System Control

4. Transient and Steady-State Response Analysis

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Systems

- **Linear Time Invariant System**

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Where A,B,C and D are constant matrix

- **Linear Time Varying System**

$$\dot{x} = A(t)x + B(t)u$$
$$y = C(t)x + D(t)u$$

- **Nonlinear System**

$$\dot{x} = f(x(t), y(t), u(t), t)$$
$$y = h(x(t), u(t), y(t), t)$$

Time Invariant System

- **The Laplace Transform of Linear Time Invariant System**

$$sX(s) - x(0) = AX(s) + BU(s)$$
$$sX(s) - AX(s) = x(0) + BU(s)$$
$$(sI - A)X(s) = x(0) + BU(s)$$
$$\therefore X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$

Transfer function is derived from zero-initial condition

$$Y(s) = [C(sI - A)^{-1}B + D]U(s)$$

, thus the Transfer function G(s) is

$$\therefore G(s) = C(sI - A)^{-1}B + D$$

In general the Transfer function is expressed as follows

$$\frac{Y(s)}{U(s)} = G(s) = \frac{b_1 s^m + \cdots + b_{m+1}}{s^n + a_1 s^{n-1} + \cdots}(m \le n)$$
$$= \frac{k_1}{s + p_1} + \frac{k_2}{s + p_2} + \frac{c_1 s + c_2}{s^2 + as + b} + \cdots$$

System Response

- **Transient response**

  - Response goes from the initial state to the final state

- **Steady state response**

  -The manner in which the system output behaves as t approaches infinity

let $G(s) = \dfrac{Q(s)}{P(s)}$ then

$P(s) = 0$: the characteristic equation
s_i : such that $P(s) = 0$ is characteristic roots or poles
$Q(s) = 0$: such that s_k are called zeros

$$Y(s) = G(s)U(s)$$

➔ Partial Fraction ={ G(s) terms }+ U(s)
➔ Poles $s_1, s_2, s_3(real), \sigma_1 \pm j\omega_1, \sigma_2 \pm j\omega_2$
➔ Then the transient response becomes
➔ $C_1 e^{s_1 t} + C_2 e^{s_2 t} + \cdots + D_1 e^{\sigma_1 t} \sin \omega_1 t + \cdots$

Stability

- **Stable**

If $\lim_{t \to \infty} x(t) = 0$ with no zero initial condition

A linear time invariant system is "stable" if the output eventually comes back to equilibrium state when the system is subject to an initial condition

- **Equilibrium :** $\dot{x} = 0$

With no disturbance and input, the output stays in the same state, which is called equilibrium.

- **Stable condition**

$\mathrm{Re}(s_i) < 0$ for all s_i , where s_i is poles

- **Critically stable**

Oscillations of the output continue forever some $\mathrm{Re}(s_i) = 0$

- **Unstable**

The output diverges without bound from its equilibrium state (when the system subjected to an initial conditions)

Stability

- **Absolute Stability**

Whether the system is stable or unstable

- **Relative Stability**

- Transient response
- Damped Oscillation

- **Steady-state Error**

The output does not exactly agree with the input
(Concerned with the Accuracy of the system)

First Order System

- **First Order System** $$G(s)=\frac{C(s)}{R(s)}=\frac{1}{Ts+1}$$

When $R(s)=\frac{1}{s}$; step input (r(t)=u(t))

$$c(s)=\frac{1}{Ts+1}\cdot\frac{1}{s}=\frac{-T}{Ts+1}+\frac{1}{s}$$

$$c(t)=1-\exp(-\frac{1}{T}t) \qquad \dot{c}(t)=\frac{1}{T}e^{-\frac{1}{T}t}$$

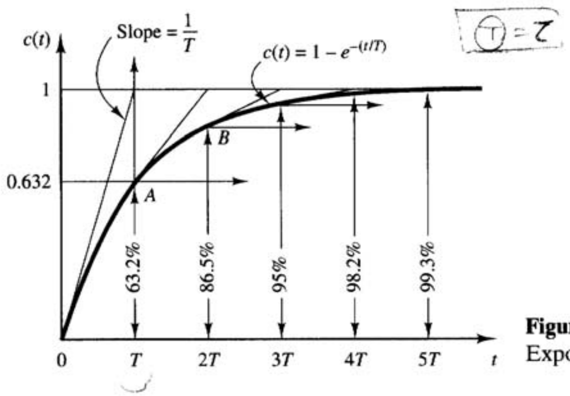


Figure 4–2 Exponential response curve.

