2. Stress and infinitesimal strain

2.1 Problem definition

- Shallow depth – Block movement in jointed rock mass
  under low-stress environment

  Deep mining – Stress is main concern for the mine stability.
  Rock mass behaves as continuum.

- Force
  - Body force: Gravity, magnetic, inertia (F/V)
  - Surface force: Conveyed by physical contact (F/A)

2.2 Force and stress

- Force is a vector with magnitude and direction.

  Ex.) F_x, F_y, F_z, F_1, F_2,…

- Stress is a tensor with magnitude, direction and surface orientation.

  Ex.) σ_{xx} $(=\sigma_x)$, $\sigma_{xy}(=\tau_{xy})$, σ_{yz},…

  surface orientation (normal to the surface)

  stress direction

2.2 Force and stress

- Traction is a force per unit area acting on a surface (Fig.2.1(b)).

- *"*Tensor *is a further extension of ideas we already use when defining quantities like scalars and vectors."*

  Ex.) Scalar → rank zero

  Vector → rank one

  Stress → rank two

  ⋮

2.2 Force and stress

- Tensor notation

- Introduced in deriving the theory of relativity by Einstein.
- Basic idea is to make summation for the repeatedly used subscripts.

Ex.) $a_i b_i = a_x b_x + a_y b_y + a_z b_z$

$a_{ii} b_k = a_{xx} b_k + a_{yy} b_k + a_{zz} b_k$ *(i: dummy index, k: free index)*

$$\begin{aligned} a_{ik} b_i c_k &= a_{xk} b_x c_k + a_{yk} b_y c_k + a_{zk} b_z c_k \\ &= a_{xx} b_x c_x + a_{xy} b_x c_y + a_{xz} b_x c_z \\ &\quad + a_{yx} b_y c_x + a_{yy} b_y c_y + a_{yz} b_y c_z \\ &\quad + a_{zx} b_z c_x + a_{zy} b_z c_y + a_{zz} b_z c_z \end{aligned}$$

2.2 Force and stress

- Important symbols in tensor notation

- Kronecker delta

$\delta_{ij} = 1$ if $i=j$.

$= 0$ if $i \neq j$.

Ex.) $\delta_{xx} = 1, \delta_{zz} = 1, \delta_{xy} = 0, \delta_{yz} = 0$

$\delta_{ij}\, a_i b_j = a_x b_x + a_y b_y + a_z b_z = a_i b_i$

- Permutation symbol

$\varepsilon_{ijk} = 0$ if any pair of subscripts are identical.

$= (-1)^p$, where p is the number of subscript transposition.

Ex.) $\varepsilon_{xyz} = 1, \varepsilon_{xyx} = 0, \varepsilon_{yyz} = 0, \varepsilon_{xzz} = 0$

$\varepsilon_{xzy} = -1, \varepsilon_{zxy} = 1, \varepsilon_{yxz} = -1, \varepsilon_{zyx} = -1$

2.2 Force and stress

- Linear algebra in tensor notation

- Matrix operation

$Ax = b \rightarrow a_{ij}x_j = b_i$

$AB = Y \rightarrow a_{ij}b_{jk} = Y_{ik}$

$A^TB = Z \rightarrow a_{ji}b_{jk} = Z_{ik}$

$x^TBx = c \rightarrow x_ib_{ij}x_j = c$

- Derivatives

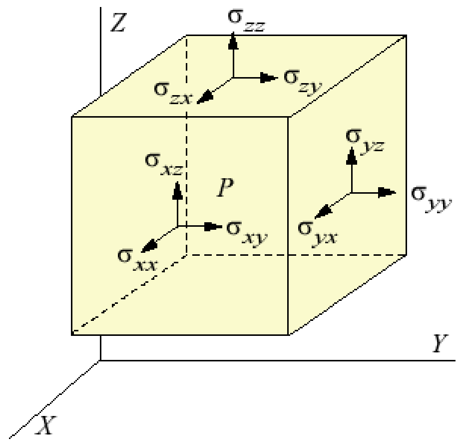
$$Grad\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{e}_x + \frac{\partial\phi}{\partial y}\hat{e}_y + \frac{\partial\phi}{\partial z}\hat{e}_z = \phi_{,i}$$

$$Div\,\vec{V} = \nabla\bullet\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = V_{i,i}$$

