System Control

6-2. Motor Control

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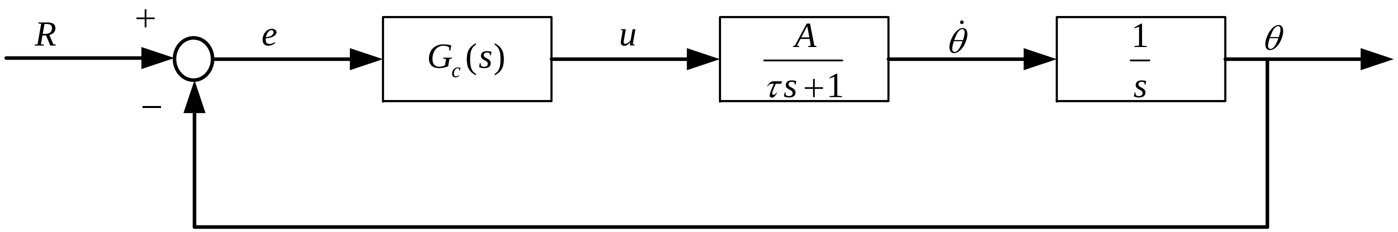
Motor Position Control



1. Proportional

$$D = \frac{A}{s(\tau s+1)}$$

$$G_c(s) = K_p$$

$$e = \frac{1}{1+G_c(s)\frac{A}{s(\tau s+1)}} \cdot R$$

$$= \frac{1}{1+\frac{K_p A}{s(\tau s+1)}} \cdot R$$

: Type 1 $\quad DG = \frac{K_p A}{s(\tau s+1)}$

Motor Position Control

- **Unit step response : R=1 [rad]**

$$e_{ss} = \lim_{s\to 0} s \frac{1}{1+DG}\frac{1}{s} = \frac{1}{1+DG(0)}$$

$$= \frac{1}{1+\infty} = 0$$: no steady state position error

- **Ramp input response : R=t [rad]**

$$e_{ss} = \lim_{s\to 0} s \frac{1}{1+DG}\frac{1}{s^2} = \lim_{s\to 0} \frac{1}{1+\frac{K_p A}{s(\tau s+1)}}\frac{1}{s}$$

$$= \lim_{s\to 0} \frac{1}{s+\frac{K_p A}{\tau s+1}} = \frac{1}{K_p A}$$: finite error

2. PI control

$$D = G(s) = K_P + \frac{K_I}{s}$$

$$DG = \frac{\left(K_P s + K_I\right) A}{s^2 \left(\tau s + 1\right)}$$: Type 2

· **Ramp input response : R=t [rad]**

$$e = \frac{1}{1+DG} R = \frac{1}{1 + \frac{\left(K_p s + K_I\right) A}{s^2 \left(\tau s + 1\right)}} \frac{1}{s^2}$$

$$= \frac{1}{s^2 + \frac{\left(K_p s + K_I\right) A}{\tau s + 1}}$$

$$= \frac{1}{s^2 \left(\tau s + 1\right) + \left(K_p s + K_I\right) A}$$

$$e_{ss} = \lim_{s \to 0} s e(s) = 0$$: no steady state position error

Ch 4. Basic properties of Feedback

Feedback control system

1. Stability

2. Performance
   · Disturbance Rejection
   · Steady-state error
   · transient (dynamic properties)

3. Robustness : sensitivity of the system to change in model parameter

Stability of Feedback Systems

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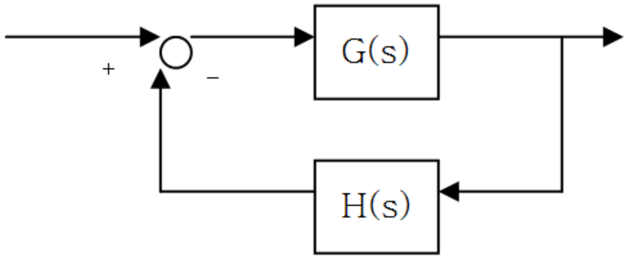

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{P(s)}{Q(s)}$$

Poles : s_i

$\mathrm{Re}(s_i) < 0$ **; stable**

Characteristic equation $Q(s) = 0$

$$b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0 = 0$$

Routh's Stability Criterion

Characteristic equation

$$b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0 = 0$$

=

s^n	b_n	b_{n-2}	b_{n-4}	$\cdots$
s^{n-1}	b_{n-1}	b_{n-3}	b_{n-5}	$\cdots$
s^{n-2}	c_1	c_2		
s^{n-3}	d_1	d_2		
.	.	.		
s_0				

$$c_1 = \frac{b_{n-1}b_{n-2} - b_n b_{n-3}}{b_{n-1}} \quad , \quad c_2 = \frac{b_{n-1}b_{n-4} - b_n b_{n-5}}{b_{n-1}}$$

$$d_1 = \frac{c_1 b_{n-3} - b_{n-1} c_2}{c_1} \quad , \quad d_2 = \frac{c_1 b_{n-5} - b_{n-1} c_3}{c_1}$$

- Rooth's Criterion

  Number of the characteristic equation with positive real parts

  =Number of sign changes of the first column have the same sign

Routh's Stability Criterion

Example 1)

=

$$\frac{C}{R} = \frac{P(s)}{s^5 + s^4 + 10s^3 + 72s^2 + 152s + 240}$$

s^5	1	10	152
s^4	1	72	240
s^3	−62	−88	
s^2	70.6	240	
s^1	122.6		
s^0	240		

Two sign change

➔ Q(s)=0 has two roots with positive real parts

➔ unstable

Routh's Stability Criterion

Example 2)

=

s^4	1	2	5
s^3	1	2	
s^2	0		
s^1	?		

➔ prevents completion of the array

method 1) $s = \dfrac{1}{x}$

$$Q(s) = \frac{5x^4 + 2x^3 + 2x^2 + x + 1}{x^4}$$

x^4	5	2	1
x^3	2	1	
x^2	−0.5	1	
x^1	−1	2	
x^0	5		

two sign change ➔ unstable

method 2)

set $Q(s)(s+1) = 0$

Routh's Stability Criterion

Example 3)

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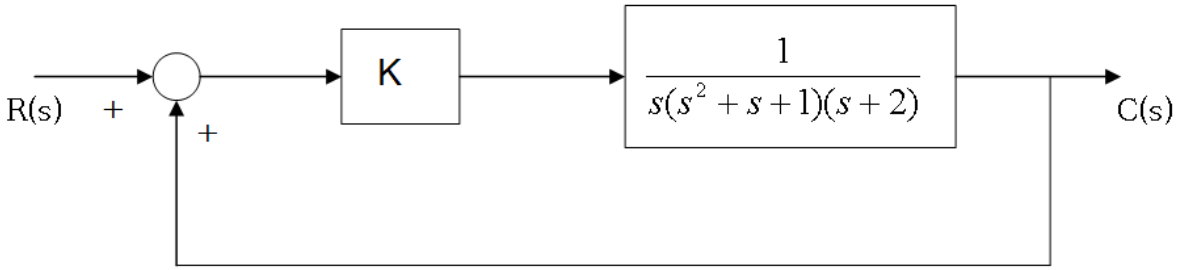


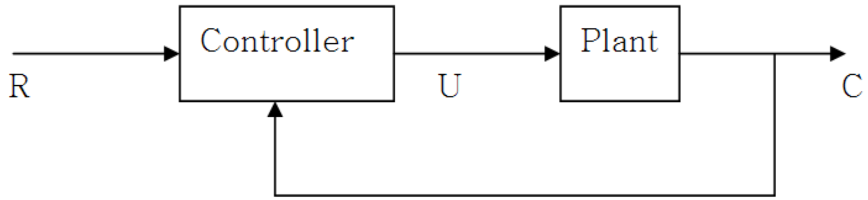
$$\frac{C(s)}{R(s)} = \frac{\dfrac{K}{s(s^2+s+1)(s+2)}}{1+\dfrac{K}{s(s^2+s+1)(s+2)}} = \frac{K}{s(s^2+s+1)(s+2)+K} = \frac{K}{s^4+3s^3+3s^2+2s+K}$$

s^4	1	3	K
s^3	3	2	
s^2	$\frac{7}{3}$	K	
s^1	$\left(2-\frac{9}{7}K\right)$		
s^0	K		