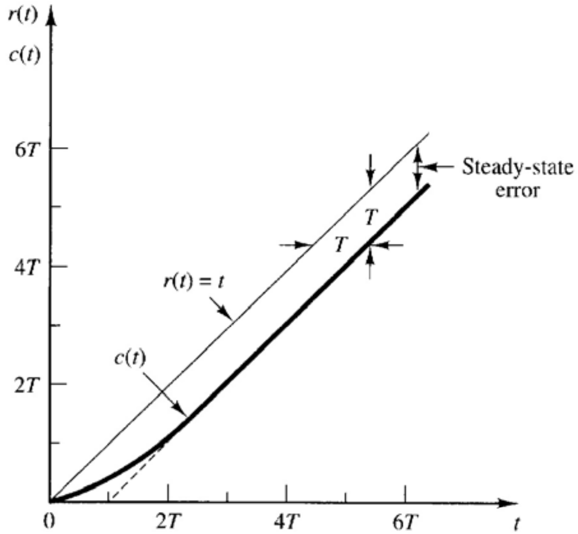
$$G(s)=\frac{b}{s+a}=\frac{b}{a}\left(\frac{1}{\frac{1}{a}s+1}\right)$$

for First Order system, the time constant is $\frac{1}{a}$

T : time constant of First order system
For large T : 응답이 느리다
For small T : 응답이 빠르다

First Order System

When $R(s) = \frac{1}{s^2}$; unit ramp input, that is, r(t)=t



$$c(s) = \frac{1}{Ts+1} \cdot \frac{1}{s^2} = \frac{T^2}{Ts+1} + \frac{1}{s^2} - \frac{T}{s}$$

$$\therefore c(t) = Te^{-t/T} + t - T$$

Second Order System

- **Second Order System**

$$\frac{C(s)}{R(s)} = G(s) = \frac{b}{s^2 + as + b}$$

where $a = 2\zeta\omega_n \quad b = \omega_n^2$

$$= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Note that poles : $-\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}\,j$

Unit step response

for $R(s) = \frac{1}{s}$; step input

$$C(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s} = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

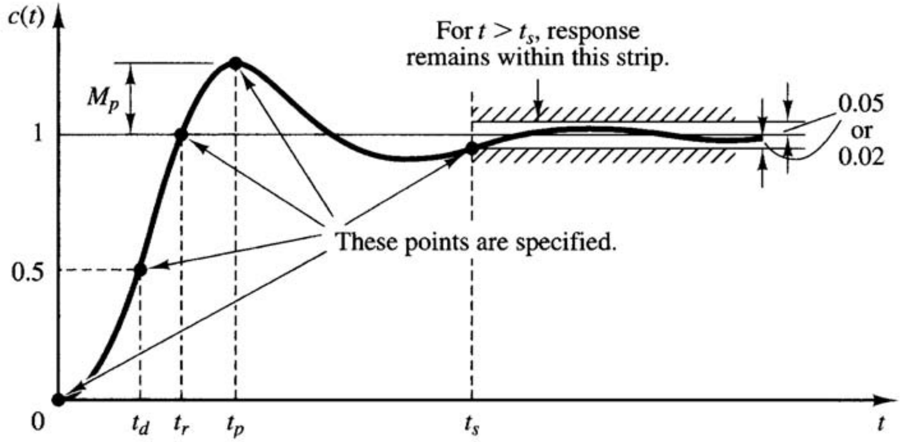
$$\therefore c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta})$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$: Damped natural frequency

ω_n : Natural frequency

ζ : Damping ration

Second Order System



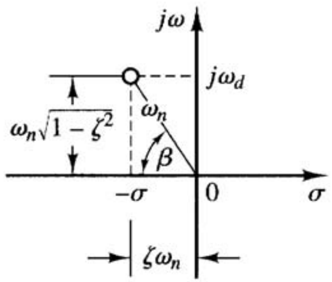


Figure 4–13
Definition of the angle β.