2.2 Force and stress

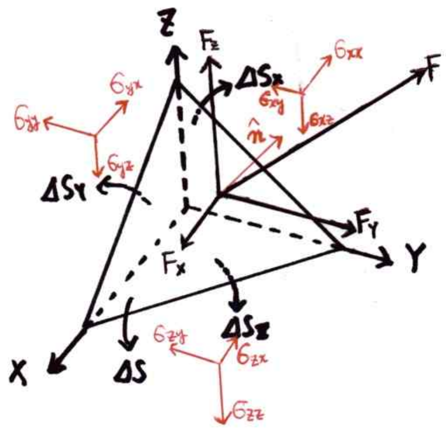
- Stress in rectangular coordinates

$$[\sigma]=\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}\end{bmatrix}=\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}\end{bmatrix}$$



- Stress in an arbitrary plane

$$\hat{n}=\left(\frac{\Delta S_x}{\Delta S},\frac{\Delta S_y}{\Delta S},\frac{\Delta S_z}{\Delta S}\right)$$



2.2 Force and stress

$$\vec{F} = \vec{F}_x + \vec{F}_y + \vec{F}_z$$

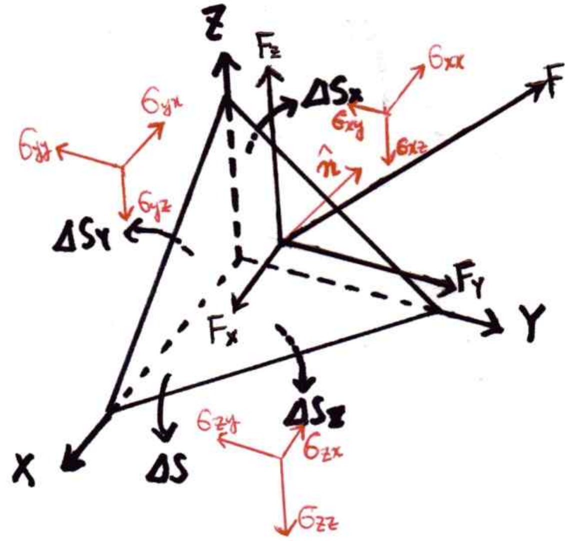
$$\Delta S t^{\vec{n}} = \Delta S t_x^{\vec{n}} + \Delta S t_y^{\vec{n}} + \Delta S t_z^{\vec{n}}$$

$$\Delta S t_x^{\hat{n}} = \Delta S_x \sigma_{xx} + \Delta S_y \sigma_{yx} + \Delta S_z \sigma_{zx}$$

$$\Delta S t_y^{\hat{n}} = \Delta S_x \sigma_{xy} + \Delta S_y \sigma_{yy} + \Delta S_z \sigma_{zy}$$

$$\Delta S t_z^{\hat{n}} = \Delta S_x \sigma_{xz} + \Delta S_y \sigma_{yz} + \Delta S_z \sigma_{zz}$$

$$t_i^{\hat{n}} = n_j \sigma_{ji} \left(= \sigma_{ij} n_j \right)$$

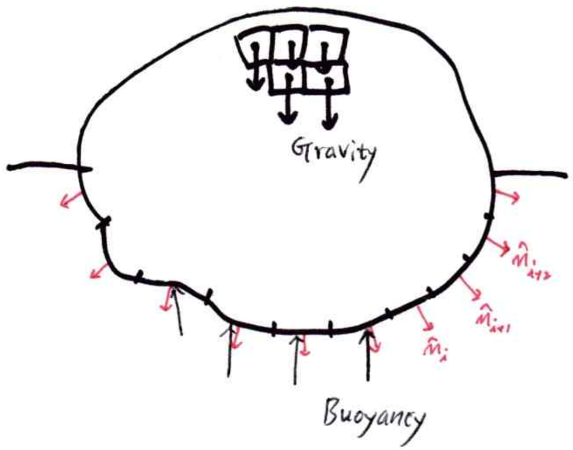


2.2 Force and stress

- Equilibrium conditions

- Force equilibrium & moment equilibrium

$$\sum \vec{F} = 0, \quad \sum \vec{M} = 0$$



- Force equilibrium

$$\int_s t_i^{\hat{n}} ds + \int_v \rho b_i dv = 0$$

$$\int_s n_j \sigma_{ji} ds = \int_v \sigma_{ji,j} dv = 0 \quad (Divergence\ theorem)$$

