Introduction to Nuclear Fusion (409.308A, 3 Credits)

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Plasmas as Fluids

- **Two-fluid equations**

The continuity equation will apply separately to each of the different species. The momentum balance equation must consider the fact that particles of one species can collide with particles of another species, thereby transferring momentum between the different species.

$$\vec{R}_{\alpha\beta} = -m_\alpha n_\alpha \nu_{\alpha\beta}(\vec{u}_\alpha - \vec{u}_\beta)$$

The rate at which momentum per unit volume is gained by species α due to collisions with species β.

$\nu_{\alpha\beta}$: collision frequency of α on β

$$m_\alpha n_\alpha \left(\frac{\partial \vec{u}_\alpha}{\partial t} + (\vec{u}_\alpha \cdot \nabla)\vec{u}_\alpha \right) = n_\alpha q_\alpha (\vec{E} + \vec{u}_\alpha \times \vec{B}) - \nabla \cdot \overleftrightarrow{P}_\alpha + \sum_\beta \vec{R}_{\alpha\beta}$$

$$\vec{R}_{\beta\alpha} = -\vec{R}_{\alpha\beta}$$

The rate at which momentum per unit volume is gained by species β due to collisions with species α.

$$m_\alpha n_\alpha \nu_{\alpha\beta} = m_\beta n_\beta \nu_{\beta\alpha}$$

Plasmas as Fluids

- **Plasma resistivity**

The acceleration of electrons by an electric field applied along (or in the absence of) a magnetic field is impeded by collisions with non-accelerated particles, in particular the ions, which, because of their much larger mass, are relatively unresponsive to the applied electric field. Collisions between electrons and ions, acting in this way to limit the current that can be driven by an electric field, give rise to an important plasma quantity, namely its electrical resistivity, η.

$$\vec{R}_{ei} = -m_e n_e \nu_{ei} (\vec{u}_e - \vec{u}_i)$$

$$m_\alpha n_\alpha \left(\frac{\partial \vec{u}_\alpha}{\partial t} + (\vec{u}_\alpha \cdot \nabla) \vec{u}_\alpha \right) = n_\alpha q_\alpha (\vec{E} + \vec{u}_\alpha \times \vec{B}) - \nabla \cdot \overleftrightarrow{P}_\alpha + \sum_\beta \vec{R}_{\alpha\beta}$$

Homogeneous (neglecting the electron pressure and velocity gradients along **B**)

$$0 = -n_e e E_{\|} + R_{ei\|}$$

Plasmas as Fluids

- **Plasma resistivity**

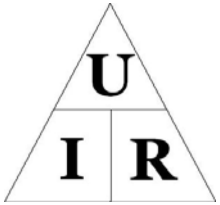
$$0 = -n_e e E_{\|} + R_{ei\|} \qquad \vec{R}_{ei} = -m_e n_e \nu_{ei} (\vec{u}_e - \vec{u}_i)$$

$$J_{\|} = -n_e e (u_{e\|} - u_{i\|})$$

$$E_{\|} = -\frac{m_e \nu_{ei}}{e}(u_{e\|} - u_{i\|}) = \frac{m_e \nu_{ei}}{n_e e^2} J_{\|} = \eta J_{\|}$$

Simplified Ohm's law

$$\eta = \frac{m_e \langle \nu_{ei} \rangle}{n_e e^2}$$



Ohm's triangle

Momentum gained by electrons due to collisions with ions

$$\vec{R}_{ei} = -m_e n_e \langle \nu_{ei} \rangle (\vec{u}_e - \vec{u}_i) = -\eta n_e^2 e^2 (\vec{u}_e - \vec{u}_i) = \eta n_e e \vec{J}$$

Plasmas as Fluids

- **Single-fluid magnetohydrodynamics (MHDs)**

A single-fluid model of a fully ionised plasma, in which the plasma is treated as a single hydrogynamic fluid acted upon by electric and magnetic forces.

- **The magnetohydrodynamic (MHD) equation**

$$\rho = n_i M + n_e m \approx n(M+m) \approx nM$$ mass density

Hydrogen plasma, charge neutrality assumed

$$\sigma = (n_i - n_e)e$$ charge density

$$\vec{v} = (n_i M\vec{u}_i + n_e m\vec{u}_e)/\rho \approx (M\vec{u}_i + m\vec{u}_e)/(M+m) \approx \vec{u}_i + (m/M)\vec{u}_e$$ mass velocity

$$\vec{J} = e(n_i\vec{u}_i - n_e\vec{u}_e) \approx ne(\vec{u}_i - \vec{u}_e)$$

$$\vec{u}_i \approx \vec{v} + \frac{m}{M}\frac{\vec{J}}{ne}, \quad \vec{u}_e \approx \vec{v} - \frac{\vec{J}}{ne}$$

electron inertia neglected: electrons have an infinitely fast response time because of their small mass

Plasmas as Fluids

- **The magnetohydrodynamic (MHD) equation**

$$\frac{\partial n_{i,e}}{\partial t} + \nabla \cdot (n_{i,e}\vec{u}_{i,e}) = 0$$

multiplied by M and m, respectively and added together

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

Mass continuity equation

$$\frac{\partial \sigma}{\partial t} + \nabla \cdot \vec{J} = 0$$

Charge continuity equation

$$Mn_i \frac{d\vec{u}_i}{dt} = en_i(\vec{E} + \vec{u}_i \times \vec{B}) - \nabla p_i + \vec{R}_{ie}$$

$$mn_e \frac{d\vec{u}_e}{dt} = -en_e(\vec{E} + \vec{u}_e \times \vec{B}) - \nabla p_e + \vec{R}_{ei}$$

