System Control

10. State Feedback and Pole Placement

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State Feedback

- **State Feedback**

$$\dot{x} = Ax + Bu$$
$$u = -Kx$$
$$\dot{x} = (A - BK)x$$

(A, B) **Controllable : arbitrary pole assignment**

In general, full states are not available ⟶ **Observer.**

$$y = Cx$$
$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$
$$e = x - \hat{x}$$
$$\dot{e} = (A - LC)e$$

(A, C) **Observable : arbitrary pole assignment**

$$\hat{x} \to x$$
$$u = -K\hat{x} \to \text{2n system}$$
$$\dot{x} = Ax + Bu$$
$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$
$$y = Cx$$
$$u = -K\hat{x}$$

$2n$ eigenvalues.

$(A - BK), (A - LC)$: separation property

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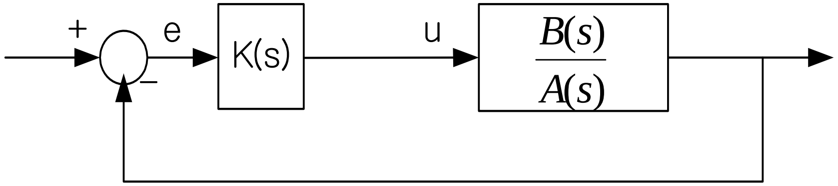

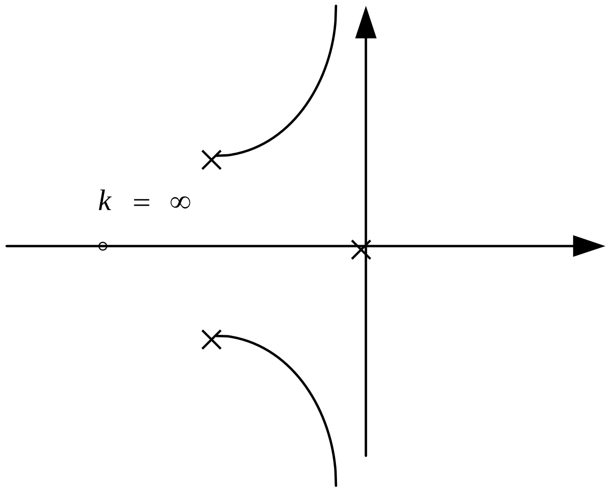

Location of closed loop poles are restricted in root locus.

Q: Is it possible to assign arbitrary closed loop poles?
A: Yes, If the system is controllable & all state variables are feed back single input system in controllable canonical form.

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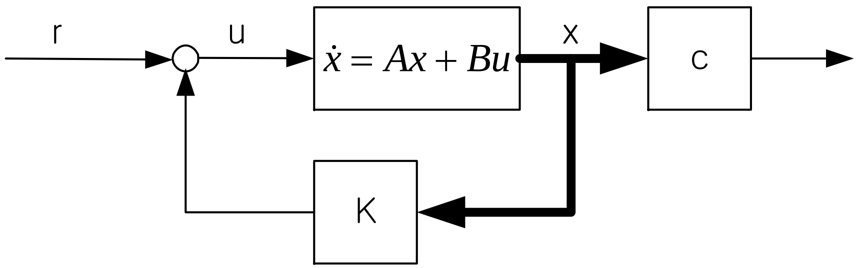
$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \cdots & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$\det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_n = 0$$



$$u = -Kx + r$$

$$= -\begin{bmatrix} k_1 & k_2 & k_3 & \cdots & k_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + r$$

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Closed loop System

$$
\begin{aligned}
\dot{x} &= Ax + B[-Kx + r] \\
&= [A - BK]x + Br \\
&= \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ k_1 & \cdots & \cdots & k_n \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & & & 1 \\ -(a_n + k_1) & \cdots & \cdots & \cdots & -(a_1 + k_n) \end{bmatrix} x + Br
\end{aligned}
$$

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Closed loop Ch. Eq

$$s^n + (a_1 + k_n)s^{n-1} + \cdots + (a_{n-1} + k_2)s + (a_n + k_1) = 0$$

$\{\lambda_1 \;,\; \lambda_2 \;,\; \cdots \lambda_n\}$: Desired closed loop poles

