Block Diagram

ex1. First order

$$\dot{y} + ay = u$$

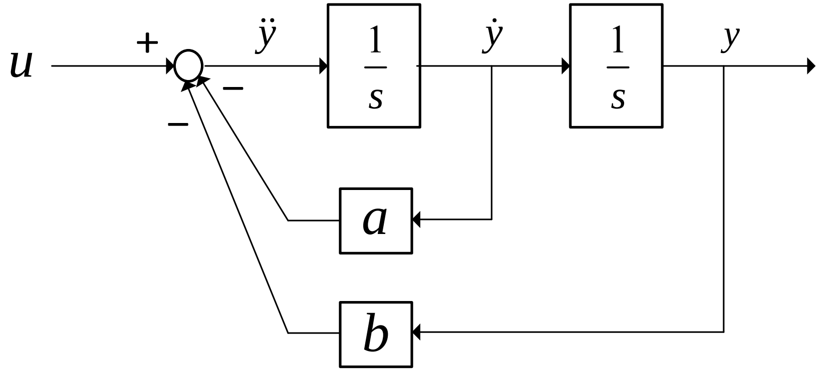
$$\frac{Y}{U} = \frac{1}{s+a}$$



ex2. Second order

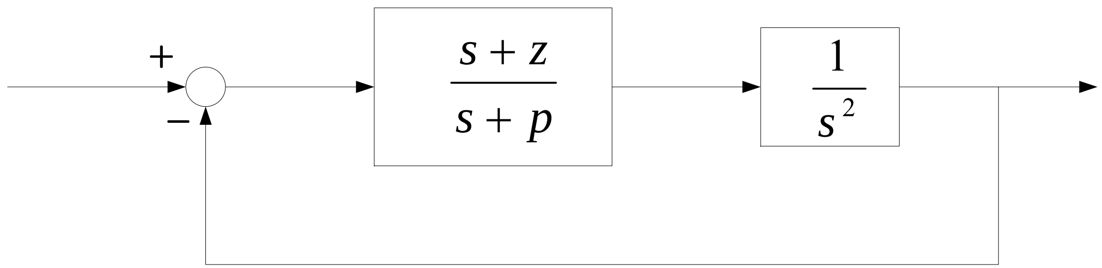
$$\ddot{y} + a\dot{y} + by = u$$

$$\frac{Y}{U} = \frac{1}{s^2 + as + b} = \frac{1}{s(s+a)+b}$$



Block Diagram

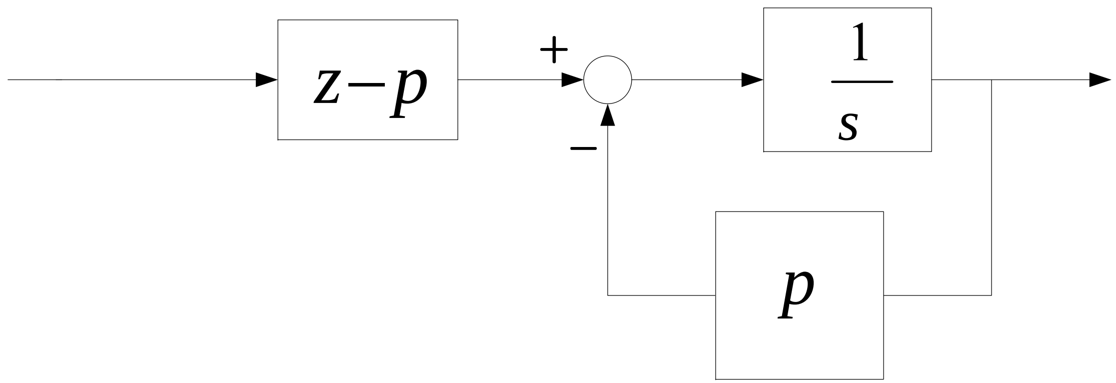
ex3.



