Chapter 5 Flow of an Incompressible Ideal Fluid

Objectives:

- Apply Newton's 2nd law to derive equation of motion, Euler's equation
- Introduce the important Bernoulli and work-energy equations, which permit us to predict pressures and velocities in a flowfield
- Derive Bernoulli equation and more general work-energy equation based on a control volume analysis

우리가 물이 되어 만난다면
가문 어느 집에선들 좋아하지 않으랴
우리가 키 큰 나무와 함께 서서
우르르 우르르 비 오는 소리로 흐른다면

흐르고 흘러서 저물녘엔
저 혼자 깊어지는 강물에 누워
죽은 나무 뿌리를 적시기도 한다면
…

만리 밖에서 기다리는 그대여
저 불 지난 뒤에
흐르는 물로 만나자

강은교 <우리가 물이 되어> 중에서

▪ What is ideal fluid?

- An ideal fluid is a fluid assumed to be inviscid.
- In such a fluid there are no frictional effects between moving fluid layers or between these layers and boundary walls.
- There is no cause for eddy formation or energy dissipation due to friction.
- Thus, this motion is analogous to the motion of a solid body on a frictionless plane.

[Cf] The real fluid – viscous fluid

▪ Why we first deal with the flow of ideal fluid instead of real fluid?

- Under the assumption of frictionless motion, equations are considerably simplified and more easily assimilated by the beginner in the field.
- These simplified equations allow solution of engineering problems to accuracy entirely adequate for practical use in many cases.
- The frictionless assumption gives good results in real situations where the actual effects of friction are small.

  [Ex] the lift on a wing

▪ Incompressible fluid; $\dfrac{\partial \rho}{\partial (t, x, y, z)} = 0$

~ constant density

~ negligibly small changes of pressure and temperature

~ thermodynamic effects are disregarded

5.1 Euler's Equation

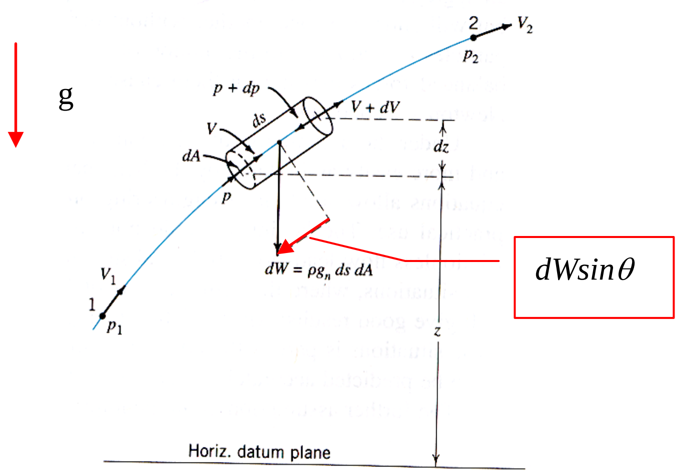


Fig. 5.1

Euler (1750) first applied Newton's 2nd law to the motion of fluid particles.

Consider a streamline and select a small cylindrical fluid system

Pressure force

Gravitational force

$$\sum \vec{F} = m\vec{a}$$

(i) $dF = pdA - (p+dp)dA - dW\sin\theta$

$$= -dp\,dA - \rho g dA ds\,\frac{dz}{ds}$$

$\sin\theta = \dfrac{dz}{ds}$

$$= -dp\,dA - \rho g\,dA dz$$

(ii) $dm = \rho\,dAds$ (density × volume)

(iii) $a = \dfrac{dV}{dt} = \dfrac{dV}{ds}\dfrac{ds}{dt} = V\dfrac{dV}{ds}$

$$\therefore -dpdA - \rho gdAdz = (\rho dsdA)V\frac{dV}{ds}$$

Dividing by ρdA gives the one-dimensional Euler's equation

$$\frac{dp}{\rho} + VdV + gdz = 0$$

Divide by g

$$\frac{dp}{\gamma} + \frac{1}{g}VdV + dz = 0$$

$$\boxed{d(V^2) = 2V\,dV}$$

$$\frac{dp}{\gamma} + d\left(\frac{V^2}{2g}\right) + dz = 0$$

For incompressible fluid flow,

$$\boxed{d\left(\frac{p}{\gamma} + \frac{V^2}{2g} + z\right) = 0}$$

→ 1-D Euler's equation (Eq. of motion)

5.2 Bernoulli's Equation

For incompressible fluid flow, integrating 1-D Euler's equation yields

$$\boxed{\frac{p}{\gamma}+\frac{V^2}{2g}+z=\text{const.}=H} \tag{5.1}$$

where H = total head

$\rightarrow$ Bernoulli equation

Between two points on the streamline, (5.1) gives

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+z_2$$

$\frac{p}{\gamma}$ = pressure head $\qquad \frac{\text{kg}\cdot\text{m/s}^2}{\text{m}^2}\Big/\frac{\text{kg}\cdot\text{m/s}^2}{\text{m}^3}=\text{m}$

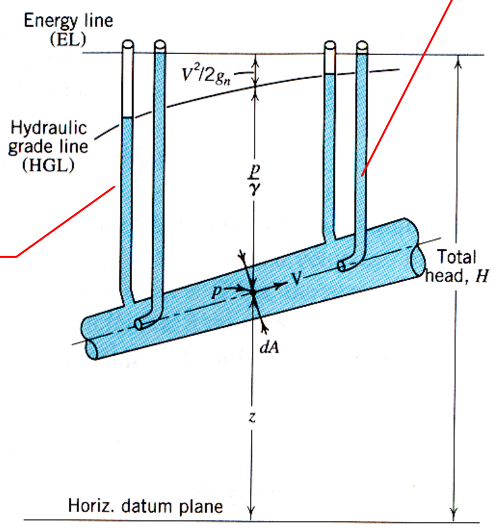
z = potential head (elevation head), m

$\frac{V^2}{2g}$ = velocity head $\qquad \frac{(\text{m/s})^2}{\text{m/s}}=\text{m}$

Henri de Pitot (1695~1771)

Pitot tube

manometer

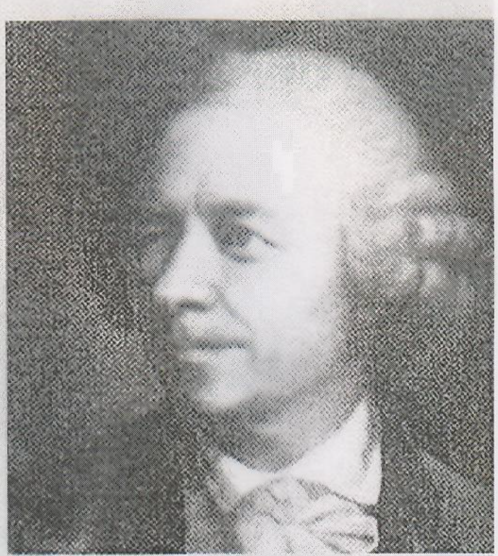


Bernoulli Family:
Jacob
Johann - Nikolaus
Daniel

DANIEL BERNOULLI
1700 Groningen NL – 1782 Basel

Study of mathematics and medicin in Basel
1725 Appointment to the Academy of Science at St. Petersburg by Katherine I (1684–1727)
1733 Return to Basel
1750 Professor for physics

evements
"Hydrodynamica", 1738 written during activities in St. Petersburg. The first rk on fluid mechanics based on thematical principles
Promotion of Leonard Euler
Genius in the fields of mechanics, /sics and mathematics



