System Control

5. Basic Control Actions and Response of Control Systems

(1) PID Control

(2) Routh's stability tests

(3) Systems Type

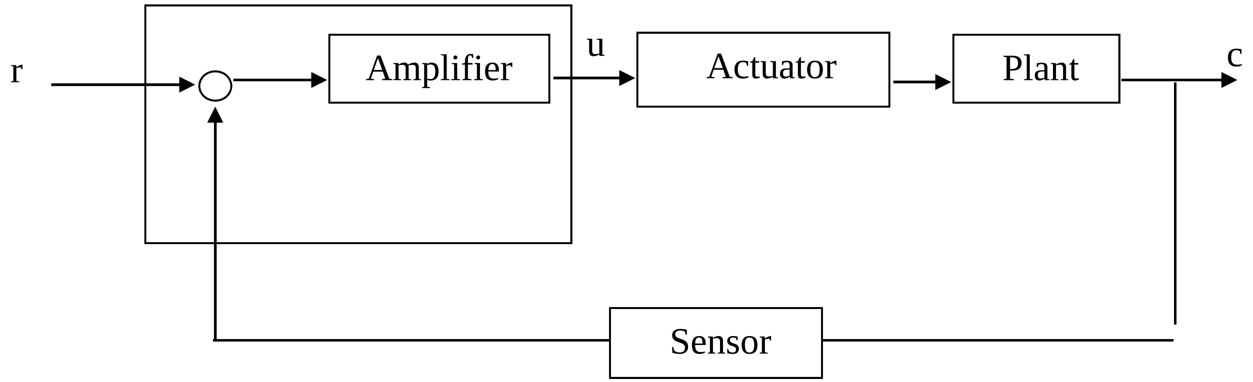
Open loop and closed loop : model sensitivity

Professor Kyongsu Yi

Vehicle Dynamics and Control Laboratory

Seoul National University

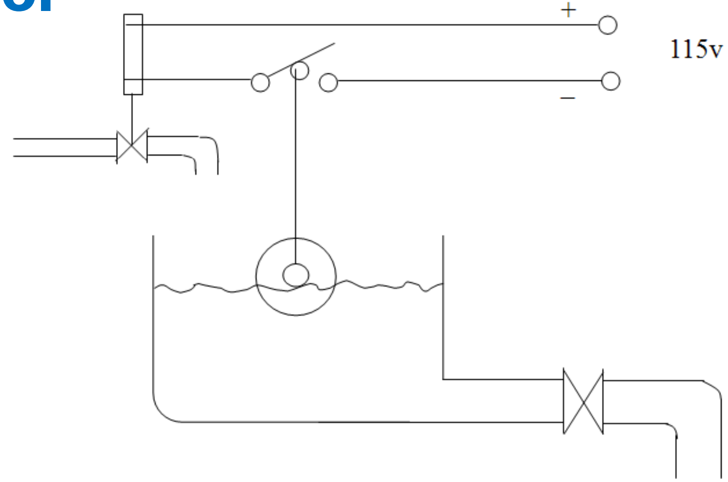
Control Systems

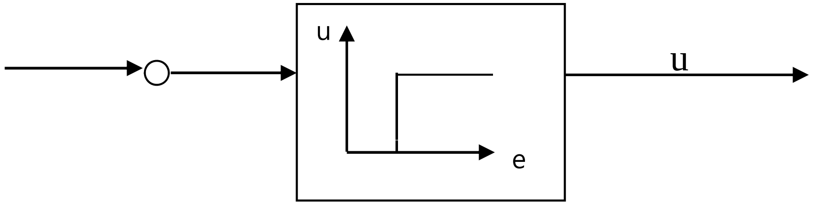


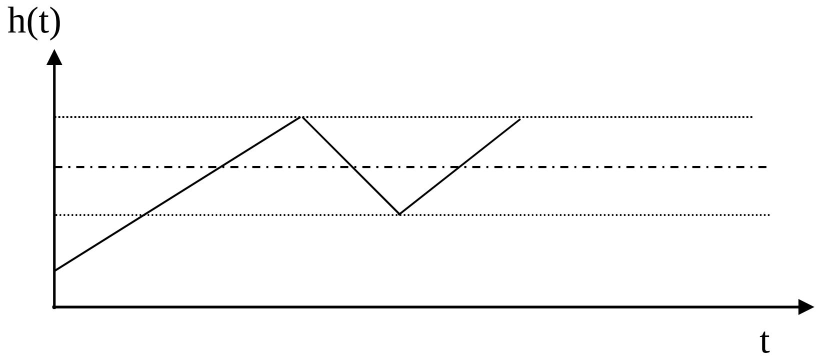
$$u=f(r,c)$$

Classifications of controllers

1. ON-OFF control



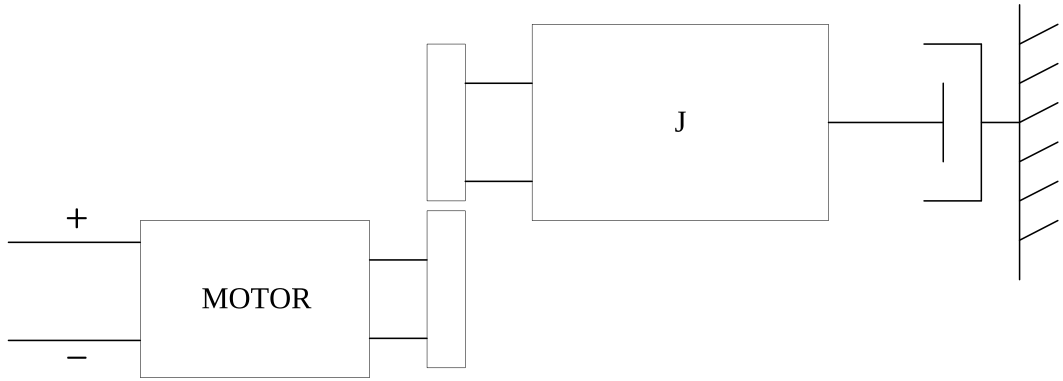


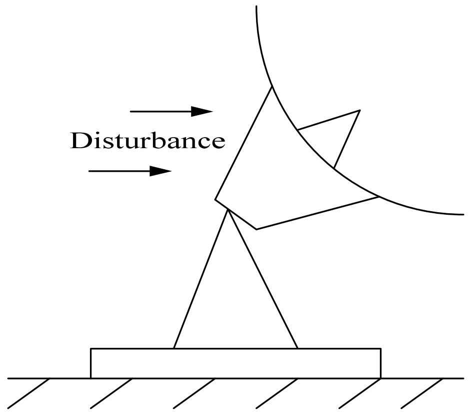


Classifications of controllers

2. Proportional Control

$U = k_p e$ k_p : The proportional Gain



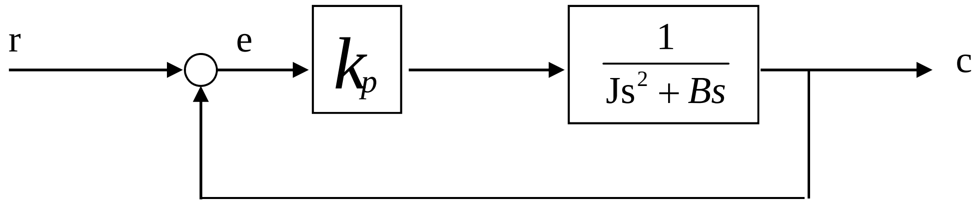


Antenna Position
Control System

Classifications of controllers

2. Proportional Control

$U = k_p e$ $\quad$ k_p : The proportional Gain



$$\frac{c(s)}{r(s)} = \frac{\dfrac{K_p}{Js^2 + Bs}}{1 + \dfrac{K_p}{Js^2 + Bs}} = \frac{K_p}{Js^2 + Bs + K_p}$$

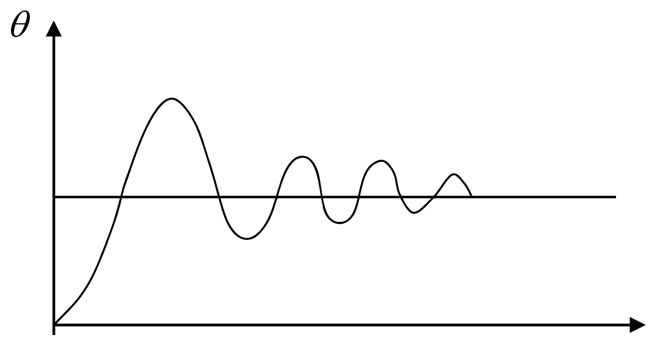
Classifications of controllers

2. Proportional Control

- Step input response $r(s) = \dfrac{1}{s}$

- steady state response $c(t) = \lim_{s \to 0} sC(s)$

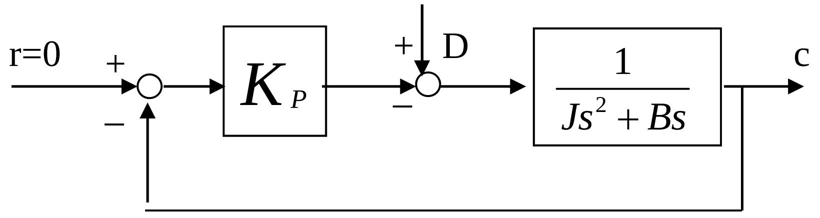
$$= \lim_{s \to 0} s \frac{K_p}{Js^2 + Bs + K_p} \frac{1}{s} = 1$$