Equations of motion:
isotropic pressure assumed

In the case, where a plasma is nearly Maxwellian (or at least nearly isotropic), the pressure tensor term can be replaced by the gradient of a scalar pressure, ∇p

$$\rho \frac{d\vec{v}}{dt} = \rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}\right) = \sigma \vec{E} + \vec{J} \times \vec{B} - \nabla p$$

Single-fluid equation of motion

Plasmas as Fluids

$$\vec{R}_{ei} = mn\langle \nu_{ei} \rangle (\vec{u}_i - \vec{u}_e) = \eta n^2 e^2 (\vec{u}_i - \vec{u}_e) = \eta n e \vec{J}$$

$$\vec{E} + \vec{u}_e \times \vec{B} = \eta \vec{J} - \frac{\nabla p_e}{ne}$$

$$\vec{E} + \vec{v} \times \vec{B} = \eta \vec{J} + \frac{\vec{J} \times \vec{B} - \nabla p_e}{ne} \quad \longleftarrow \quad \vec{u}_e \approx \vec{v} - \frac{\vec{J}}{ne}$$

Generalized Ohm's law

Neglect electron inertia entirely: valid for phenomena that are sufficiently slow that electrons have time to reach dynamical equilibrium in regard to their motion along the magnetic field

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Maxwell equation

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \cdot (\varepsilon_0 \vec{E}) = \sigma$$

Plasmas as Fluids

- **Ideal MHD model**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0$$

Mass continuity equation

$$\rho \frac{d\vec{v}}{dt} = \vec{J} \times \vec{B} - \nabla p$$

Single-fluid equation of motion

$$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0$$

Energy equation (equation of state):
adiabatic evolution

$$\vec{E} + \vec{v} \times \vec{B} = 0$$

Ohm's law: perfect conductor → "ideal" MHD

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

$$\nabla \cdot \vec{B} = 0$$

Maxwell equations

$\varepsilon_0 \rightarrow 0$ assmued
(Full → low-frequency Maxwell's equations)
Displacement current, net charge neglected

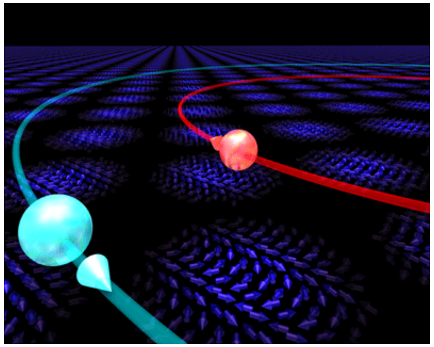
Plasmas as Fluids

- **Ideal MHD**

- Single-fluid model

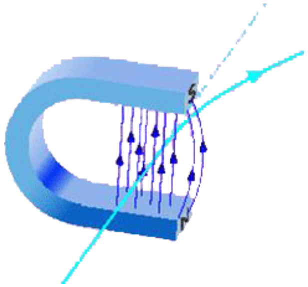
- Ideal:

    Perfect conductor with zero resistivity



- MHD:

    Magnetohydrodynamic (magnetic fluid dynamic)

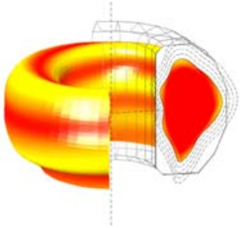


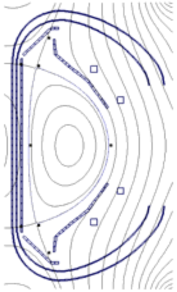
- Assumptions:

    Low-frequency, long-wavelength
    collision-dominated plasma

- Applications:

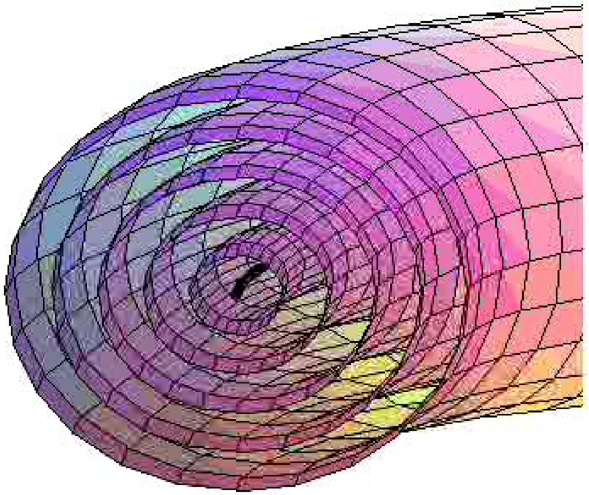
    Equilibrium and stability in fusion plasmas



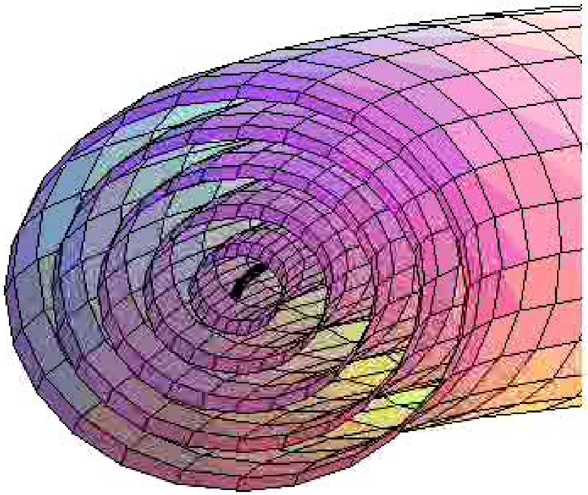


What is Ideal MHD?