$$\rightarrow \sigma_{ji,j} + \rho b_i = 0$$

2.2 Force and stress

- Moment equilibrium

$$\vec{M} = \vec{r} \times \vec{F} \left(= \varepsilon_{ijk} r_j F_k \right)$$

$$\int_s \vec{r} \times t^{\hat{n}} ds + \int_v \vec{r} \times \rho \vec{b} \, dv = 0$$

$$\rightarrow \varepsilon_{ijk} \sigma_{jk} = 0$$

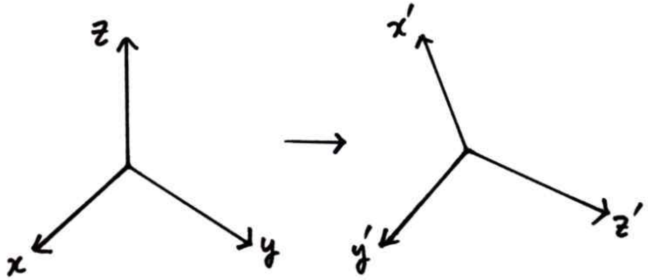
$$i = x: \quad \sigma_{yz} - \sigma_{zy} = 0$$

$$i = y: \quad \sigma_{zx} - \sigma_{xz} = 0 \quad \rightarrow \quad \sigma_{ij} = \sigma_{ji}$$

$$i = z: \quad \sigma_{xy} - \sigma_{yx} = 0$$

2.3 Stress transformation

- Vector transformation



	x	y	z
x'	$\cos\theta_{x'x}$	$\cos\theta_{x'y}$	$\cos\theta_{x'z}$
y'	$\cos\theta_{y'x}$	$\cos\theta_{y'y}$	$\cos\theta_{y'z}$
z'	$\cos\theta_{z'x}$	$\cos\theta_{z'y}$	$\cos\theta_{z'z}$

$$\rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

2.3 Stress transformation

- Stress transformation

$$u_i' = a_{ip}u_p, \quad v_j' = a_{jq}v_q \quad (vector\ transformation)$$

$$u_i'v_j' = \sigma_{ij}' \quad (outer\ product)$$

$$u_i'v_j' = a_{ip}u_p a_{jq}v_q = a_{ip}a_{jq}u_p v_q = a_{ip}a_{jq}\sigma_{pq} = a_{ip}\sigma_{pq}a_{jq}$$

$$\rightarrow \sigma_{ij}' = a_{ip}\sigma_{pq}a_{jq} \quad \left(\Sigma' = A\cdot\Sigma\cdot A^T\right)$$

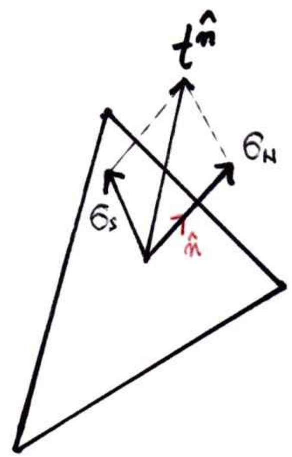
2.4 Principal stress and stress invariants

- Principal stress

- Traction in a surface can be decomposed into normal and shear stresses which are in the same plane with the traction.

$$\sigma_N = \hat{n} \cdot t^{\hat{n}}$$

$$\sigma_S = t^{\hat{n}} - \sigma_N$$



2.4 Principal stress and stress invariants

- The surface on which shear component of the traction is zero is a principal plane and its normal component is called principal stress.

$$t^{\hat{n}} = \Sigma \cdot \hat{n} = \sigma \hat{n}$$

$$\sigma_{ij} n_j - \sigma n_i = 0$$

$$\left(\sigma_{ij} - \sigma \delta_{ij} \right) n_j = 0$$

$$\begin{bmatrix} \sigma_{xx} - \sigma & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \sigma & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \sigma \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