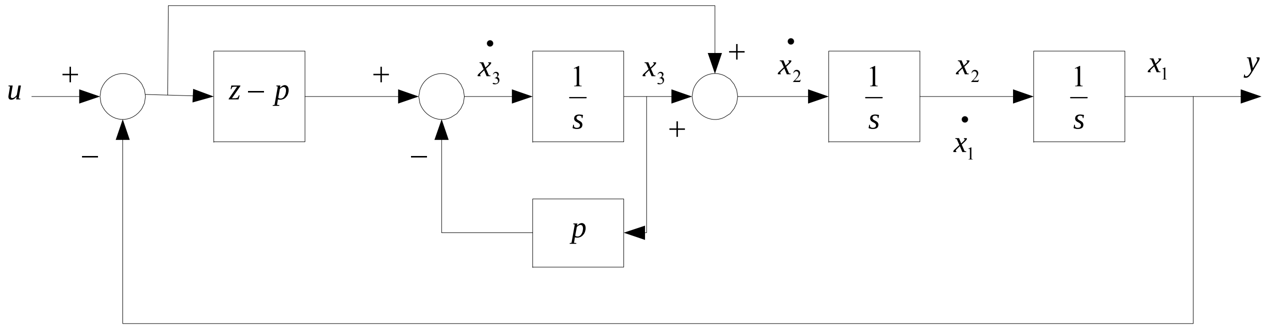
$$\frac{s+z}{s+p} = \frac{s+p+z-p}{s+p} = 1 + \frac{z-p}{s+p}$$

let $\quad \frac{y}{u} = \frac{z-p}{s+p}$

then $\quad sy + py = (z-p)u$



Block Diagram



Thus,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3 + (u - x_1)$$
$$\dot{x}_3 = (z - p)\cdot(u - x_1) - px_3$$

The state representation is as follows

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ p - z & 0 & -p \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ z - p \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x_1$$

Canonical Forms

- **Canonical Forms (T.F. → State Eq.)**

Controllable Canonical Form	← Direct Programming Method
Observable Canonical Form	← Nested Programming Method
Diagonal (Jordan) Canonical Form	← Partial Fraction Expansion

- **Controllable Canonical Form**

$$G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

Ex) n=3

$$G(s) = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

$$= b_0 + \frac{(b_1 - b_0 a_1) s^2 + (b_2 - b_0 a_2) s + (b_3 - b_0 a_3)}{s^3 + a_1 s^2 + a_2 s + a_3}$$

$$\begin{cases} b_1' = b_1 - b_0 a_1 \\ b_2' = b_2 - b_0 a_2 \\ b_3' = b_3 - b_0 a_3 \end{cases}$$