Classifications of controllers

2. Proportional Control

- Response to torque disturbance



$$C(s)=\frac{K_p}{Js^2+Bs+K_p}r(s)+\frac{1}{Js^2+Bs+K_p}D(s)$$

Assume that $D(s)=\frac{1}{s}$ steady state response

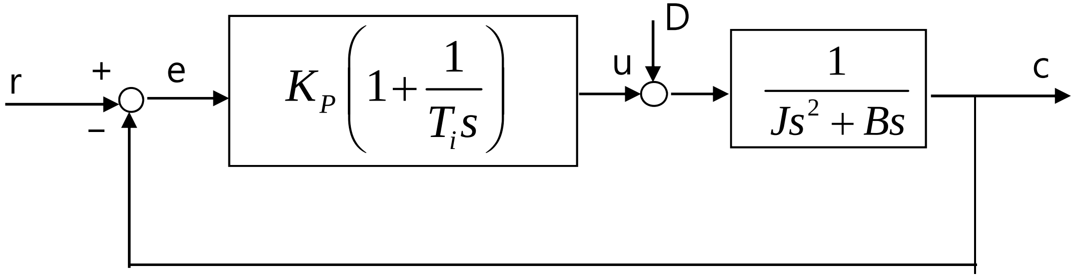
$$c(t)=\lim_{s->\infty} s\cdot\frac{1}{Js^2+Bs+K_p}\cdot\frac{1}{s}=\frac{1}{K_p}$$: steady-state error

Large K_p
→ small steady-state error
→ large motor power is needed
→ oscillations
→ large w_n $\left(w_n=\sqrt{\frac{K_p}{J}}\right)$
→ small damping ratio $\zeta=\frac{B}{2\sqrt{JK_p}}$

Classifications of controllers

3. Proportional-Integral Control (PI control)

- Response to torque disturbance



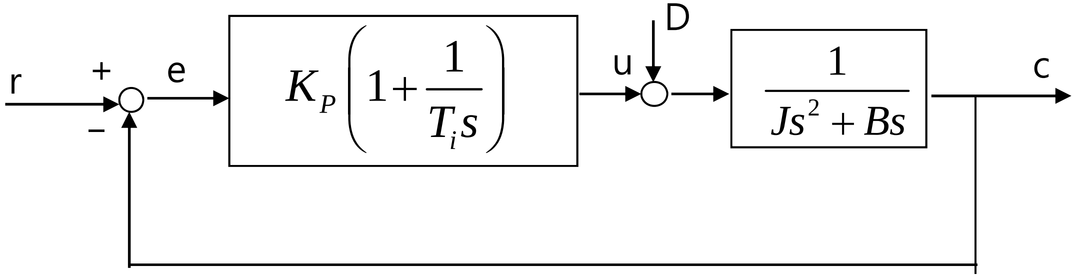
$$\frac{U(s)}{E(s)} = K_P\left(1+\frac{1}{T_i s}\right)$$

$$u(t) = K_p e(t) + \frac{K_P}{T_i}\int_0^t e(t)dt$$

$$\frac{C(s)}{R(s)} = \frac{K_P s + \frac{1}{T_i}K_P}{Js^3 + Bs^2 + K_P s + \frac{K_P}{T_i}}$$

No steady-state error for reference input

3. Proportional-Integral Control (PI control)



$$\frac{C(s)}{D(s)} = \frac{\dfrac{1}{Js^2+Bs}}{1+K_P(1+\dfrac{1}{T_i s})\dfrac{1}{Js^2+Bs}}$$

$$= \frac{s}{Js^3+Bs^2+K_P s+\dfrac{K_P}{T_i}}$$

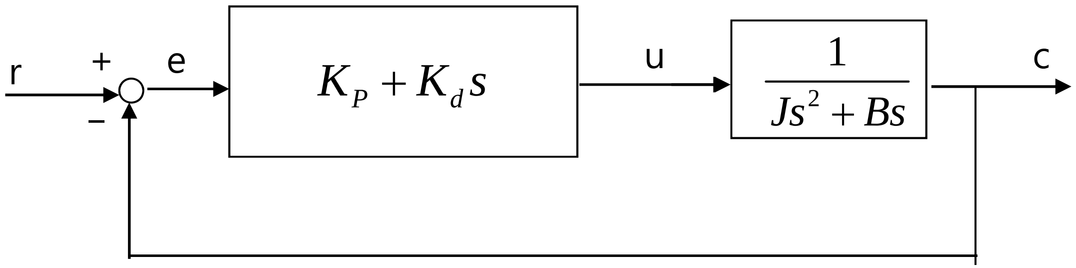
$$D(s) = \frac{1}{s}$$

$$C(t) = \lim_{s\to 0} s \frac{s}{Js^3+Bs^2+K_P s+\dfrac{K_P}{T_i}} \frac{1}{s} = 0$$

No steady-state error for step disturbance

Classifications of controllers

4. Proportional-Derivative Control (PD control)



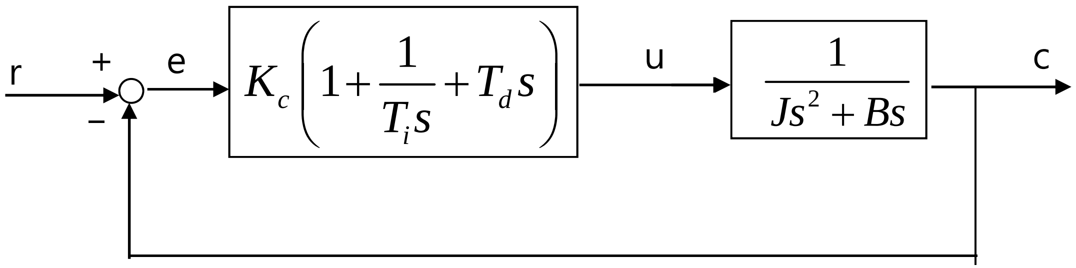
$$u(t) = K_P e(t) + K_d \frac{d}{dt} e(t)$$

$$\frac{C(s)}{R(s)} = \frac{K_P + K_d s}{Js^2 + (B + K_d)s + K_P}$$

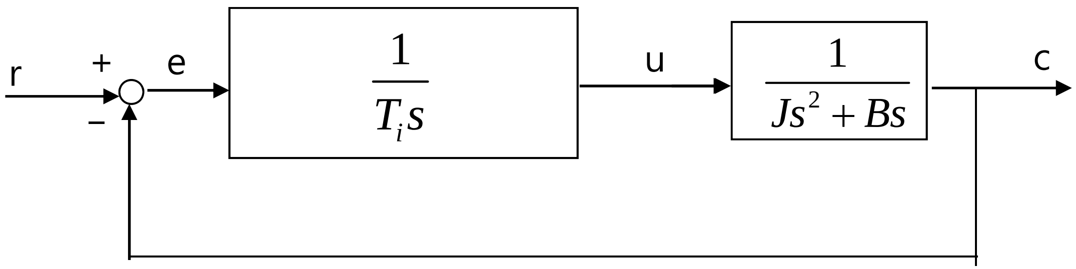
$$\zeta = \frac{B + K_d}{2\sqrt{JK_P}}$$: increased effective damping