1. Rise time t_r : 10% → 90%
   5% → 95%

2. Max. Overshoot, M_p

3. Settling time, t_s : 2% criterion $t_s = 4/\omega_n\zeta$
   5% criterion $t_s = 3/\omega_n\zeta$

4. Delay time, t_d = 50%

5. Peak time, t_p

Second Order System

- **Step Response of Second Order System**

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

(1) Under damped $: 0 < \zeta < 1$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)} \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

Step response $R(s) = \frac{1}{s}$

$$c(t) = L^{-1}[C(s)] = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\cos\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right)$$

Second Order System

- **Step Response of Second Order System**

(2) Critically damped : $\zeta = 1$

$$C(s) = \frac{\omega_n^2}{(s+\omega_n)^2 s}$$

$$c(t) = 1 - e^{-\omega_n t}(1+\omega_n t)$$

(3) Critically damped : $\zeta > 1$

$$C(s) = \frac{\omega_n^2}{\left(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}\right)\left(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}\right)s}$$

$$c(t) = 1 + \frac{1}{2\sqrt{\zeta^2-1}\left(\zeta+\sqrt{\zeta^2-1}\right)}e^{-\left(\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} - \frac{1}{2\sqrt{\zeta^2-1}\left(\zeta-\sqrt{\zeta^2-1}\right)}e^{-\left(\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}$$

$$= 1 + \frac{1}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right) \qquad s_1 = \left(\zeta+\sqrt{\zeta^2-1}\right)\omega_n \quad s_2 = \left(\zeta-\sqrt{\zeta^2-1}\right)\omega_n$$

$$|s_1| << |s_2|$$

The effect of $-s_1$ on the response is much smaller than that of $-s_2$

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Effect of Pole Locations

(1) First Order System

$$\frac{Y}{R} = G(s) = \frac{\sigma}{s+\sigma}; = \frac{1}{Ts+1}$$

Step response : $R(s) = \dfrac{1}{s}$

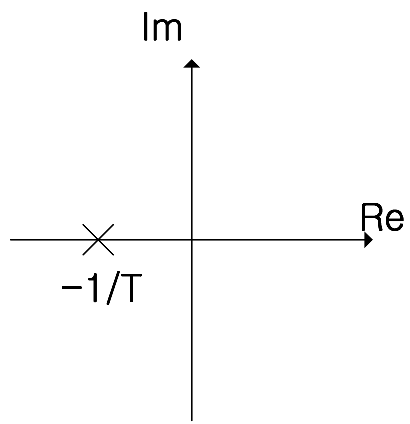
$$Y(s) = \frac{1}{Ts+1}\cdot\frac{1}{s}$$

$$= \frac{1}{s} - \frac{T}{Ts+1}$$

$$= \frac{1}{s} - \frac{1}{s+\left(1/T\right)}$$

$$y(t) = 1 - e^{-\frac{1}{T}t} \quad \text{for } t \geq 0$$

Pole : $s = -\sigma = -1/T$



Effect of Pole Locations

(2) Second Order System

$$\frac{Y}{R} = \frac{b}{s^2 + as + b} \qquad \text{where} \quad a = 2\zeta\omega_n \quad b = \omega_n^2$$

$$= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{\omega_n^2}{\left(s + \zeta\omega_n\right)^2 + \omega_n^2\left(1 - \zeta^2\right)}$$