When $\sigma_1 \geq \sigma_2 \geq \sigma_3$

σ_1 : major principal stress

σ_2 : intermediate principal stress

σ_3 : minor principal stress

2.4 Principal stress and stress invariants

- Stress invariants

$$\begin{vmatrix} \sigma_{xx}-\sigma & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy}-\sigma & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}-\sigma \end{vmatrix} = 0 \quad for\ \hat{n} \neq (0,0,0)$$

$$\rightarrow \sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

$$I_1 = \sigma_{ii} \left(= \sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)$$

$$I_2 = \frac{1}{2}\left[\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}\right]\left(= \sigma_{xx}\sigma_{yy} + \sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz} - \sigma_{xy}\sigma_{xy} - \sigma_{xz}\sigma_{xz} - \sigma_{yz}\sigma_{yz}\right)$$

$$I_3 = \left|\sigma_{ij}\right|$$

CUBIC EQUATION: $x^3 + a_1x^2 + a_2x + a_3 = 0$

Let $$Q = \frac{3a_2 - a_1^2}{9}, \quad R = \frac{9a_1a_2 - 27a_3 - 2a_1^3}{54},$$

$$S = \sqrt[3]{R + \sqrt{Q^3 + R^2}}, \quad T = \sqrt[3]{R - \sqrt{Q^3 + R^2}}$$

Solutions:
$$\begin{cases} x_1 = S + T - \frac{1}{3}a_1 \\ x_2 = -\frac{1}{2}(S+T) - \frac{1}{3}a_1 + \frac{1}{2}i\sqrt{3}\,(S - T) \\ x_3 = -\frac{1}{2}(S+T) - \frac{1}{3}a_1 - \frac{1}{2}i\sqrt{3}\,(S - \end{cases}$$

If a_1, a_2, a_3 are real and if $D = Q^3 + R^2$ is the *discriminant*, then

(i) one root is real and two complex conjugate if $D > 0$

(ii) all roots are real and at least two are equal if $D = 0$

(iii) all roots are real and unequal if $D < 0$.

If $D < 0$, computation is simplified by use of trigonometry.

Solutions if $D < 0$:
$$\begin{cases} x_1 = 2\sqrt{-Q}\cos\left(\frac{1}{3}\theta\right) - \frac{1}{3}a_1 \\ x_2 = 2\sqrt{-Q}\cos\left(\frac{1}{3}\theta + 120^\circ\right) - \frac{1}{3}a_1 \\ x_3 = 2\sqrt{-Q}\cos\left(\frac{1}{3}\theta + 240^\circ\right) - \frac{1}{3}a_1 \end{cases} \quad \text{where } \cos\theta = R/\sqrt{-Q^3}$$

$$x_1 + x_2 + x_3 = -a_1, \quad x_1x_2 + x_2x_3 + x_3x_1 = a_2, \quad x_1x_2x_3 = -a_3$$

where x_1, x_2, x_3 are the three roots.

2.5 Differential equations of static equilibrium

Refer to section 2.2

$$\sigma_{ji,j} + \rho b_i = 0$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho b_x = 0$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + \rho b_y = 0$$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho b_z = 0$$

2.6 Plane problems and biaxial stress

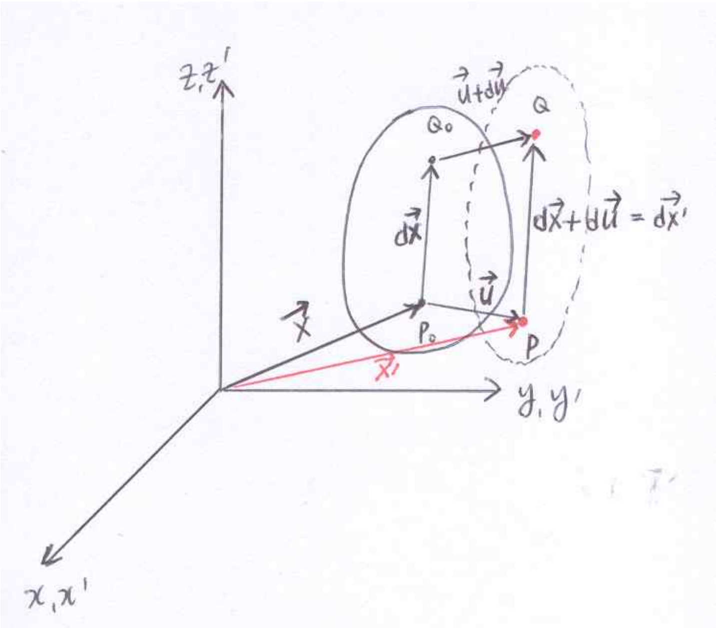
Apply the principles in 3D to 2D problems