LEONHARD EULER
1707 Basel – 1783 St. Petersburg

1722 Mathematics and physics studi under Johann Bernoulli
1727 Emigration to St. Petersburg. T book on mechanics and the fo ing of the Modern Theory of N bers
1741 Emigration to Berlin, work on (ferential calculus, but also tech expertises and research on canals ship's rudders, pumps, turbines guide vanes
1766 Return to St. Petersburg. Becar blind

Achievements
- Perfect connection between under- standing and intuition
- Theory of structural strengths of tubes
- Development of the Euler equation the basis of modern hydrodynamic

5.3 The One-Dimensional Assumption for Streamtubes of Finite Cross Section

Bernoulli Eq. is valid for a single streamline or infinitesimal streamtube across which variation of p, V and z is negligible.

This equation can also be applied to large stream tubes such as pipes, canals.

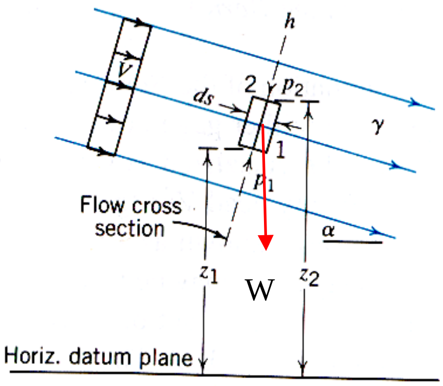


Fig. 5.3

Consider a cross section of large flow through which all streamlines are precisely straight and parallel.

i) Forces, normal to the streamlines, on the element of fluid are in equilibrium

→ acceleration toward the boundary is zero.

$$\sum \vec{F} = 0$$

$$(p_1 - p_2)ds - \gamma h\, ds \cos\alpha = 0$$

$$\boxed{\cos\alpha = (z_2 - z_1)/h}$$

$$\therefore \quad (p_1 - p_2)ds = \gamma(z_2 - z_1)ds$$

$$\frac{p_1}{\gamma}+z_1=\frac{p_2}{\gamma}+z_2 \tag{2.6}$$

→ the same result as that in Ch. 2

→ quantity $\left(z+\frac{p}{\gamma}\right)$ is constant over the flow cross section normal to the streamlines when they are straight and parallel.

→ This is often called a <u>hydrostatic pressure distribution</u> ($z+\frac{p}{\gamma}$ = const. for fluid at rest).

ii) In ideal fluid flows, distribution of velocity over a cross section of a flow containing straight and parallel streamlines is <u>uniform</u> because of the absence of friction.

→ All fluid particles pass a given cross section at the same velocity, V (average velocity)

$$V_1=V_2$$

Combine (i) and (ii)

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+z_2$$

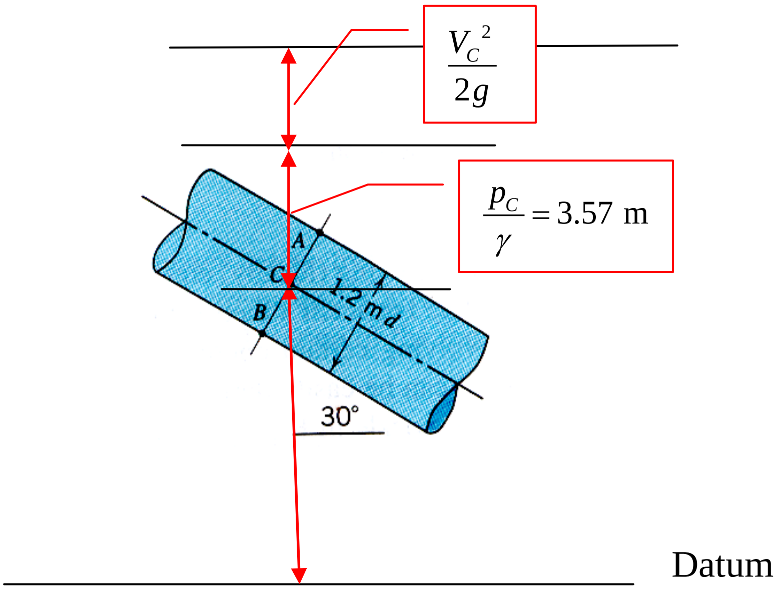
→ Bernoulli equation can be extended from infinitesimal to the finite streamtube.

→ Total head H is the same for every streamline in the streamtube.

→ Bernoulli equation of single streamline may be extended to apply to 2- and 3-dimensional flows.

[IP 5.1] p. 129

Water is flowing through a section of cylindrical pipe. $p_C = 35\,\text{kPa}$, $\gamma = 9.8\times10^3\ \text{N/m}^3$



[Sol]

$$\frac{p_A}{\gamma} + z_A = \frac{p_B}{\gamma} + z_B = \frac{p_C}{\gamma} + z_C$$

$$p_A = p_C + \gamma(z_C - z_A) = 35\times10^3 - (9.8\times10^3)\left(\frac{1.2}{2}\right)\cos 30^\circ = 29.9\ \text{kPa}$$

$$p_B = p_C + \gamma(z_C - z_B) = 35\times10^3 + (9.8\times10^3)\left(\frac{1.2}{2}\right)\cos 30^\circ = 40.1\ \text{kPa}$$

→ The hydraulic grade line is $\dfrac{p_C}{\gamma} = \dfrac{35\times10^3}{9.8\times10^3} = 3.57$ m above point C.

5.4 Applications of Bernoulli's Equation

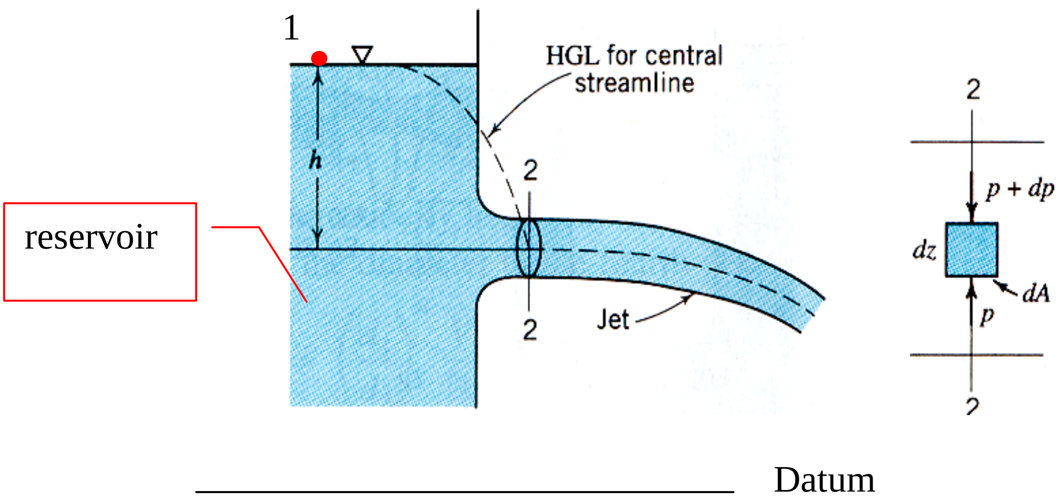
• Bernoulli's equation

$$\frac{p}{\gamma}+\frac{V^2}{2g}+z=H=\text{const.}$$

→ where velocity is high, pressure is low.