- Ideal MHD: $\eta = 0$

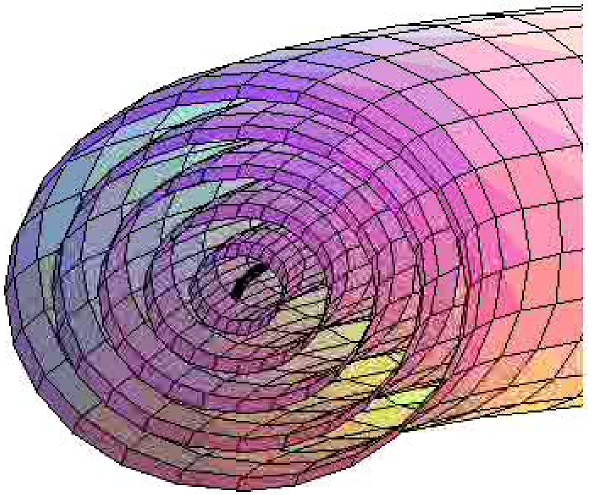


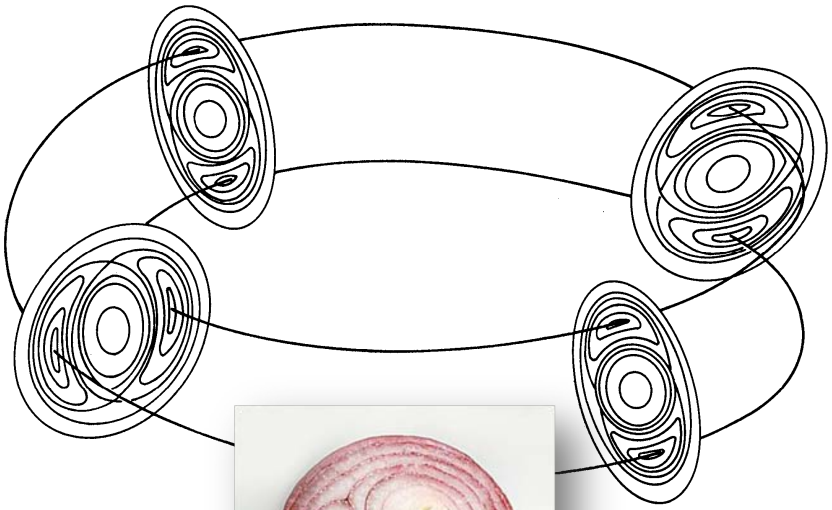
- Resistive MHD: $\eta \neq 0$



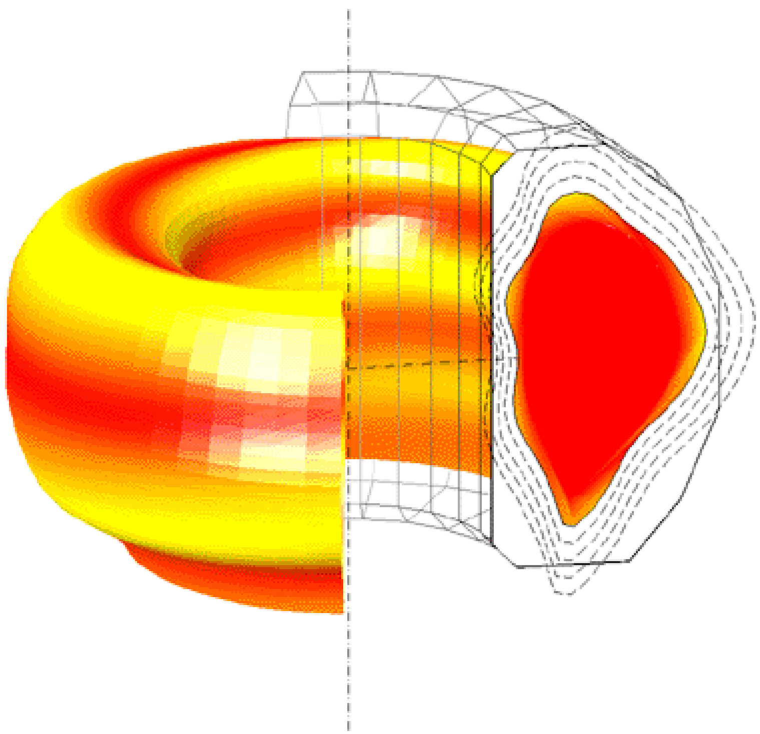
What is Ideal MHD?