Classifications of controllers

5. Proportional-Integral-Derivative Control (PID control)

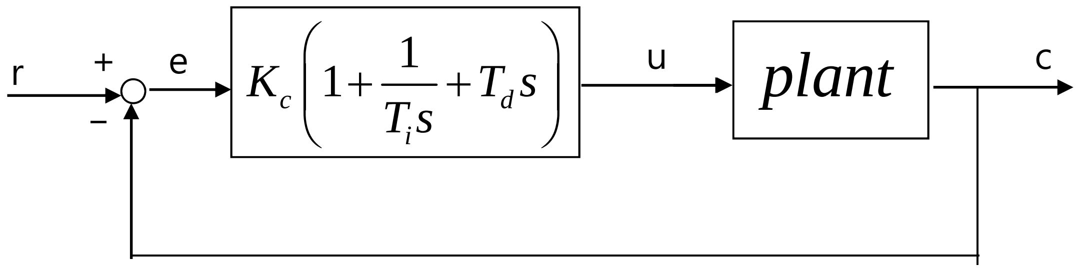


6. I control

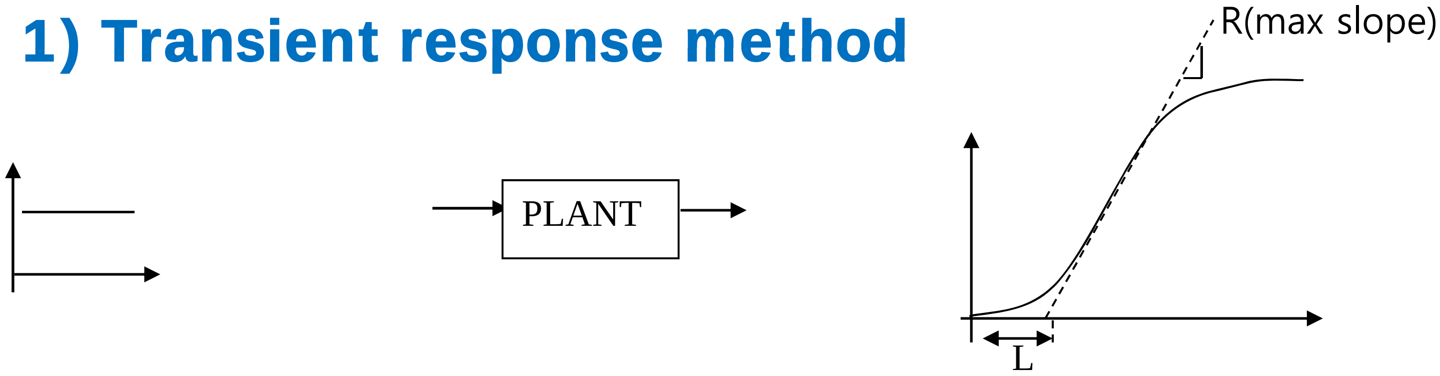


$$\frac{C(s)}{R(s)}=\frac{\frac{1}{T_i s}\frac{1}{Js^2+Bs}}{1+\frac{1}{T_i s}\frac{1}{Js^2+Bs}}=\frac{\frac{1}{T_i}}{Js^3+Bs^2+\frac{1}{T_i}}$$

Ziegler-Nicholas Tuning Rules for PID controllers



Method 1) Transient response method

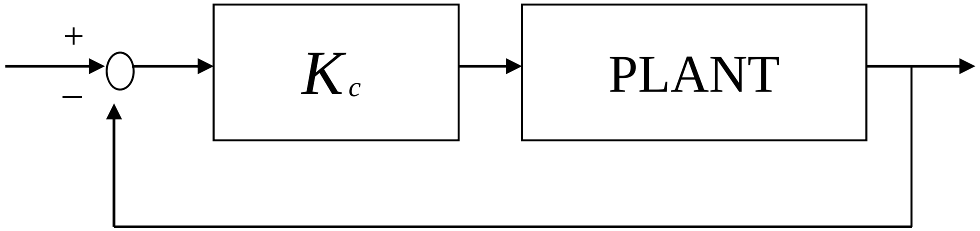


$$\text{Step input}\begin{cases} K_c=\dfrac{1}{RL} \quad for \quad P-control \\ K_c=\dfrac{0.9}{RL},\quad T_i=3.3L \quad for \quad PI-control \\ K_c=\dfrac{1.2}{RL},\quad T_i=2L,\quad T_d=0.5L \quad for \quad PID-control \end{cases}$$

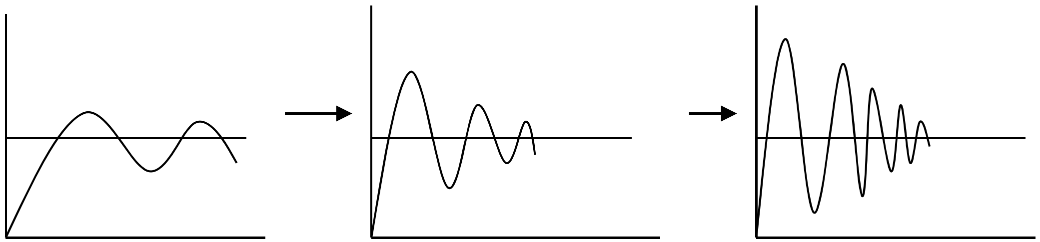
- This method works good if the unit step response is ∫ (sigmod) shaped

Ziegler-Nicholas Tuning Rules for PID controllers

Method 2) Ultimate sensitivity method



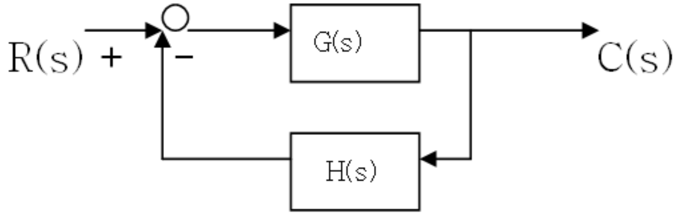
$$K_c \gg 1,\ \omega_n \rightarrow increase$$
$$\zeta \rightarrow decrease$$



- Increase K_c until you hit the stability limit

$$\begin{cases} K_c = 0.5K_U \quad for \quad P-control \\ K_c = 0.45K_U, \quad T_i = 0.83P_U \quad for \quad PI-control \\ K_c = 0.6K_U, \quad T_i = 0.5P_U \quad T_i = 0.125P_U \quad for \quad PID-control \end{cases}$$

Higher Order Systems



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

- unit step response $R(s) = \frac{1}{s}$

$$C(s) = \frac{b_0 s^m + \cdots + b_m}{a_0 s^n + \cdots + a_0} \cdot \frac{1}{s}$$

characteristic equation

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n = 0$$

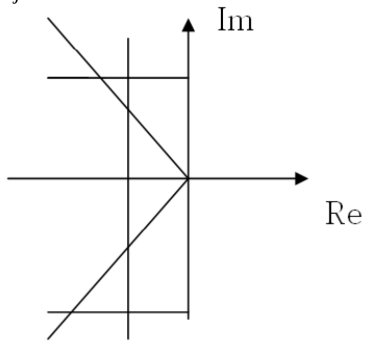
$$s = p_i \qquad i = 1, \cdots q$$

$$s = -\zeta_k \omega_k \pm \sqrt{1-\zeta_k^2}\,\omega_k j \qquad k = 1, \cdots r$$

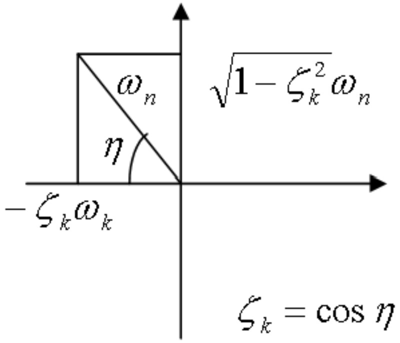
zero ; $s = Z_i \quad i = 1, \cdots m$

$$C(s) = \frac{k\prod_{i=1}^{m}(s-z_i)}{s\prod_{j=1}^{q}(s-p_i)\prod_{k=1}^{r}(s^2+2\zeta_k\omega_k+\omega_k{}^2)} = \frac{a}{s} + \sum_{j=1}^{q}\frac{a_j}{s-p_i} + \sum_{r=1}^{r}\frac{b_k(s+\zeta_k\omega_k) + C_k\omega_k\sqrt{1-\zeta_k^2}}{s^2+2\zeta_k\omega_k s+\omega_k^2}$$

$$C(t) = a + \sum_{j=1}^{q} a_j e^{p_i t} + \sum_{k=1}^{r} b_k e^{-\zeta_k\omega_k t}\cos\omega_k\sqrt{1-\zeta_k^2}\,t + \sum_{k=1}^{r} C_k e^{-\zeta_k\omega_k t}\sin\omega_k\sqrt{1-\zeta_k^2}\,t$$