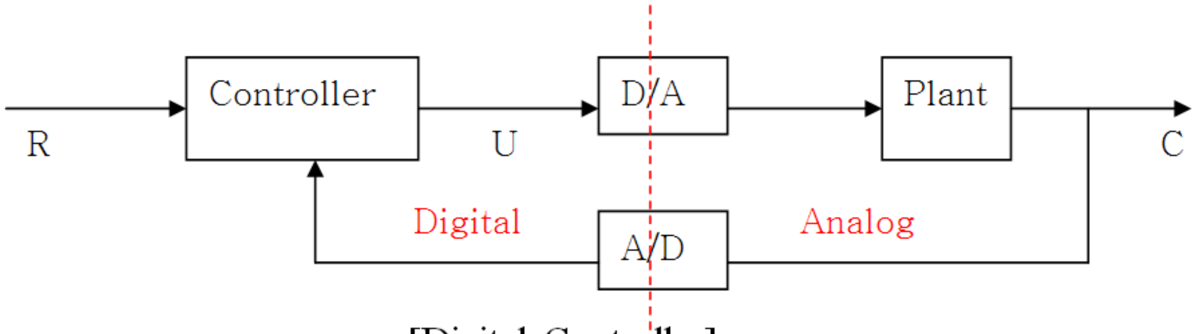
$$2-\frac{9}{7}K>0 \Rightarrow K<\frac{14}{9} \qquad K>0$$

$$\therefore \; 0<K<\frac{14}{9}$$

Analog / Digital Controller



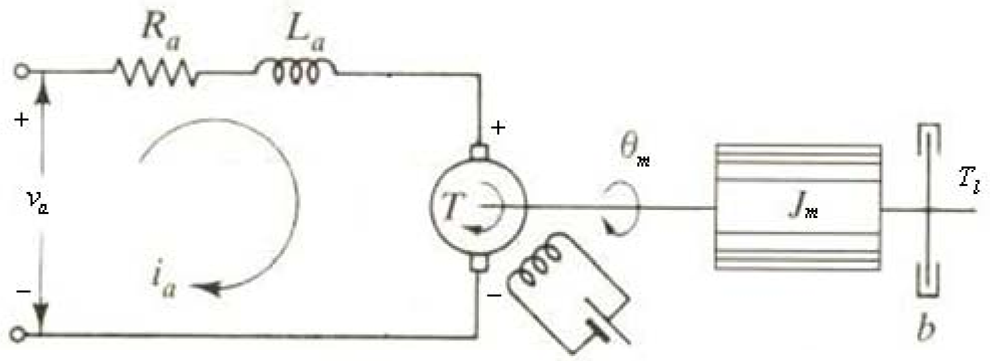
[Analog Controller]



[Digital Controller]

Analog Controller	Digital Controller
Pneumatic controller (compressed air) Hydraulic controller (oil) Electronic controller	Microprocessor -flexible -modern control algorithm -reliable

DC motor



$$J_m\ddot{\theta}_m + b\dot{\theta}_m = K_t i_a - T_l$$

$$L_a\frac{di_a}{dt} + R_a i_a = v_a - K_e\dot{\theta}_m$$

Negligible armature inductance $\left(L_a \approx 0\right)$ **and load torque disturbance** T_l

$$J_m\ddot{\theta}_m + b\dot{\theta}_m = K_t i_a - T_l$$

$$K_e\dot{\theta}_m + R_a i_a = v_a$$

DC motor

Speed control

$$\dot{\theta}_m = \omega_m, \quad \Omega_m(s) = s\theta(s)$$

$$\Rightarrow \left(\underbrace{\frac{J_m R_a}{bR_a + K_t K_e}}_{\tau} s + 1 \right) \Omega_m(s) = \frac{K_t}{bR_a + K_t K_e} V_a(s) - \frac{R_a}{bR_a + K_t K_e} T_l(s)$$

$$\Rightarrow (\tau s + 1)\Omega_m(s) = AV_a(s) - BT_l(s)$$

τ : **Time constant**

A: **Voltage gain, rad / (volt sec)**

B: **torque gain, rad / (N m sec)**

DC motor

Speed control

$$J_m s \cdot \Omega_m(s) + b\Omega_m(s) = K_t I_a - T_l(s)$$

$$K_e \Omega_m(s) + R_a I_a(s) = V_a(s)$$

$$\rightarrow I_a(s) = \left[\left(J_m s + b\right)\Omega_m(s) + T_l(s)\right] / K_t$$

$$\rightarrow K_e \Omega_m(s) + \frac{R_a}{K_t}\left[\left(J_m s + b\right)\Omega_m(s)\right] + \frac{R_a}{K_t} T_l(s) = V_a(s)$$

$$\rightarrow \left[K_e K_t + bR_a + R_a J_m s\right]\Omega_m(s) = K_t V_a(s) - R_a T_l(s)$$

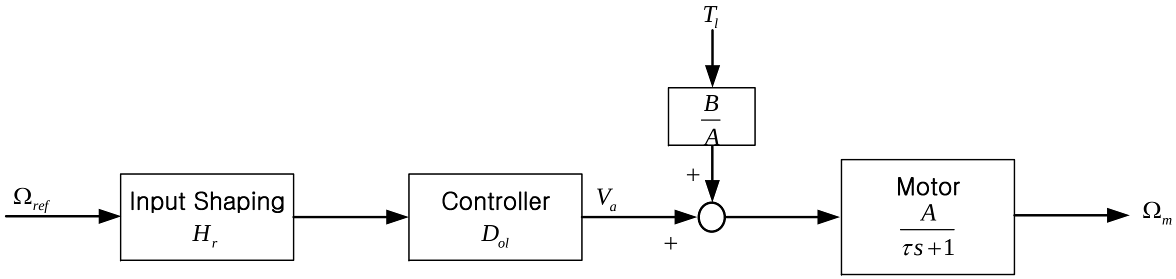
$$\rightarrow \left(\frac{J_m R_a}{bR_a + K_t K_e} s + 1\right)\Omega_m(s) = \frac{K_t}{bR_a + K_t K_e} V_a(s) - \frac{R_a}{bR_a + K_t K_e} T_l(s)$$

$$\Omega_m(s) = \frac{A}{\tau s + 1} V_a(s) - \frac{B}{\tau s + 1} T_l(s)$$

Steady –state, $v_a = a,\ T_l = b$

$$w_{ss} = Aa - Bb$$

Open loop control



$$H_r = 1\frac{volt \cdot \sec}{rad} \qquad D_{ol} = open - loop\ controller = \frac{K_{ol} \cdot volt}{volt}$$

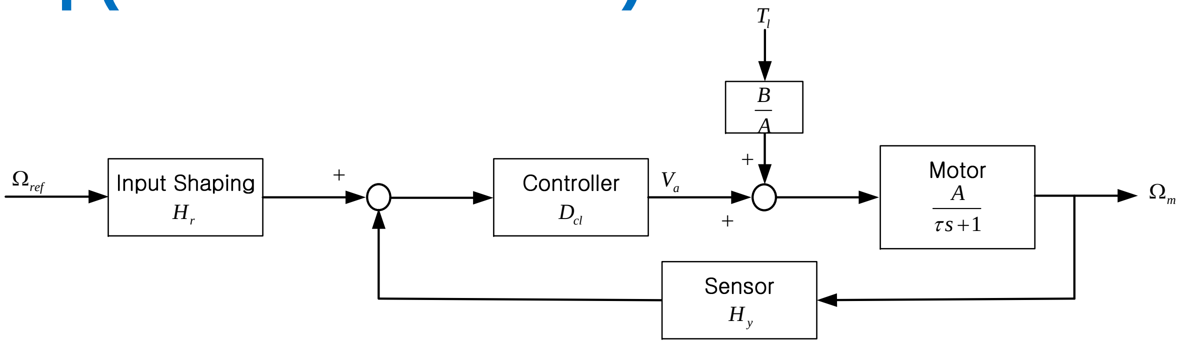
$$\frac{\Omega_m}{\Omega_{ref}} = H_r K_{ol} \frac{A}{\tau s + 1}$$ **(open-loop transfer function)**

$$\begin{cases} v_a = D_{ol} \cdot \omega_{ref} = K_{ol} \cdot \omega_{ref} \\ w_{ss} = A \cdot K_{ol} \cdot \omega_{ref} \\ \Rightarrow K_{ol} = \dfrac{1}{A} \end{cases}$$

