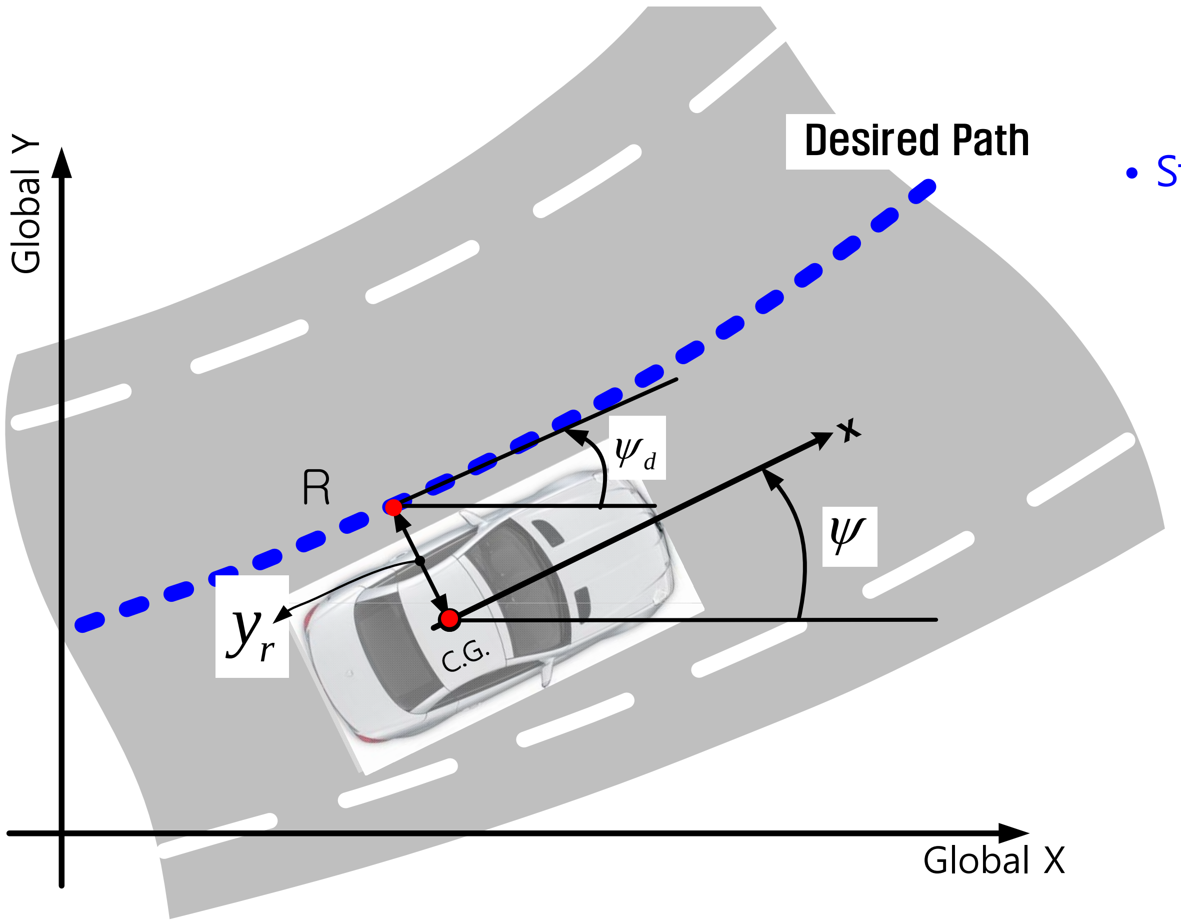
Part.1

Lateral Vehicle Dynamics

1. Vehicle Dynamic Model
2. Planar Model
3. Tire Models
4. Bicycle Model
5. Understeer/oversteer
6. **Dynamic model in terms of error w.r.t. road**
7. lane keeping model
8. Vehicle Stability Control

6. Dynamic model in terms of error w.r.t. road

6. Dynamic model in terms of error w.r.t. road



- State definition :
  - Lateral Position Error(y_r)

    $$\triangle y_y \simeq v_y \cdot \triangle t + v_x \cdot \triangle t \cdot (\psi - \psi_d)$$

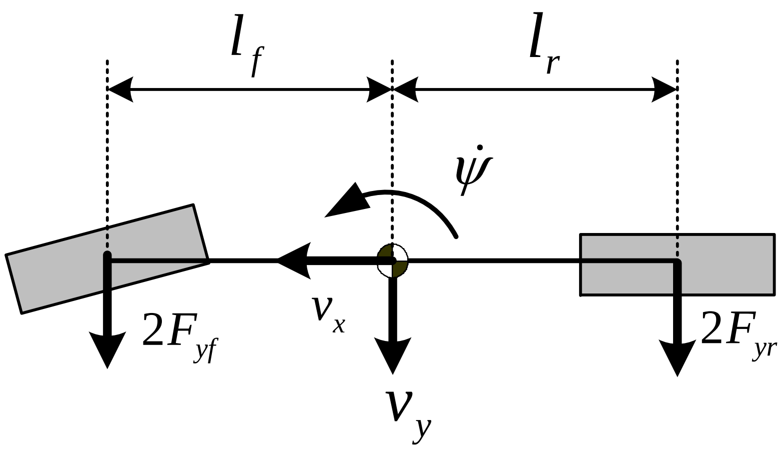
    $$\dot{y}_r = v_y + v_x \cdot (\psi - \psi_d)$$

  - yaw angle Error($\psi - \psi_d$)

    $$\dot{\psi}_d = \frac{v_x}{\rho}$$

6. Dynamic model in terms of error w.r.t. road

- 2-DOF Bicycle model :



- Dynamic Equation of Bicycle model

$$\sum F_y = m \cdot a_y = m \cdot (\dot{v}_y + v_x \cdot \dot{\psi}) = 2 \cdot F_{yf} + 2 \cdot F_{yr}$$

$$\sum M_z = I_z \cdot \ddot{\psi} = 2 \cdot l_f \cdot F_{yf} - 2 \cdot l_r \cdot F_{yr}$$

- Linearization of Lateral Tire Model

  - Front Lateral Tire Force

$$F_{yf} = C_f \cdot (\delta_f - \frac{v_y + l_f \cdot \dot{\psi}}{v_x})$$

$$= C_f \cdot \{\delta_f - \frac{\dot{y}_r + l_f \cdot (\dot{\psi} - \dot{\psi}_d)}{v_x} + (\psi - \psi_d) - \frac{l_f}{v_x} \cdot \dot{\psi}_d\}$$