• Torricelli's theorem (1643)

~ special case of the Bernoulli equation.



Apply Bernoulli equation to points 1 and 2

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+z_2$$

$V_1 \cong 0$ (for very large reservoir); $p_1 = p_{atm} = 0$

$$z_1 = z_2+\frac{V_2^2}{2g}+\frac{p_2}{\gamma}$$

$$z_1 - z_2 = h = \frac{p_2}{\gamma}+\frac{V_2^2}{2g} \qquad \text{(a)}$$

Apply Newton's 2nd law in the vertical direction at section 2

$$\sum F = ma$$

$$dF = -(p + dp)dA + pdA - \gamma dAdz = -dpdA - \gamma dAdz$$

$$dm = \rho\, dAdz$$

$$a = -g$$

$$\therefore \quad -dAdp - \gamma dAdz = -\left(\rho dAdz\right) g$$

$$-dp - \gamma dz = -\gamma\, dz$$

$$\therefore \quad dp = 0$$

→ no pressure gradient across the jet at section 2.

→ $p_A = p_B = p_C = p_2$

$$\therefore p_A = p_{atm} = 0 \text{ (gage)} \qquad \text{(b)}$$

Thus, combining (a) and (b) gives

$$h = \frac{V_2^2}{2g}$$

$$\rightarrow V_2 = \sqrt{2gh}$$

~ equal to solid body falling from rest through a height h.

[IP 5.2] p.131

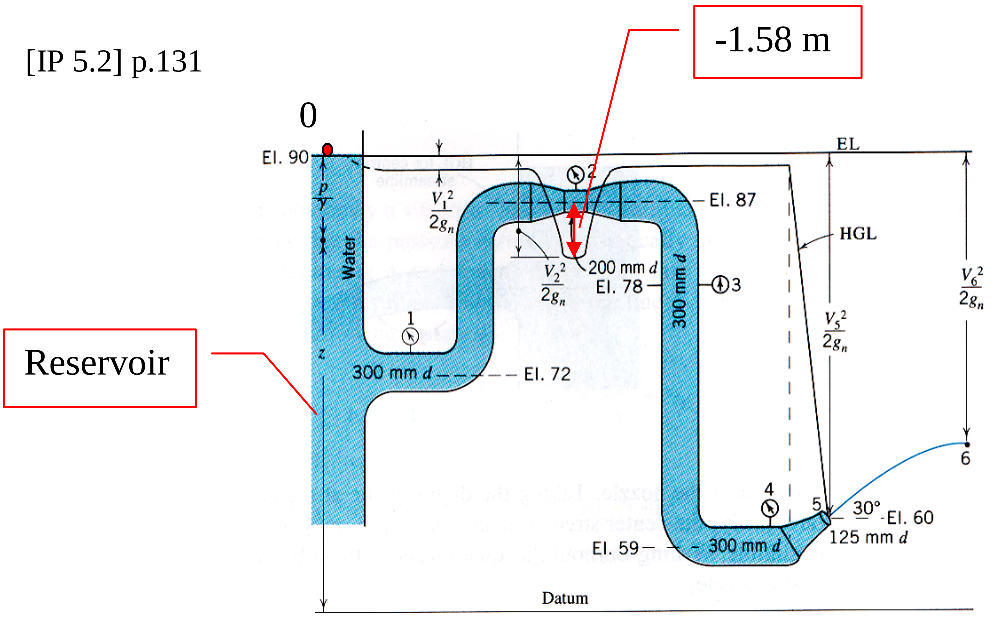


Fig. Problem 5.2

Find: p_1, p_2, p_3, p_4 and eleveation at point 6

Sol]

(i)Bernoulli's Eq. between ⓞ & ⑤

$$\frac{p_0}{\gamma}+\frac{V_0^2}{2g}+z_0=\frac{p_5}{\gamma}+\frac{V_5^2}{2g}+z_5$$

$$p_0=p_5=p_{atm}=0,\ \ V_0=0$$

$$\rightarrow\quad 90=60+\frac{V_5^2}{2g}$$

$$V_5=24.3\text{ m/s}$$

Calculate Q using Eq. (4.4)

$$Q=AV=24.3\times\frac{\pi}{4}(0.125)^2=0.3\text{ m}^3/\text{s}$$

(ii) Apply Continuity equation, Eq. (4.5)

$$A_1V_1 = Q = A_5V_5 \quad \therefore \quad V_1 = \left(\frac{125}{300}\right)^2 V_5 \tag{4.5}$$

$$\therefore \quad \frac{V_1^2}{2g} = \left(\frac{125}{300}\right)^4 \frac{V_5^2}{2g} = \left(\frac{125}{300}\right)^4 (30) = 0.9\,\text{m}$$

Continuity equation

$$V_1 = \sqrt{0.9(2\times 9.8)} = 4.2 \text{ m/s} = V_3 = V_4$$

$$\frac{V_2^2}{2g} = \left(\frac{125}{200}\right)^4 \frac{V_5^2}{2g} = \left(\frac{125}{200}\right)^4 (30) = 4.58\,\text{m},$$

$$V_2 = \sqrt{4.58(2\times 9.8)} = 9.5 \text{ m/s}$$

(iii) B. E. ⓞ & ①

$$90 = \frac{p_1}{\gamma} + \frac{V_1^2}{2g} + 72 \quad \text{of } H_2O \leftarrow \text{head}$$

$$\therefore \quad \frac{p_1}{\gamma_w} = 18 - 0.9 = 17.1 \text{ m} \quad \text{of } H_2O \leftarrow \text{head}$$

$$p_1 = 17.1(9.8\times 10^3) = 167.5 \text{ kPa} \tag{5.1}$$

(iv) B. E. ⓞ & ②

$$90 = \frac{p_2}{\gamma} + 87 + 4.58 \quad \therefore \frac{p_2}{\gamma} = -1.58 \text{ m}$$

$$p_2 = -1.58(9.8\times 10^3) = -15.48 \text{ kPa} = \frac{-15.48\times 10^3}{133.3} = 116 \text{ mmHg} \;\; \text{vacuum}$$

$$\rightarrow 15.48 \text{ kPa below } p_{atm}$$

[Re] 1 bar=1000 mb(millibar)=100 kPa=100 $\mathrm{kN/m^2}$ =10^5 $\mathrm{N/m^2}$

$$p_{atm} \simeq 760 \text{ mmHg} = 101.325 \text{ kPa}(10^5 \text{pascal}) = 1013 \text{ mb} = 29.92 \text{ in. Hg}$$