No steady-state error when load torque $T_l = 0$

DC motor

Closed-loop(transfer function) control



$$D_{cl} = closed-loop\ controller = \frac{K_{cl}\cdot volt}{volt}$$

$$H_y = sensor = 1\cdot\frac{volt\cdot \sec}{rad}$$

$$H_r = Input\ shaping = 1\cdot\frac{volt\cdot \sec}{rad}$$

$$\Omega_m(s) = \frac{K_{cl}\cdot A}{\tau s+1+K_{cl}\cdot A}\Omega_{ref}(s) - \frac{B}{\tau s+1+K_{cl}\cdot A}T_l(s)$$

$$\frac{\Omega_m(s)}{\Omega_{ref}(s)} = T_{cl}(s) = H_r\frac{D_{cl}\cdot\frac{A}{\tau s+1}}{1+D_{cl}H_y\cdot\frac{A}{\tau s+1}} = \frac{H_r\cdot D_{cl}\cdot A}{\tau s+1+D_{cl}\cdot H_y\cdot A}$$

$$\omega_{m,ss} = \frac{A\cdot K_{cl}}{1+A\cdot K_{cl}}\omega_{ref}$$

$A\cdot K_{cl}\gg 1$ **, then** $\omega_{m,ss}\approx\omega_{ref}$

DC motor

Disturbance Rejection (non zero T_l)

Open-loop

$$\omega_{ss} = A \cdot K_{ol} \cdot \omega_{ref} - B \cdot T_l$$
$$= \omega_{ref} - B \cdot T_l$$

Closed-loop

$$\omega_{ss} = \frac{A \cdot K_{cl}}{1 + A \cdot K_{cl}} \omega_{ref} - \frac{B}{1 + A \cdot K_{cl}} \cdot T_l$$

Numerical Examples

$$\tau = \frac{1}{60}\sec,\ \ A = 10,\ \ B = 50,\ \ \omega_{ref} = 100\ rad/\sec,\ T_l = 0.1$$

Open-loop

$$\omega_{ss} = 100(50)(0.1) = 95\,rad/\sec\ \rightarrow 5\%\ error$$

Closed-loop

$$T_l = 0,\ \ \omega_{ss} = \frac{A \cdot K_{cl}}{1 + A \cdot K_{cl}}\omega_{ref} = \frac{99}{100}100 = 99\ rad/\sec$$

$$\omega_{ss} = \frac{A \cdot K_{cl}}{1 + A \cdot K_{cl}}\omega_{ref} - \underbrace{\frac{B}{1 + A \cdot K_{cl}}}_{0.5\text{로 만들자}} \cdot T_l = 99 - 0.05 = 98.95$$

Reject disturbances by a factor of at least 100 compared with the open loop system,

$$1 + A \cdot K_{cl} = 100$$

$$K_{cl} = 9.9$$

0.051% variation from the speed w/o disturbance

Sensitivity of System Gain to Parameter Changes

Motor Control

Open-loop

$$\Omega_m(s) = \frac{A}{\tau s + 1} \cdot K_{ol} \cdot \Omega_{ref}(s) - \frac{B}{\tau s + 1} \cdot T_l(s)$$

Closed-loop

$$\Omega_m(s) = \underbrace{\frac{K_{cl} \cdot A}{\tau s + 1 + K_{cl} \cdot A}}_{T_{cl}(s)} \cdot \Omega_{ref}(s) - \frac{B}{\tau s + 1 + K_{cl} \cdot A} \cdot T_l(s)$$

DC motor

Sensitivity of System Gain to Parameter Changes

Motor Gain Variations $A \rightarrow A + \delta A$

Open-loop

: the nominal gain

$$T_{ol} + \delta T_{ol} = K_{ol}\left(A + \delta A\right) = K_{ol}A + K_{ol}\delta A = T_{ol} + K_{ol}\delta A$$

$$T_{ol} = T_{ol}(s=0)$$

$$\delta T_{ol} = K_{ol} \cdot \delta A$$

$$\frac{\delta T_{ol}}{T_{ol}} = \frac{\delta A}{A}$$

; → 10 % error in A would yield a 10 % error in T_{ol}

$$\frac{\delta T_{ol}}{T_{ol}} \Big/ \frac{\delta A}{A} = 1$$

; sensitivity, s, of the gain from Ω_{ref} **to** Ω_m **with respect to the parameter A**

Sensitivity of System Gain to Parameter Changes

Motor Gain Variations $A \to A + \delta A$

Closed-loop $T_{cl} = \dfrac{A \cdot K_{cl}}{1 + A \cdot K_{cl}}$ $\qquad T_{cl} + \delta T_{cl} = \dfrac{(A + \delta A) \cdot K_{cl}}{1 + (A + \delta A) \cdot K_{cl}}$

$$\delta T_{cl} = \frac{dT_{cl}}{dA} \cdot \delta A \qquad \frac{\delta T_{cl}}{T_{cl}} = \frac{A}{T_{cl}} \frac{dT_{cl}}{dA} \frac{\delta A}{A}$$

$$\frac{\delta T_{cl}}{T_{cl}} \Big/ \frac{\delta A}{A} = \frac{A}{T_{cl}} \frac{dT_{cl}}{dA}$$ **; sensitivity**

$$S_A^{T_{cl}} \triangleq sensitivity\ of\ T_{cl}\ with\ respect\ to\ A$$

$$\triangleq \frac{A}{T_{cl}} \frac{dT_{cl}}{dA}$$

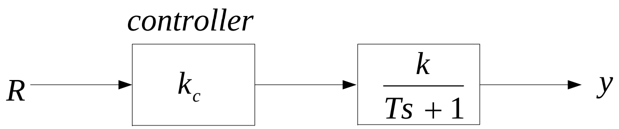
$$= \frac{A}{A \cdot K_{cl} \big/ (1 + A \cdot K_{cl})} \cdot \frac{(1 + A \cdot K_{cl}) \cdot K_{cl} - K_{cl}(A \cdot K_{cl})}{(1 + A \cdot K_{cl})^2}$$

$$= \frac{1}{1 + A \cdot K_{cl}}$$ **; less sensitive compared to open loop**

$$\left(S_A^{T_{ol}} = 1\right); = \frac{1}{1 + 10(9.9)} = \frac{1}{100}$$

Open/Closed loop control systems

open loop



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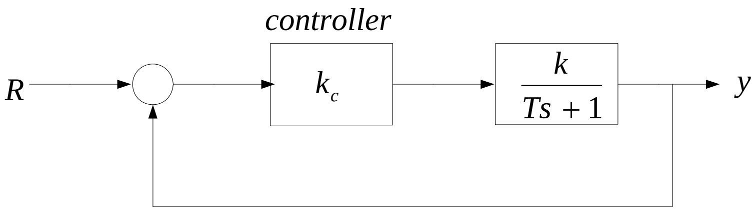
closed loop



Open loop Transfer function 에서

$$\frac{y}{R} = k_c \frac{1}{Ts+1} \rightarrow k_c = \frac{1}{k}$$ 이 되어야 **error=0** 이 된다.

Closed loop Transfer function 에서

$$\frac{y}{R} = \frac{k_p \dfrac{k}{Ts+1}}{1 + k_p \dfrac{k}{Ts+1}} = \frac{k_p k}{Ts + 1 + k_p k}$$

Steady/ state error for $R(s) = \frac{1}{s}$

Open loop Transfer function 에서 $e_{ss} = 0$

Closed loop Transfer function 에서 $e_{ss} = 1 - \frac{k_p k}{1 + k_p k} = \frac{1}{1 + k_p k}$

Open/Closed loop control systems

예외의 경우에 대한 예>

$k = 10, \Delta k = 1$ 인 model error 가 있을 경우

Open loop Transfer function 에서

$$\frac{y}{R} = \frac{1}{k_c}\frac{k+\Delta k}{Ts+1}, \quad y(t) = \frac{k+\Delta k}{k} = 1.1$$

(10% steady state error)

Closed loop Transfer function 에서

$$e_{ss} = \frac{1}{1+k_p k} \qquad \text{let} \quad k_p = \frac{100}{k} \Rightarrow e_{ss} = \frac{1}{1+\frac{100}{k}k} = \frac{1}{101}$$

$$k+\Delta k, \quad e_{ss} = \frac{1}{1+\frac{100}{k}(k+\Delta k)} = \frac{1}{1+10(11)} = \frac{1}{111} = 0.009$$