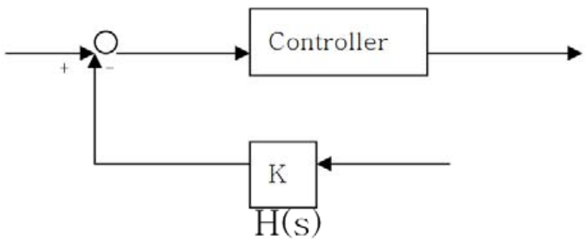


$\omega_{dk} = \sqrt{1-\zeta_k^2}\,\omega_k$



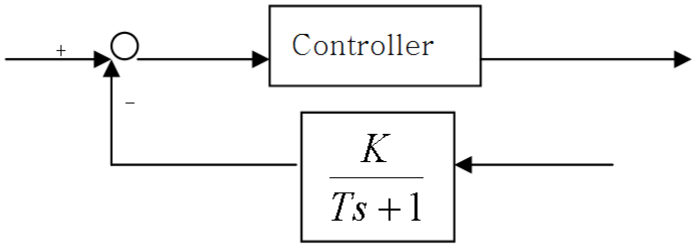
$\zeta_k = \cos\eta$

Effect of sensors on system performance

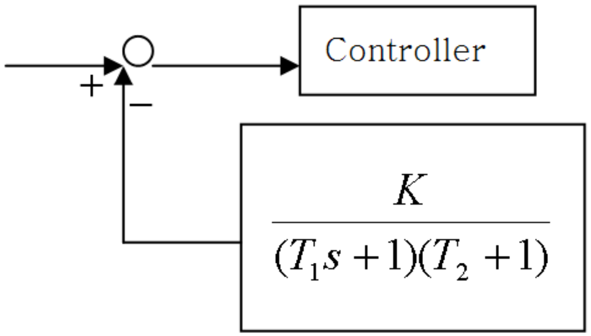


- **Fast sensor dynamics** ; H(s)=constant

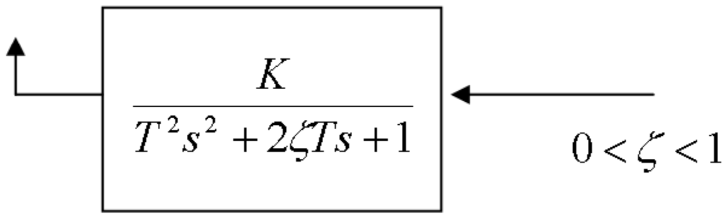
  - First order sensor



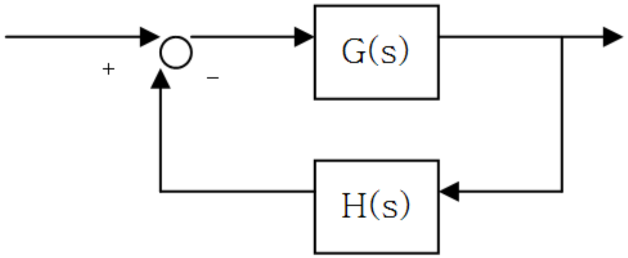
  - Overdamped second order



  - underdamped second order



Stability of Feedback Systems



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{P(s)}{Q(s)}$$

Poles : s_i

$\mathrm{Re}(s_i) < 0$; stable

Characteristic equation $Q(s) = 0$

$$b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0 = 0$$

Routh's Stability Criterion

Characteristic equation

$$b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0 = 0$$

$$\begin{array}{c|cccc}
s^n & b_n & b_{n-2} & b_{n-4} & \cdots \\
s^{n-1} & b_{n-1} & b_{n-3} & b_{n-5} & \cdots \\
\hline
s^{n-2} & c_1 & c_2 & & \\
s^{n-3} & d_1 & d_2 & & \\
. & . & . & & \\
s_0 & & & &
\end{array}$$

$$c_1 = \frac{b_{n-1}b_{n-2} - b_n b_{n-3}}{b_{n-1}} \quad , \quad c_2 = \frac{b_{n-1}b_{n-4} - b_n b_{n-5}}{b_{n-1}}$$

$$d_1 = \frac{c_1 b_{n-3} - b_{n-1}c_2}{c_1} \quad , \quad d_2 = \frac{c_1 b_{n-5} - b_{n-1}c_3}{c_1}$$

- Rooth's Criterion

  Number of the characteristic roots with positive real parts
  =Number of sign changes of the first column have the same sign

Routh's Stability Criterion

Example 1)

$$\frac{C}{R} = \frac{P(s)}{s^5 + s^4 + 10s^3 + 72s^2 + 152s + 240}$$

s^5	1	10	152
s^4	1	72	240
s^3	−62	−88	
s^2	70.6	240	
s^1	122.6		
s^0	240		

Two sign change

➔ Q(s)=0 has two roots with positive real parts

➔ unstable

Routh's Stability Criterion

Example 2)

s^4	1	2	5
s^3	1	2	
s^2	0		
s^1	?		

➔ prevents completion of the array

method 1) $s = \dfrac{1}{x}$

$$Q(s) = \frac{5x^4 + 2x^3 + 2x^2 + x + 1}{x^4}$$

x^4	5	2	1
x^3	2	1	
x^2	-0.5	1	
x^1	-1	2	
x^0	5		

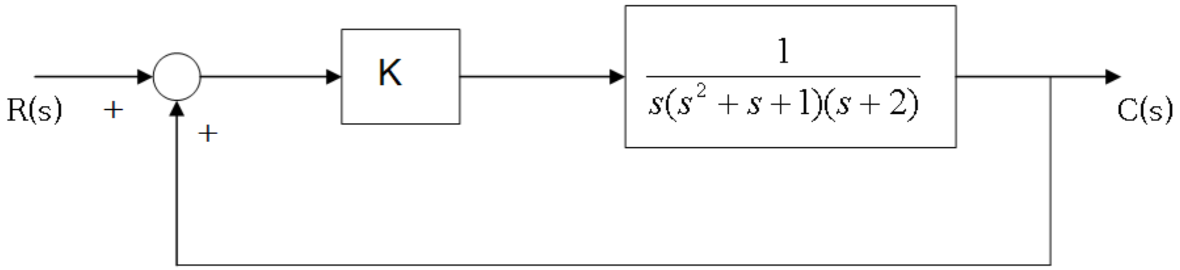
two sign change ➔ unstable

method 2)

set $Q(s)(s+1) = 0$

Routh's Stability Criterion

Example 3)



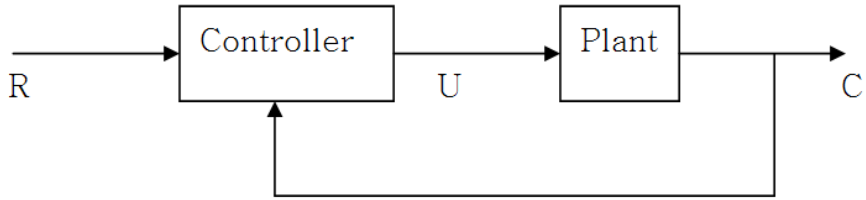
$$\frac{C(s)}{R(s)} = \frac{\dfrac{K}{s(s^2+s+1)(s+2)}}{1+\dfrac{K}{s(s^2+s+1)(s+2)}} = \frac{K}{s(s^2+s+1)(s+2)+K} = \frac{K}{s^4+3s^3+3s^2+2s+K}$$

s^4	1	3	K
s^3	3	2	
s^2	$\frac{7}{3}$	K	
s^1	$\left(2-\frac{9}{7}K\right)$		
s^0	K		

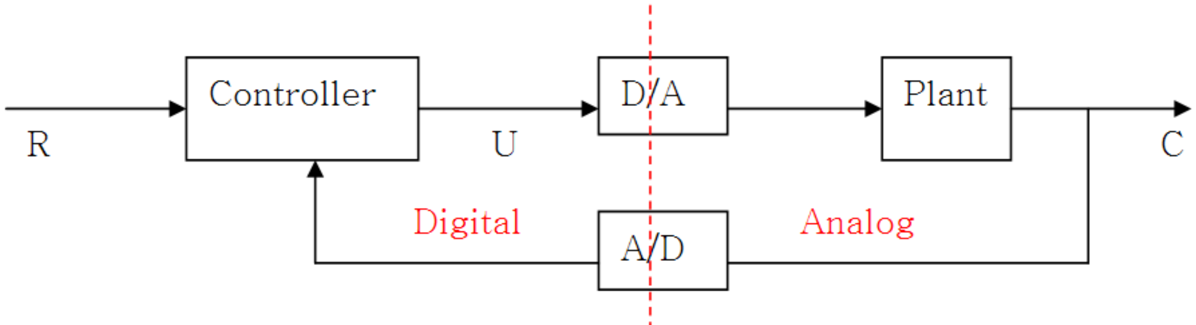
$$2-\frac{9}{7}K>0 \implies K<\frac{14}{9} \qquad K>0$$

$$\therefore\ 0<K<\frac{14}{9}$$

Analog / Digital Controller



[Analog Controller]



[Digital Controller]

Analog Controller	Digital Controller
Pneumatic controller (compressed air)	Microprocessor
Hydraulic controller (oil)	-flexible -modern control algorithm -reliable
Electronic controller	

Pneumatic vs Hydraulic

① Air and gases are compressible, whereas oil is incompressible.
②Air lacks lubricating property,
Oil functions as a hydraulic fluid as well as a lubricator.
③Operating pressure
- Pneumatic ; low
- Hydraulic ; high

④ Output power
- Pneumatic ; low
- Hydraulic ; high

⑤Accuracy
- Pneumatic ; poor at low velocity
- Hydraulic ; satisfactory at all velocities

⑥Pneumatic
- External leakage : permissible
- Internal leakage ; make be avoid

Hydraulic
- External leakage ; make be avoid
- Internal leakage ; permissible

⑦ Pneumatic ; no return pipe
Hydraulic ; always needed
⑧Pneumatic ; 5。 ~ 60。 C (0。 ~ 200。 C) insensitive
Hydraulic ; Hydraulic : 20。 ~70。 C sensitive (viscosity)
⑨Pneumatic ; fire-and explosion-proof
Hydraulic ; not

Hydraulic Control Systems

- Characteristics

① high horsepower-to-weight
② accurate
③ fast response
④ machine tool, aircraft, industrial

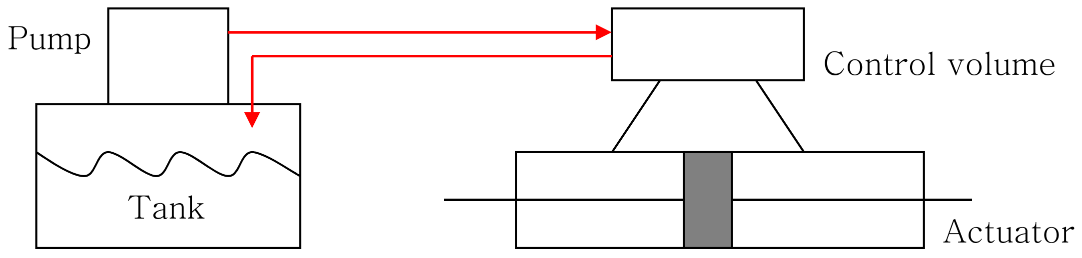
- Advantages

①hydraulic fluid ; lubricant, heat exchanger
②small size – large force actuators
③higher speed of response,
fast starts, stops and speed reversals.
④hydraulic actuators can be operated under continuous, intermittent, reversing and stalled conditions without damage
⑤linear and rotary actuators ; flexibility in design
⑥low leakage in actuators
⇒ speed drop when loads are applied is small