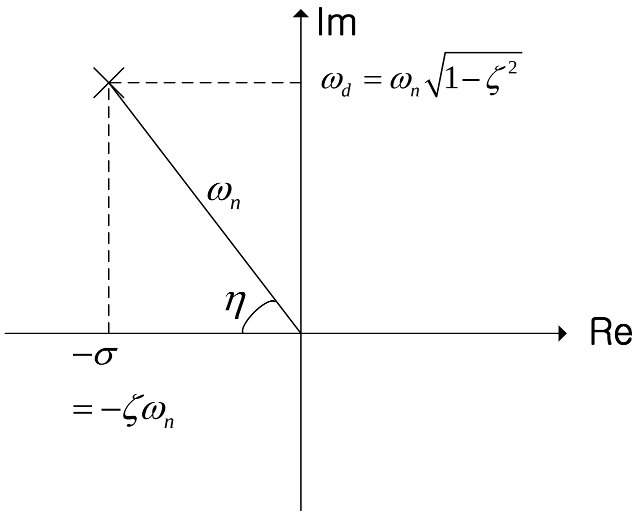
Poles : $-\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}\,j$

Step response : $R(s) = \frac{1}{s}$

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s}$$

$$= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$



ζ : damping ratio

ω_n : natural freq.

ω_d : damped freq.

$$\omega_n \cos\eta = \zeta\omega_n$$

$$\zeta = \cos\eta$$

Pole Locations and Transient Response (Impulse)