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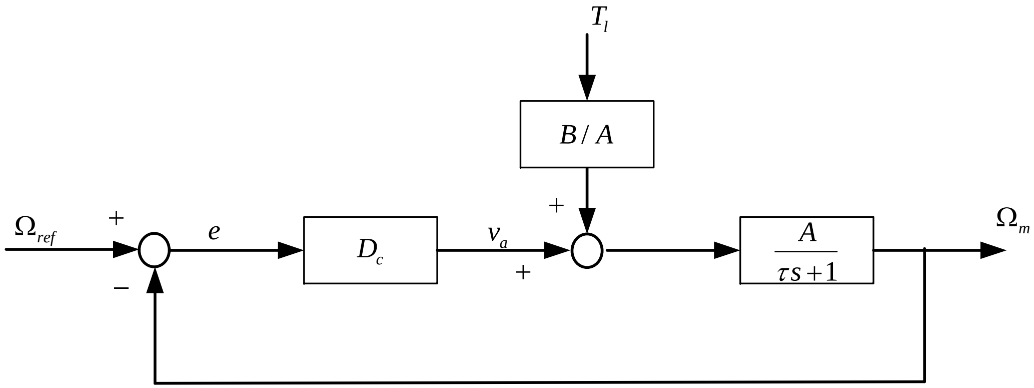
The Classical Three-term Controller (PID controller)

Motor Speed Control (unity feedback)



$$v_a = K_P \cdot e + K_I \int_{t_0}^{t} e(\tau)d\tau + K_D \frac{de}{dt}$$

$$\frac{U(s)}{E(s)} = \frac{V_a(s)}{E(s)} = D_c(s) = K_P + \frac{K_I}{s} + K_D s$$

P- control

$$\Omega_m(s) = \frac{A \cdot K_{cl}}{\tau s + 1 + A \cdot K_{cl}} \Omega_{ref}(s) + \frac{B}{\tau s + 1 + A \cdot K_{cl}} T_l(s)$$

$$K_{cl} = K_P,\ K_I = 0,\ K_D = 0$$

PI- control

$$v_a = K_P \left(\Omega_{ref} - \Omega_m\right) + K_I \frac{\Omega_{ref} - \Omega_m}{s}$$

$$D_c(s) = K_P + \frac{K_I}{s}$$

$$\Rightarrow \Omega_m(s) = \frac{A \cdot (K_P s + K_I)}{\tau s^2 + (A \cdot K_P + 1)s + A \cdot K_I} \Omega_{ref}(s) + \frac{Bs}{\tau s^2 + (A \cdot K_P + 1)s + A \cdot K_I} T_l(s)$$

Steady state $\omega_m = \omega_{ref} + 0 \cdot T_l = \omega_{ref}$

The Classical Three-term Controller (PID controller)

Motor Speed Control (unity feedback)

PID- control $$D_c(s)=\frac{U(s)}{E(s)}=K_P+\frac{K_I}{s}+K_D s$$
$$=K_P\left[1+\frac{1}{T_I s}+T_D s\right]$$

Non-negligible armature inductance $L_a \neq 0$

$$\left[\left(L_a\cdot s+R_a\right)\left(J_m s+b\right)+K_t K_e\right]\Omega_m=K_t v_a-\left(L_a\cdot s+R_a\right)T_l$$
$$\Rightarrow\left(a_0 s^2+a_1 s+a_2\right)\Omega_m(s)=K_t v_a-\left(L_a\cdot s+R_a\right)T_l$$

PID control, $V_a=\left[K_P+\frac{K_I}{s}+K_D s\right]\left(\Omega_{ref}-\Omega_m\right)$

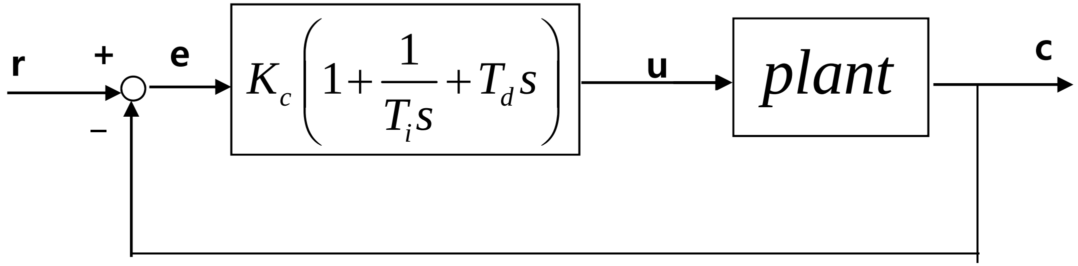
$$\Rightarrow\left(a_0 s^2+a_1 s+a_2+K_t\left(K_P+\frac{K_I}{s}+K_D s\right)\right)\Omega_m(s)=K_t\left(K_P+\frac{K_I}{s}+K_D s\right)\Omega_{ref}(s)-\left(L_a\cdot s+R_a\right)T_l$$

$$\Rightarrow\Omega_m(s)=\frac{K_t K_D s^2+K_t K_P s+K_t K_I}{a_0 s^3+\left(a_1+K_t K_D\right)s^2+\left(a_2+K_t K_P\right)s+K_t K_I}$$
$$+\frac{-\left(L_a s^2+R_a s\right)}{a_0 s^3+\left(a_1+K_t K_D\right)s^2+\left(a_2+K_t K_P\right)s+K_t K_I}T_l$$

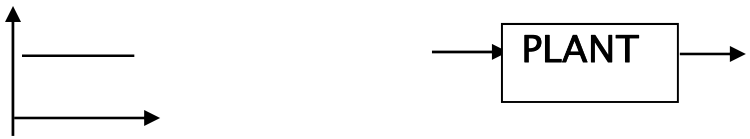
Ex 4.6) Ch.eq. $a_0 s^3+\left(a_1+K_t K_D\right)s^2+\left(a_2+K_t K_P\right)s+K_t K_I=0$

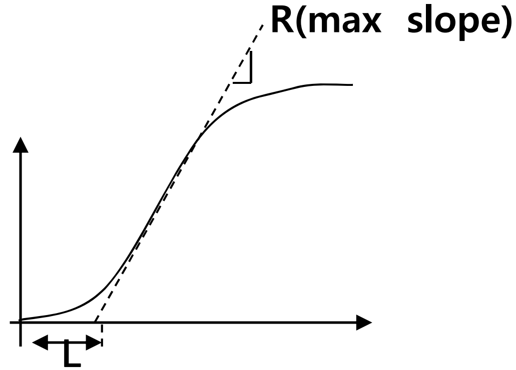
➔ 3 parameters for 3 ch. Roots ; arbitrary

Ziegler-Nicholas Tuning Rules for PID controllers



Method 1) Transient response method





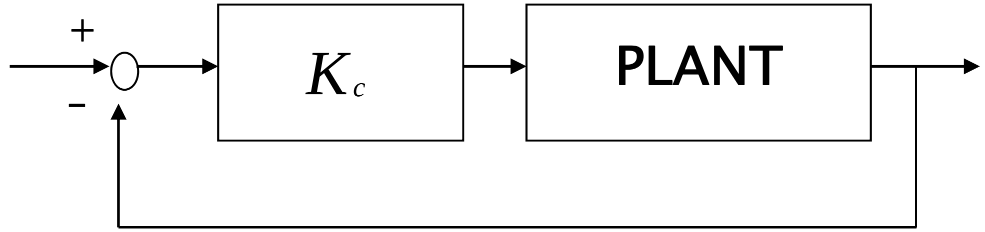
Step input

$$
\begin{cases}
K_c = \frac{1}{RL} \quad for \quad P-control \\
K_c = \frac{0.9}{RL}, \quad T_i = 3.3L \quad for \quad PI-control \\
K_c = \frac{1.2}{RL}, \quad T_i = 2L, \quad T_d = 0.5L \quad for \quad PID-control
\end{cases}
$$

- **This method works good if the unit step response is (sigmod) shaped**

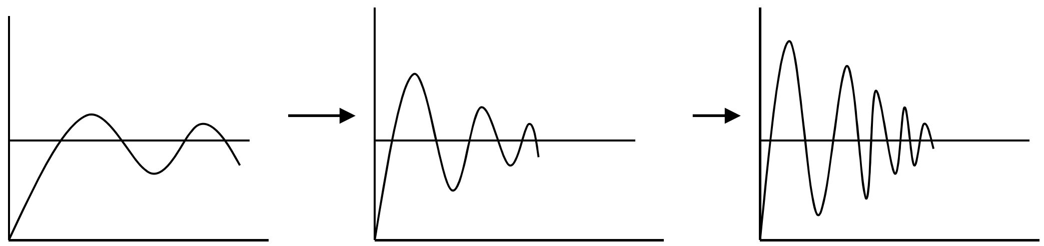
Ziegler-Nicholas Tuning Rules for PID controllers