  - Rear Lateral Tire Force

$$F_{yr} = C_r \cdot (-\frac{v_y - l_r \cdot \dot{\psi}}{v_x})$$

$$= C_r \cdot \{-\frac{\dot{y}_r - l_r \cdot (\dot{\psi} - \dot{\psi}_d)}{v_x} + (\psi - \psi_d) + \frac{l_r}{v_x} \cdot \dot{\psi}_d\}$$

- y-axis Motion Dynamic Equation

$$\dot{v}_y + v_x \cdot \dot{\psi} = \frac{2 \cdot F_{yf} + 2 \cdot F_{yr}}{m}$$

$$\underbrace{\dot{v}_y + v_x \cdot (\dot{\psi} - \dot{\psi}_d)}_{\ddot{y}_r} = \frac{2 \cdot F_{yf} + 2 \cdot F_{yr}}{m} - v_x \cdot \dot{\psi}_d$$

- State definition :

$$x = [y_r \quad \dot{y}_r \quad \psi - \psi_d \quad \dot{\psi} - \dot{\psi}_d]^T$$
$$= [x_1 \quad x_2 \quad x_3 \quad x_4]^T$$

- Substituting the linear tire model into the above equation

$$\dot{v}_y + v_x \cdot (\dot{\psi} - \dot{\psi}_d) = \ddot{y}_r = \left[\begin{array}{l} \frac{2 \cdot C_f}{m} \cdot \left\{ \delta_f - \frac{\dot{y}_r + l_f \cdot (\dot{\psi} - \dot{\psi}_d)}{v_x} + (\psi - \psi_d) - \frac{l_f}{v_x} \cdot \dot{\psi}_d \right\} \\ + \frac{2 \cdot C_r}{m} \cdot \left\{ - \frac{\dot{y}_r - l_r \cdot (\dot{\psi} - \dot{\psi}_d)}{v_x} + (\psi - \psi_d) + \frac{l_r}{v_x} \cdot \dot{\psi}_d \right\} \end{array} - v_x \cdot \dot{\psi}_d \right]$$

$$= \left\{ \begin{array}{l} - \frac{2 \cdot (C_f + C_r)}{m \cdot v_x} \dot{y}_r + \frac{2 \cdot (C_f + C_r)}{m} \cdot (\psi - \psi_d) - \frac{2 \cdot (l_f \cdot C_f - l_r \cdot C_r)}{m \cdot v_x} \cdot (\dot{\psi} - \dot{\psi}_d) \\ + \frac{2 \cdot C_f}{m} \cdot \delta_f + \left(-v_x - \frac{2 \cdot (l_f \cdot C_f - l_r \cdot C_r)}{m \cdot v_x} \right) \cdot \dot{\psi}_d \end{array} \right\}$$

$$\dot{x}_2 = \left\{ \begin{array}{l} - \frac{2 \cdot (C_f + C_r)}{m \cdot v_x} x_2 + \frac{2 \cdot (C_f + C_r)}{m} \cdot x_3 - \frac{2 \cdot (l_f \cdot C_f - l_r \cdot C_r)}{m \cdot v_x} \cdot x_4 \\ + \frac{2 \cdot C_f}{m} \cdot \delta_f + \left(-v_x - \frac{2 \cdot (l_f \cdot C_f - l_r \cdot C_r)}{m \cdot v_x} \right) \cdot \dot{\psi}_d \end{array} \right\}$$

- yaw-axis Motion Dynamic Equation

$$\ddot{\psi} = \frac{2 \cdot l_f \cdot F_{yf} - 2 \cdot l_r \cdot F_{yr}}{I_z}$$

$$\ddot{\psi} - \ddot{\psi}_d = \frac{2 \cdot l_f \cdot F_{yf} - 2 \cdot l_r \cdot F_{yr}}{I_z} - \ddot{\psi}_d$$

- State definition :

$$x = [y_r \quad \dot{y}_r \quad \psi - \psi_d \quad \dot{\psi} - \dot{\psi}_d]^T$$
$$= \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$$

- Substituting the linear tire model into the above equation

$$\ddot{\psi} - \ddot{\psi}_d = \frac{2 \cdot l_f}{I_z} \cdot F_{yf} - \frac{2 \cdot l_r}{I_z} \cdot F_{yr} - \ddot{\psi}_d$$

$$= \left[\begin{array}{l} \frac{2 \cdot l_f \cdot C_f}{I_z} \cdot \{\delta_f - \frac{\dot{y}_r + l_f \cdot (\dot{\psi} - \dot{\psi}_d)}{v_x} + (\psi - \psi_d) - \frac{l_f}{v_x} \cdot \dot{\psi}_d\} \\ -\frac{2 \cdot l_r \cdot C_r}{I_z} \cdot \{-\frac{\dot{y}_r - l_r \cdot (\dot{\psi} - \dot{\psi}_d)}{v_x} + (\psi - \psi_d) + \frac{l_r}{v_x} \cdot \dot{\psi}_d\} - \ddot{\psi}_d \end{array} \right]$$

$$= \left[\begin{array}{r} -\frac{2 \cdot (l_f \cdot C_f - l_r \cdot C_r)}{I_z \cdot v_x} \cdot \dot{y}_r + \frac{2 \cdot (l_f \cdot C_f - l_r \cdot C_r)}{I_z} \cdot (\psi - \psi_d) - \frac{2 \cdot (l_f^2 \cdot C_f + l_r^2 \cdot C_r)}{I_z \cdot v_x} \cdot (\dot{\psi} - \dot{\psi}_d) \\ + \frac{2 \cdot C_f \cdot l_f}{I_z} \delta_f - \frac{2 \cdot (l_f^2 \cdot C_f + l_r^2 \cdot C_r)}{I_z v_x} \dot{\psi}_d - \ddot{\psi}_d \end{array} \right]$$

$$\dot{x}_4 = \left[\begin{array}{r} -\frac{2 \cdot (l_f \cdot C_f - l_r \cdot C_r)}{I_z \cdot v_x} \cdot x_2 + \frac{2 \cdot (l_f \cdot C_f - l_r \cdot C_r)}{I_z} \cdot x_3 - \frac{2 \cdot (l_f^2 \cdot C_f + l_r^2 \cdot C_r)}{I_z \cdot v_x} \cdot x_4 \\ + \frac{2 \cdot C_f \cdot l_f}{I_z} \delta_f - \frac{2 \cdot (l_f^2 \cdot C_f + l_r^2 \cdot C_r)}{I_z v_x} \dot{\psi}_d - \ddot{\psi}_d \end{array} \right]$$