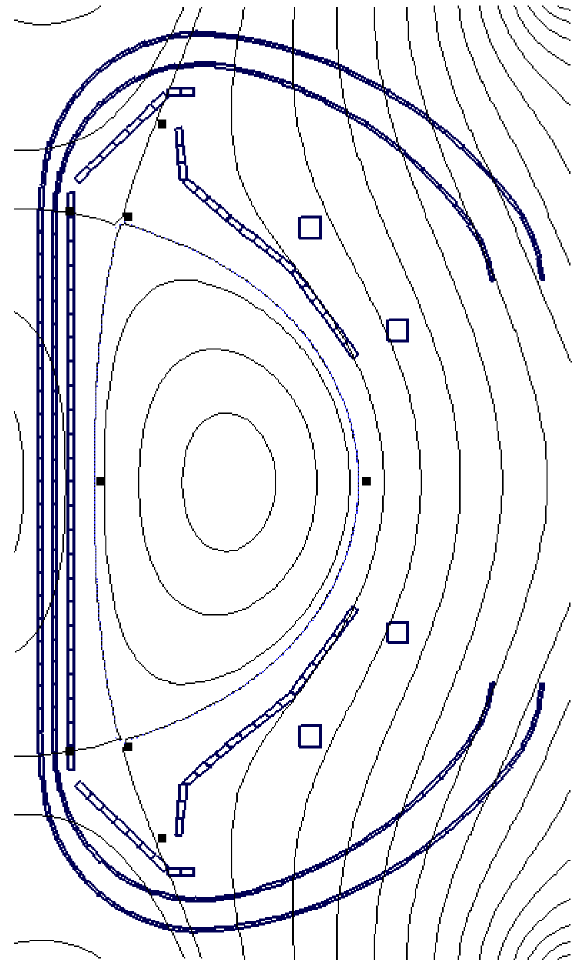
- Ideal MHD: $\eta = 0$
- Resistive MHD: $\eta \neq 0$