Method 2) Ultimate sensitivity method



$$K_c \rangle\rangle 1,\ \omega_n \rightarrow increase$$

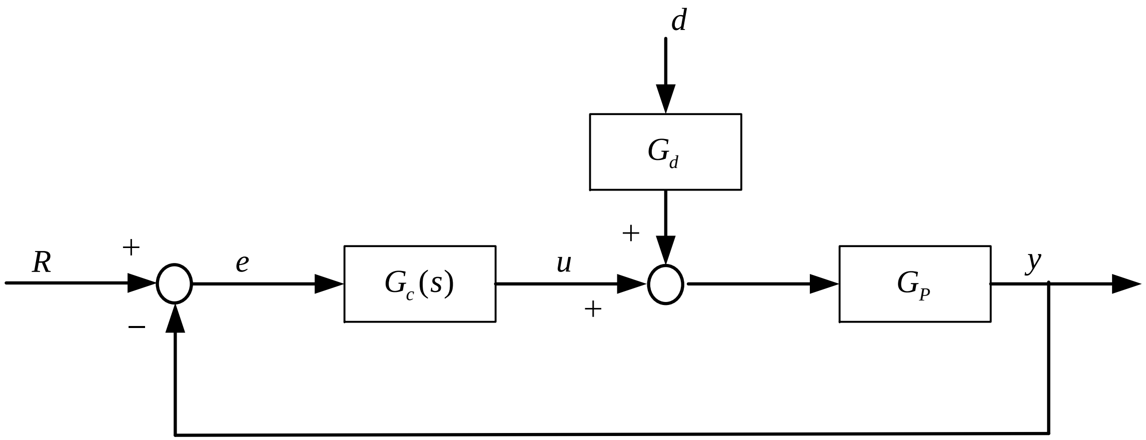
$$\zeta \rightarrow decrease$$



- **Increase** K_c **until you hit the stability limit**

$$\begin{cases} K_c = 0.5K_U \quad for \quad P-control \\ K_c = 0.45K_U, \quad T_i = 0.83P_U \quad for \quad PI-control \\ K_c = 0.6K_U, \quad T_i = 0.5P_U \quad T_i = 0.125P_U \quad for \quad PID-control \end{cases}$$

PID controllers – reviews, gain tuning



1)

$$u(t) = K_P e(t) + K_I \int_{t_0}^{t} e(t)dt + K_D \frac{de}{dt}$$

$$G_c(s) = K_P + \frac{K_I}{s} + K_D s$$

$$= K_P \left[1 + \frac{1}{T_i s} + T_D s \right]$$

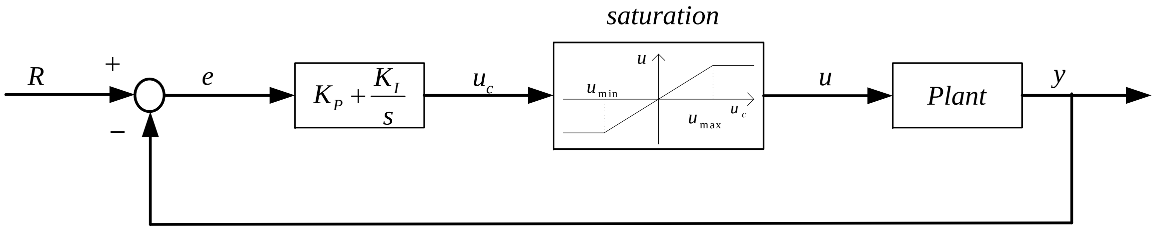
2) Ziegler-Nichols Tuning Rules

3) I - control : no steady-state error, disturbance rejection

D - control : additional damping

Integrator Anti-windup

- **PI- control**



1. **Large R → Large u_c → u saturation → integration of error, Large u_c**
   u_c increases until the sign of the error changes ➔ very large overshoot
   (wind up) poor transient response

2. **Solution to this problem : integrator anti-windup**
   - " turn off" the integral action when the actuator saturates
   Digital control : $"if\ |u| = u_{max},\ K_I = 0"$
   (implementation-next page) Fig 4.22

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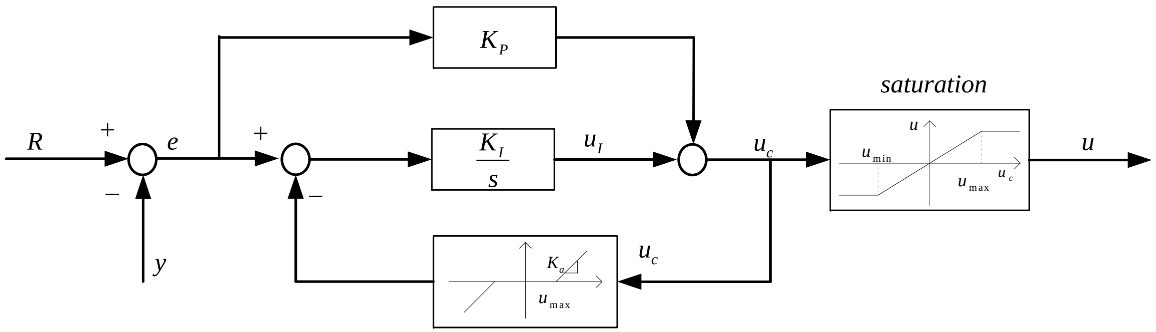
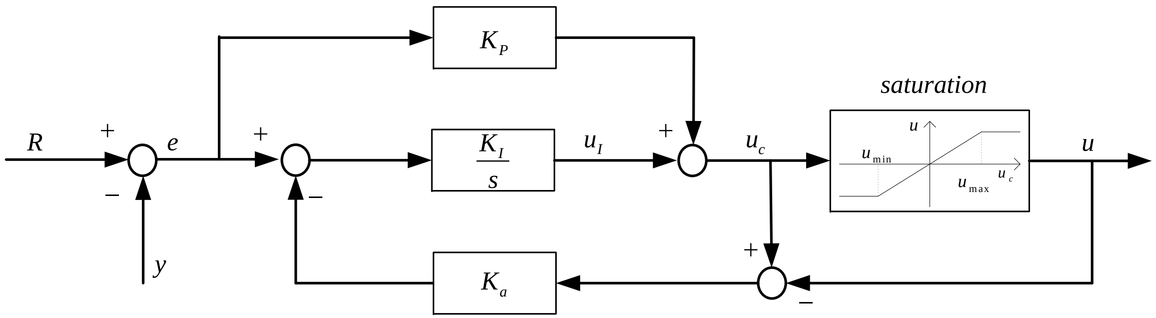
Integrator Anti-windup

- **PI- controller with anti-windup (Fig 4.22)**



(a)

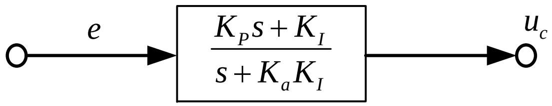


(b) Implementation of anti-windup with a single non-linearity

saturation : $u = \pm u_{\max}$

Integrator Anti-windup

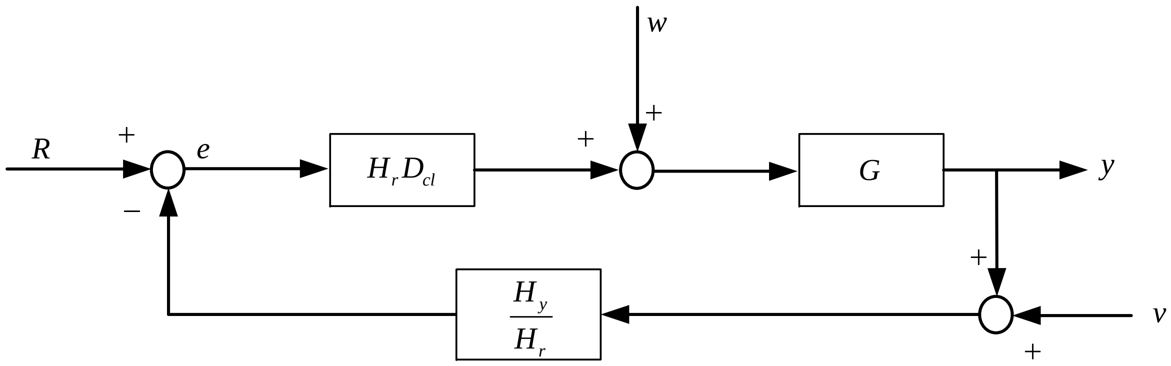
- **First order lag equivalent (during saturation)**