6. Dynamic model in terms of error w.r.t. road

State equation (1)

$$\dot{x} = A \cdot x + B \cdot \delta_f + F_d \cdot w_d$$

$$x = [y_r \quad \dot{y}_r \quad \psi - \psi_d \quad \dot{\psi} - \dot{\psi}_d]^T$$

where,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{A_1}{v_x} & -A_1 & \frac{A_2}{v_x} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{A_3}{v_x} & -A_3 & \frac{A_4}{v_x} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ B_1 \\ 0 \\ B_2 \end{bmatrix}$$

$$F_d = \begin{bmatrix} 0 & 0 \\ -v_x + \frac{A_2}{v_x} & 0 \\ 0 & 0 \\ \frac{A_4}{v_x} & -1 \end{bmatrix}$$

State equation (2)

$$w_d = \begin{bmatrix} \dot{\psi}_d \\ \ddot{\psi}_d \end{bmatrix}$$

▲ The disturbance is defined using the road information

$$A_1 = -\frac{2 \cdot (C_f + C_r)}{m}$$

$$A_2 = -\frac{2 \cdot (l_f \cdot C_f - l_r \cdot C_r)}{m} \qquad B_1 = \frac{2 \cdot C_f}{m}$$

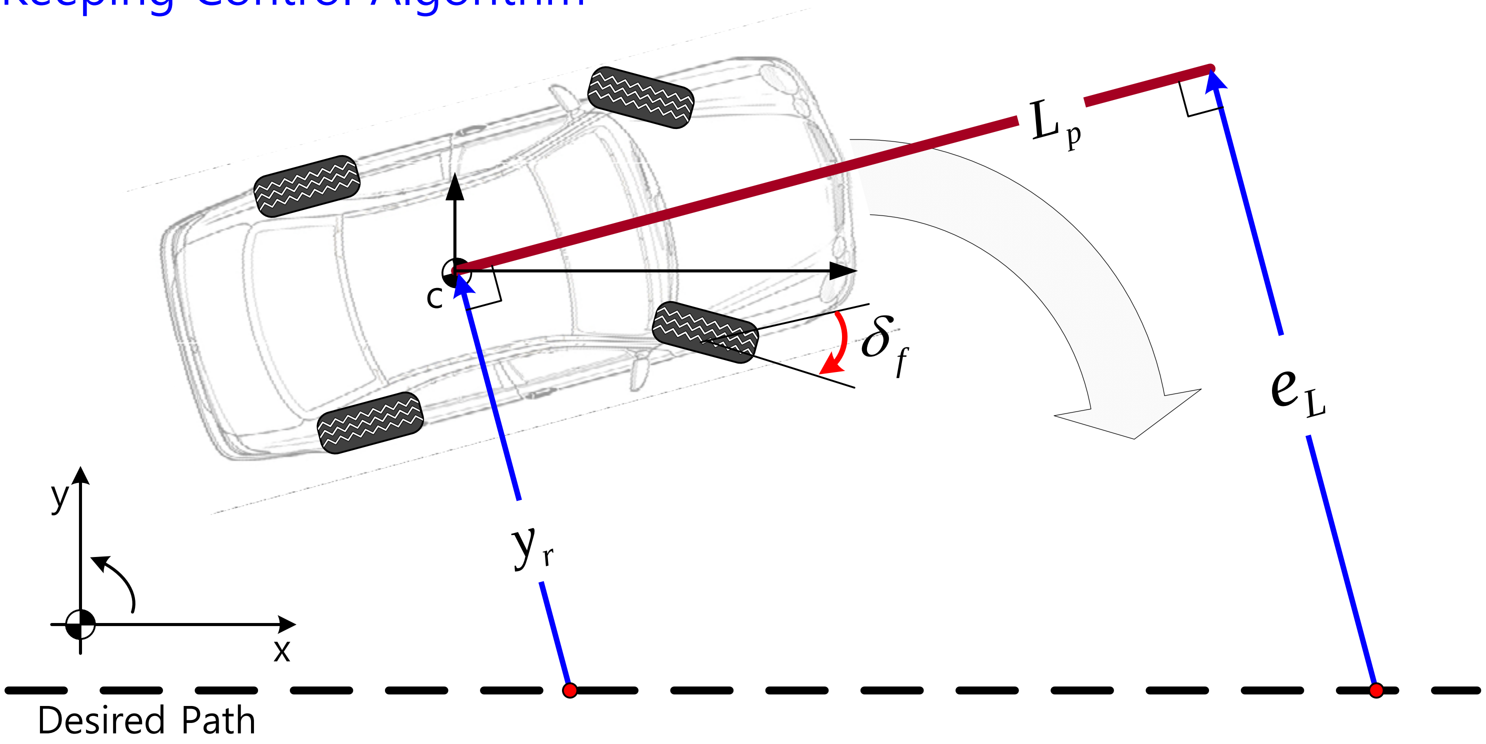
$$A_3 = -\frac{2 \cdot (l_f \cdot C_f - l_r \cdot C_r)}{I_z} \qquad B_2 = \frac{2 \cdot C_f \cdot l_f}{I_z}$$

$$A_4 = -\frac{2 \cdot (l_f^2 \cdot C_f + l_r^2 \cdot C_r)}{I_z}$$

7. lane keeping model

7.1 Design of Lane Keeping Control Algorithm

▪ Lane Keeping Control Algorithm



▪ Control Strategy

Control Error
Lateral Position Error (y_r)
Preview Distance Error(e_L)