Canonical Forms

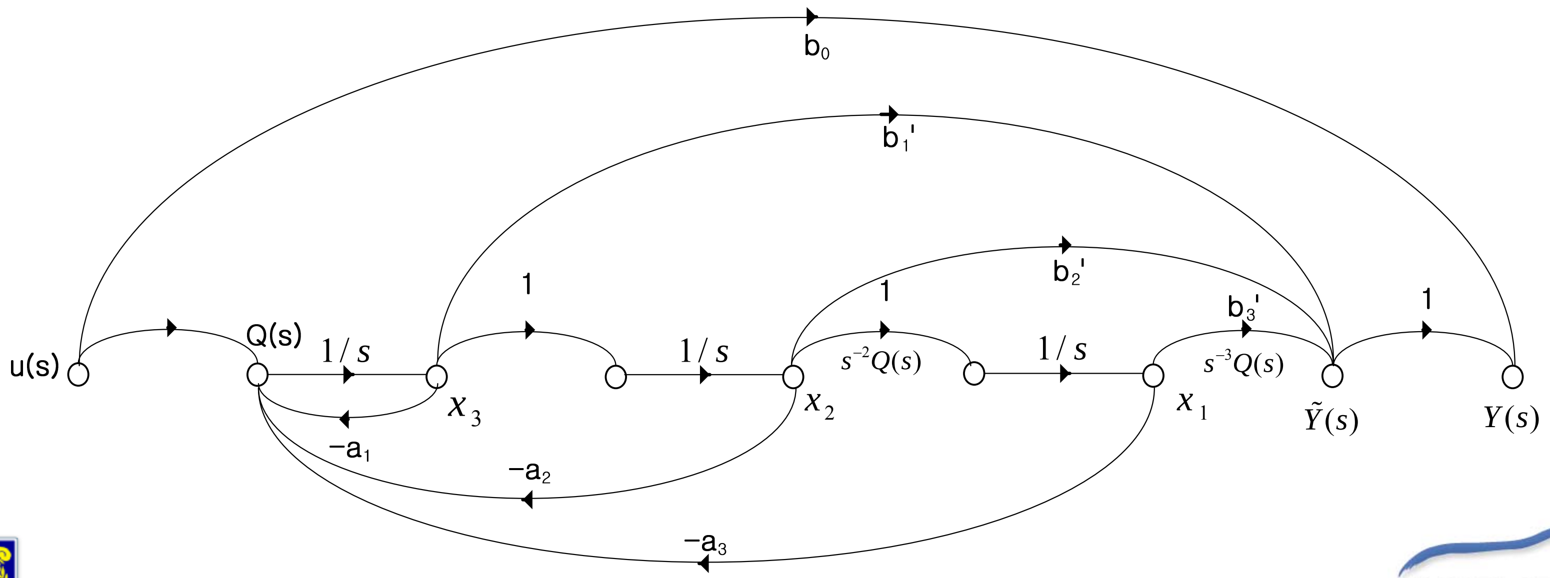
$$\tilde{Y}(s) = \left(\frac{b_1' s^2 + b_2' s + b_3'}{s^3 + a_1 s^2 + a_2 s + a_3} \right) u(s) \times \frac{s^{-3}}{s^{-3}} \qquad Y(s) = b_0 u(s) + \tilde{Y}(s)$$

$$Y(s) = \left[C\left(sI - A\right)^{-1} B + D \right] u(s)$$

$$\Rightarrow \frac{\tilde{Y}(s)}{b_1' s^{-1} + b_2' s^{-2} + b_3' s^{-3}} = \frac{u(s)}{1 + a_1 s^{-1} + a_2 s^{-2} + a_3 s^{-3}} = Q(s)$$

$$Q(s) = u(s) - a_1 s^{-1} Q(s) - a_2 s^{-2} Q(s) - a_3 s^{-3} Q(s)$$

$$\tilde{Y}(s) = \left(b_1' s^{-1} + b_2' s^{-2} + b_3' s^{-3} \right) Q(s)$$



Canonical Forms

- **Controllable Canonical Form**

$$\frac{dx_1}{dt} = x_2$$
$$\frac{dx_2}{dt} = x_3$$
$$\frac{dx_3}{dt} = -a_3 x_1 - a_2 x_2 - a_1 x_3 + u$$
$$y = b_3' x_1 + b_2' x_2 + b_1' x_3 + b_0 u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_3' & b_2' & b_1' \end{bmatrix} x + \begin{bmatrix} b_0 \end{bmatrix} u$$

Canonical Forms

- **Observable Canonical Form (Nested Programming)**

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = x_3$$

$$\frac{dx_3}{dt} = -a_3 x_1 - a_2 x_2 - a_1 x_3 + u$$

$$y = b_3' x_1 + b_2' x_2 + b_1' x_3 + b_0 u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_1' \\ b_2' \\ b_3' \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} b_0 \end{bmatrix} u$$

(Note : x in the controllable canonical form $\neq$ x in the observable canonical form)