(d)

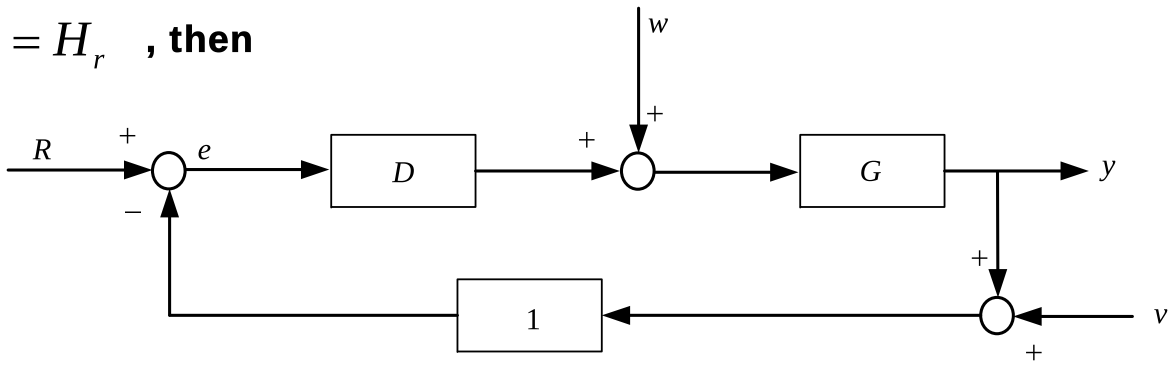
➔ K_a **Should be large enough**

The anti-windup : provide local feedback to make the controller stable alone when the main loop is opened by signal saturation (and any circuit which does this will perform as anti-windup)



Integrator Anti-windup

If $H_y = H_r$ **, then**



Integrator Anti-windup

Ex) The Effect of anti-windup

Consider a plant with the following transfer function for small signals

$$G(s) = \frac{1}{s}$$

and a PI controller,

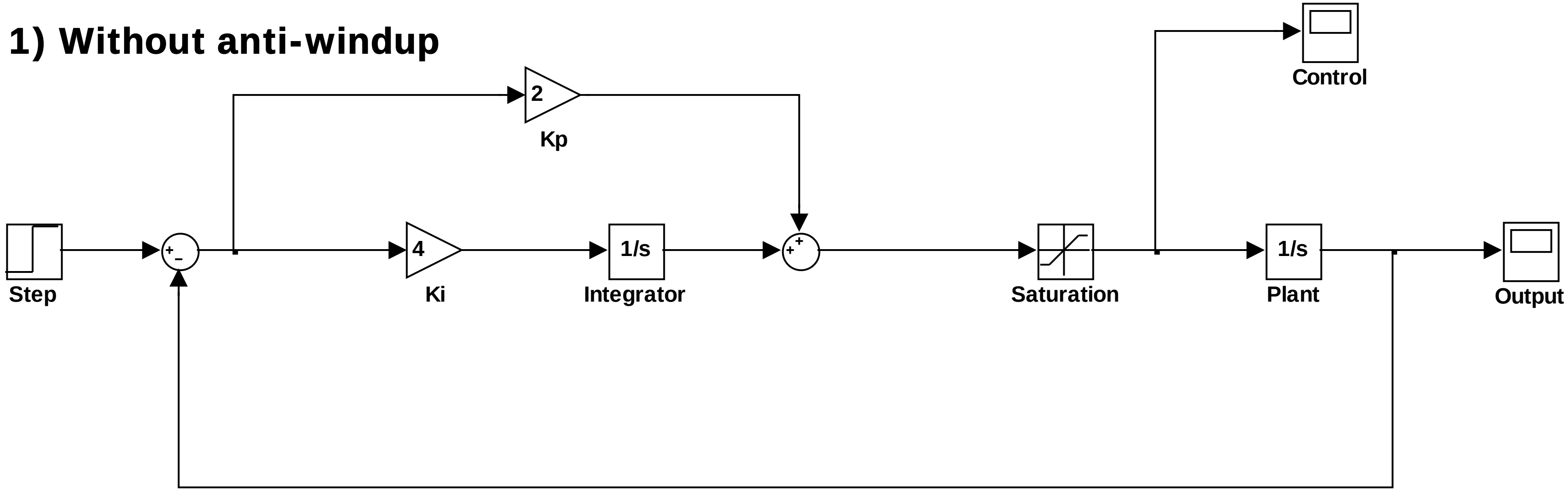
$$D_c(s) = k_P + \frac{k_I}{s} = 2 + \frac{4}{s}$$

in the unity feedback configuration. The input to the plant is limited to ± 1.0 **.**

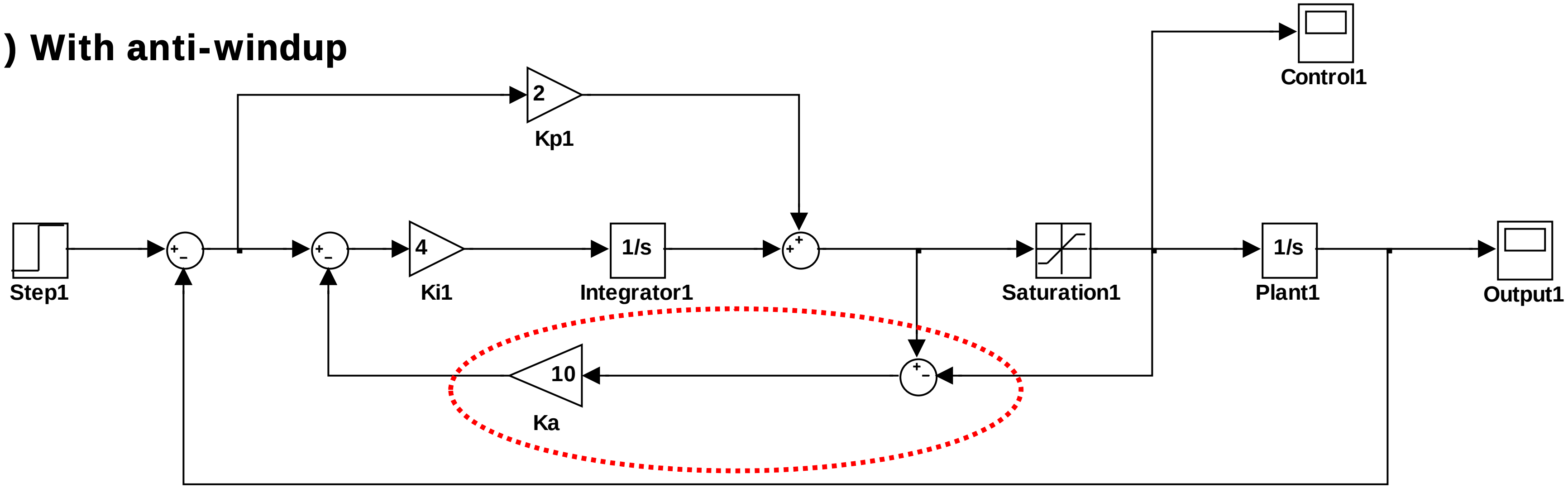
Study the effect of anti-windup on the response of the system