Hydraulic Control Systems

- Disadvantages

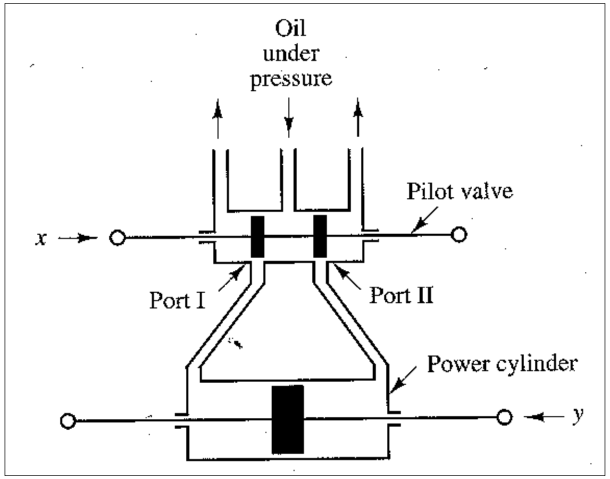
① hydraulic power is not readily available compared to electric power



② cost ; higher than a comparable electric system performing a similar function
③ ford and explosion hazards exist unless fire-resistant fluids are used
④ system tends to be messy (leakage)
⑤ contaminated oil ⇒ failure
⑥ highly nonlinear – sophisticated design
⑦hydraulic systems – poor damping characteristics
if properly design ⇒ some unstable phenomena may occur or disappear depending on operating condition.

Hydraulic Control Systems

- Hydraulic Integral Controller



$$q = K_1 x \quad [kg/\sec]$$

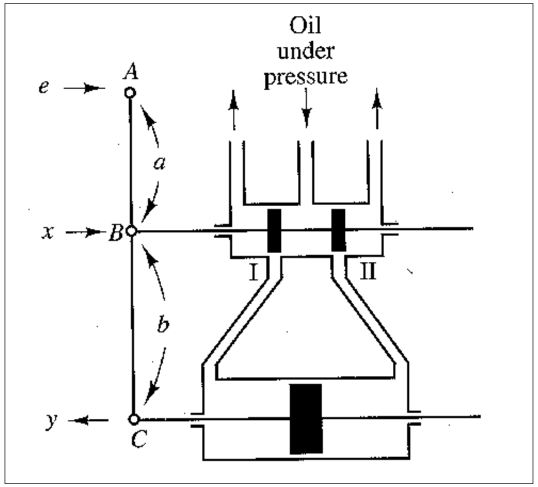
$$qdt = \rho A dy$$

$$\frac{dy}{dt} = \frac{1}{\rho A} q = \frac{K_1}{\rho A} x = Kx$$

$$sY(s) = KX(s) \quad \Rightarrow \quad \frac{Y(s)}{X(s)} = \frac{K}{s}$$

Hydraulic Control Systems

- Hydraulic Proportional Controller



$$\frac{y}{e} = \frac{b}{a} \qquad \text{Steady State}$$

$$x = \frac{b}{a+b}e - \frac{a}{a+b}y$$

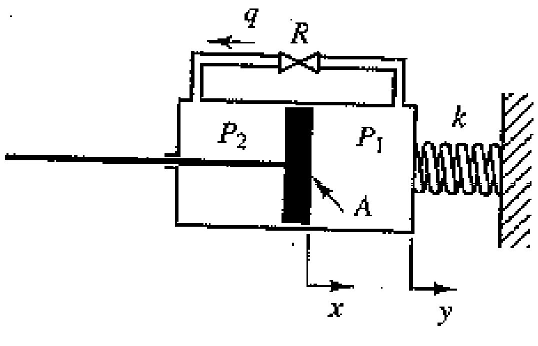
$$\frac{Y(s)}{E(s)} = \frac{\frac{b}{a+b}\cdot\frac{K}{s}}{1+\frac{K}{s}\cdot\frac{a}{a+b}} = \frac{bK}{s(a+b)+Ka} = \frac{\frac{bK}{a+b}}{s+\frac{aK}{a+b}} \cong \frac{b}{a}$$

Time constant $\frac{a+b}{Ka}$ (large K)

Unit Step Response $E(s) = \frac{1}{s}$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\cdot\frac{bK}{s(s+b)+Ka}\cdot\frac{1}{s} = \frac{b}{a}$$

Hydraulic Control Systems



(a)

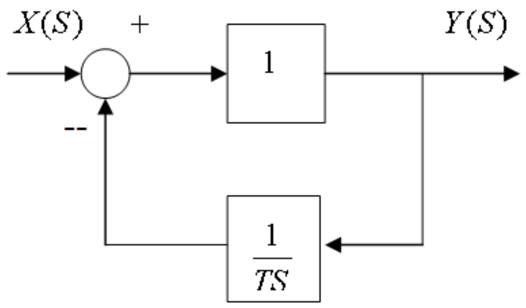
$$q = \frac{P_1 - P_2}{R}$$

$$F = A(P_1 - P_2) = ky$$

$$q = A\frac{d(x-y)}{dt}\rho$$

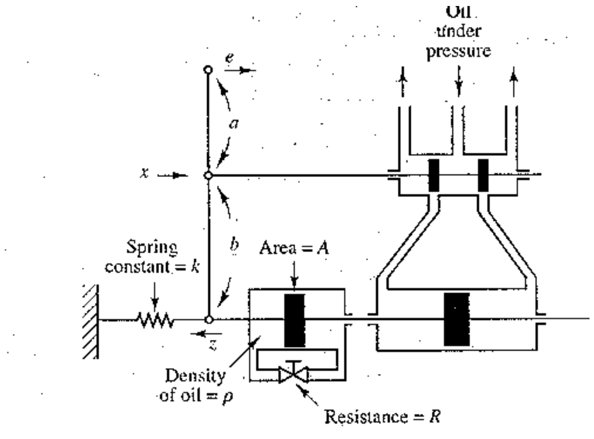
$$\frac{dx}{dt} - \frac{dy}{dt} = \frac{1}{A\rho}q = \frac{1}{A\rho}\frac{1}{R}(P_1 - P_2) = \frac{k}{A^2\rho R}y$$

$$sY(s) + \frac{1}{T}Y(s) = sX(s) \qquad \frac{Y(s)}{X(s)} = \frac{s}{s+\frac{1}{T}} = \frac{1}{1+\frac{1}{Ts}}$$

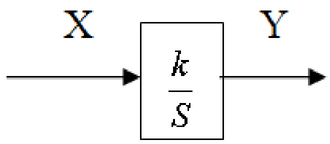


Hydraulic Control Systems

- Hydraulic PI Controller

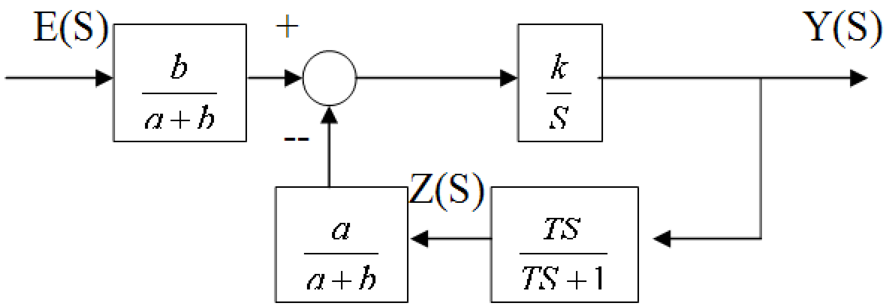


(a)



$$x = \frac{b}{a+b}e - \frac{a}{a+b}z$$

$$\frac{Z(s)}{Y(s)} = \frac{Ts}{Ts+1}$$

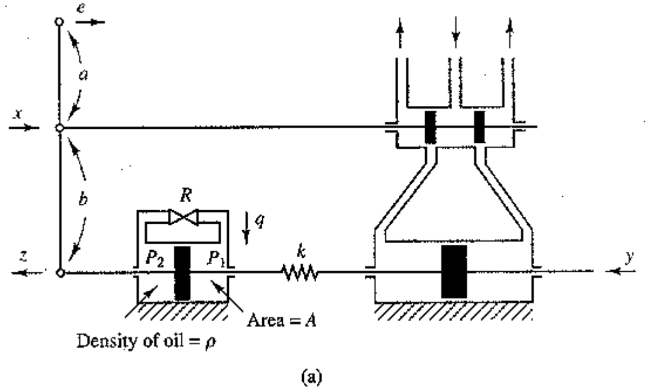


$$\frac{Y}{E} = \frac{\frac{b}{a+b}\frac{k}{S}}{1+\frac{ka}{a+b}\frac{k}{S}\frac{TS}{TS+1}} \qquad k = \frac{k_1}{A\rho} \qquad A\rho\frac{dy}{dt} = k_1 x$$

$$\Rightarrow \left|\frac{kaT}{(a+b)(Ts+1)}\right| >> 1 \qquad \frac{Y}{E} = \frac{\frac{b}{a+b}\frac{k}{s}}{\frac{a}{a+b}k\frac{T}{Ts+1}} = k_p\left(1+\frac{1}{T_i s}\right) \qquad \left(k_p = \frac{b}{a},\ T_i = T = \frac{RA\rho^2}{k}\right)$$