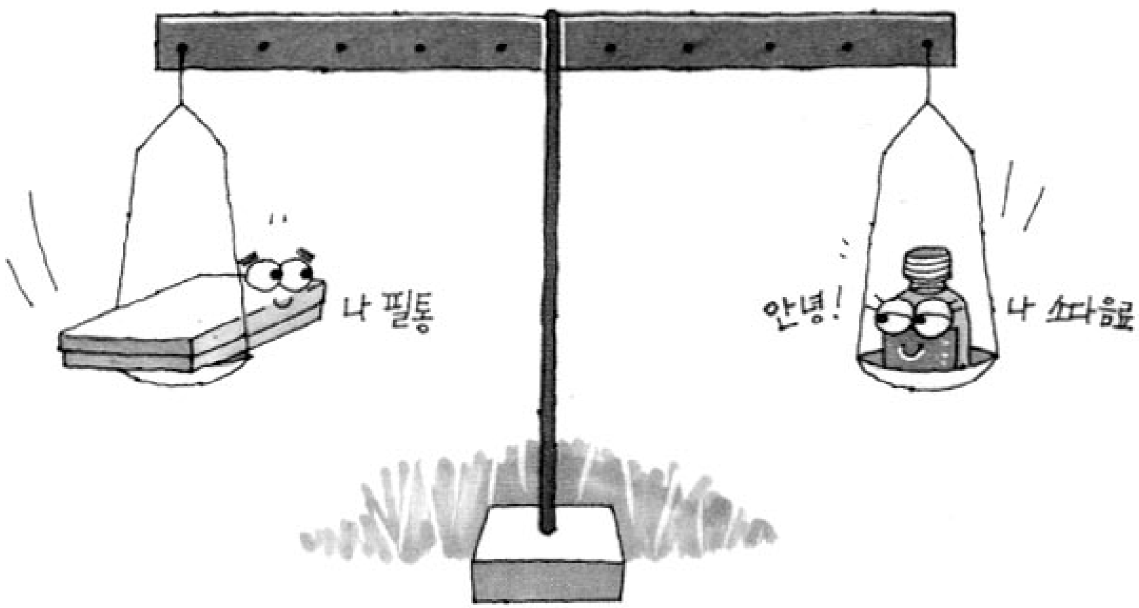
Applications





Plasma Eqauilibrium and Stability

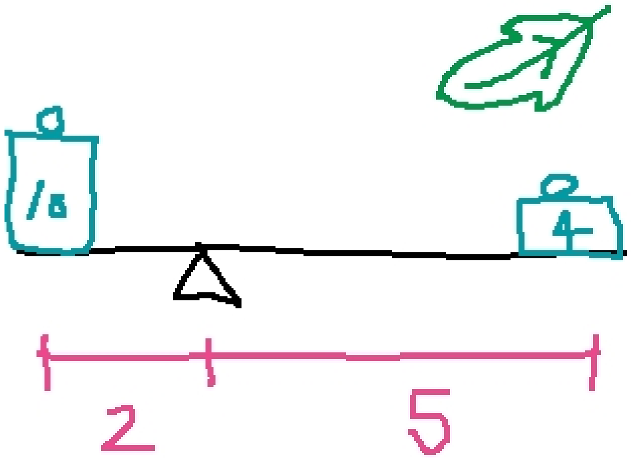
Equilibrium and Stability

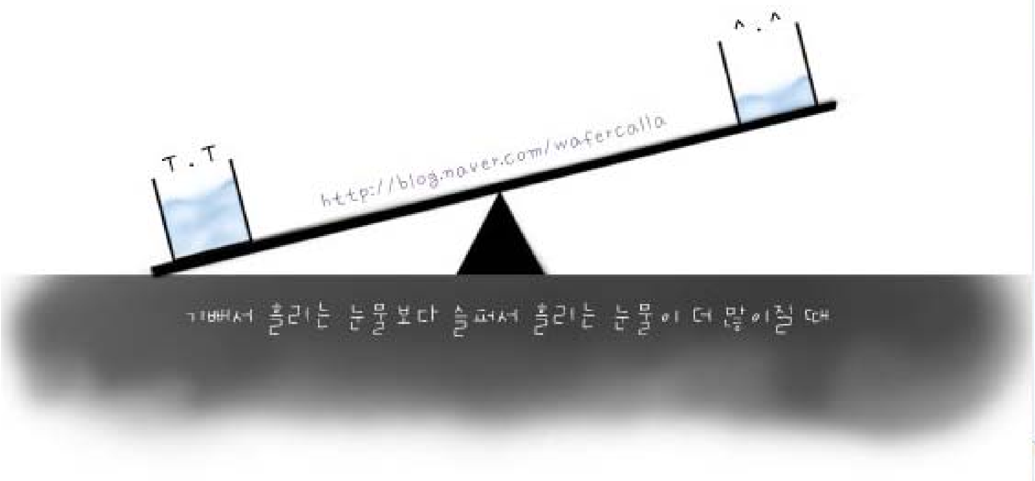


Equilibrium? Yes! Forces are balanced

Stable? No!

Equilibrium and Stability





Equilibrium? Yes! Forces are balanced

Stable? No! The system cannot recover.

Equilibrium

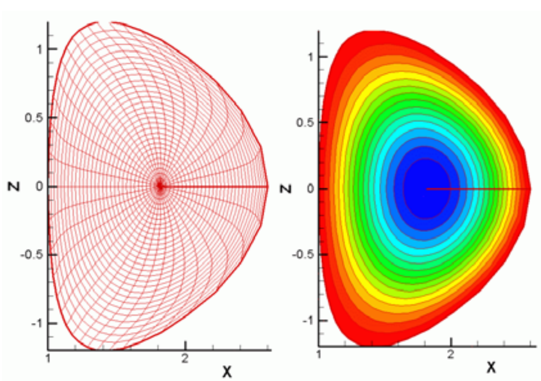
- **Basic Equations**
  - MHD equilibrium equations:
    time-independent with **v** = 0 (static)

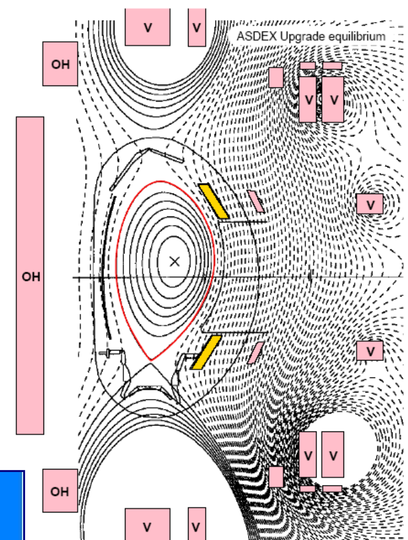
$$\boxed{\begin{aligned} \nabla p &= \vec{J}\times\vec{B} \\ \nabla\times\vec{B} &= \mu_0\vec{J} \\ \nabla\cdot\vec{B} &= 0 \end{aligned}}$$

→ Force balance

→ Ampere's law

→ Closed magnetic field lines





ASDEX Upgrade IPP

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\rho\vec{v}=0$$

$$\rho\frac{d\vec{v}}{dt}=\vec{J}\times\vec{B}-\nabla p$$

$$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right)=0$$

$$\vec{E}+\vec{v}\times\vec{B}=0$$

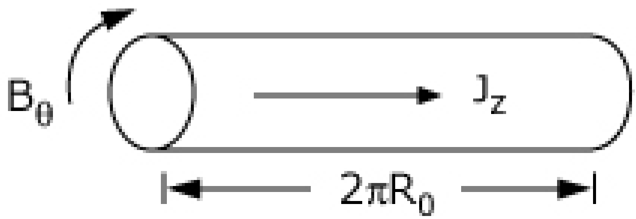
$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\times\vec{B}=\mu_0\vec{J}$$

$$\nabla\cdot\vec{B}=0$$

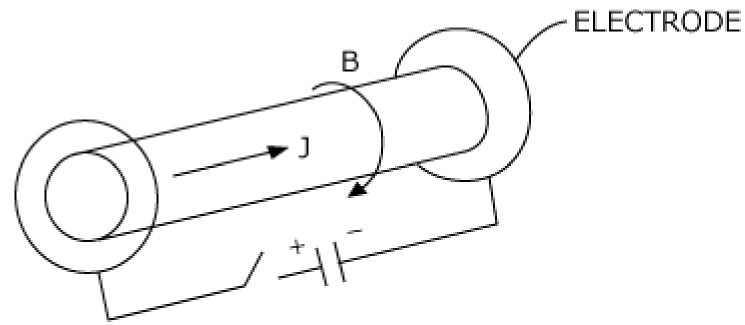
Equilibrium: 1-D Configurations

- **The *Z* Pinch**
  - 1-D toroidal configuration with purely poloidal field



self field induced by J_Z

longitudinal plasma current



- Sequence of solution of the MHD equilibrium equations

1. The $\nabla \cdot \mathbf{B} = 0$
2. Ampere's law: $\mu_0 \mathbf{J} = \nabla \times \mathbf{B}$
3. The momentum equation: $\mathbf{J} \times \mathbf{B} = \nabla p$