λ : symmetric with respect to the real axis of s-plane

$$(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_m) = 0 \Rightarrow$$

$$s^n + [-(\lambda_1 + \lambda_2 + \cdots + \lambda_n)]s^{n-1} + [\lambda_1\lambda_2 + \cdots]s^{n-2} + [(-\lambda_1)(-\lambda_2)\cdots(-\lambda_n)] = 0$$

$$\Rightarrow s^n + a_{c1}s^{n-1} + a_{c2}s^{n-2} + \cdots + a_{cn} = 0$$

$$a_{n+1-i} + k_i = a_{c(n+1-i)}$$

k_i: State feed back control gain

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Single-Input Controllable System not in controllable canonical form

$$\dot{x} = Ax + Bu, \quad rank[\underbrace{B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B}_{\Rightarrow M}] = \underbrace{n}_{, M^{-1} exist}$$

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ \vdots & & a_1 & 1 & 0 \\ a_2 & \iddots & \iddots & 0 & 0 \\ a_1 & \iddots & & \vdots & \vdots \\ 1 & & & 0 & 0 \end{bmatrix}$$

$$T = MW \qquad T^{-1} = W^{-1}M^{-1} \qquad x = Tx_c$$

$$T\dot{x} = ATx_c + Bu$$

$$T\dot{x}_c = T^{-1}ATx_c + T^{-1}Bu \qquad : \text{Controllable Canonical form}$$

$$= A_c x_c + B_c u$$

$$u = -K_c x_c + V$$

$$= -\underbrace{K_c T^{-1}}_{K} x + V$$

Controllability $\Rightarrow$ arbitrary
$\Leftarrow$ pole assignment.
Necessary and sufficient condition

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<u>**Discrete Time Case**</u>

$$x(k+1) = Gx(k) + Hu(k) \quad , \qquad u(k) = -Kx(k)$$

$$= (G - HK)x(k)$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ & & \ddots & & 0 \\ & & & & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} x(k)$$

Nilpotent matrix

- Characteristic Equation

$$z^n = (z-0)(z-0)\cdots(z-0) = 0$$

$$(G - HK)^n = 0$$

- Dead beat response (Finite Settling Time control)

$$x(n) = (G - HK)^n x(0) = 0 \qquad for \quad any \quad x(0)$$

In general, Dead beat response should not be attempted with a small sampling time.

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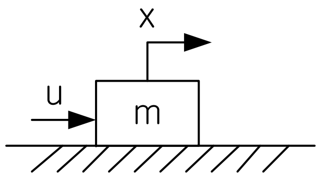
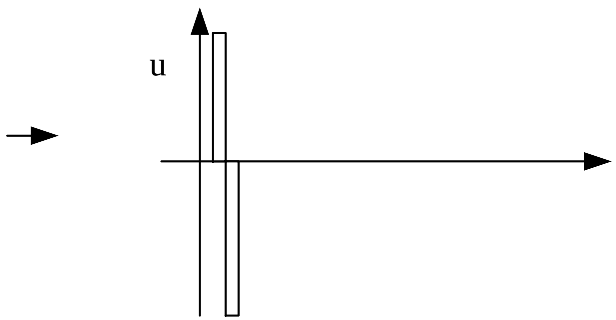
<u>**Ex**</u>



$$\frac{x}{u} = \frac{1}{ms^2} \rightarrow \frac{T^2(z+1)}{2(z-1)^2}$$

If T=0.0000001 sec



Q: Is X directly accessible for state feed back?
A: No, in general.

State Feedback Control and Pole Placement

Observer / State estimator

$$\hat{x};\quad state \quad estimator$$

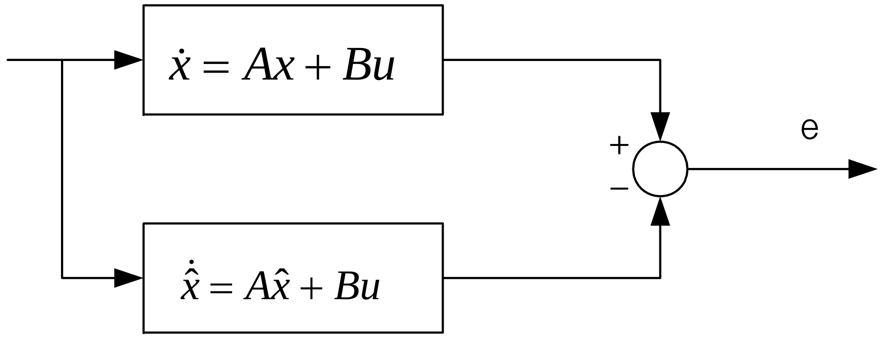
$$\hat{x} \to x \quad in \quad some \quad sense$$