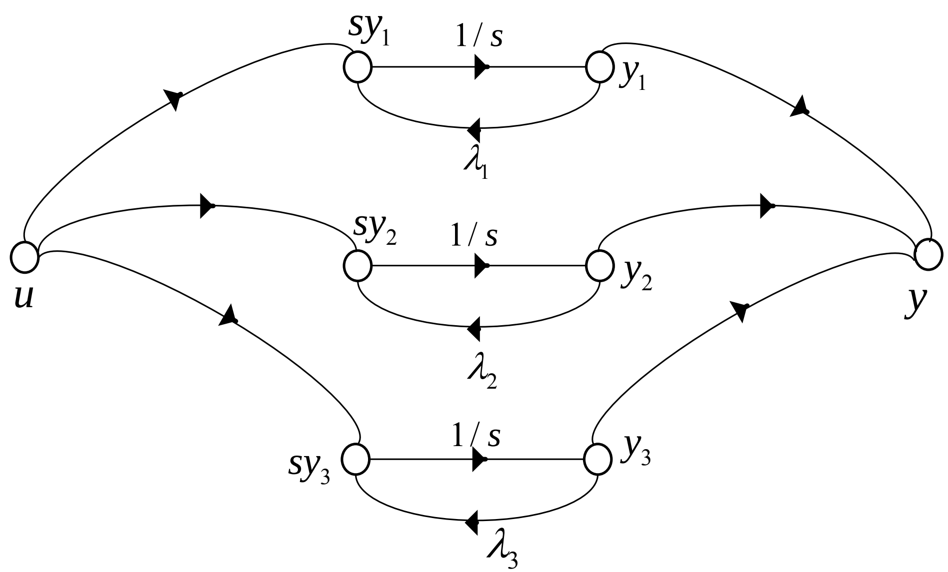
Canonical Forms

- **Diagonal (or Jordan) Canonical Form** **(Partial Fraction Expansion)**

Case 1. Distinct Roots $\left(\lambda_1 \neq \lambda_2 \neq \lambda_3\right)$

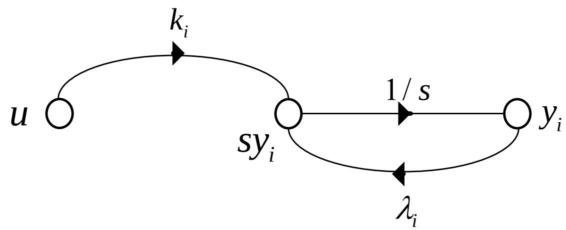
$$G(s) = \frac{B(s)}{A(s)} = \frac{K_1}{s-\lambda_1} + \frac{K_2}{s-\lambda_2} + \frac{K_3}{s-\lambda_3}$$

$$Y(s) = \sum_{i=1}^{3} \frac{K_i}{s-\lambda_i} u(s) = y_1 + y_2 + y_3$$



$$y_i = \frac{K_i}{s-\lambda_i} u$$

$$sy_i = \lambda_i y_i + K_i u$$



let $x_1 = y_1, x_2 = y_2, x_3 = y_3$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x$$

Canonical Forms

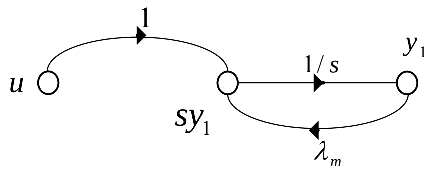
• Diagonal (or Jordan) Canonical Form (Partial Fraction Expansion)

Case 2. Multiple Roots

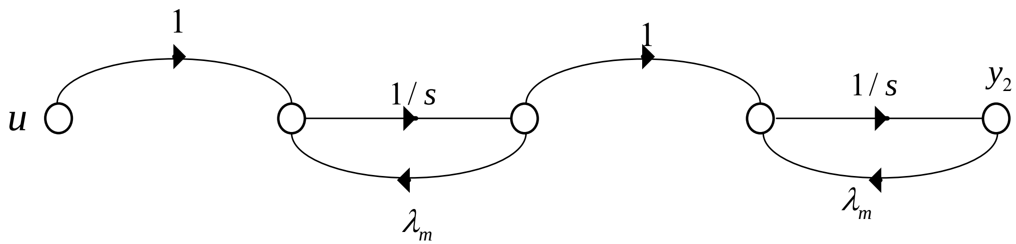
$$G(s) = \frac{B(s)}{(s-\lambda_m)^3} = \frac{K_1}{s-\lambda_m} + \frac{K_2}{(s-\lambda_m)^2} + \frac{K_3}{(s-\lambda_m)^3} \qquad \lambda_1 = \lambda_2 = \lambda_3 = \lambda_m$$