Equilibrium: 1-D Configurations

- **The *Z* Pinch**

- Sequence of solution of the MHD equilibrium equations

1. The $\nabla\cdot\mathbf{B} = 0$

$$\frac{1}{r}\frac{\partial B_\theta}{\partial \theta} = 0$$

2. Ampere's law: $\mu_0\mathbf{J} = \nabla\times\mathbf{B}$

$$J_z = \frac{1}{\mu_0 r}\frac{d}{dr}(rB_\theta)$$

3. The momentum equation: $\mathbf{J}\times\mathbf{B} = \nabla p$

$$J_z B_\theta = -\frac{dp}{dr}$$

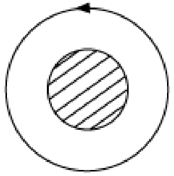
$$\frac{dp}{dr} + \frac{B_\theta}{\mu_0 r}\frac{d}{dr}(rB_\theta) = 0 \qquad \frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_0}\right) + \frac{B_\theta^2}{\mu_0 r} = 0$$

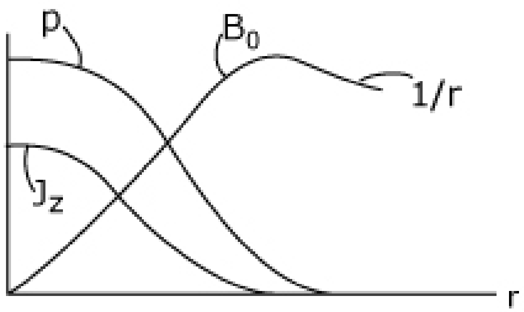
particle pressure + magnetic pressure force

tension force by the curvature of the magnetic field lines

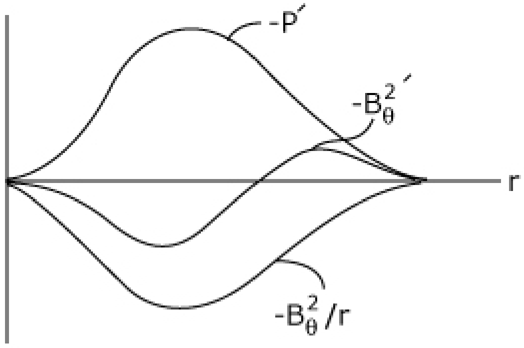
Equilibrium: 1-D Configurations

- **The *Z* Pinch**
  - It is the tension force and not the magnetic pressure gradient that provides radial confinement of the plasma.





FIELDS



FORCES

$$B_\theta = \frac{\mu_0 I_0}{2\pi}\frac{r}{r^2 + r_0^2}$$

$$J_z = \frac{I_0}{\pi}\frac{r_0^2}{(r^2 + r_0^2)^2}$$

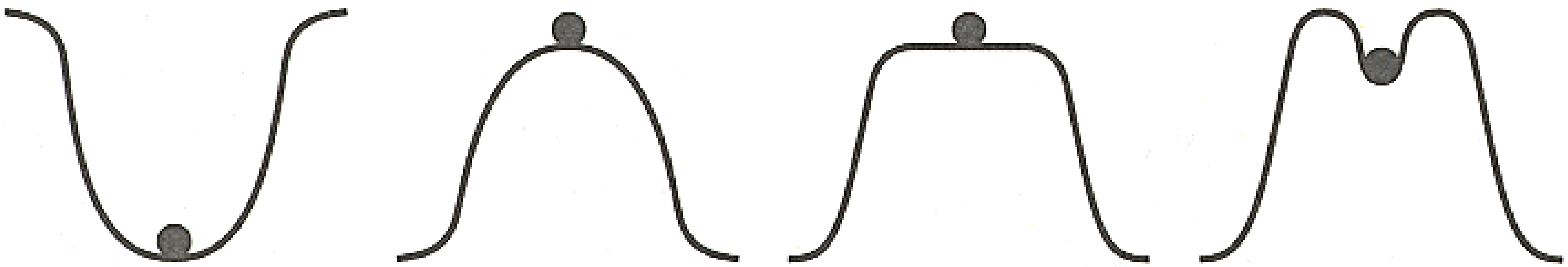
$$p = \frac{\mu_0 I_0^2}{8\pi^2}\frac{r_0^2}{(r^2 + r_0^2)^2}$$

Bennett profiles
(Bennett, 1934)

Stability: General Considerations

- **Definition of Stability**

  - Suggested by physics, where stability means, roughly speaking, that a small change (disturbance) of a physical system at some instant changes the behavior of the system only slightly at all future times *t*.
  - The fact that one can find an equilibrium does not guarantee that it is stable. Ball on hill analogies:



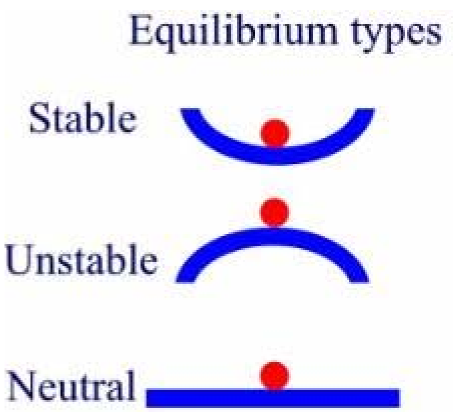
stable | linear unstable | metastable | non-linear unstable