Hydraulic Control Systems

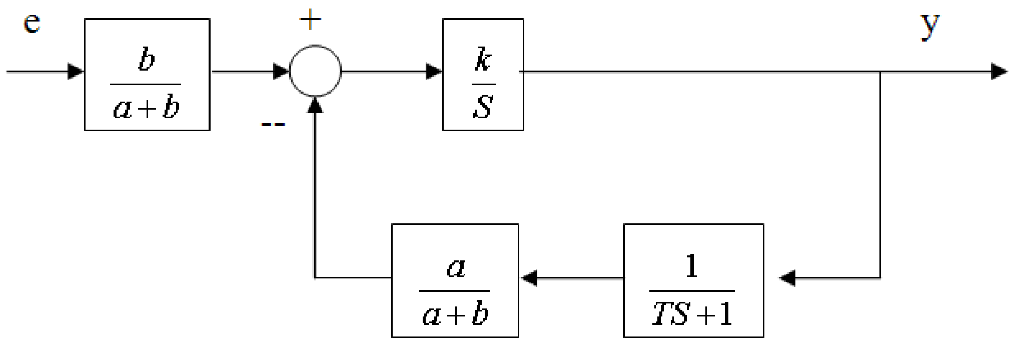
- Hydraulic PD Controller



$$A(P_2 - P_1) = k(y - z)$$

$$q = \frac{P_2 - P_1}{R} \qquad q = A\rho\frac{dz}{dt}$$

$$\frac{Z(s)}{y(s)} = \frac{1}{Ts+1} \qquad T = \frac{RA\rho^2}{k}$$

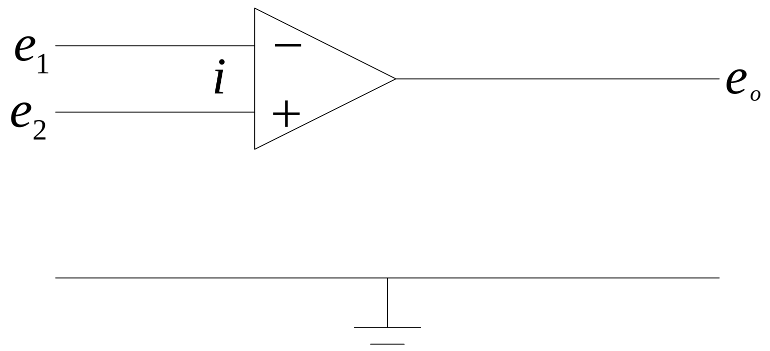


$$x = \frac{b}{a+b}e - \frac{a}{a+b}z$$

$$\frac{Y(s)}{E(s)} = \frac{\frac{b}{a+b}\frac{k}{s}}{1+\frac{a}{a+b}\frac{k}{s}\frac{1}{Ts+1}} \approx k_p(1+Ts) \qquad \left(k_p = \frac{b}{a}, T = \frac{RA\rho^2}{k}\right)$$

Electronic Controllers

- **Operation Amplifiers**



$$e_0 \approx k(e_2 - e_1)$$ k:very large-10^5~10^6 (for signals less than 10 Hz)

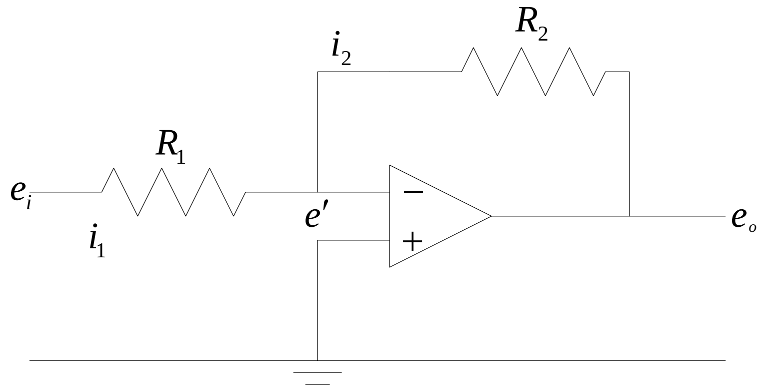
$$i \approx 0$$

$$e_1 \approx e_2 (k = \infty)$$

Electronic Controllers

• Operation Amplifiers

- Inverting Amplifiers

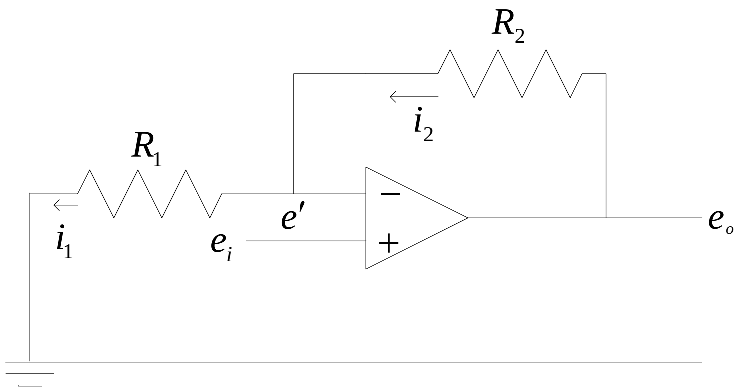


$$i_1 = i_2, e' = 0$$

$$i_1 = \frac{e_i}{R_1} = i_2 = \frac{0 - e_0}{R_2}$$

$$e_0 = -\frac{R_2}{R_1} e_i$$

- Non-inverting Amplifiers



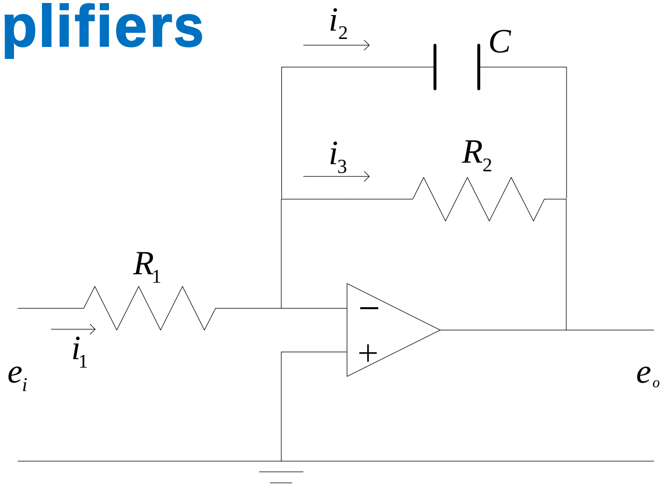
$$i_1 = i_2$$

$$i_1 = \frac{e_i}{R_{1\,2}} = \frac{e_o - e_i}{R_2}$$

$$e_0 = (1 + \frac{R_2}{R_1}) \cdot e_i$$

• Operation Amplifiers

Example)



$$i_1 = i_2 + i_3$$

$$i_1 = \frac{e_i}{R_1} \quad \text{2)} \ \frac{dv_c}{dt} = \frac{1}{c} i_2 = -\frac{de_0}{dt} \quad \text{3)} \ i_3 = \frac{0 - e_0}{R_2}$$

$$\frac{e_i}{R_1} = -C\frac{de_0}{dt} + \frac{-e_0}{R_2}$$

Laplace → $\frac{1}{\sqrt{R_1}} E_i(s) = -(cs + \frac{1}{R_2})E_0(s)$

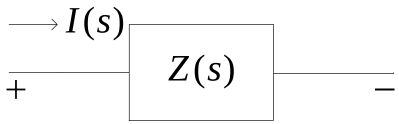
$$\therefore \frac{E_0(s)}{E_i(s)} = -\frac{\frac{1}{R_1}}{cs + \frac{1}{R_2}} = -\frac{R_2}{R_1}\frac{1}{R_2 cs + 1}$$

VDCL Vehicle Dynamics & Control Laboratory

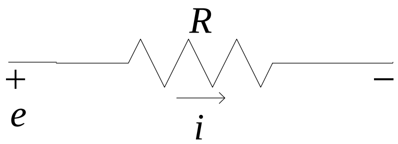
Electronic Controllers

• Impedance approach for obtaining transfer functions

- Impedance



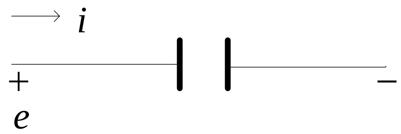
Laplace tranceform 된 I(s) 와 E(s)의 relation을 impedance라 한다.