Looks reasonable to replace

$$u = -Kx \quad by \quad u = -K\hat{x}$$

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

$$\dot{\hat{x}} = A\hat{x} + Bu$$



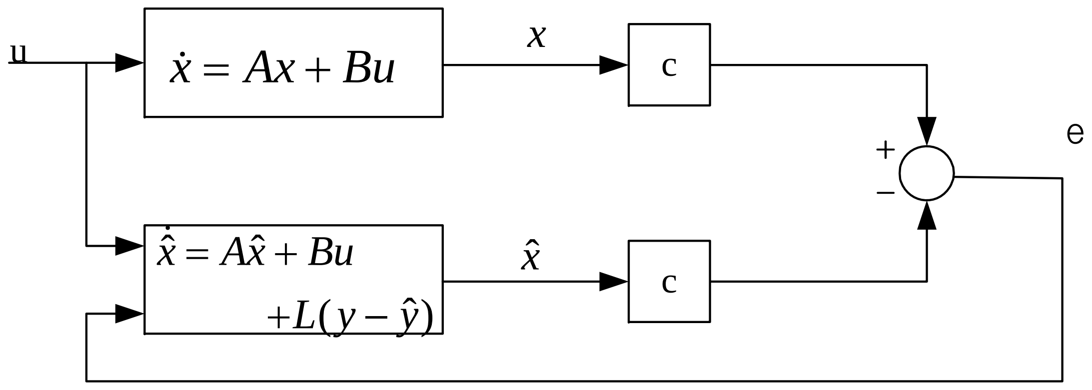
$$e = x - \hat{x}$$

$$\dot{e} = (\dot{x} - \dot{\hat{x}}) = A(x - \hat{x}) = Ae$$

State Feedback Control and Pole Placement

If A is asymptotically stable, then $\lim_{t\to\infty} e(t) = 0$

If A is not stable, then e $\to \infty$ **are t** $\to \infty$



$$\dot{\hat{x}} = A\hat{x} + Bu + L(cx - c\hat{x}) = (A - LC) = (A - LC)\hat{x} + Bu + LCx$$

$$\dot{e} = (A - LC)e$$

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System in obs. Canonical form

$$A = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & & \\ -a_3 & & & \ddots & \\ \vdots & & & & 1 \\ -a_n & 0 & & & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \qquad L = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{bmatrix}$$

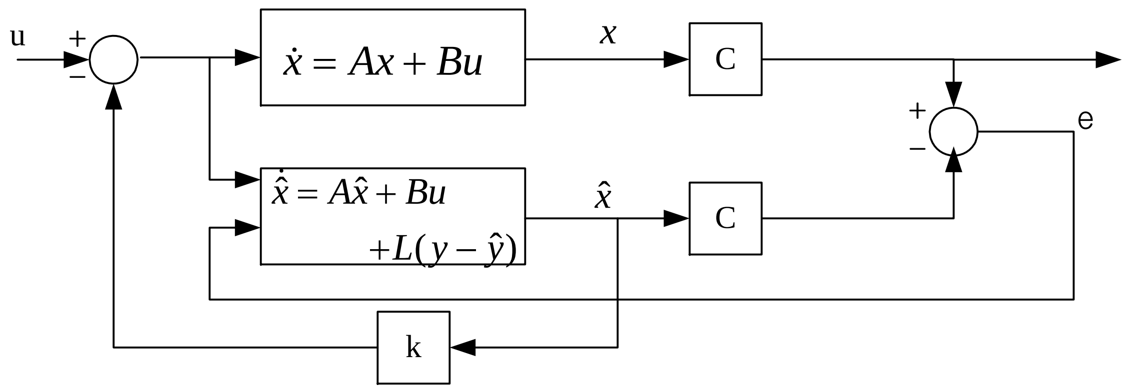
$$A - LC = \begin{bmatrix} -(a_1 + l_1) & 1 & 0 & \cdots & 0 \\ -(a_2 + l_2) & 0 & 1 & & \\ \vdots & & & \ddots & \\ \vdots & & & & 1 \\ -a_n & 0 & & & 0 \end{bmatrix}$$