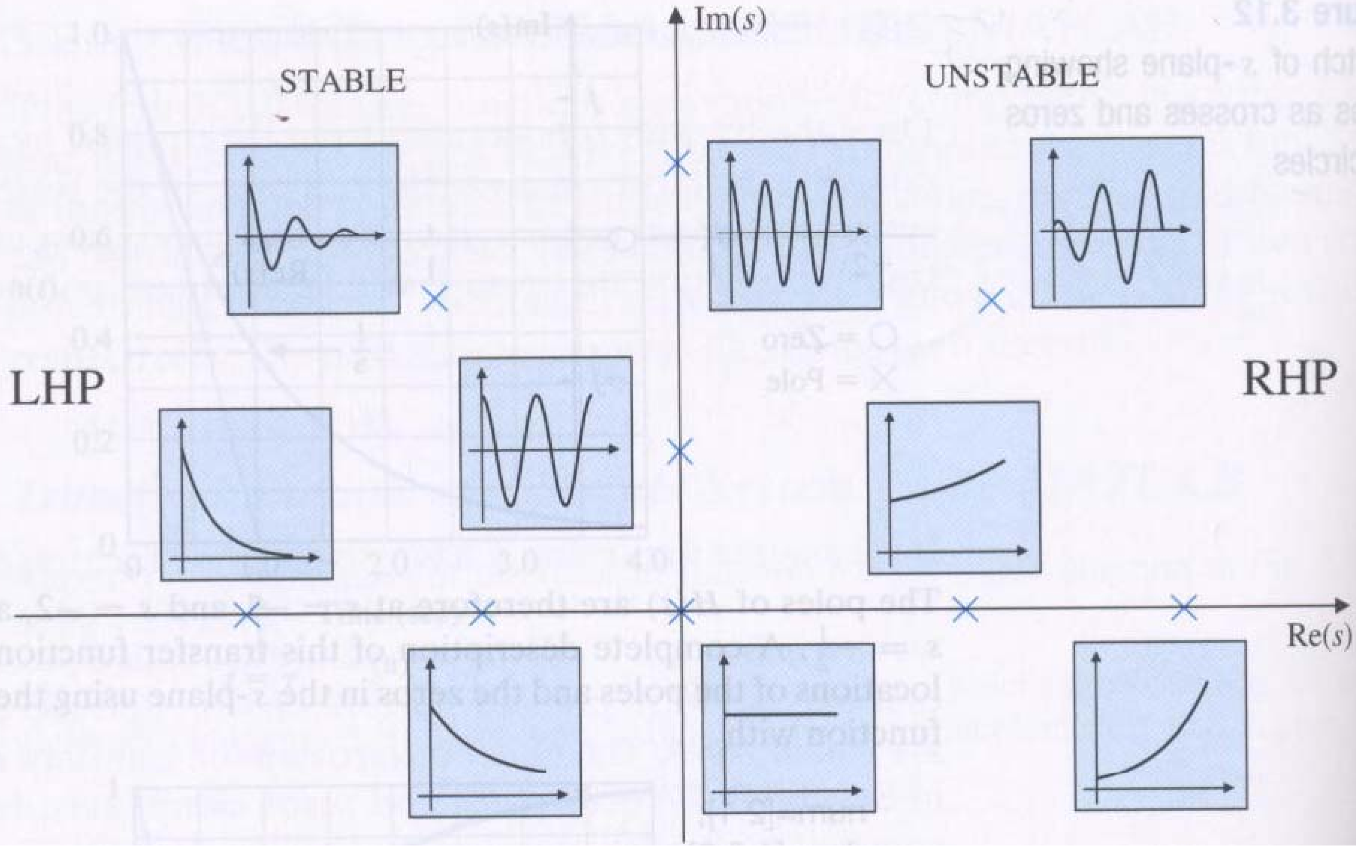


Figure 3.13
Time functions associated with points in the s-plane (LHP, left half-plane; RHP, right half-plane)

Effects of Zeros

1. The effect of zero near poles (cancel the pole response)

$$H_1(s) = \frac{2}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{2}{s+2}$$

$$H_2(s) = \frac{2(s+1.1)}{1.1(s+1)(s+2)} = \frac{2}{1.1}\left(\frac{0.1}{s+1} + \frac{0.9}{s+2}\right) = \frac{0.18}{s+1} + \frac{1.64}{s+2}$$

- If we put the zero exactly at s=-1, this term will vanish completely
- The coefficient of the term (s+1) has been modified from 2 in $H_1(s)$ to 0.18 in $H_2(s)$

In general, a zero near a pole reduces the amount of that term in the total response

coefficient $c_1(s) = (s - p_1)F(s)\big|_{s=p_1}$

zero near the pole P_1, F(s) will be small

Effects of Zeros

2. Effect of zeros on the transient response

Two complex poles and one zero

$$H(s)=\frac{(s/\alpha\zeta\omega_n)+1}{(s/\omega_n)^2+2\zeta(s/\omega_n)+1}=\frac{\dfrac{\omega_n}{\alpha\zeta}s+\omega_n{}^2}{s^2+2\zeta\omega_n s+\omega_n{}^2}$$

poles : $s=-\zeta\omega_n\pm\omega_n\sqrt{1-\zeta^2}\,j$

zero : $s=-\alpha\zeta\omega_n$

$\alpha\cong 1$: the value of the zero will be close to that of the real part of the poles

$\alpha\geq 3$: very little effect on Mp

$\alpha\leq 3$: increasing effect as α decreases below 3

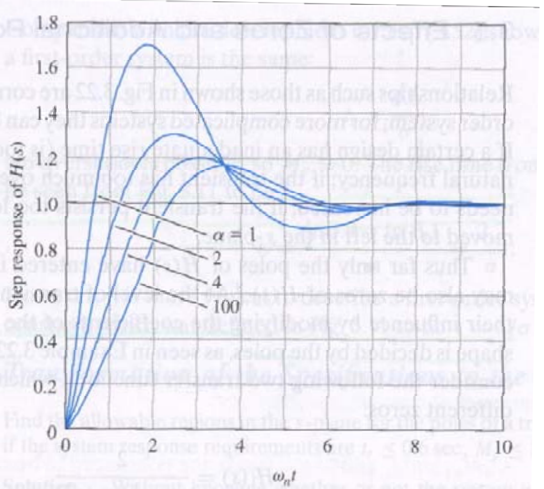


Figure 3.24 Plots of the step response of a second-order system with a zero ($\zeta = 0.5$)