Problem

- Calculate traction, normal stress and shear stress in a surface whose normal vector is as follow (Σ is stress on the object).

$$\hat{n} = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}, \right) \quad \Sigma = \begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{bmatrix}$$

2.7 Displacement and strain



$$\vec{X}=(x,y,z) \quad \vec{X}'=(x',y',z')$$

$$\vec{u}=\vec{X}'-\vec{X} \quad d\vec{X}'=d\vec{X}+d\vec{u}$$

$$(dX')^2=d\vec{X}'\cdot d\vec{X}'\rightarrow dX_k^{'}dX_k^{'}$$

$$dX_k^{'}=\frac{\partial X_k^{'}}{\partial X_j}dX_j$$

$$(dX')^2-(dX)^2=\frac{\partial X_k^{'}}{\partial X_i}\frac{\partial X_k^{'}}{\partial X_j}dX_idX_j-\delta_{ij}dX_idX_j$$

$$=\left(\frac{\partial X_k^{'}}{\partial X_i}\frac{\partial X_k^{'}}{\partial X_j}-\delta_{ij}\right)dX_idX_j$$

2.7 Displacement and strain

$$L_{ij} = \frac{1}{2}\left(\frac{\partial X'_k}{\partial X_i}\frac{\partial X'_k}{\partial X_j} - \delta_{ij}\right)$$

$$X'_k = u_k + X_k$$

$$L_{ij} = \frac{1}{2}\left(\left(\frac{\partial u_k}{\partial X_i} + \delta_{ki}\right)\left(\frac{\partial u_k}{\partial X_j} + \delta_{kj}\right) - \delta_{ij}\right)$$

$$= \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j} + \delta_{ij} - \delta_{ij}\right)$$

$$\approx \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \quad \text{for infinitesimal displacement}$$

$$\therefore (dX')^2 - (dX)^2 = 2L_{ij}dX_i dX_j$$

2.7 Displacement and strain

$$L_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$$

$$\left[L_{ij}\right] = \begin{bmatrix} \frac{\partial u_x}{\partial X_x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial X_y} + \frac{\partial u_y}{\partial X_x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial X_z} + \frac{\partial u_z}{\partial X_x}\right) \\ \frac{1}{2}\left(\frac{\partial u_y}{\partial X_x} + \frac{\partial u_x}{\partial X_y}\right) & \frac{\partial u_y}{\partial X_y} & \frac{1}{2}\left(\frac{\partial u_y}{\partial X_z} + \frac{\partial u_z}{\partial X_y}\right) \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial X_x} + \frac{\partial u_x}{\partial X_z}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial X_y} + \frac{\partial u_y}{\partial X_z}\right) & \frac{\partial u_z}{\partial X_z} \end{bmatrix}$$

…..Linear (Infinitesimal) strain tensor

2.7 Displacement and strain

• Normal strain

$$(dX')^2 - (dX)^2 = (dX' - dX)(dX' + dX)$$

$$\approx 2(dX' - dX)dX = 2L_{ij}dX_i dX_j$$

$$dX' - dX = \frac{L_{ij}dX_i dX_j}{dX}$$

$$\frac{dX' - dX}{dX} = \frac{dX_i}{dX} L_{ij} \frac{dX_j}{dX} = \hat{n} \cdot L \cdot \hat{n} \quad (\text{normal strain})$$