$$1 \text{ mmHg} = 133.3 \text{ Pa} = 133.3 \text{ N/m}^2$$

(v) B. E. ⓞ & ③

$$90 = \frac{p_3}{\gamma} + 0.9 + 78$$

$$\therefore \ \frac{p_3}{\gamma} = 12 - 0.9 = 11.1\text{m}$$

$$p_3 = 108.8 \text{ kPa} \qquad (5.1)$$

(vi) B. E. ⓞ & ④

$$\frac{p_4}{\gamma} = 31 - 0.9 = 30.1\text{m}$$

$$p_4 = 295.0 \ kPa \qquad (5.1)$$

(vii) Velocity at the top of the trajectory

$\rightarrow \ V_6 = 24.3 \cos 30^\circ = 21.0$ m/s

Apply B. E. ⓞ & ⑥

$$\therefore \quad El. = 90 - \frac{21.0^2}{2g} = 67.5\text{m}$$

	Point 0	Point 1	Point 2	Point 3	Point 4
Pressure, kPa	0	167.5	-15.8	108.7	294.9
Velocity, m/s	0	4.22	4.61	4.22	4.22
Elevation, m	90	72	87	78	59

- Cavitation

As velocity or potential head increase, the pressure within a flowing fluid drops.

~ Pressure does not drop below the absolute zero of pressure.

$$(p_{atm} \approx 10^3 \text{millibar} = 100 \text{ kPa} \quad \therefore \; p_{abs} = 0 \Rightarrow p_{gage} = -100 \text{ kPa})$$

~ Actually, in liquids the absolute pressure can drop only to the vapor pressure of the liquid.

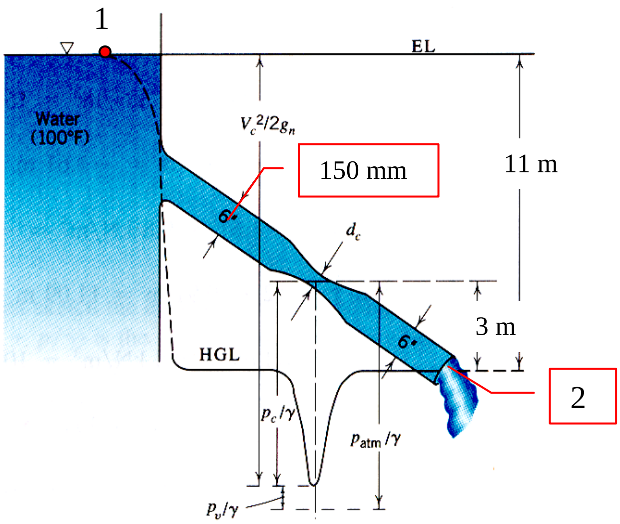
For water, p_v

p_v
1.23kPa @ 10°C
1.70kPa @ 15°C
2.34kPa @ 20°C

[IP 5.3] p.134

p_B = 96.5 kPa = barometric pressure.

What diameter of constriction can be expected to produce incipient cavitation at the throat of the constriction?



Water at 40$^\circ C$

$$\gamma = 9.73 \text{ kN/m}^3; \quad p_v = 7.38 \text{ kPa}$$

$$\frac{p_v}{\gamma} = \frac{7.38\times10^3 \text{ N/m}^2}{9.73\times10^3 \text{ N/m}^3} = 0.76 \text{ m}$$

$$\frac{p_B}{\gamma} = \frac{p_{atm}}{\gamma} = \frac{96.5\times10^3 \text{ N/m}^2}{9.73\times10^3 \text{ N/m}^3} = 9.92 \text{ m}$$

(i) Bernoulli Eq. between ① and ©

$$z_1 + \frac{p_1}{\gamma} + \frac{V_1^2}{2g} = z_c + \frac{p_c}{\gamma} + \frac{V_c^2}{2g}$$

$$V_1 \approx 0, \quad p_1 = p_B, \quad p_c = p_v$$

Incipient cavitation

$$\therefore \quad 11 + 9.92 + 0 = 3 + 0.76 + \frac{V_c^2}{2g}$$

$$\frac{V_c^2}{2g} = 17.16 \text{ m} \quad \rightarrow V_c = 18.35 \text{ m/s}$$

(ii) Bernoulli Eq. between ① and ②

$$z_1 + \frac{p_1}{\gamma} + \frac{V_1^2}{2g} = z_2 + \frac{p_2}{\gamma} + \frac{V_2^2}{2g}$$

$$V_1 \approx 0, \;\; p_1 = p_2 = p_B$$

$$11 + 9.92 + 0 = 0 + 9.92 + \frac{V_2^2}{2g}$$

$$V_2 = 14.69 \text{ m/s}$$

(iii) <u>Continuity</u> between ② and ©

$$Q = A_2 V_2 = A_c V_c$$

$$\frac{\pi}{4}(0.15)^2(14.69) = \frac{\pi}{4} d_c^2 (18.35)$$

$$\therefore d_c = 0.134 \text{ m} = 134 \text{ mm}$$

[Cp] For incipient cavitation,

critical gage pressure at point C is

$$\left.\frac{p_c}{\gamma}\right)_{gage} = -\left(\frac{p_{atm}}{\gamma} - \frac{p_v}{\gamma}\right) = -(9.92 - 0.76) = -9.16 \text{ m}$$

▪Bernoulli Equation in terms of pressure

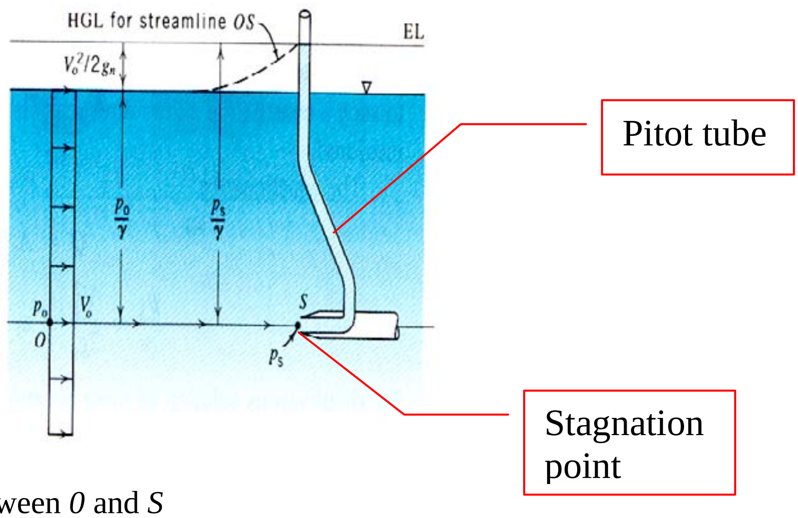
$$\boxed{p_1 + \frac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \gamma z_2}$$

p_1 = static pressure

$\frac{1}{2}\rho V_1^2$ = dynamic pressure

γz = potential pressure

▪ Stagnation pressure, p_S



Apply Bernoulli equation between *0* and *S*

$$p_0 + \frac{1}{2}\rho V_0^2 + \gamma z_0 = p_S + \frac{1}{2}\rho V_S^2 + \gamma z_S$$