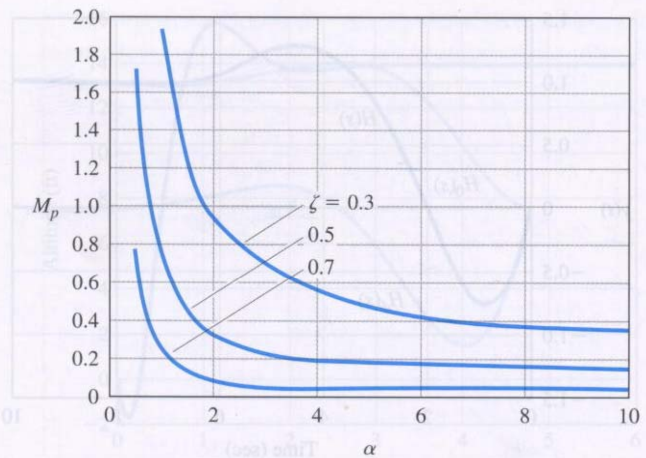


Figure 3.25 Plot of overshoot M_p as a function of normalized zero location α. At $\alpha = 1$, the real part of the zero equals the real part of the poles

Effects of Zeros

2. Effect of zeros (L.T. Analysis)

Replacing s/ω_n with s

$$H(s) = \frac{s/\alpha\zeta + 1}{s^2 + 2\zeta s + 1}$$

$$= \frac{1}{s^2 + 2\zeta s + 1} + \frac{1}{\alpha\zeta}\frac{s}{s^2 + 2\zeta s + 1}$$

$$= \underbrace{H_0(s)}_{h_0(t)} + \frac{1}{\alpha\zeta}\underbrace{H_d(s)}_{\frac{d}{dt}h_0(t)}$$

: produce overshoot

3. Nonminimum-phase zero

$\alpha<0$: the zero is in the RHP where s>0

; RHP zero

nonminimum-phase zero

Figure 3.26
Second-order step responses $y(t)$ of the transfer functions $H(s)$, $H_0(s)$, and $H_d(s)$

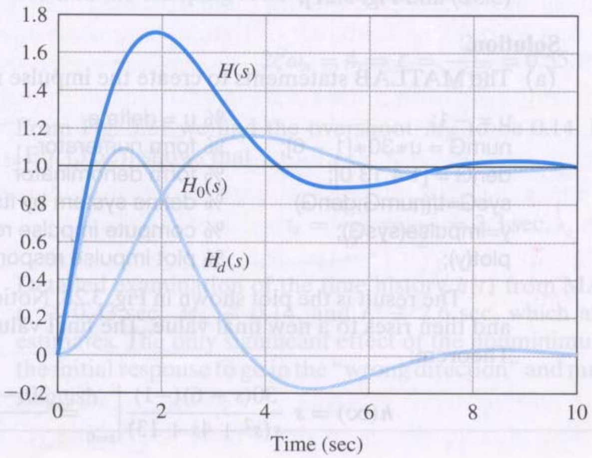


Figure 3.27
Step responses $y(t)$ of a second-order system with a zero in the RHP: a nonminimum-phase system

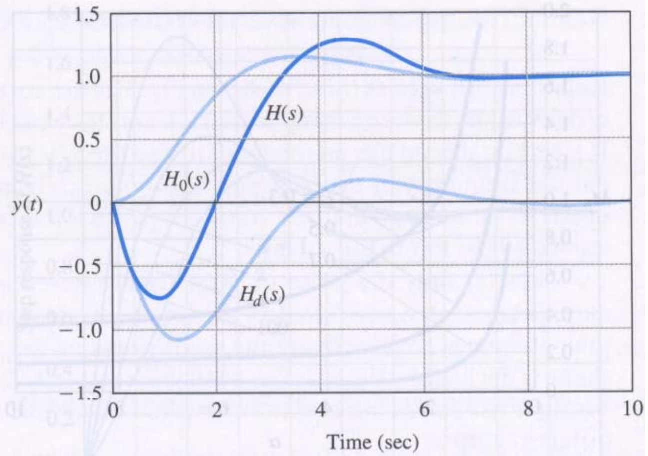
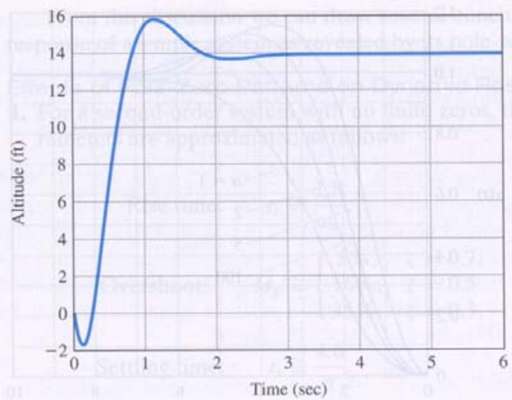