$$e = iR$$

$$E(s) = I(s)R$$

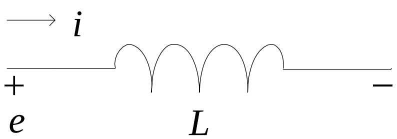
$$Z(s) = \frac{E(s)}{I(s)} = R$$



$$\frac{de}{dt} = \frac{1}{c}i$$

$$sE(s) = \frac{1}{c}I(s)$$

$$Z(s) = \frac{E(s)}{I(s)} = \frac{1}{cs}$$



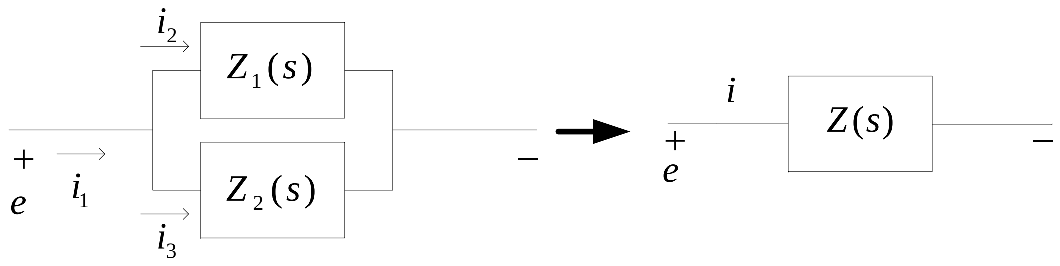
$$e = L\frac{di}{dt}$$

$$E(s) = LsI(s)$$

$$Z(s) = \frac{E(s)}{I(s)} = Ls$$

• Impedance approach for obtaining transfer functions

Example 1)



$$I_2(s) = \frac{E(s)}{Z_1(s)},\ I_3(s) = \frac{E(s)}{Z_2(s)}$$

$$i_1 = i_2 + i_3$$

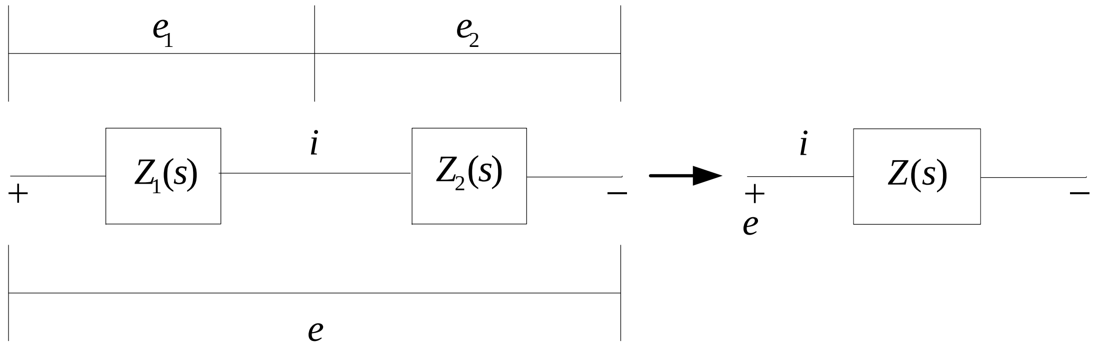
$$I_1(s) = \frac{E(s)}{Z(s)}$$

$$I_1(s) = \frac{E(s)}{Z(s)} = \frac{E(s)}{Z_1(s)} + \frac{E(s)}{Z_2(s)}$$

$$\frac{1}{Z(s)} = \frac{1}{Z_1(s)} + \frac{1}{Z_2(s)}$$

- **Impedance approach for obtaining transfer functions**

Example 2)

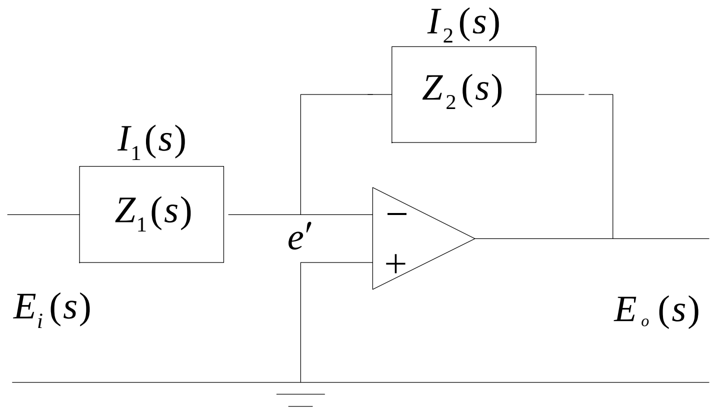


$$
\begin{aligned}
&E(s) = E_1(s) + E_2(s) \\
&I(s)\cdot Z(s) = Z_1(s)\cdot I(s) + Z_2(s)\cdot I(s) \\
&\therefore\ Z(s) = Z_1(s) + Z_2(s)
\end{aligned}
$$

Electronic Controllers

- **Impedance approach for obtaining transfer functions**

Example 3)



$$I_1(s) = I_2(s)$$

$$\frac{E_i(s)}{Z_1(s)} = -\frac{E_0(s)}{Z_2(s)}$$

$$\frac{E_0(s)}{E_i(s)} = -\frac{Z_2(s)}{Z_1(s)}$$

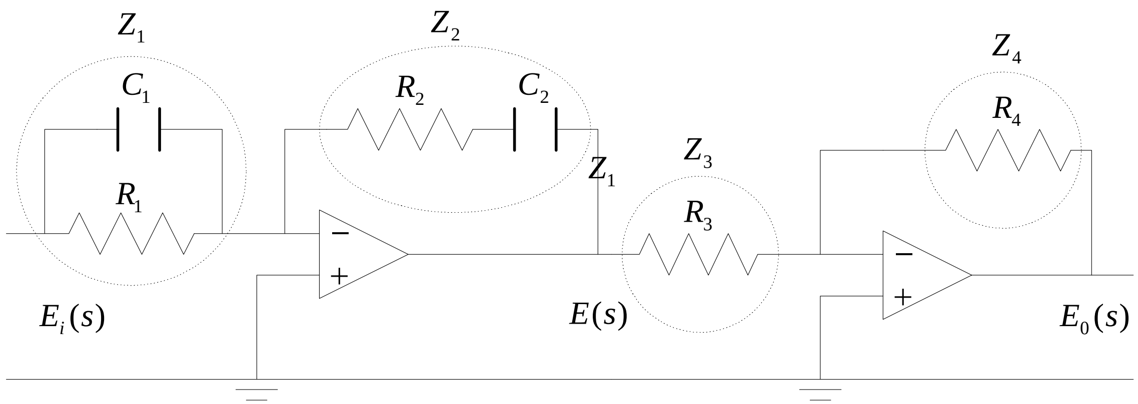
앞의 예에서

$$Z_1(s) = R_1$$

$$\frac{1}{Z_2(s)} = \frac{1}{R_2} + \frac{1}{1/CS}$$

$$\frac{E_0(s)}{E_i(s)} = -\frac{1/(CS + 1/R_2)}{R_1} = -\frac{R_2}{R_1} \cdot \frac{1}{R_2 CS + 1}$$

- **OP-amp 로 PID 만들기**



$$\frac{1}{Z_1(s)} = \frac{1}{R_1} + \frac{1}{1/C_1S}$$

$$Z_2(s) = R_2 + \frac{1}{C_2S}$$

$$Z_3(s) = R_3$$

$$Z_4(s) = R_4$$

$$\frac{E(s)}{E_i(s)} = -\frac{Z_2(s)}{Z_1(s)}, \qquad \frac{E_0(s)}{E(s)} = -\frac{Z_4(s)}{Z_3(s)}$$

양변을 곱하면

$$\frac{E_0(s)}{E_i(s)} = \frac{Z_2(s)Z_4(s)}{Z_1(s)Z_3(s)} = k_p\left(1+\frac{1}{T_i}\frac{1}{s}+T_d s\right)$$

$$K_p = \frac{R_4(R_1C_1+R_2C_2)}{R_3R_1C_2}, \quad T_i = R_1C_1+R_2C_2, \quad T_d = \frac{R_1C_1R_2C_2}{R_1C_1+R_2C_2}$$

→ PID

Phase Lead and Phase Lag in Sinusoidal Response

Linearsystem

$$\begin{array}{c} x(t) \\ X(s) \end{array} \longrightarrow \boxed{G(s)} \longrightarrow \begin{array}{c} y(t) \\ Y(s) \end{array}$$

$$G(s) = \frac{P(s)}{g(s)} = \frac{P(s)}{(S+S_1)(S+S_2)+\cdots+(S+S_n)}$$

위 system에 $x(t) = X\sin\omega t$ 를 넣어주면 $y(t)$는 ?

$$X(s) = \frac{X\omega}{s^2+\omega^2}$$

$$\Rightarrow Y(s) = G(s)X(s) = G(s)\frac{X\omega}{s^2+\omega^2}$$

$$= \frac{b_1}{s+s_1} + \frac{b_2}{s+s_2} + \cdots + \frac{b_n}{s+s_n} + \frac{a}{s+j\omega} + \frac{\bar{a}}{s-j\omega}$$