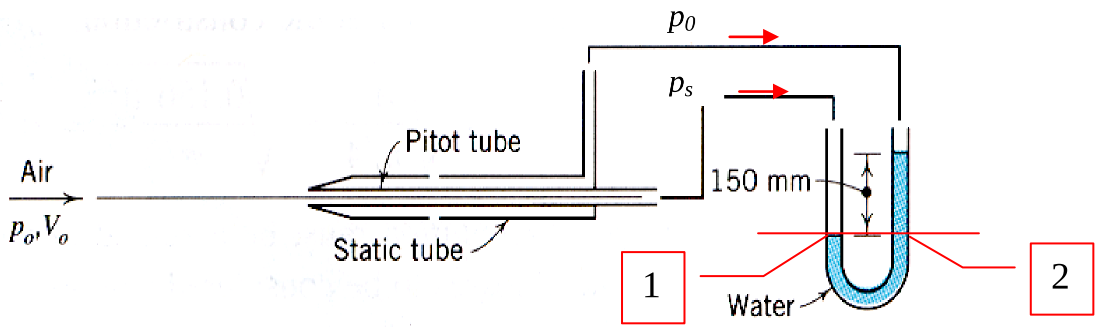
$$z_0 = z_S;\; V_S \approx 0$$

$$p_0 + \frac{1}{2}\rho V_0^2 = p_S + 0$$

$$\boxed{V_0 = \sqrt{\frac{2(p_S - p_0)}{\rho}}}$$

[IP 5.4] p.136

What is the velocity of the airstream, V_0?



$$\rho_{air} = 1.23\,\mathrm{kg/m^3} \quad \gamma_W = 9810\,\mathrm{N/m^3}$$

$$V_0 = \left[\frac{2}{\rho_a}(p_S - p_0)\right]^{\frac{1}{2}}$$

By the way,

$$p_1 = p_2$$

$$p_1 = p_S + 0.15\rho_{air}g; \;\; p_2 = p_0 + 0.15\gamma_w$$

$$\therefore\; p_S - p_0 = 0.15(\gamma_w - \rho_{air}g) = 0.15(9{,}810 - 1.23\times 9.81) = 1{,}469.7 \text{ pa}$$

$$V_0 = \sqrt{\frac{2}{1.23}(1{,}469.7)} = 48.9 \text{ m/s}$$

[Cf] If $\gamma_{air} = \gamma_w = \gamma$

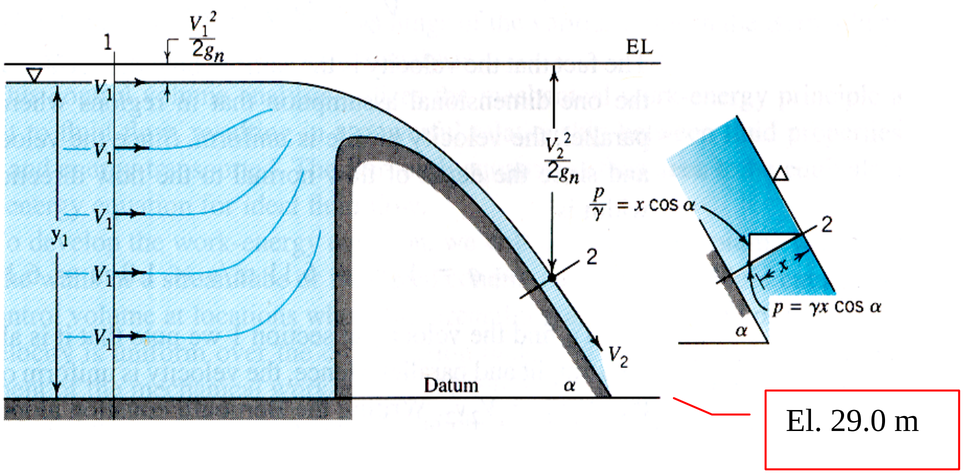
Then, $p_S - p_0 = \gamma h$

$\therefore\; V_0 = \sqrt{2gh}$

- Bernoulli principle for open flow
  - Flow over the spillway weir: a moving fluid surface in contact with the atmosphere and dominated by gravitational action
  - At the upstream of the weir, the streamlines are straight and parallel and velocity distribution is uniform.
  - At the chute way, Section 2, the streamlines are assumed straight and parallel, the pressures and velocities can be computed from the one-dimensional assumption.

[IP 5.6] p.139

At section 2, the water surface is at elevation 30.5 m and the 60° spillway face is at elevation 30.0 m. The velocity at the water surface at section 2 is 6.11 m/s.



[Sol]

Thickness of sheet flow = $(30.5-30)/\cos 60^{\circ} = 1\ \text{m}$

Apply 1-D assumption across the streamline at section ②

$$\cancel{\frac{p_{w.s.}}{\gamma}} + z_{w.s.} = \frac{p_b}{\gamma} + z_b$$

$$\therefore\ p_b = \gamma(z_{w.s.} - z_b) = 9.8\times10^3(0.5) = 4.9\ \text{kPa}$$

Elevation of energy line $H = 30.5 + \dfrac{6.1^2}{2g} = 32.4\ \text{m}$

Apply B.E. between ② and ⓑ

Velocity is the same at both the surface and the bottom

$$\frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 = \frac{p_b}{\gamma} + \frac{V_b^2}{2g} + z_b$$

$$32.4 = \frac{4.9}{9.8} + \frac{V_b^2}{2g} + 30.0 \quad \therefore\ V_b = 6.11\,\text{m/s}$$

$q = h_2V_2 = 1\times6.11 = 6.11\ \text{m}^2/\text{s}$ per meter of spillway length

Apply Bernoulli equation between ① and ②

$$y_1 + 29.0 + \frac{1}{2g}\left(\frac{6.11}{y_1}\right)^2 = 32.4$$

$$y_1 = 3.22\ \text{m}$$

$$V_1 = \frac{q}{h_1} = \frac{6.11}{3.22} = 1.9\ \text{m/s}$$

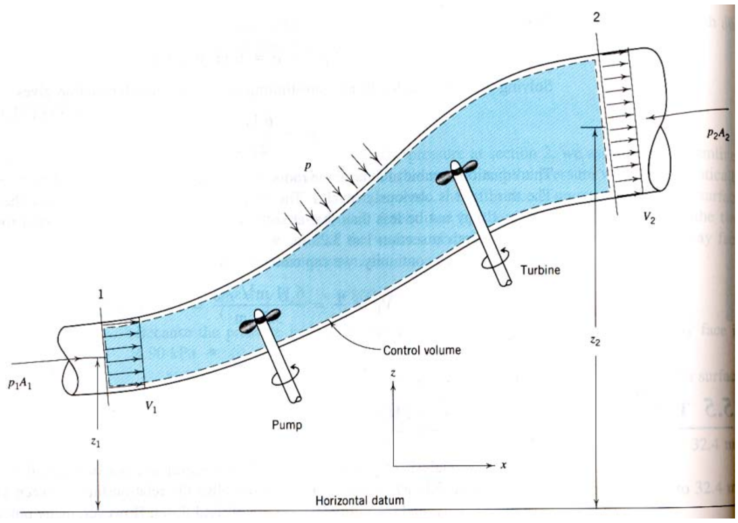
$h_1 = y_1$

5.5 The Work-Energy Equation

For pipelines containing pumps and turbines, the mechanical work-energy equation can be derived via a control volume analysis.