(A,C) observable ⇔ Arbitrary pole assignment for (A-LC)

State Feedback Control / Observer

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$$
$$\hat{y} = C\hat{x}$$
$$u \quad = -K\hat{x} + r$$



$$\dot{x} = Ax + B(-K\hat{x} + r)$$
$$\dot{\hat{x}} = A\hat{x} + B(-K\hat{x} + r) + L(Cx - C\hat{x})$$

State Feedback Control / Observer

The order of the overall system is 2n $\Rightarrow$ 2n eigenvalues

$$\frac{d}{dt}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A-BK-LC \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} r$$

$$e = x - \hat{x}$$

$$\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

$$\dot{x} = Ax + B(-K\hat{x}) \pm BKx$$

$$= (A - BK)x + BK(e)$$

$$\dot{e} = (A - LC)e$$

$$\frac{d}{dt}\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r$$

State Feedback Control / Observer

- Eigenvalues are form

$$\det(sI - A) = \det\begin{bmatrix} sI-(A-BK) & -BK \\ 0 & sI-(A-LC) \end{bmatrix}$$

$$= \det(sI-(A-BK))\det(sI-(A-LC))$$

$$= 0 \qquad\qquad observerpoles$$

- n eigenvalues = regulator poles

$$x = Ax + Bu$$

$$u = -Kx \qquad \Rightarrow \text{ separation property}$$

- Selection of Best L

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$$

$$\dot{\hat{x}} = (A - LC)\hat{x}$$

$$\dot{e} = (A - LC) - Ln$$

State Feedback Control / Observer

- In general

i) For fast error convergence $\Rightarrow$ Large L
ii) If y is not reliable, contaminated by disturbance, and measurement $\Rightarrow$ small L

-Thus,
i) Select several , L is choose several pole locations of the observer
ii) Simulation
iii) Choose the best L , from the view point of overall system performance.

➔Compromise between speedy response and sensitivity to disturbances and noise

➔Kalman Filter

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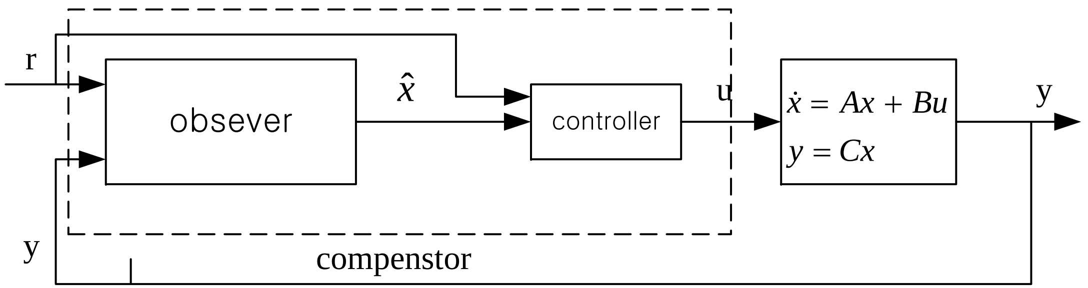
State Feedback Control / Observer

$$\dot{x} = Ax + Bu$$
$$u = -K\hat{x} + r$$
$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$$



$$\dot{\hat{x}} = A\hat{x} + B(-K\hat{x} + r) + LC(y - C\hat{x})$$
$$(sI - A + BK + LC)\hat{x}(s) = -L(-Y(S))$$

$$U = -K\hat{x}(s) + r$$
$$= K(sI - A + BK + LC)^{-1}(-Y) + \cdots$$

Discrete Time Observer

$$x(k+1) = G \cdot x(k) + H \cdot u(k)$$
$$y(k) = Cx(k)$$

Observer

$$\hat{x}(k+1) = G \cdot \hat{x}(k) + H \cdot u(k) + K_e\left[y(k) - C\hat{x}(k)\right]$$
$$e(k) = x(k) - \hat{x}(k)$$
$$e(k+1) = (G - K_e C)e(k)$$

$\hat{x}(k+1)$ is given in terms of $\underbrace{\{y(0), y(1), \ldots, y(k)\}}_{y_k}$

$$\because \begin{pmatrix} \hat{x}(k+1|k): \text{estimate of } x(k+1) \text{ based on } y_k \\ \hat{x}(k+1|k+1): \text{estimate of } x(k+1) \text{ based on } y_{k+1} \end{pmatrix}$$

$$-\hat{x}(k+1|k+1) = \hat{x}(k+1|k) + K_e[y(k+1) - C\hat{x}(k+1|k)] : \text{Current Observer.}$$