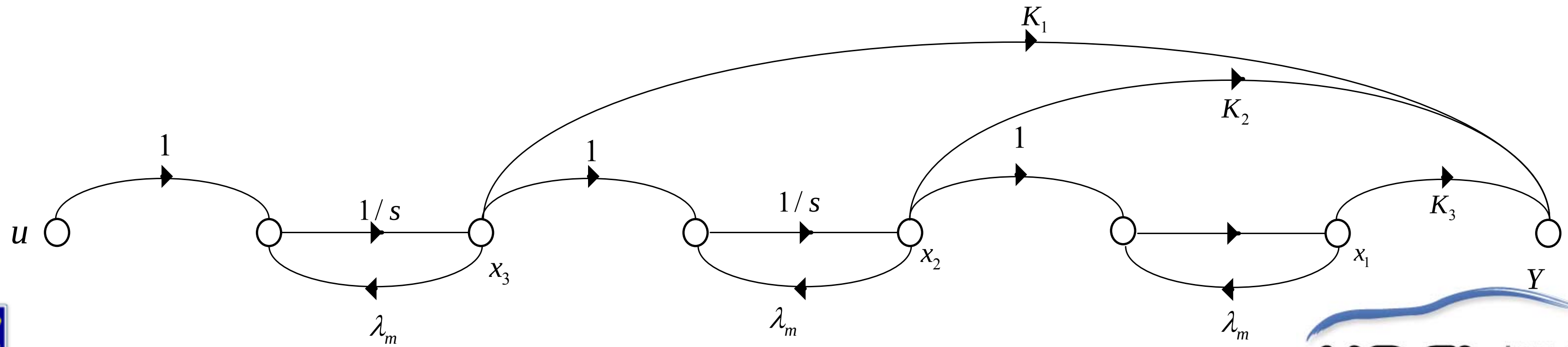
$$y_1 = \frac{1}{s-\lambda_m}u$$

$$sy_1 = \lambda_m y_1 + u$$



$$y_2 = \frac{1}{(s-\lambda_m)^2}u$$





Canonical Forms

- **Diagonal (or Jordan) Canonical Form** **(Partial Fraction Expansion)**

Case 2. Multiple Roots

$$\dot{x}_1 = \lambda_m x_1 + x_2$$
$$\dot{x}_2 = \lambda_m x_2 + x_3$$
$$\dot{x}_3 = \lambda_m x_3 + u$$
$$y = K_3 x_1 + K_2 x_2 + K_1 x_3$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} K_3 & K_2 & K_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Canonical Forms

• Diagonal (or Jordan) Canonical Form (Partial Fraction Expansion)

Case 3. Complex Roots

$$G(s) = \frac{1}{s^2 - 2\sigma s + \sigma^2 + \omega^2} = \frac{1}{\left[s - (\sigma + j\omega)\right]\left[s - (\sigma - j\omega)\right]}$$

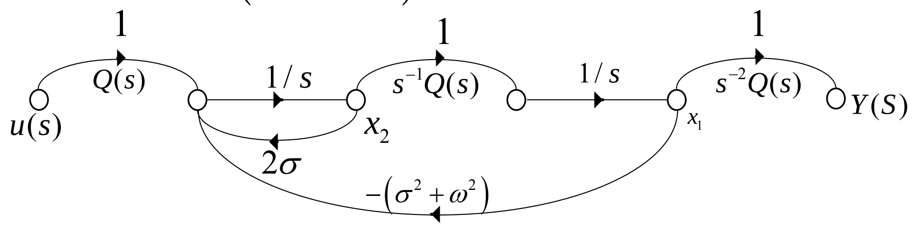
$$\frac{Y}{u} = G(s) = \frac{1}{s^2 - 2\sigma s + \sigma^2 + \omega^2}$$

$$= \frac{s^{-2}}{1 - 2\sigma s^{-1} + \left(\sigma^2 + \omega^2\right) s^{-2}}$$

$$\frac{Y}{s^{-2}} = \frac{u}{1 - 2\sigma s^{-1} + \left(\sigma^2 + \omega^2\right) s^{-2}} = Q(s)$$

$$\begin{cases} Y = s^{-2} Q(s) \\ Q(s)\left(1 - 2\sigma s^{-1} + \left(\sigma^2 + \omega^2\right) s^{-2}\right) = u \end{cases}$$

$$Q(s) = u + 2\sigma s^{-1} Q(s) - \left(\sigma^2 + \omega^2\right) s^{-2} Q(s)$$