• pump = add energy to the fluid system
  turbine = extract energy from the fluid system

• Bernoulli equation = mechanical work-energy equation for ideal fluid flow



Apply mechanical work-energy principle to fluid flow

→ work done on a fluid system is exactly balanced by the change in the sum of the kinetic energy(KE) and potential energy(PE) of the system.

$$dW = dE \tag{1}$$

where dW = the increment of work done; dE = resulting incremental change in energy

~ Heat transfer and internal energy are neglected.

[Cf] The first law of Thermodynamics

~ Heat transfer and internal energy are included.

Dividing (1) by *dt* yields

$$\frac{dW}{dt} = \frac{dE}{dt} \tag{2}$$

(i)Apply the Reynolds Transport Theorem to evaluate the rate of change of an extensive property, in this case energy

→ steady state form of the Reynolds Transport Theorem

$$\frac{dE}{dt} = \iint_{c.s.out} i\rho\vec{v}\cdot\overrightarrow{dA} + \iint_{c.s.in} i\rho\vec{v}\cdot\overrightarrow{dA} \tag{3}$$

where i = energy per unit mass

$$i = gz + \frac{V^2}{2} \tag{4}$$

Potential energy

Kinetic energy

Substituting (4) into (3) gives

$$\frac{dE}{dt} = \iint_{c.s.out} \left(gz + \frac{V^2}{2}\right)\rho\vec{v}\cdot\overrightarrow{dA} + \iint_{c.s.in} \left(gz + \frac{V^2}{2}\right)\rho\vec{v}\cdot\overrightarrow{dA} \tag{5}$$

where $\dfrac{dE}{dt}$ = the rate of energy increase for the fluid system

→ Even in steady flow, the fluid system energy can change with time because the system moves through the control volume where both velocity and elevation can change.

Since the velocity vector is normal to the cross sectional area and the velocity is uniform over the two cross sections, integration of RHS of (5) yields

$$\begin{aligned}\frac{dE}{dt} &= \rho\left(gz_2 + \frac{V_2^{\,2}}{2}\right)V_2A_2 - \rho\left(gz_1 + \frac{V_1^{\,2}}{2}\right)V_1A_1 \\ &= \rho g\left(z_2 + \frac{V_2^{\,2}}{2g}\right)V_2A_2 - \rho g\left(z_1 + \frac{V_1^{\,2}}{2g}\right)V_1A_1\end{aligned} \tag{6}$$

Continuity equation is

$$Q = V_2A_2 = V_1A_1 \tag{7}$$

Substituting the Continuity equation into (6) gives

$$\frac{dE}{dt} = Q\gamma\left[\left(z_2 + \frac{V_2^{\,2}}{2g}\right) - \left(z_1 + \frac{V_1^{\,2}}{2g}\right)\right] \tag{5.4}$$

(ii)Now, evaluate the work done by the fluid system (dW)

1) Flow work done via fluid entering or leaving the control volume

→ Pressure work $= p \times \text{Area} \times \text{Distance}$

2) Shaft work done by pump and turbine

3) Shear work done by shearing forces action across the boundary of the system

→ $W_{shear} = 0$ for <u>inviscid fluid</u>

• Pressure work

~ consider only pressure forces at the control surface, p_1A_1 and p_2A_2

→ Net <u>pressure work rate</u> = pressure force x distance / time = pressure force x velocity

$= \; p_1 A_1 V_1 - p_2 A_2 V_2$ (8)

• Shaft work

$W_T \geq 0$ (energy is extracted from the system)

$W_p \leq 0$ (energy is put in)

→ Net shaft work rate = $Q\gamma E_P - Q\gamma E_T$ (9)

where $E_P \; (E_T) =$ work done per unit weight of fluid flowing

Combining the two net-work-rate equations, Eqs. (8) and (9), yields

$$\text{Net work rate} = Q\gamma\left(\frac{p_1}{\gamma} - \frac{p_2}{\gamma} + E_P - E_T\right) \tag{5.5}$$

Equating Eqs. (5.4) and (5.5), we get

$$Q\gamma\left[\left(z_2 + \frac{V_2^{\,2}}{2g}\right) - \left(z_1 + \frac{V_1^{\,2}}{2g}\right)\right] = Q\gamma\left(\frac{p_1}{\gamma} - \frac{p_2}{\gamma} + E_P - E_T\right) \tag{5.6}$$

Head, m

Collecting terms with like subscripts gives

$$\boxed{z_1 + \frac{p_1}{\gamma} + \frac{V_1^{\,2}}{2g} + E_P = z_2 + \frac{p_2}{\gamma} + \frac{V_2^{\,2}}{2g} + E_T} \tag{5.7}$$

→ Work-energy equation

~ used in real fluid flow situations

~ Work-energy W/O E_p and E_T is identical to the Bernoulli equation for ideal fluid.

• Addition of mechanical energy (E_P) or extraction (E_T) cause abrupt rises of falls of energy line.

• Power of machines

$$\text{Power} = \frac{W}{t} = \frac{\text{work}}{\text{time}} = \frac{\text{Force} \times \text{distance}}{\text{time}} = \frac{m\,g \times E}{t} = \frac{\rho vol.\,g \times E}{t} = \gamma\left(\frac{vol.}{t}\right) \times E = E\,\gamma\,Q$$

Kilowatts (kW) of machine = $\gamma Q \dfrac{E_P \;\; or \;\; E_T}{1000}$ (5.8a)

Horsepower (hp) of machine = $\gamma Q \dfrac{E_P \;\; or \;\; E_T}{550}$ (5.8b)

$\rightarrow$ $1\ \text{hp} = 0.746\ k\text{W}$

[IP 5.7] p.145

The pump delivers a flowrate of 0.15 m^3/s of water. How much power must the pump supply to the water to maintain gage readings of 250 mm of mercury vacuum on the suction side of the pump and 275 kPa of pressure on the discharge side?