linear: with small perturbation
non-linear: with large perturbation

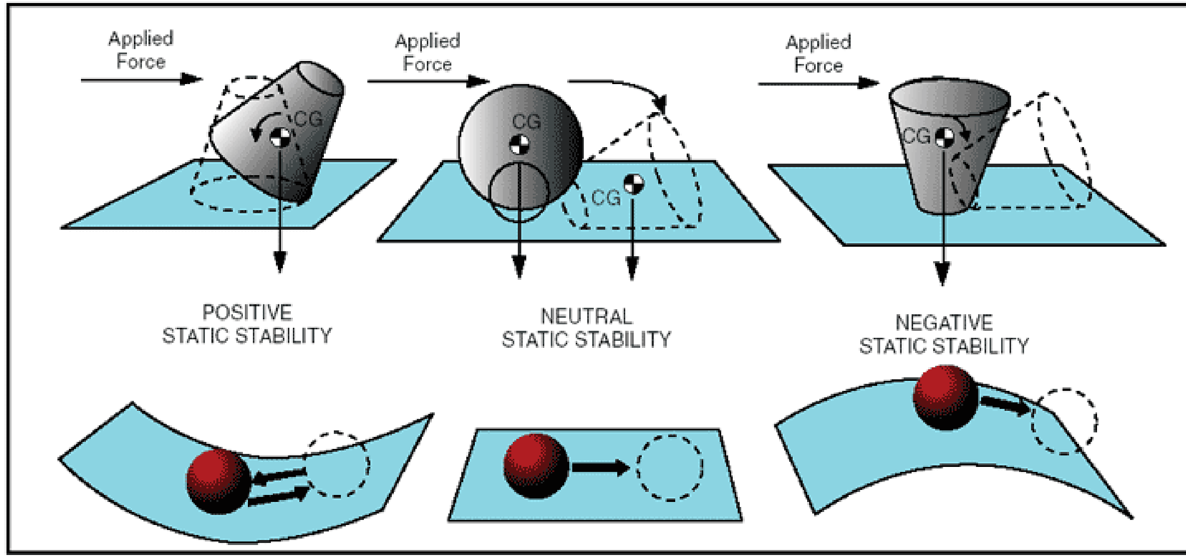
- Generation of instability is the general way of redistributing energy which was accumulated in a non-equilibrium state.

Stability

- **Definition of Stability**



Marginally stable



- assuming all quantities of interest linearised about their equilibrium values.

$$Q(\vec{r},t) = Q_0(\vec{r}) + \tilde{Q}_1(\vec{r},t)$$

small 1st order perturbation

$$\tilde{Q}_1 / |Q_0| << 1 \qquad \tilde{Q}_1(\vec{r},t) = Q_1(\vec{r})\varepsilon^{-i\omega t}$$

Im ω > 0: exponential instability

Im $\omega \leq$ 0: exponential stability

Stability

- **Various Approaches for Stability Analyses**

1. Initial value problem using the general linearised equations of motion
2. Normal-mode eigenvalue problem
3. Variational principle
4. Energy Principle

Stability

- Initial Value Formulation

$$\vec{J}_0 \times \vec{B}_0 = \nabla p_0$$

$$\mu_0 \vec{J}_0 = \nabla \times \vec{B}_0$$

$$\nabla \cdot \vec{B}_0 = 0$$

$$\vec{v}_0 = 0$$

$$Q(\vec{r},t) = Q_0(\vec{r}) + \tilde{Q}_1(\vec{r},t) \qquad \tilde{Q}_1 / |Q_0| << 1$$

linearized

$$\tilde{v}_1 = \frac{\partial \xi}{\partial t}$$

ξ: displacement of the plasma away from its equilibrium position

Aim: to express all perturbed quantities in terms of ξ and then obtain a single equation describing the time evolution of ξ

Stability

- Initial Value Formulation

$$\vec{J}_0 \times \vec{B}_0 = \nabla p_0$$

$$\mu_0 \vec{J}_0 = \nabla \times \vec{B}_0$$

$$\nabla \cdot \vec{B}_0 = 0$$

$$\vec{v}_0 = 0$$

$$Q(\vec{r},t) = Q_0(\vec{r}) + \tilde{Q}_1(\vec{r},t) \quad \tilde{Q}_1 / |Q_0| << 1$$

linearized

$$\tilde{v}_1 = \frac{\partial \xi}{\partial t}$$

ξ: displacement of the plasma away from its equilibrium position

$$\rho \frac{\partial^2 \xi}{\partial t^2} = \vec{F}(\xi)$$

momentum equation

$$\vec{F}(\xi) = \vec{J} \times \tilde{B}_1 + \tilde{J}_1 \times \vec{B} - \nabla \tilde{p}_1$$

force operator

$$= \frac{1}{\mu_0}(\nabla \times \vec{B}) \times \tilde{Q} + \frac{1}{\mu_0}(\nabla \times \tilde{Q}) \times \vec{B} + \nabla(\xi \cdot \nabla p + \gamma p \nabla \cdot \xi)$$

$$\xi(\vec{r},0) = 0, \quad \frac{\partial \xi(\vec{r},0)}{\partial t} = \tilde{v}_1(\vec{r},0)$$

+ Boundary conditions

Formulation of the generalized stability equations as an initial value problem

Stability

- Normal-Mode Formulation

$$\tilde{Q}_1(\vec{r},t) = Q_1(\vec{r})\exp(-i\omega t)$$

$$\rho_1 = -\nabla\cdot(\rho\xi)$$ conservation of mass

$$p_1 = -\xi\cdot\nabla p - \gamma p\nabla\cdot\xi$$ conservation of energy

$$\vec{Q} \equiv \vec{B}_1 = \nabla\times(\xi\times\vec{B})$$ Faraday's law

$$-\omega^2\rho\xi = \vec{F}(\xi)$$ normal-mode formulation

$$\vec{F}(\xi) = \frac{1}{\mu_0}(\nabla\times\vec{B})\times\tilde{Q} + \frac{1}{\mu_0}(\nabla\times\tilde{Q})\times\vec{B} + \nabla(\xi\cdot\nabla p + \gamma p\nabla\cdot\xi)$$

- An eigenvalue problem for the eigenvalue ω^2

Stability

- Variational Principle

  Classic eigenvalue problem

$$\frac{d}{dx}\left(f\frac{\partial y}{\partial x}\right) + (\lambda - g)y = 0$$

λ: eigenvalue

$$y(0) = y(1) = 0$$

$$\lambda = \frac{\int (fy'^2 + gy^2)dx}{\int y^2 dx}$$

Multiplied by y
and integrated over the region $0 \le x \le 1$

Why is this variational?