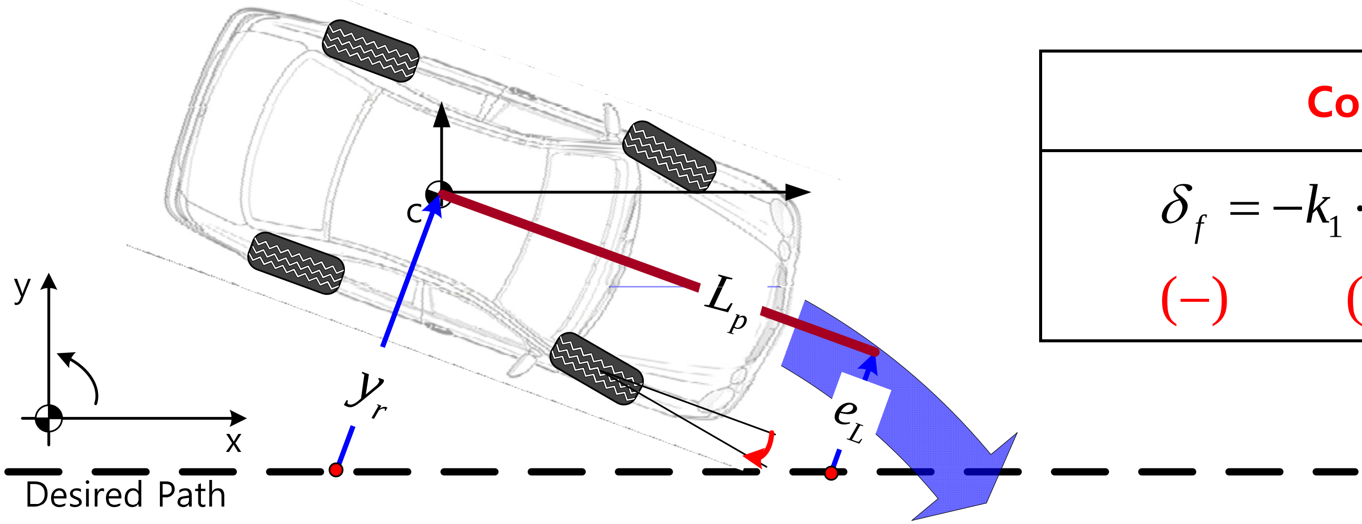
$$y_r \to 0$$
$$e_L \to 0$$

Control Input
$\delta_f = -k_1 \cdot y_r \quad -k_2 \cdot e_L$ $(-) \qquad (-)$

7.1 Design of Lane Keeping Control Algorithm

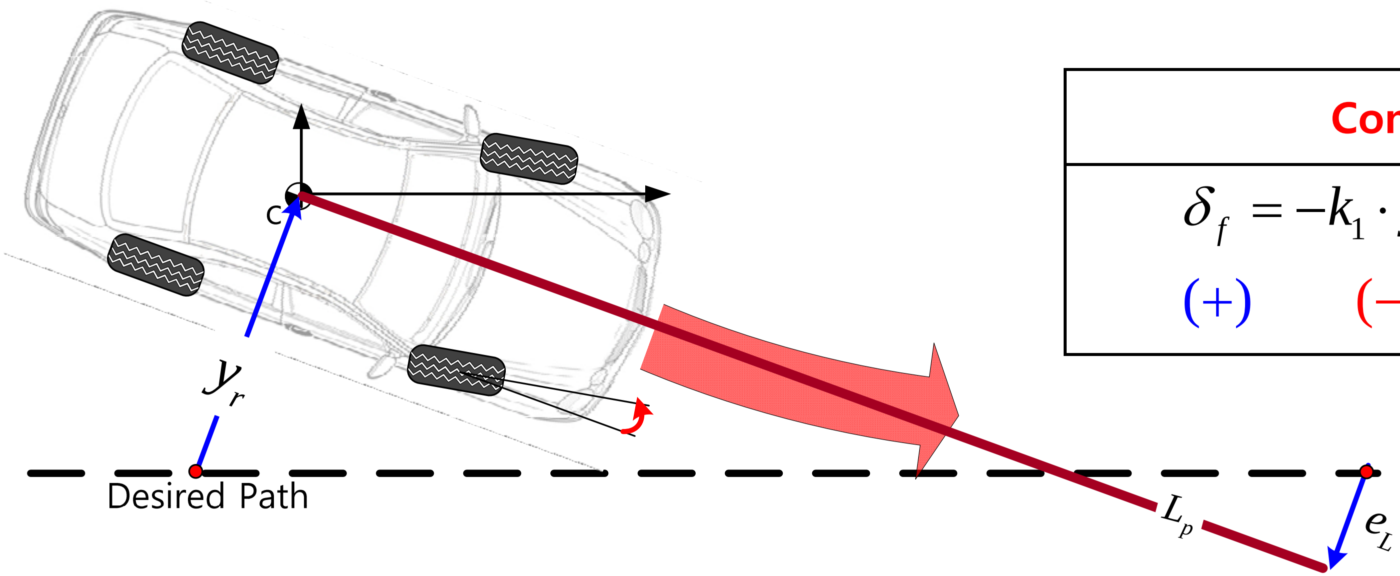
- Effectiveness of Preview Distance (L_p)

Short Preview Distance



Control Input
$\delta_f = -k_1 \cdot y_r \quad -k_2 \cdot e_L$ (−) (−) (−)

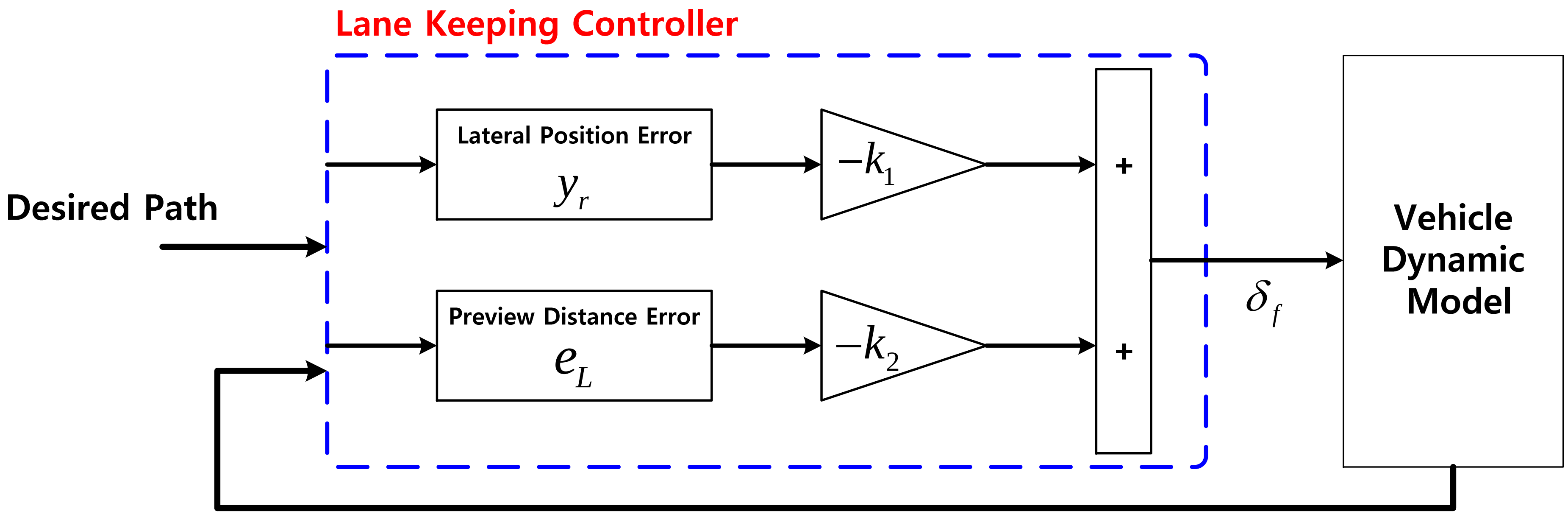
Long Preview Distance



Control Input
$\delta_f = -k_1 \cdot y_r \quad -k_2 \cdot e_L$ (+) (−) (+)