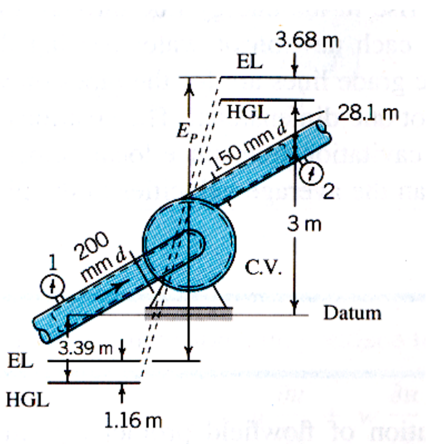


Fig. Problem 5.7

[Sol]

$$p_1 = -250 \text{ mm of Hg} < 760 \text{ mmHg}$$

$$= -250 \times 133.3 \text{ N/m}^2 = -33,325 \text{ N/m}^2$$

$$\frac{p_1}{\gamma} = \frac{-33,325}{9800} = -3.39 \text{ m}$$

$$p_2 = 275 \text{ kPa} > 100 \text{ kPa}$$

$$\frac{p_2}{\gamma} = \frac{275 \times 10^3}{9800} = 28.1 \text{ m}$$

Apply Continuity Equation

$$Q = A_1 V_1 = A_2 V_2$$

$$V_1 = \frac{0.15}{\frac{\pi}{4}(0.2)^2} = 4.8 \text{ m/s}$$

$$\therefore \ \frac{V_1^2}{2g} = \frac{4.8^2}{2 \times 9.8} = 1.16 \text{ m}$$

$$V_2 = \frac{0.15}{\frac{\pi}{4}(0.15)^2} = 8.5 \text{ m/s}$$

$$\therefore \ \frac{V_2^2}{2g} = \frac{8.5^2}{2 \times 9.8} = 3.68 \text{ m}$$

Apply Work-Energy equation between ① & ②

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 + E_p = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + E_T \qquad (5.7)$$

$$-3.39 + 1.16 + 0 + E_p = 28.1 + 3.68 + 3$$

$$\therefore E_p = 37.0 \text{ m}$$

Pump power $= \dfrac{Q\gamma(E_p)}{1000} = \dfrac{0.15(9800)(37.0)}{1000} = 54.4 \ k\text{W}$ (5.8b)

- The local velocity in the pump passage may be considerably larger than the average velocity in the pipes.

  → There is no assurance that the pump will run <u>cavitation-free</u>.

5.6 Euler's Equations for Two-Dimensional Flow

• Two-Dimensional Flow

~ The solution of flowfield problems is much more complex than the solution of 1D flow.

~ Partial differential equations for the motion for real fluid are usually solved by computer-based numerical methods.

~ present an introduction to certain essentials and practical problems

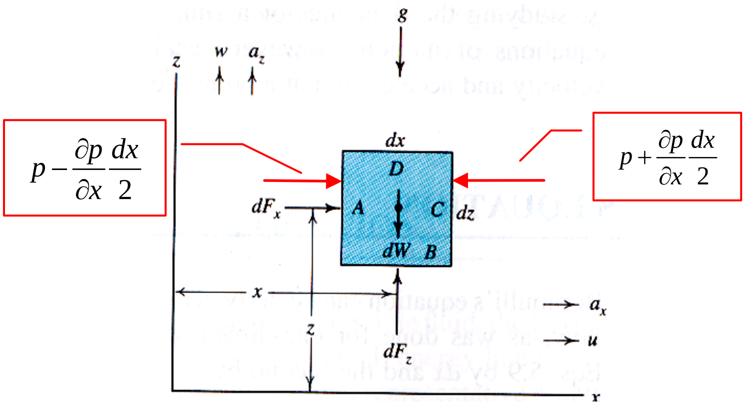


Fig. 5.9

• Euler's equations for a vertical two-dimensional flowfield may be derived by applying Newton's 2nd law of motion to differential system $dxdz$.

$$\sum \vec{F} = m\vec{a}$$

Force:

$$dF_x = -\frac{\partial p}{\partial x} dxdz$$

$$dF_z = -\frac{\partial p}{\partial z} dxdz - \rho g dxdz$$

Acceleration for steady flow:

$$a_x = u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}$$

$+\dfrac{\partial u}{\partial t}$ for unsteady flow

$$a_z = u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}$$

x - direction: $-\dfrac{\partial p}{\partial x}dxdz = \rho dxdz\left(u\dfrac{\partial u}{\partial x} + w\dfrac{\partial u}{\partial z}\right)$

z - direction: $-\dfrac{\partial p}{\partial z}dxdz - \rho g dxdz = \rho dxdz\left(u\dfrac{\partial w}{\partial x} + w\dfrac{\partial w}{\partial z}\right)$

Euler's equation for 2-D flow

$$\boxed{-\frac{1}{\rho}\frac{\partial p}{\partial x} = u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}} \tag{5.9a}$$

$$\boxed{-\frac{1}{\rho}\frac{\partial p}{\partial z} = u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + g} \tag{5.9b}$$

•Equation of Continuity for 2-D flow of ideal fluid

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0} \tag{4.11}$$

Unknowns: $p,\ u,\ w$

Equations: 3

→ simultaneous solution for non-linear PDE

5.7 Bernoulli's Equation for Two-Dimensional Flow

Bernoulli's equation can be derived by integrating the Euler's equations for a uniform density flow.

$$dx \times \left(-\frac{1}{\rho}\frac{\partial p}{\partial x} \right) = \left(u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} \right) \times dx \qquad \text{(a)}$$

$$dz \times \left(-\frac{1}{\rho}\frac{\partial p}{\partial z} \right) = \left(u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + g \right) \times dz \qquad \text{(b)}$$

(a)+(b): $$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial z}dz \right) = u\frac{\partial u}{\partial x}dx + w\frac{\partial u}{\partial z}dx + u\frac{\partial w}{\partial x}dz + w\frac{\partial w}{\partial z}dz + gdz$$

$$= \left(u\frac{\partial u}{\partial x}dx + u\frac{\partial u}{\partial z}dz \right) + \left(w\frac{\partial w}{\partial x}dx + w\frac{\partial w}{\partial z}dz \right)$$

$u\,du$

$$+u\frac{\partial w}{\partial x}dz - u\frac{\partial u}{\partial z}dz + w\frac{\partial u}{\partial z}dx - w\frac{\partial w}{\partial x}dx + gdz$$

$w\,dw$

$$(udz - wdx)\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \right) = (udz - wdx)\xi$$

By the way,

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial z}dz$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial z}dz$$

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial z}dz$$

$$\xi = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}$$

$$\frac{d(u^2)}{2} = \frac{2u\,du}{2} = u\frac{\partial u}{\partial x}dx + u\frac{\partial u}{\partial z}dz$$

Incorporating these terms and dividing by g gives

$$-\frac{dp}{\gamma} = \frac{1}{2g}d(u^2 + w^2) + \frac{1}{g}(udz - wdx)\xi + dz \qquad \text{(c)}$$

Integrating (c) yields

$$\frac{p}{\gamma} + \frac{1}{2g}(u^2 + w^2) + z = H - \frac{1}{g}\int \xi(udz - wdx) \qquad \text{(d)}$$

where $H =$ constant of integration

Substituting resultant velocity, V

$$V^2 = u^2 + w^2$$

$$\boxed{\frac{p}{\gamma} + \frac{V^2}{2g} + z = H - \frac{1}{g}\int \xi(udz - wdx)} \qquad (5.10)$$

(i) For irrotational (potential) flow $\xi = 0$

$$\therefore \frac{p}{\gamma} + \frac{V^2}{2g} + z = H$$

→ Constant H is the same to <u>all streamlines of the 2-D flowfield</u>.