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Error Dynamics

$$
\begin{aligned}
e(k+1|k) &= x(k+1) - \hat{x}(k+1|k) \\
&= Gx(k) + Hu(k) - [Gx(k|k) + Hu(k)] \\
&= Gx(k) - G[\hat{x}(k|k-1) + K_e[Cx(k) - C\hat{x}(k|k-1)]] \\
&= G(I - K_eC)\cdot e(k|k-1)
\end{aligned}
$$

$$
\begin{aligned}
e(k+1|k+1) &= x(k+1) - \hat{x}(k+1|k+1) \\
&= \underbrace{(I - K_eC)G}_{G - K_e \underbrace{C\cdot G}_{C^*}} \cdot e(k|k)
\end{aligned}
$$

$:(G, C^*)$Observable

$\rightarrow$ arbitrary pole assignment

$y = Cx + n$ $\quad$ (n : measurement noise)

$$\dot{x} = Ax + Bu$$

$$y = Cx + n$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L\cdot(y - C\hat{x})$$

$$\dot{e} = (A - LC)e - L\cdot n$$

$\rightarrow$: Optimal compromise between convergence speed and noise effects $\rightarrow$ Kalman Filter. $\rightarrow$ $\min E[(x-\hat{x})^T(x-\hat{x})]$

$$u = -Kx$$

Vector Differential

(1) $\underbrace{f(x)}_{n\times1} : \underbrace{x}_{n\times1} \rightarrow \underbrace{y}_{n\times1}$

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_1} \\ \vdots & & & \\ \vdots & & & \\ \frac{\partial f_1}{\partial x_1} & \cdots & \cdots & \frac{\partial f_1}{\partial x_1} \end{bmatrix}$$

(2) $R \cdot u \qquad R : n \times n, \quad u : n \times 1$

$$\frac{\partial Ru}{\partial u} = R$$

$$\frac{\partial u^{\mathrm{T}} R}{\partial u} = \left[\frac{\partial R^{\mathrm{T}} u}{\partial u} \right] = \left[R^{\mathrm{T}} \right]^{\mathrm{T}} = R$$

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Vector Differential

(3) $\underbrace{u^{\mathrm{T}}}_{1\times n} \cdot \underbrace{R}_{n\times n} \cdot \underbrace{u}_{n\times n}$

$$\frac{\partial u^{\mathrm{T}} R u}{\partial u} = 2u^{\mathrm{T}} R$$

$$\underbrace{x}_{1\times n}{}^{\mathrm{T}} \underbrace{u}_{n\times 1} \quad \frac{\partial x^{\mathrm{T}} u}{\partial u} = x^{\mathrm{T}}$$

$$u^{\mathrm{T}} x \qquad \frac{\partial u^{\mathrm{T}} x}{\partial u} = x^{\mathrm{T}}$$

Review

1. Key Concepts

- Modeling
- Control System Design
- Controller Design
- Linear, Nonlinear Systems
- Design Approach

2. System Representations

- Laplace Transform, Z-Transform
- Graphical representation, block diagram, signal flow graph
- Transfer Function
- State Equation (cont. discrete.)
- State Eq. T.F.
- Canonical forms
- Solution of Linear state Eq. : cont. discrete.
- Discrete Representation of cont. time systems
- Relationships between cont. and discrete time eigenvalues
- Sampling of cont. time systems

Review

3. Dynamic Properties of the L.T.I Systems

- Stability. Definitions
- Routh stability Criterion
- Bilinear Transformation
- Lyapunov function
- Stability / In-stability theorems
- Controllability / Observability definitions : cont. discrete   Sufficient Conditions for controllability and observability
- Pole – Zero cancellations

4. Analysis and Design

- Root locus. Design concepts. Pole-Zero additions
- Frequency response
- Nyquist stability Criterion
- Lead – Lag compensators
- Robustness to modeling errors – Nyquist stability Criterion
- MIMO Nyquist stability Criterion
- Singular values
- Pole placement – state feedback
- Observer / State estimation
- Discrete time current observer
- Separation property
- LQ optimal control

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