Integrator Anti-windup

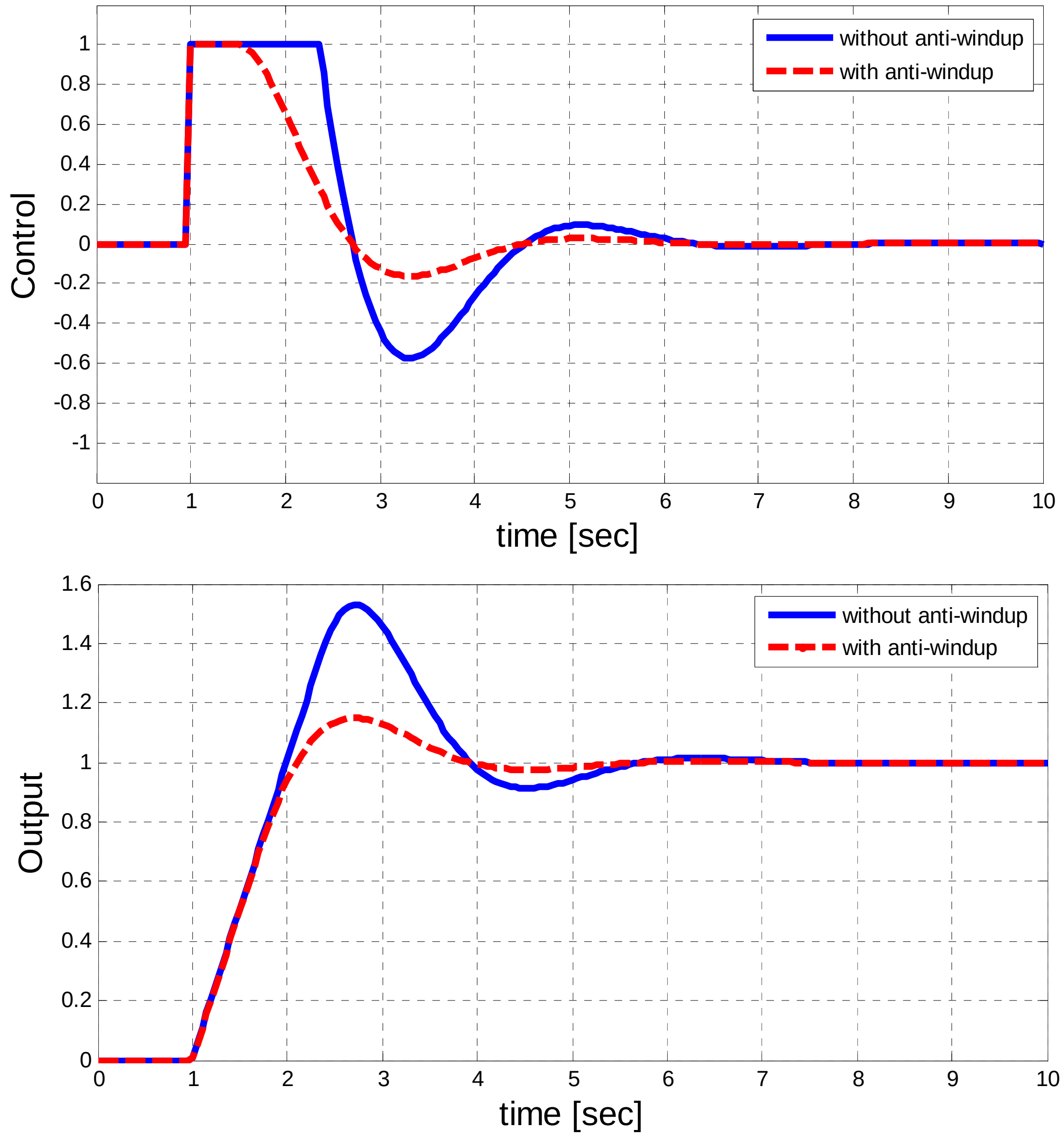
1) Without anti-windup



2) With anti-windup



Integrator Anti-windup



Steady-state Tracking and System Type

Fig 4.26

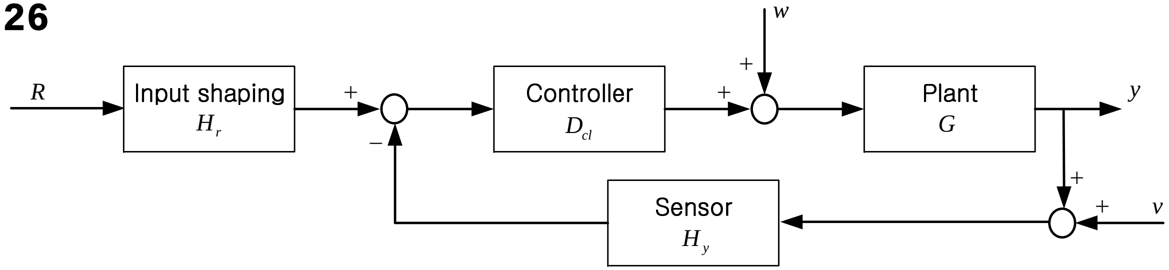
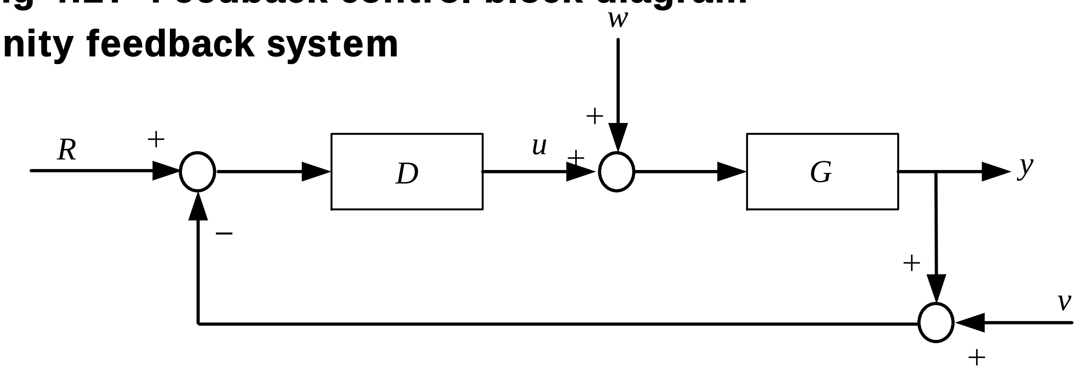


Fig 4.26, Fig 4.27 Feedback control block diagram

Fig 4.28 unity feedback system



$$Y = \frac{DG}{1+DG}R + \frac{G}{1+DG}W - \frac{DG}{1+DG}V$$

$$U = \frac{D}{1+DG}R - \frac{DG}{1+DG}W - \frac{D}{1+DG}V$$

Actual error ➔ $E = R - Y = \frac{1}{1+DG}R - \frac{G}{1+DG}W + \frac{DG}{1+DG}V$

(not error signal in feedback loop)

Steady-state Tracking and System Type

- **Definition**

  The sensitivity : $S = \dfrac{1}{1+DG}$

  The complementary sensitivity : $T = 1 - S = \dfrac{DG}{1+DG}$

  ➔ $E = SR - SGW + TV$

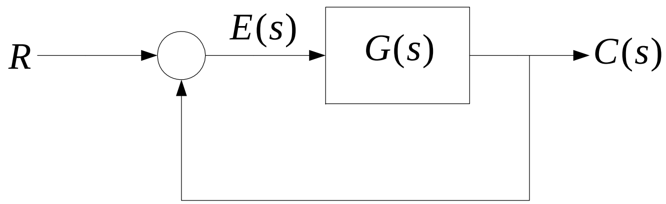
- **A major goal of control**

  to keep the error small for any input and the face of parameter changes

  both S **and** T **should be kept small**

  ~ frequency band

Steady state Error in Unity Feedback control systems



=

$$G(s) = \frac{K(T_a s + 1)(T_b s + 1)\cdots(T_n s + 1)}{S^N (T_1 s + 1)(T_2 s + 1)\cdots(T_p s + 1)}$$

N=0 ; Type 0,　　N=1 ; Type 1,　　N=2 ; Type 2

Steady state Error 를 생각해보자

$$E(s) = \frac{1}{1+G(s)} R(s)$$

1) $R(s) = \frac{1}{s}$ **일 경우 (unit step input),**

$$e_{ss} = \lim_{s->0} sE(s) = \lim_{s->0} s\frac{1}{1+G(s)}\frac{1}{s} = \frac{1}{1+G(0)}$$

Static position error constant k_p

$$k_p = \lim_{s->0} G(s) = G(0) = \begin{cases} k & ;Type\ 0\ \ system \\ \infty & ;Type\ 1\ \ system \\ \infty & ;Type\ 2\ \ system \end{cases}$$

$$e_{ss} = \frac{1}{1+k_p} = \begin{cases} \frac{1}{1+k_p} & ;Type\,0\ \ system \\ 0 & ;Type\ 1,\ 2,..\ system \end{cases}$$

Steady state Error in Unity Feedback control systems

2) $R(s)=\frac{1}{s^2}$ 일 경우 (unit step input),

=

$$e_{ss}=\lim_{s->0} s\frac{1}{1+G(s)}\frac{1}{s^2}=\lim_{s->0}\frac{1}{s(G(s)+1)}=\lim_{s->0}\frac{1}{sG(s)}$$

Static velocity error constant k_v

$$k_v=\lim_{s->p} SG(s)=G(0)=\begin{cases}0 & ;Type\ 0\ system\\ k & ;Type\ 1\ system\\ \infty & ;Type\ 2,3..\ system\end{cases}$$

$$e_{ss}=\frac{1}{k_v}=\begin{cases}\infty & ;Type\ 0\ system\\ \frac{1}{k} & ;Type\ 1\ system\\ 0 & ;Type\ 2,3,..\ system\end{cases}$$