- Substitute all allowable trial function $y(x)$ into the equation above.
- When resulting λ exhibits an extremum (maximum, minimum, saddle point) then λ and y are actual eigenvalue and eigenfunction.

$$\delta\lambda \approx -\frac{2\int \delta y[(fy_0')' + (\lambda_0 - g)y_0]dx}{\int y_0^2 dx}$$

Stability

- Variational Principle

$$-\omega^2 \rho \xi = \vec{F}(\xi)$$ normal-mode formulation

$$\omega^2 = \frac{\delta W(\xi^*, \xi)}{K(\xi^*, \xi)}$$ dot product with ξ^* then integrated over the plasma volume

$$\lambda = \frac{\int (fy'^2 + gy^2)dx}{\int y^2 dx}$$

$$\delta W(\xi^*, \xi) = -\frac{1}{2}\int \xi^* \cdot \vec{F}(\xi) d\vec{r}$$

$$= -\frac{1}{2}\int \xi^* \cdot [\frac{1}{\mu_0}(\nabla \times \vec{Q}) \times \vec{B} + \frac{1}{\mu_0}(\nabla \times \vec{B}) \times \vec{Q} + \nabla(\gamma p \nabla \cdot \xi + \xi \cdot \nabla p)] d\vec{r}$$

$$K(\xi^*, \xi) = \frac{1}{2}\int \rho |\xi|^2 d\vec{r}$$

Any allowable function ξ for which ω^2 becomes an extremum is an eigenfunction of the ideal MHD normal mode equations with eigenvalue ω^2.

Stability

- Variational Principle

$$-\omega^2 \rho \xi = \vec{F}(\xi)$$ normal-mode formulation

$$\omega^2 = \frac{\delta W(\xi^*, \xi)}{K(\xi^*, \xi)}$$ dot product with ξ^* then integrated over the plasma volume

$$\lambda = \frac{\int (fy'^2 + gy^2)dx}{\int y^2 dx}$$

$$-\omega^2 K + \delta W = 0$$ Conservation of energy

Kinetic energy

- Change in potential energy associated with the perturbation
  - Equal to the work done against the force **F**(ξ) in displacing the plasma by an amount ξ.

Stability

- Energy Principle

$$\omega^2 = \frac{\delta W}{K} \geq 0 \qquad \text{stable}$$

$$\delta W \geq 0 \qquad \text{stable}$$

$$\delta W = \delta W_F + \delta W_S + \delta W_V$$

$$\delta W_F = \frac{1}{2}\int_P d\vec{r}\left[\frac{\left|\vec{Q}\right|^2}{\mu_0} - \xi_\perp^* \cdot (\vec{J} \times \vec{Q}) + \gamma p\left|\nabla \cdot \xi\right|^2 + (\xi_\perp \cdot \nabla p)\nabla \cdot \xi_\perp^*\right]$$

$$\delta W_S = \frac{1}{2}\int_S d\vec{S}\left|\vec{n} \cdot \xi_\perp\right|^2 \vec{n} \cdot [[\nabla\left(p + B^2 / 2\mu_0\right)]]$$

$$\delta W_V = \frac{1}{2}\int_V d\vec{r}\,\frac{\left|\hat{B}_1\right|^2}{\mu_0}$$

Boundary conditions on trial functions

$$\vec{n} \cdot \hat{B}_1\Big|_{r_w} = 0 \qquad \vec{n} \cdot \hat{B}_1\Big|_{r_p} = \hat{B}_1 \cdot \nabla(\vec{n} \cdot \xi_\perp) - (\vec{n} \cdot \xi_\perp)[\vec{n} \cdot (\vec{n} \cdot \nabla)\hat{B}_1]\Big|_{r_p}$$

Stability

$$\delta W_F = \frac{1}{2}\int_P d\vec{r}\left[\underbrace{\frac{|\vec{Q}_\perp|^2}{\mu_0} + \frac{B^2}{\mu_0}\left|\nabla\cdot\xi_\perp + 2\xi_\perp\cdot\vec{\kappa}\right|^2 + \gamma p\left|\nabla\cdot\xi\right|^2}_{\text{stabilising}} \overbrace{- 2(\xi_\perp\cdot\nabla p)(\vec{\kappa}\cdot\xi_\perp^*) - J_\parallel(\xi_\perp^*\times\vec{b})\cdot\vec{Q}_\perp}^{\text{destabilising}}\right]$$

Pressure-driven modes (+ or -)

current-driven (kinks) modes (+ or -)

Energy required to bend magnetic field lines: dominant potential energy contribution to the shear Alfvén wave

Energy necessary to compress the magnetic field: major potential energy contribution to the compressional Alfvén wave

Energy required to compress the plasma: main source of potential energy for the sound wave

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