Canonical Forms

- **Diagonal (or Jordan) Canonical Form** **(Partial Fraction Expansion)**

Case 3. Complex Roots

$$AP = P\Lambda$$

Note : Complex Roots, Complex State x

$$\dot{x} = \Lambda x + bu$$
$$y = Cx$$

→ Complex case의 diagonalization 방법 이용

$$\Lambda K = KJ$$

$$\Lambda = \begin{bmatrix} \sigma + j\omega & 0 \\ 0 & \sigma - j\omega \end{bmatrix} \qquad K = \begin{bmatrix} \frac{1}{2} & -\frac{j}{2} \\ \frac{1}{2} & \frac{j}{2} \end{bmatrix} \qquad K^{-1} = \frac{2}{j}\begin{bmatrix} \frac{j}{2} & \frac{j}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$J = K^{-1}\Lambda K$$
$$= K^{-1}P^{-1}APK \qquad J = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

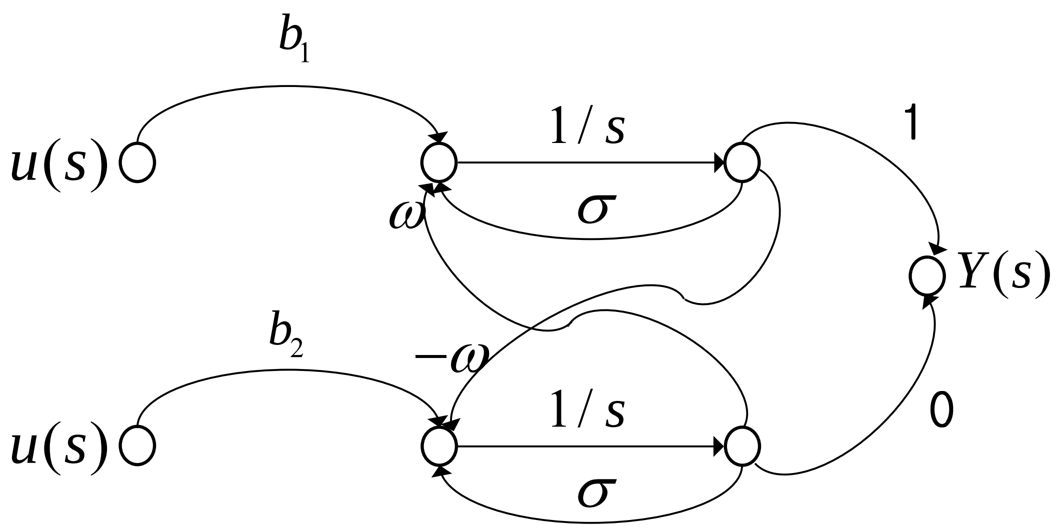
Canonical Forms

- **Diagonal (or Jordan) Canonical Form (Partial Fraction Expansion)**

Case 3. Complex Roots

Ex) $$\dot{z} = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} z + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} z$$



Canonical Forms

- **Diagonal (or Jordan) Canonical Form (Partial Fraction Expansion)**

Case 3. Complex Roots

Step 1 $\dot{x} = Ax + Bu$

Step 2 let $x = P\xi$

$$\dot{\xi} = \underbrace{P^{-1}AP}_{\Lambda}\,\xi + P^{-1}Bu$$

: diagonal

Step 3 let $\xi = Kz$

$$\dot{z} = \underbrace{K^{-1}\Lambda K}_{J}\,z + K^{-1}P^{-1}Bu$$

$$= \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} z + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u$$

$$\begin{cases} x = P\xi = PKz \\ \dot{z} = \underbrace{K^{-1}\Lambda K}_{J}\,z + K^{-1}P^{-1}bu \\ y = CPKz \end{cases}$$