7.1 Design of Lane Keeping Control Algorithm

- Block Diagram of Lane Keeping Control Algorithm



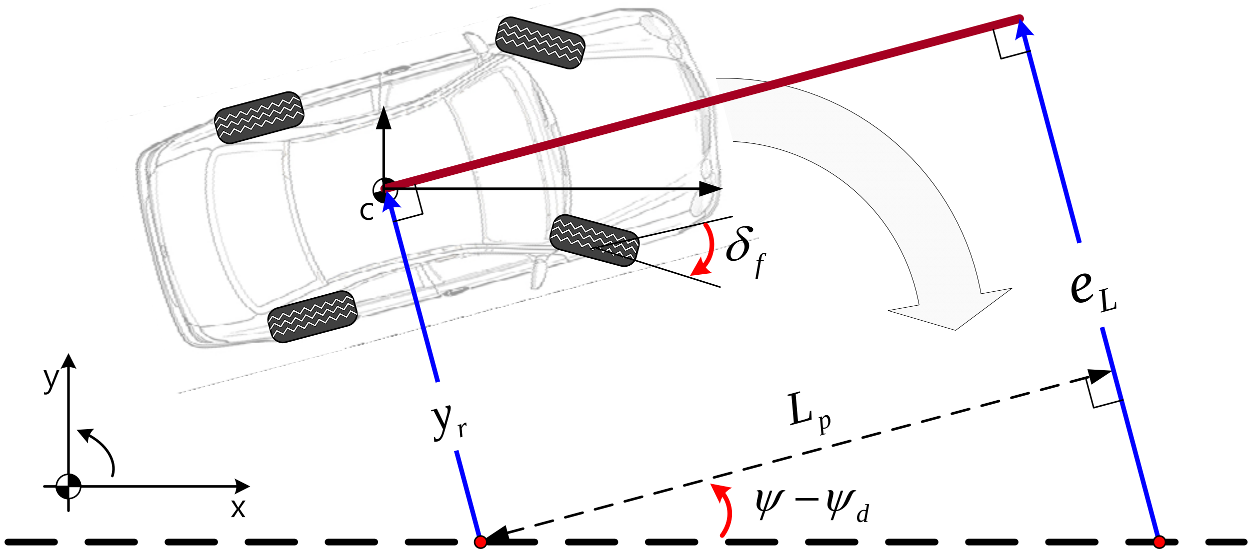
- Steering Control Input: $\delta_f = -k_1 \cdot y_r \quad -k_2 \cdot e_L$

- Design Parameter

Design Parameter	Symbol
Preview Distance	L_p
Control Gain of Lateral Position Error	k_1
Control Gain of Preview Distance Error	k_2

7.1 Design of Lane Keeping Control Algorithm

- Lane Keeping Control Algorithm



- Closed Loop System

$$\dot{x} = A \cdot x + B \cdot \delta_f + F_d \cdot w_d$$

where, $x = [y_r \quad \dot{y}_r \quad \psi - \psi_d \quad \dot{\psi} - \dot{\psi}_d]^T$

$$\delta_f = -k_1 \cdot y_r \quad -k_2 \cdot e_L$$

$$w_d = [\dot{\psi}_d \quad \ddot{\psi}_d]^T$$

- Assuming that: $\dot{\psi}_d(t) = \text{constant}, \qquad t_0 \le t \le t_f$

$$e_L = y_r + L_p \sin(\psi - \psi_d) = x_1 + L_p \sin(x_3)$$

$$\simeq x_1 + L_p \cdot x_3$$

$$e_L = e_L(L_p,\ x_1,\ x_3)$$

7.1 Design of Lane Keeping Control Algorithm

▪ Closed Loop System

- Using the above Equation, control input for lane keeping can be rewritten as follows:

$$\begin{aligned}\delta_f &= -k_1 \cdot y_r - k_2 \cdot e_L = -k_1 \cdot x_1 - k_2 \cdot \left(x_1 + L_p \cdot x_3 \right) \\ &= -\left(k_1 + k_2 \right) \cdot x_1 - k_2 \cdot L_p \cdot x_3 \\ &= -K \cdot x\end{aligned}$$

where, $K = \begin{bmatrix} k_1 + k_2 & 0 & k_2 \cdot L_p & 0 \end{bmatrix}$

- The closed loop system can be rewritten using the control input as follows:

$$\begin{aligned}\dot{x} &= A \cdot x + B \cdot \delta_f + F_d \cdot w_d \\ &= A \cdot x - B \cdot K \cdot x + F_d \cdot w_d \\ &= \left(A - B \cdot K \right) \cdot x + F_d \cdot w_d \\ &= A_c \cdot x + F_d \cdot w_d\end{aligned}$$

▲ Is the closed loop system stable ?

7.2 Control Stability based Lyapunov Equation

- Lyapunov Theorem for Linear Time-Invariant System

1) Given a linear system of the form $\dot{x} = A \cdot x$, let us consider a quadratic Lyapunov function candidate

$V = x^T \cdot P \cdot x > 0$ where, P is a symmetric positive definite matrix.

2) Differentiating the Lyapunov function,

$$
\begin{aligned}
\dot{V} &= \dot{x}^T \cdot P \cdot x + x^T \cdot P \cdot \dot{x} \\
&= \left(x^T \cdot A^T\right) \cdot P \cdot x + x^T \cdot P \cdot \left(A \cdot x\right) = x^T \cdot \left(A^T \cdot P + P \cdot A\right) \cdot x \\
&= -x^T \cdot Q \cdot x
\end{aligned}
$$

If Q is semi-positive definite matrix, the system is stable.