(ii) For rotational flow ($\xi \neq 0$) : $\int \xi (udz - wdx) \neq 0$

However, along a streamline for steady flow,

$$\frac{w}{u} = \frac{dz}{dx} \rightarrow udz - wdx = 0 \qquad \text{(e)}$$

Substituting (e) into (5.10) gives

$$\frac{p}{\gamma} + \frac{V^2}{2g} + z = H$$

→ H is <u>different for each streamline</u>.

[Re]

For ideal incompressible fluid, for larger flow through which all streamlines are straight and parallel (irrotational flow)

→ Bernoulli equation can be applied to any streamline.

5.9 Stream Function and Velocity Potential

The concepts of the stream function and the velocity potential can be used for developing of differential equations for two-dimensional flow.

5.9.1 Stream function

Definition of the stream function is based on the continuity principle and the concept of the streamline.

→ provides a mathematical means of solving for two-dimensional steady flowfields.

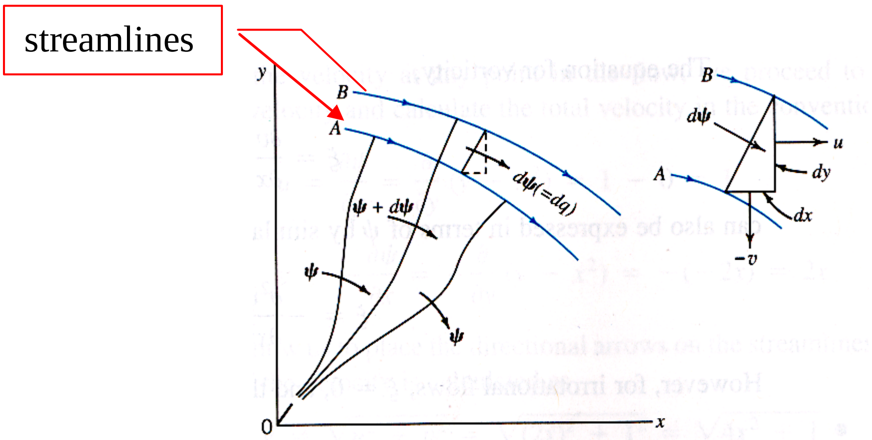


Fig. 5.18 Definition of the stream function.

Consider streamline A: no flow crosses it

→ the flowrate ψ across all lines OA is the same.

→ ψ is a constant of the streamline.

→ If ψ can be found as a function of x and y, the streamline can be plotted.

The flowrate of the adjacent streamline B will be $\psi + d\psi$

The flowrates into and out of the elemental triangle are equal from continuity concept.

$$d\psi = -v dx + u dy \tag{a}$$

Total derivative of $\psi(x, y)$ is

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy \tag{5.14}$$

Compare (a) & (5.14)

$$\boxed{u = \frac{\partial \psi}{\partial y}} \tag{5.15a}$$

$$\boxed{v = -\frac{\partial \psi}{\partial x}} \tag{5.15b}$$

where ψ = stream function

→ If ψ is known *u*, *v* can be calculated.

Integrate (5.14)

$$\psi = \int \frac{\partial \psi}{\partial x} dx + \int \frac{\partial \psi}{\partial y} dy + C$$

$$= \int -v dx + \int u dy + C \tag{b}$$

→ If *u*, *v* are known ψ can be calculated.

- Property of stream function

1) The equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{4.11}$$

Substitute (5.15) into (4.11)

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right)-\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right)=0$$

$$\therefore \quad \frac{\partial^2 \psi}{\partial x \partial y}=\frac{\partial^2 \psi}{\partial y \partial x}$$

→ Flow described by a stream function satisfies the continuity equation.

2) The equation of vorticity

$$\xi=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} \tag{3.10}$$

Substitute (5.15) into (3.10)

$$\xi=\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right)-\frac{\partial}{\partial y}\left(+\frac{\partial \psi}{\partial y}\right)=-\frac{\partial^2 \psi}{\partial x^2}-\frac{\partial^2 \psi}{\partial y^2}$$

For irrotational flow, $\zeta=0$

$$\therefore \quad \frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}=\nabla^2 \psi=0 \quad \rightarrow \text{ Laplace Eq.}$$

→ The stream function of all <u>irrotational flows</u> must satisfy the Laplace equation.

5.9.2 Velocity Potential

Suppose that another function $\phi(x, y)$ is defined as

$$\vec{V} \equiv -\nabla\phi \equiv grad\ \phi = -\left[\frac{\partial\phi}{\partial x}\vec{e_x} + \frac{\partial\phi}{\partial y}\vec{e_y}\right] \qquad \text{(a)}$$

By the way,

$$\vec{V} = u\vec{e_x} + v\vec{e_y} \qquad \text{(b)}$$

Comparing (a) and (b) gives

$$\boxed{\begin{aligned} u &= -\frac{\partial\phi}{\partial x} \\ v &= -\frac{\partial\phi}{\partial y} \end{aligned}} \qquad (5.16)$$

where $\phi =$ velocity potential

- Property of stream function

1) The equation of continuity

Substitute Eq. (5.16) into continuity Eq.

$$\frac{\partial}{\partial x}\left(-\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial\phi}{\partial y}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \qquad \rightarrow \text{ Laplace Eq.} \qquad (5.18)$$

→ All practical flows which conform to the continuity Eq. must satisfy the Laplace equation in terms of ϕ.

2) Vorticity Eq.

Substitute Eq. (5.16) into vorticity eq.

$$\xi = \frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial x}\right) = -\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y \partial x} = 0$$

→ The vorticity must be zero for the existence of a velocity potential.

→ irrotational flow = potential flow

→ Only irrotational flowfields can be characterized by a velocity potential ϕ.

[IP 5.14] p164

A flowfield is described by the equation $\psi = y - x^2$.

1) Sketch streamlines $\psi = 0,\ 1,\ 2$.

2) Derive an expression for the velocity V at any point.

3) Calculate the vorticity.

[Sol]

1) $\psi = 0 \rightarrow 0 = y - x^2$

$\therefore y = x^2 \qquad \rightarrow$ parabola

$\psi = 1 \rightarrow y = x^2 + 1$

$\psi = 2 \rightarrow y = x^2 + 2$

2) $u = \dfrac{\partial \psi}{\partial y} = \dfrac{2}{\partial y}(y - x^2) = 1$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}(y - x^2) = 2x$$

$$\therefore \quad V = \sqrt{u^2 + v^2} = \sqrt{(2x)^2 + 1^2} = \sqrt{4x^2 + 1}$$

3) $\xi = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} = \dfrac{\partial}{\partial x}(2x) - \dfrac{\partial}{\partial y}(1) = 2(s^{-1})$

$\therefore \quad \xi \neq 0 \rightarrow$ The flowfield is rotational.

Homework Assignment # 5

Due: 1 week from today

Prob. 5.6

Prob. 5.11

Prob. 5.24

Prob. 5.30

Prob. 5.46

Prob. 5.48

Prob. 5.59

Prob. 5.89

Prob. 5.98

Prob. 5.104

Prob. 5.119

Prob. 5.123

Prob. 5.149

Prob. 5.157