3) $R(s)=\frac{1}{s^3}$ 일 경우 (unit step input),

$$e_{ss}=\lim_{s->0} s\frac{1}{1+G(s)}\frac{1}{s^3}=\lim_{s->0}\frac{1}{s^2G(s)}$$

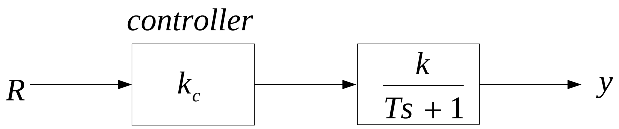
Static acceleration error constant k_a

$$k_a=\lim_{s->0} S^2G(s)=G(0)=\begin{cases}0 & ;Type\ 0\ system\\ 0 & ;Type\ 1\ system\\ k & ;Type\ 2\ system\\ \infty & ;Type\ 3\ system\end{cases}$$

$$e_{ss}=\frac{1}{k_a}=\begin{cases}\infty & ;Type\ 0,1\ system\\ \frac{1}{k} & ;Type\ 2\ system\\ 0 & ;Type\ 3,4,..\ system\end{cases}$$

Open/Closed loop control systems

open loop



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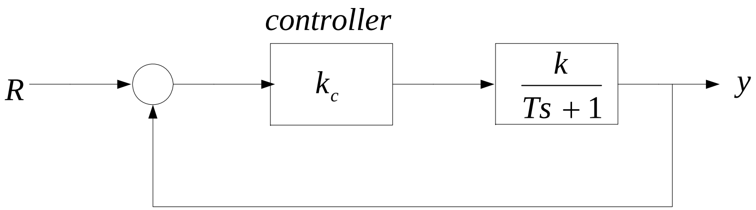
closed loop



Open loop Transfer function 에서

$$\frac{y}{R} = k_c \frac{1}{Ts+1} \rightarrow k_c = \frac{1}{k}$$ 이 되어야 **error=0** 이 된다.

Closed loop Transfer function 에서

$$\frac{y}{R} = \frac{k_p \frac{k}{Ts+1}}{1 + k_p \frac{k}{Ts+1}} = \frac{k_p k}{Ts + 1 + k_p k}$$

Steady/ state error for $R(s) = \frac{1}{s}$

Open loop Transfer function 에서 $e_{ss} = 0$

Closed loop Transfer function 에서 $e_{ss} = 1 - \frac{k_p k}{1 + k_p k} = \frac{1}{1 + k_p k}$

Open / Closed loop control systems

예외의 경우에 대한 예>

= $k = 10, \Delta k = 1$ 인 model error 가 있을 경우

Open loop Transfer function 에서

$$\frac{y}{R} = \frac{1}{k_c}\frac{k+\Delta k}{Ts+1}, \quad y(t) = \frac{k+\Delta k}{k} = 1.1$$ (10% steady state error)

Closed loop Transfer function 에서

$$e_{ss} = \frac{1}{1+k_p k} \quad \text{let} \quad k_p = \frac{100}{k} \Rightarrow e_{ss} = \frac{1}{1+\frac{100}{k}k} = \frac{1}{101}$$

$$k+\Delta k, \quad e_{ss} = \frac{1}{1+\frac{100}{k}(k+\Delta k)} = \frac{1}{1+10(11)} = \frac{1}{111} = 0.009$$

System Type for Reference Tracking

- **System type : the degree of the polynomial for which the steady-state system error is a nonzero finite constant**

- **type 0 : finite error to be a step input which is an input polynomial of zero degree**

Reference input $\left(W = V = 0\right)$

$$E = \frac{1}{1 + DG} R = SR$$

$$r(t) = t^k 1(t) \qquad R = \frac{1}{s^{k+1}}$$

Steady state error

$$\lim_{t \to \infty} e(t) = e_{ss} = \lim_{s \to 0} sE(s)$$

$$= \lim_{s \to 0} s \frac{1}{1 + DG} R(s)$$

$$= \lim_{s \to 0} s \frac{1}{1 + DG} \frac{1}{s^{k+1}}$$

System Type for Reference Tracking

Step input

if DG has no pole at the origin, i.e. $DG(0) \neq 0$

$$e_{ss} = \lim_{s \to 0} s \frac{1}{1+DG} \frac{1}{s} = \frac{1}{1+DG(0)} \neq 0$$

Type 0 , $K_P \triangleq DG(0)$ **: “position error constant**

Type 1 , $DG = \frac{1}{s} G_0(s)$

Type 2 , $DG = \frac{1}{s^2} G_0(s)$

System Type for Reference Tracking

System

$$DG = \frac{G_0(s)}{s^n} = \frac{1}{s^n} G_0(s)$$ **:** $G_0(s)$ **no pole at the origin**

$$G_0(s) = K_n$$

$$e_{ss} = \lim_{s\to 0} s \frac{1}{1+\frac{1}{s^n}G_0(s)} \frac{1}{s^{k+1}}$$

$$= \lim_{s\to 0} \frac{s^n}{s^n + G_0(s)} \frac{1}{s^k}$$

If $n > k$ **, then** $e = 0$

$n < k$ **, then** $e \to \infty$

$n = k \neq 0$ **, then** $e = \frac{1}{K_n}$ $\quad$ *Type n*

$n = k = 0$ **, then** $e = \frac{1}{1+K_0}$ $\quad$ *Type 0*

Type 0 ; $n = 0,\ k = 0$

$$e_{ss} = \frac{1}{1+G_0(s)} = \frac{1}{1+K_0}$$ $\quad K_0 : position\ error\ const\ (= K_p)$

Type 1 ; $n = 1,\ k = 1$

$$e_{ss} = \frac{1}{G_1(s)} = \frac{1}{K_1} = \frac{1}{K_v}$$ $\quad K_v : velocity\ const$

Type 2 ; $n = 2,\ k = 2$

$$e_{ss} = \frac{1}{G_0(s)} = \frac{1}{K_2} = \frac{1}{K_a}$$ $\quad K_a : acceleration\ const$

System Type for Reference Tracking

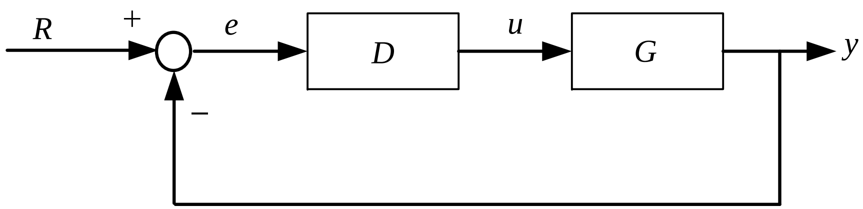
Table 4.3

Type	Step (position)	Ramp (velocity)	Parabola (acceleration)
0	$\frac{1}{1+K_p}$	∞	∞
1	0	$\frac{1}{K_v}$	∞
2	0	0	$\frac{1}{K_a}$

System Type for Reference Tracking

EX 4.10 Motor

$$G = \frac{A}{\tau s + 1}$$



Proportional

$$D = K_P$$

$$e = \frac{1}{1+DG} R$$

$$DG = \frac{K_P A}{\tau s + 1} = \frac{1}{s^0} \frac{K_P A}{\tau s + 1}$$

: type 0

Unit step response

$$e_{ss} = \lim_{s \to 0} s \frac{1}{1+DG} \frac{1}{s} = \frac{1}{1+DG(0)} = \frac{1}{1+K_0}$$

: finite error

System Type for Reference Tracking

EX 4.11 PI control $G = \dfrac{A}{\tau s + 1}$

$$D = K_P + K_I / s$$

$$DG = \frac{K_P s + K_I}{s(\tau s + 1)} A \quad \textbf{: type 1}$$

Unit step response ; $R = \dfrac{1}{s}$

$$e_{ss} = \lim_{s \to 0} s \frac{1}{1 + \dfrac{K_P s + K_I}{s(\tau s + 1)} A} \frac{1}{s} = 0$$

Unit ramp response ; $R = \dfrac{1}{s^2}$

$$e_{ss} = \lim_{s \to 0} s \frac{1}{1 + \dfrac{K_P s + K_I}{s(\tau s + 1)} A} \frac{1}{s^2} = \lim_{s \to 0} \frac{1}{sDG(s)} = \frac{1}{K_I A}$$