Figure 3.28
Response of an airplane's altitude to an impulsive elevator input



The Effect of an extra pole

- **Effect on the Standard Second-order step response**

$$H(s) = \frac{1}{\left(s/\alpha\zeta\omega_n + 1\right)\left[\left(s/\omega_n\right)^2 + 2\zeta\left(s/\omega_n\right) + 1\right]}$$

$s = -\alpha\zeta\omega_n$ α : big, far left poles

- DC gain of a system
  : the ratio of the output of a system to its input (presumed constant) after all transients have decayed

DC gain $= \lim_{s\to 0} s \cdot G(s)\frac{1}{s} = \lim_{s\to 0} G(s)$

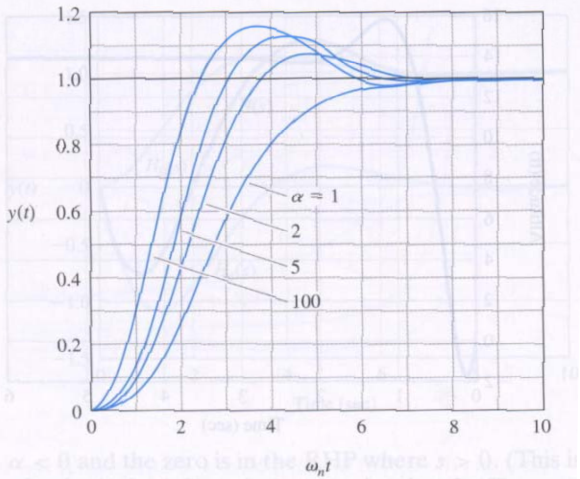


Figure 3.29
Step responses for several third-order systems with $\zeta = 0.5$

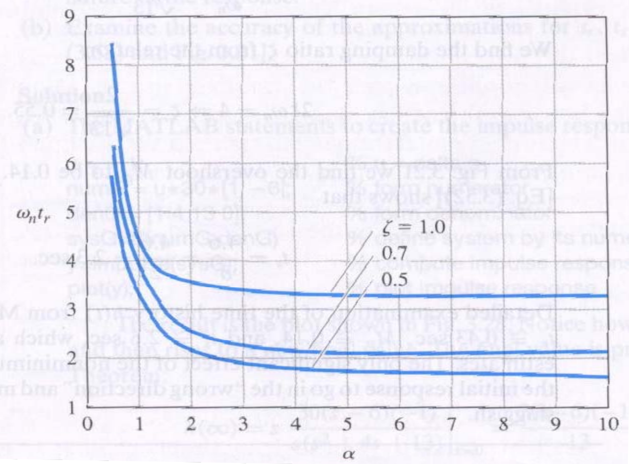


Figure 3.30
Normalized rise time for several locations of an additional pole

major effect : increase the rise time

Effect of Poles-Zeros on Dynamic System

1. 2nd order system with no finite zeros

Rise time : $t_r \cong \dfrac{1.8}{\omega_n}$ Overshoot : $M_p \cong \begin{cases} 5\%, & \zeta = 0.7 \\ 16\%, & \zeta = 0.5 \\ 35\%, & \zeta = 0.3 \end{cases}$

Settling time : $t_s \cong \dfrac{4.6}{\sigma}$ $\sigma = \zeta\omega_n$

2. A Zero in the LHP

Increase the overshoot
(if the zero is within a factor of 4 of the real part of the complex poles)

3. A Zero in the RHP (nonminimum-phase zero)

- Depress the overshoot
- May cause the step response to start out in the wrong direction

4. An additional pole in the LHP

- Increase the rise time significantly if the extra pole is within a factor of 4 of the real part of the complex poles