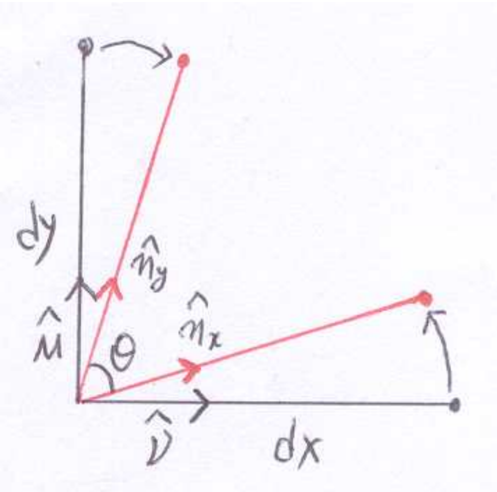
$$\hat{e}_x \cdot L \cdot \hat{e}_x = L_{xx}$$

$$\hat{e}_y \cdot L \cdot \hat{e}_y = L_{yy}$$

$$\hat{e}_z \cdot L \cdot \hat{e}_z = L_{zz}$$

2.7 Displacement and strain

- Shear strain



$$\hat{n}_x \cdot \hat{n}_y = \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right) = \sin\gamma \approx \gamma \quad (\text{shear strain})$$

$$L_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) = \frac{1}{2}(J + J_c)$$

$$d\vec{u} = \frac{\partial u_i}{\partial X_j} dX_j = J \cdot d\vec{X}$$

$$\hat{n}_x \approx \hat{\upsilon} + J \cdot \hat{\upsilon}, \quad \hat{n}_y \approx \hat{\mu} + J \cdot \hat{\mu}$$

$$\gamma = \hat{n}_x \cdot \hat{n}_y = (\hat{\upsilon} + J \cdot \hat{\upsilon}) \cdot (\hat{\mu} + J \cdot \hat{\mu})$$

$$= \hat{\upsilon} \cdot \hat{\mu} + \hat{\upsilon} \cdot (J + J_c) \cdot \hat{\mu} + \hat{\upsilon} \cdot J_c \cdot J \cdot \hat{\mu}$$

$$= \hat{\upsilon} \cdot 2L \cdot \hat{\mu}$$

$$\hat{e}_x \cdot L \cdot \hat{e}_y = L_{xy} = \frac{1}{2}\gamma_{xy} \quad \hat{e}_y \cdot L \cdot \hat{e}_z = L_{yz} = \frac{1}{2}\gamma_{yz}$$

$$\hat{e}_z \cdot L \cdot \hat{e}_x = L_{zx} = \frac{1}{2}\gamma_{zx}$$

2.7 Displacement and strain

$$L_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right)$$

$$\left[L_{ij} \right] = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{bmatrix}$$

2.7 Displacement and strain

•Decomposition of relative displacement (*du*)

$$du_i = \frac{\partial u_i}{\partial X_j} dX_j$$

$$= \left[\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i} \right) \right] dX_j$$

$$= \left[L_{ij} + \Omega_{ij} \right] dX_j$$

$$\left[\Omega_{ij} \right] = \begin{bmatrix} 0 & \omega_{xy} & \omega_{xz} \\ -\omega_{xy} & 0 & \omega_{yz} \\ -\omega_{xz} & -\omega_{yz} & 0 \end{bmatrix} \quad \text{.... rotation tensor}$$

$$= \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \quad \left(\vec{\Omega} = \frac{1}{2} \vec{\nabla} \times \vec{u} = \frac{1}{2} \begin{bmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x & u_y & u_z \end{bmatrix} \right)$$

2.8 Principal strains, strain transformation, volumetric strain and deviator strain

Principal strains: Eigen values/vectors of strain tensor

Strain transformation: analogous to the stress transformation

$$\text{Volumetric strain} = \frac{V - V_0}{V_0} = \frac{(1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3)\,dX_1 dX_2 dX_3}{dX_1 dX_2 dX_3} - 1$$

$$= \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1 + \varepsilon_1\varepsilon_2\varepsilon_3$$

$$\approx \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

$$\text{Deviator strain} = \begin{bmatrix} \varepsilon_x - \Delta/3 & \gamma_{xy} & \gamma_{xz} \\ \gamma_{xy} & \varepsilon_y - \Delta/3 & \gamma_{yz} \\ \gamma_{xz} & \gamma_{yz} & \varepsilon_z - \Delta/3 \end{bmatrix} \quad \Delta : \text{volumetric strain}$$