$\dot{V} = -x^T \cdot Q \cdot x < 0$ for $x \neq 0$

- Lyapunov Equation

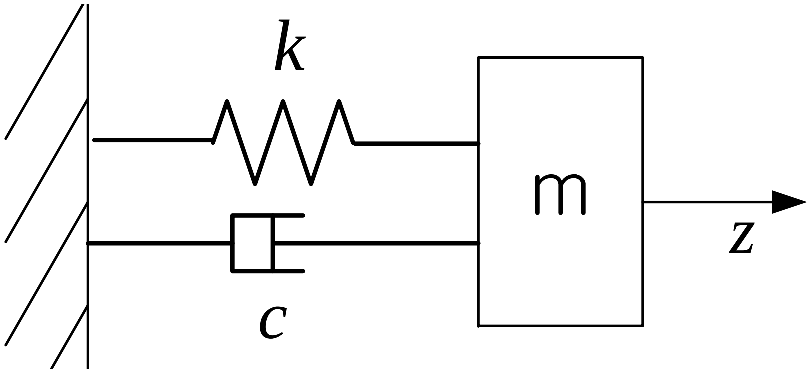
$A^T \cdot P + P \cdot A = -Q$ where, P is a symmetric positive definite matrix.

Q is a semi- positive definite matrix.

7.2 Control Stability based Lyapunov Equation

- Lyapunov Theorem for Linear Time-Invariant System (Example)

  - Consider spring-mass system as follows:



- Dynamic Equation:

$$m \cdot \ddot{z} = -k \cdot z - c \cdot \dot{z}$$

- Define the state of system

$$x = \begin{bmatrix} z & \dot{z} \end{bmatrix}^T = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$$

- State Equation:

$$\dot{x} = \begin{bmatrix} \dot{z} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{k}{m} \cdot x_1 - \frac{c}{m} \cdot x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \cdot x$$

7.2 Control Stability based Lyapunov Equation

- Lyapunov Theorem for Linear Time-Invariant System (Example)

1) Consider a quadratic Lyapunov function candidate (Energy Function) as follows:

$$V = \frac{1}{2}k \cdot z^2 + \frac{1}{2}m \cdot \dot{z}^2 = x^T \cdot \underbrace{\begin{bmatrix} \frac{k}{2} & 0 \\ 0 & \frac{m}{2} \end{bmatrix}}_{=P} \cdot x > 0$$

▲ P is a positive definite matrix.

2) Differentiating the Lyapunov function,

$$\dot{V} = k \cdot z \cdot \dot{z} + m \cdot \dot{z} \cdot \ddot{z} = k \cdot x_1 \cdot x_2 + m \cdot x_2 \cdot \dot{x}_2$$

$$= k \cdot x_1 \cdot x_2 + m \cdot x_2 \cdot \left(-\frac{k}{m} \cdot x_1 - \frac{c}{m} \cdot x_2 \right)$$

$$= -c \cdot x_2^2 = -x^T \cdot \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}}_{=Q} \cdot x < 0 \quad \left(for\ x \neq 0 \right)$$

▲ Q is a semi- positive definite matrix.

Therefore, the system is stable.

7.3 Control Stability of Lane Keeping Controller

1) Consider Closed System of Lane Keeping System

$$\dot{x} = A \cdot x + B \cdot \delta_f + F_d \cdot w_d = (A - B \cdot K) \cdot x + F_d \cdot w_d$$
$$= A_c \cdot x + F_d \cdot w_d$$

Eq.1

where, $x = [y_r \quad \dot{y}_r \quad \psi - \psi_d \quad \dot{\psi} - \dot{\psi}_d]^T$

$$K = \begin{bmatrix} k_1 + k_2 & 0 & k_2 \cdot L_p & 0 \end{bmatrix}$$

$$\|F_d \cdot w_d\| \leq \gamma \cdot \|x\|$$

▲ Disturbance term is bounded.

2) Consider a quadratic Lyapunov function as follows:

$$V = x^T \cdot P \cdot x > 0$$

where, P is a symmetric positive definite matrix. Eq.2

3) Differentiating the Lyapunov function,

$$\dot{V} = \dot{x}^T \cdot P \cdot x + x^T \cdot P \cdot \dot{x}$$
$$= (A_c \cdot x + F_d \cdot w_d)^T \cdot P \cdot x + x^T \cdot P \cdot (A_c \cdot x + F_d \cdot w_d)$$
$$= x^T \cdot (A_c^T \cdot P + P \cdot A_c) \cdot x + 2x^T \cdot P \cdot F_d \cdot w_d$$

Eq.3

7.3 Control Stability of Lane Keeping Controller

4) For stable system, the derivative of the Lyapunov function should be negative.

$$\begin{aligned}\dot{V} &= x^T \cdot \left(A_c^T \cdot P + P \cdot A_c\right) \cdot x + 2x^T P \cdot F_d \cdot w_d \\ &= -x^T \cdot Q \cdot x + 2x^T P \cdot F_d \cdot w_d < 0\end{aligned}$$

Eq.4

5) Substituting the below equation into the derivative of the Lyapunov function

$$i)\quad \lambda_{\min}(Q) \cdot x^2 \le x^T \cdot Q \cdot x \le \lambda_{\max}(Q) \cdot x^2$$

Eq.5

▲ Minimum eigen value of Q Matrix

$$ii)\quad 2x^T P \cdot F_d \cdot w_d \le 2x^T P \cdot \gamma \cdot x \le 2 \cdot \gamma \cdot \lambda_{\max}(P) \cdot x^2$$

Eq.6

▲ Disturbance term is bounded such that: $\left\| F_d \cdot w_d \right\| \le \gamma \cdot \left\| x \right\|$

6) Substituting Eqs (5) and (6) into Eq.(4)

$$\begin{aligned}\dot{V} &= -x^T \cdot Q \cdot x + 2x^T P \cdot F_d \cdot w_d \\ &\le -\lambda_{\min}(Q) \cdot x^2 + 2 \cdot \gamma \cdot \lambda_{\max}(P) \cdot x^2 < 0\end{aligned}$$

Eq.7

$$\gamma < \frac{\lambda_{\min}(Q)}{2 \cdot \lambda_{\max}(P)}$$

Eq.8

7.3 Control Stability of Lane Keeping Controller

9) Bounded Disturbance for stable system

$$i) \quad \left\| F_d \cdot w_d \right\| \le \gamma \cdot \left\| x \right\|$$

$$ii) \quad \gamma < \frac{\lambda_{\min}(Q)}{2 \cdot \lambda_{\max}(P)}$$

$$iii) \quad \left\| F_d \cdot w_d \right\| \le \gamma \cdot \left\| x \right\| < \frac{\lambda_{\min}(Q)}{2 \cdot \lambda_{\max}(P)} \left\| x \right\|$$

$$\Rightarrow \quad \frac{2 \cdot \lambda_{\max}(P)}{\lambda_{\min}(Q)} \cdot \left\| F_d \cdot w_d \right\| < \left\| x \right\| \qquad \text{Eq.9}$$