1) Transient response → Stable system인 경우 = 0

2) Stable system인 경우 Steady state response 는

$$y(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t}$$

$$a = G(s)\frac{\omega}{s^2+\omega^2}(-s+j\omega)_{s=-j\omega} = -\frac{XG(-j\omega)}{2j}$$

$$\bar{a} = G(s)\frac{X\omega}{s^2+\omega^2}(s-j\omega)_{s=j\omega} = \frac{XG(j\omega)}{2j}$$

$$G(j\omega) = |G(j\omega)|e^{j\phi}$$

$$G(-j\omega) = |G(j\omega)|e^{-j\phi}, \phi = phase$$

$$\therefore y(t) = X|G(j\omega)|\frac{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}}{2j} = X|G(j\omega)|\sin(\omega t+\phi)$$

$$= Y\sin(\omega t+\phi), Y = X|G(j\omega)|, \phi = \angle G(j\omega)$$

Phase Lead and Phase Lag in Sinusoidal Response

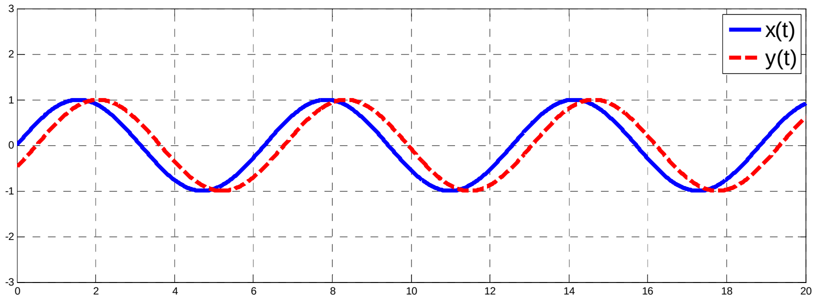
Example) First order system

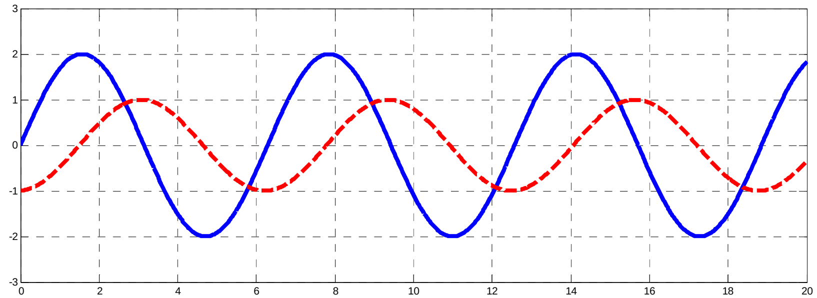
$$G(s) = \frac{K}{Ts+1}$$

$$G(j\omega) = \frac{K}{Tj\omega+1}, \left|G(j\omega)\right| = \frac{K}{\sqrt{1+T^2}}$$

$$G(j\omega) = \frac{K}{Tj\omega+1}, \left|G(j\omega)\right| = \frac{K}{\sqrt{1+T^2\omega^2}}, \angle G(j\omega) = -\tan^{-1} T\omega$$

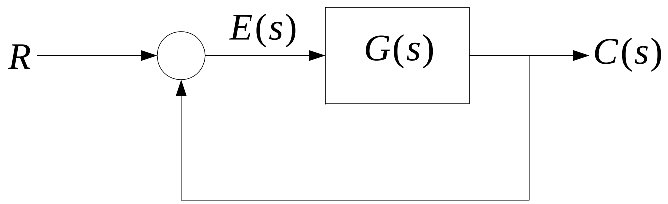
$$\therefore y_{ss}(t) = X\frac{K}{\sqrt{1+T^2\omega^2}}\sin(\omega t+\phi), \quad \phi = -\tan^{-1} T\omega$$





$\phi < 0$: phase lag, $\phi > 0$: phase lead

Steady state Error in Unity Feedback control systems



$$G(s) = \frac{K(T_a s + 1)(T_b s + 1)\cdots(T_n s + 1)}{S^N (T_1 s + 1)(T_2 s + 1)\cdots(T_p s + 1)}$$

N=0 ; Type 0, N=1 ; Type 1, N=2 ; Type 2

Steady state Error 를 생각해보자

$$E(s) = \frac{1}{1+G(s)} R(s)$$

1) $R(s) = \frac{1}{s}$ 일 경우 (unit step input),

$$e_{ss} = \lim_{s->0} sE(s) = \lim_{s->0} s \frac{1}{1+G(s)} \frac{1}{s} = \frac{1}{1+G(0)}$$

Static position error constant k_p

$$k_p = \lim_{s->0} G(s) = G(0) = \begin{cases} k & ;Type\ 0\ \ system \\ \infty & ;Type\ 1\ \ system \\ \infty & ;Type\ 2\ \ system \end{cases}$$

$$e_{ss} = \frac{1}{1+k_p} = \begin{cases} \frac{1}{1+k_p} & ;Type\ 0\ \ system \\ 0 & ;Type\ 1,\ 2,..\ system \end{cases}$$

Steady state Error in Unity Feedback control systems

2) $R(s)=\frac{1}{s^2}$ 일 경우 (unit step input),

$$e_{ss}=\lim_{s->0} s\frac{1}{1+G(s)}\frac{1}{s^2}=\lim_{s->0}\frac{1}{s(G(s)+1)}=\lim_{s->0}\frac{1}{sG(s)}$$

Static velocity error constant k_v

$$k_v=\lim_{s->p} SG(s)=G(0)=\begin{cases}0 & ;Type\ 0\ system\\ k & ;Type\ 1\ system\\ \infty & ;Type\ 2,3..\ system\end{cases}$$

$$e_{ss}=\frac{1}{k_v}=\begin{cases}\infty & ;Type\ 0\ system\\ \frac{1}{k} & ;Type\ 1\ system\\ 0 & ;Type\ 2,3,..\ system\end{cases}$$

3) $R(s)=\frac{1}{s^3}$ 일 경우 (unit step input),

$$e_{ss}=\lim_{s->0} s\frac{1}{1+G(s)}\frac{1}{s^3}=\lim_{s->0}\frac{1}{s^2G(s)}$$

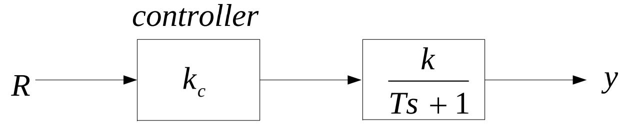
Static acceleration error constant k_a

$$k_a=\lim_{s->0} S^2G(s)=G(0)=\begin{cases}0 & ;Type\ 0\ system\\ 0 & ;Type\ 1\ system\\ k & ;Type\ 2\ system\\ \infty & ;Type\ 3\ system\end{cases}$$

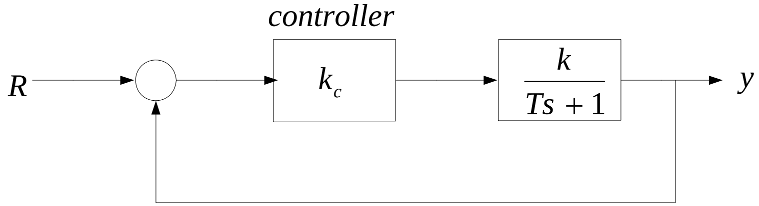
$$e_{ss}=\frac{1}{k_a}=\begin{cases}\infty & ;Type\ 0,1\ system\\ \frac{1}{k} & ;Type\ 2\ system\\ 0 & ;Type\ 3,4,..\ system\end{cases}$$

Open/Closed loop control systems

open loop



closed loop



Open loop Transfer function 에서

$$\frac{y}{R} = k_c \frac{1}{Ts+1} \rightarrow k_c = \frac{1}{k}$$ 이 되어야 error=0 이 된다.

Closed loop Transfer function 에서

$$\frac{y}{R} = \frac{k_p \frac{k}{Ts+1}}{1 + k_p \frac{k}{Ts+1}} = \frac{k_p k}{Ts + 1 + k_p k}$$

Steady/ state error for $R(s) = \frac{1}{s}$

Open loop Transfer function 에서 $e_{ss} = 0$

Closed loop Transfer function 에서 $e_{ss} = 1 - \frac{k_p k}{1 + k_p k} = \frac{1}{1 + k_p k}$

Open/Closed loop control systems

예외의 경우에 대한 예>

$k = 10, \Delta k = 1$ 인 model error 가 있을 경우

Open loop Transfer function 에서

$$\frac{y}{R} = \frac{1}{k_c} \frac{k+\Delta k}{Ts+1}, \quad y(t) = \frac{k+\Delta k}{k} = 1.1$$ (10% steady state error)

Closed loop Transfer function 에서

$$e_{ss} = \frac{1}{1+k_p k} \qquad \text{let} \quad k_p = \frac{100}{k} \Rightarrow e_{ss} = \frac{1}{1+\frac{100}{k}k} = \frac{1}{101}$$

$$k+\Delta k, \quad e_{ss} = \frac{1}{1+\frac{100}{k}(k+\Delta k)} = \frac{1}{1+10(11)} = \frac{1}{111} = 0.009$$