2.9 Strain compatibility equations

Independent strain components: 6 $(\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx})$

Independent displacement components : 3 (u_x, u_y, u_z)

Random $u_x, u_y, u_z \rightarrow \varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ (o)

Random $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx} \rightarrow u_x, u_y, u_z$ (×)

Finding restricting conditions:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial X_x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial X_y}, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial X_y} + \frac{\partial u_y}{\partial X_x}\right)$$

$$\frac{\partial^2 \varepsilon_{xx}}{\partial X_y{}^2} = \frac{\partial^3 u_x}{\partial X_x \partial X_y{}^2}, \quad \frac{\partial^2 \varepsilon_{yy}}{\partial X_x{}^2} = \frac{\partial^3 u_y}{\partial X_x{}^2 \partial X_y}, \quad \frac{\partial^2 \varepsilon_{xy}}{\partial X_x \partial X_y} = \frac{1}{2}\left(\frac{\partial^3 u_x}{\partial X_x \partial X_y{}^2} + \frac{\partial^3 u_y}{\partial X_x{}^2 \partial X_y}\right)$$

$$\therefore \frac{\partial^2 \varepsilon_{xx}}{\partial X_y{}^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial X_x{}^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial X_x \partial X_y}$$

2.9 Strain compatibility equations

$$\frac{\partial^2 \varepsilon_{xx}}{\partial X_y{}^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial X_x{}^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial X_x \partial X_y}$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial X_z{}^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial X_x{}^2} = 2\frac{\partial^2 \varepsilon_{yz}}{\partial X_y \partial X_z}$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial X_x{}^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial X_z{}^2} = 2\frac{\partial^2 \varepsilon_{zx}}{\partial X_x \partial X_z}$$

$$\frac{\partial}{\partial X_x}\left(-\frac{\partial \varepsilon_{yz}}{\partial X_x} + \frac{\partial \varepsilon_{zx}}{\partial X_y} + \frac{\partial \varepsilon_{xy}}{\partial X_z}\right) = \frac{\partial^2 \varepsilon_{xx}}{\partial X_y \partial X_z}$$

$$\frac{\partial}{\partial X_y}\left(\frac{\partial \varepsilon_{yz}}{\partial X_x} - \frac{\partial \varepsilon_{zx}}{\partial X_y} + \frac{\partial \varepsilon_{xy}}{\partial X_z}\right) = \frac{\partial^2 \varepsilon_{yy}}{\partial X_z \partial X_x}$$

$$\frac{\partial}{\partial X_z}\left(\frac{\partial \varepsilon_{yz}}{\partial X_x} + \frac{\partial \varepsilon_{zx}}{\partial X_y} - \frac{\partial \varepsilon_{xy}}{\partial X_z}\right) = \frac{\partial^2 \varepsilon_{zz}}{\partial X_x \partial X_y}$$

2.10 Stress-strain relations

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right], \quad \varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right], \quad \varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right]$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}, \quad \gamma_{yz} = \frac{1}{G}\tau_{yz}, \quad \gamma_{zx} = \frac{1}{G}\tau_{zx} \quad \left(G = \frac{E}{2(1+\nu)}\right)$$

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \nu/(1-\nu) & \nu/(1-\nu) & 0 & 0 & 0 \\ \nu/(1-\nu) & 1 & \nu/(1-\nu) & 0 & 0 & 0 \\ \nu/(1-\nu) & \nu/(1-\nu) & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-2\nu)}{2(1-\nu)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2(1-\nu)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2(1-\nu)} \end{bmatrix}\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

2.11 Cylindrical polar co-ordinates

Refer to p.37, textbook

2.12 Geomechanics conventions for displacement, strain and stress

Compressive/contractile force/displacement are positive.
Refer to Figure 2.12.

Sign convention in rock mechanical problems