Lemma.1

1) Consider a quadratic Lyapunov function candidate as follows:

$$\alpha_1(\|x\|) \le V(x) = x^T \cdot P \cdot x = P \cdot \|x\|^2 \le \alpha_2(\|x\|) \quad \text{Eq.10}$$

where, $0 < \mu \le \|x\|$

2) From Eq.12,

$$V(x) \le \varepsilon \Rightarrow \alpha_1(\|x\|) \le V(x) \le \varepsilon$$

$$\Rightarrow \|x\| \le \alpha_1^{-1}(\varepsilon) \quad \text{Eq.11}$$

3) taking $\varepsilon = \alpha_2(\mu)$

$$\|x\| \le \alpha_1^{-1}[\alpha_2(\mu)]$$

7.3 Control Stability of Lane Keeping Controller

10) Bounded tracking error

$$\underbrace{\lambda_{\min}(P)\cdot\|x\|^2}_{\alpha_1(\|x\|)} \le V(x) = x^T \cdot P \cdot x \le \underbrace{\lambda_{\max}(P)\cdot\|x\|^2}_{\alpha_2(\|x\|)} \quad \text{Eq.12}$$

where, $0 < \dfrac{2\cdot\lambda_{\max}(P)}{\lambda_{\min}(Q)}\cdot\|F_d\cdot w_d\| < \|x\|$ Eq.13

i) From Lemma.1,

$$\alpha_1(\mu) = \lambda_{\min}(P)\cdot\mu^2 \quad \Rightarrow \quad \alpha_1^{-1}(\mu) = \sqrt{\frac{\mu}{\lambda_{\min}(P)}}$$

$$\alpha_2(\mu) = \lambda_{\max}(P)\cdot\mu^2$$

$$\Rightarrow \quad \|x\| \le \alpha_1^{-1}[\alpha_2(\mu)] = \sqrt{\frac{\alpha_2(\mu)}{\lambda_{\min}(P)}} = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\cdot\mu$$

ii) taking $\mu = \dfrac{2\cdot\lambda_{\max}(P)}{\lambda_{\min}(Q)}\cdot\|F_d\cdot w_d\|$

$$\|x\| \le \frac{2\{\lambda_{\max}(P)\}^{3/2}}{\sqrt{\lambda_{\min}(P)}\cdot\lambda_{\min}(Q)}\cdot\|F_d\cdot w_d\| \quad \text{Eq.14}$$

where, $(A - B\cdot K)^T\cdot P + P\cdot(A - B\cdot K) = -Q$

▲ Bounded Tracking Error

7.4 Steady State Error From Dynamic Equations

- Closed Loop System of Lane Keeping Control Algorithm

$$\dot{x} = A \cdot x + B \cdot \delta_f + F_d \cdot w_d = (A - B \cdot K) \cdot x + F_d \cdot w_d$$
$$= A_c \cdot x + F_d \cdot w_d$$

where, $x = [y_r \quad \dot{y}_r \quad \psi - \psi_d \quad \dot{\psi} - \dot{\psi}_d]^T$

$K = \begin{bmatrix} k_1 + k_2 & 0 & k_2 \cdot L_p & 0 \end{bmatrix}$

$w_d = [\dot{\psi}_d \quad \ddot{\psi}_d]^T$

▲ Due to the presence of the disturbance term, $F_d \cdot w_d$,
the tracking error will not all converge to zero, even though $A - B \cdot K$ is asymptotically stable.

▲ In order to solve the non-zero steady state error, feedforward control input is required.

▪ Feedforward control input for zero steady state error

1) Feedforward control input

$$\delta_f = -K \cdot x + \delta_{ff} \qquad \text{where, } K = \begin{bmatrix} k_1 + k_2 & 0 & k_2 \cdot L_p & 0 \end{bmatrix}$$

$$= \begin{bmatrix} k_1' & 0 & k_3' & 0 \end{bmatrix}$$

2) Closed-loop system

$$\dot{x} = (A - B \cdot K) \cdot x + B \cdot \delta_{ff} + F_d \cdot w_d$$

$$= A_c \cdot x + B \cdot \delta_{ff} + F_d \cdot \begin{bmatrix} \dot{\psi}_d \\ \ddot{\psi}_d \end{bmatrix}$$

3) Taking Laplace transforms, assuming zero initial conditions,

$$X(s) = \left[sI - (A - B \cdot K) \right]^{-1} \cdot \left\{ B \cdot \delta_{ff}(s) + F_d \cdot \begin{bmatrix} \dot{\psi}_d(s) \\ \ddot{\psi}_d(s) \end{bmatrix} \right\}$$

4) Assuming that (steady state turning):

$$i) \ \dot{\psi}_d(t) = \text{constant} = \frac{v_x}{\rho} \quad \Rightarrow \quad \dot{\psi}_d(s) = \frac{v_x}{\rho} \cdot \frac{1}{s}$$

$$ii) \ \ddot{\psi}_d(t) = 0 \quad \Rightarrow \quad \ddot{\psi}_d(s) = 0$$

$$iii) \ \delta_{ff} = \text{constant} \quad \Rightarrow \quad \delta_{ff}(s) = \frac{\delta_{ff}}{s}$$

- Feedforward control input for zero steady state error

5) Using the Final Value Theorem, the steady state error can be obtained as follows:

$$x_{ss} = \lim_{t\to\infty} x(t) = \lim_{s\to\infty} s\cdot X(s)$$

$$= \lim_{s\to\infty} s\cdot\left[sI-\left(A-B\cdot K\right)\right]^{-1}\cdot\left\{B\cdot\delta_{ff}(s)+F_d\cdot\begin{bmatrix}\dot{\psi}_d(s)\\0\end{bmatrix}\right\}$$

$$= \lim_{s\to\infty} s\cdot\left[sI-\left(A-B\cdot K\right)\right]^{-1}\cdot\left\{B\cdot\left(\frac{\delta_{ff}}{s}\right)+F_d\cdot\begin{bmatrix}\frac{v_x}{\rho}\cdot\frac{1}{s}\\0\end{bmatrix}\right\}$$

$$x_{ss} = -\left(A-B\cdot K\right)^{-1}\cdot\left\{B\cdot\delta_{ff}+F_d\cdot\begin{bmatrix}\frac{v_x}{\rho}\\0\end{bmatrix}\right\}$$

6) Using the Symbolic Toolbox in Matlab, the above equation can be calculated as follows:

$$x_{ss} = \begin{bmatrix}\frac{\delta_{ff}}{k_1'}\\0\\0\\0\end{bmatrix}+\begin{bmatrix}-\frac{1}{k_1'}\cdot\frac{m\cdot v_x^2}{\rho\cdot\left(l_f+l_r\right)}\cdot\left\{\frac{l_r}{2C_f}-\frac{l_f}{2C_r}+\frac{l_f}{2C_r}\cdot k_3'\right\}-\frac{1}{k_1'\cdot\rho}\left\{l_f+l_r-l_r\cdot k_3'\right\}\\0\\\frac{1}{2\cdot\rho\cdot C_r\cdot\left(l_f+l_r\right)}\left\{-2l_f\cdot l_r\cdot C_r-2\cdot l_r^2\cdot C_r+l_f\cdot m\cdot v_x^2\right\}\\0\end{bmatrix}$$

- Feedforward control input for zero steady state error

7) Steady State Error of Yaw Angle Error

$$(\psi-\psi_d)_{ss}=\frac{1}{2\cdot\rho\cdot C_r\cdot\left(l_f+l_r\right)}\left\{-2l_f\cdot l_r\cdot C_r-2\cdot l_r^2\cdot C_r+l_f\cdot m\cdot v_x^2\right\}$$

$$=\frac{1}{2\cdot\rho\cdot C_r\cdot\left(l_f+l_r\right)}\left\{-2\cdot l_r\cdot C_r\cdot\left(l_f+l_r\right)+l_f\cdot m\cdot v_x^2\right\}$$

$$=-\frac{l_r}{\rho}+\frac{m\cdot l_f}{2\cdot C_r\cdot\left(l_f+l_r\right)}\cdot\frac{v_x^2}{\rho}$$

8) Steady State Error of Lateral Position Error

$$\left(y_r\right)_{ss}=\frac{\delta_{ff}}{k_1'}-\frac{1}{k_1'}\cdot\frac{m\cdot v_x^2}{\rho\cdot\left(l_f+l_r\right)}\cdot\left\{\frac{l_r}{2C_f}-\frac{l_f}{2C_r}+\frac{l_f}{2C_r}\cdot k_3'\right\}-\frac{1}{k_1'\cdot\rho}\left\{l_f+l_r-l_r\cdot k_3'\right\}$$

9) Feedforward Control input for zero steady state error of y_r

$$\delta_{ff}=\frac{m\cdot v_x^2}{\rho\cdot\left(l_f+l_r\right)}\cdot\left\{\frac{l_r}{2C_f}-\frac{l_f}{2C_r}+\frac{l_f}{2C_r}\cdot k_3'\right\}+\frac{1}{\rho}\left\{l_f+l_r-l_r\cdot k_3'\right\}$$

- Feedforward control input for zero steady state error

10) Feedforward Control input for zero steady state error of y_r

$$\delta_{ff} = \frac{m \cdot v_x^2}{\rho \cdot \left(l_f + l_r\right)} \cdot \left\{ \frac{l_r}{2C_f} - \frac{l_f}{2C_r} + \frac{l_f}{2C_r} \cdot k_3' \right\} + \frac{1}{\rho}\left\{ l_f + l_r - l_r \cdot k_3' \right\}$$

$$= \frac{m \cdot v_x^2}{\rho \cdot \left(l_f + l_r\right)} \cdot \left\{ \frac{l_r}{2C_f} - \frac{l_f}{2C_r} \right\} + \frac{m \cdot v_x^2}{\rho \cdot \left(l_f + l_r\right)} \cdot \frac{l_f}{2C_r} \cdot k_3' + \frac{1}{\rho}\left\{ l_f + l_r - l_r \cdot k_3' \right\}$$

$$= \frac{v_x^2}{\rho \cdot g} \cdot \underbrace{\frac{m \cdot g}{2\left(l_f + l_r\right)} \cdot \left\{ \frac{l_r}{C_f} - \frac{l_f}{C_r} \right\}}_{K_{us}} + \underbrace{\left\{ \frac{m \cdot l_f}{2C_r \cdot \left(l_f + l_r\right)} \cdot \frac{v_x^2}{\rho} - \frac{l_r}{\rho} \right\}}_{\left(\psi - \psi_d\right)_{ss}} \cdot k_3' + \frac{l_f + l_r}{\rho}$$

$$= \frac{v_x^2}{\rho \cdot g} \cdot K_{us} + k_3' \cdot \left(\psi - \psi_d\right)_{ss} + \frac{l_f + l_r}{\rho}$$

$$\delta_{ff} = \frac{K_{us}}{g} \cdot a_y + k_3' \cdot \left(\psi - \psi_d\right)_{ss} + \frac{l_f + l_r}{v_x} \cdot \dot{\psi}_d$$

11) Feedforward Control input for zero steady state error

$$\delta_{ff} = \frac{K_{us}}{g} \cdot a_y + \frac{l_f + l_r}{v_x} \cdot \dot{\psi}_d + k_3' \cdot \left(\psi - \psi_d\right)_{ss}$$

where, $$\left(\psi - \psi_d\right)_{ss} = \frac{m \cdot l_f}{2 \cdot C_r \cdot \left(l_f + l_r\right)} \cdot \frac{v_x^2}{\rho} - \frac{l_r}{\rho}$$

$$= \frac{m \cdot l_f}{2 \cdot C_r \cdot \left(l_f + l_r\right)} \cdot a_y - \frac{l_r}{v_x} \cdot \dot{\psi}_d$$

Finally,

$$\delta_{ff} = \frac{K_{us}}{g} \cdot a_y + \frac{l_f + l_r}{v_x} \cdot \dot{\psi}_d + k_3' \cdot \left(\frac{m \cdot l_f}{2 \cdot C_r \cdot \left(l_f + l_r\right)} \cdot a_y - \frac{l_r}{v_x} \cdot \dot{\psi}_d \right)$$

$$= \left(\frac{K_{us}}{g} + k_3' \cdot \frac{m \cdot l_f}{2 \cdot C_r \cdot \left(l_f + l_r\right)} \right) \cdot a_y + \left(\frac{l_f + l_r}{v_x} - k_3' \cdot \frac{l_r}{v_x} \right) \cdot \dot{\psi}_d$$

▲ Feedforward control input is determined by the driving situation (a_y) and the desired yaw rate of the desired path ($\dot{\psi}_d$).