Chapter 6 The Impulse-Momentum Principle

Objectives:

- Develop impulse - momentum equation, the third of three basic equations of fluid mechanics, added to continuity and work-energy principles

- Develop linear and angular momentum (moment of momentum) equations

6.0 Introduction

- Three basic tools for the solution of fluid flow problems

  - Continuity principle
  - Work-energy principle (Bernoulli equation)
  - Impulse - momentum equation (Momentum equation)

•Impulse - momentum equation

~ derived from Newton's 2nd law in vector form

$$\sum \vec{F} = m\vec{a}$$

Multifly by *dt*

$$\left(\Sigma \vec{F}\right) dt = m\vec{a}dt = d(m\vec{v_c})$$

$$\sum \vec{F} = \frac{d}{dt}(m\vec{v_c})$$

where $\vec{v_c} =$ velocity of the center of mass of the system of mass

$m\vec{v_c} =$ <u>linear momentum</u>

$$m = \int_{sys} dm$$

$$v_c = \frac{1}{m}\int_{sys} v dm$$

$(\sum F)dt =$ impulse in time dt

- Define the fluid system to include <u>all the fluid in a specified control volume</u> whereas the Euler equations was developed for a small fluid system
- Restrict the analysis to steady flow
- Shear stress is not explicitly included
- This equation will apply equally well to <u>real fluids</u> as well as ideal fluids.
- Develop linear and angular momentum (moment of momentum) equations
- Linear momentum equation: calculate magnitude and direction of resultant forces
- Angular momentum equation: calculate <u>line of action</u> of the resultant forces, rotating fluid machinery (pump, turbine)

6.1 The Linear Impulse – Momentum Equation

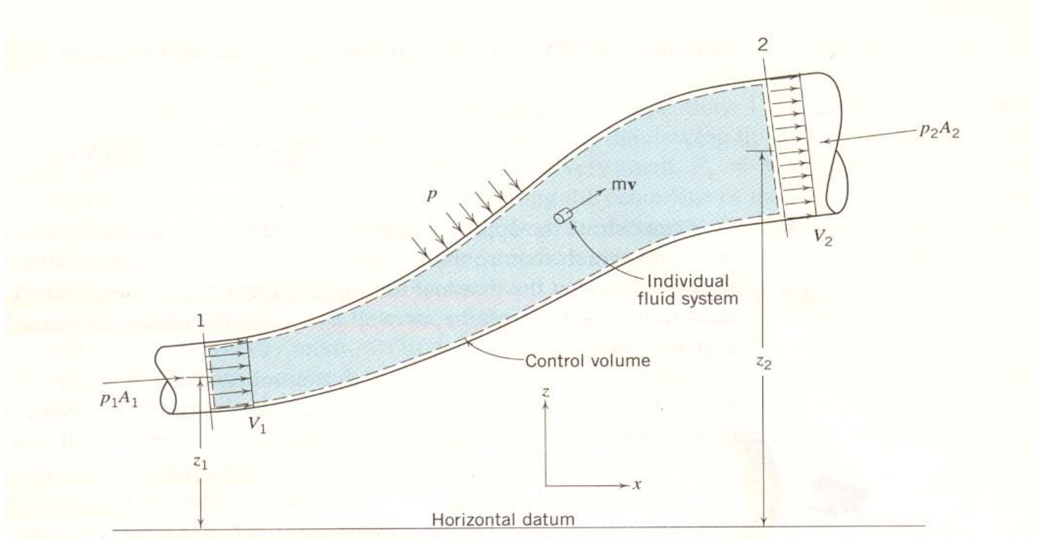


Fig. 6.1

Use the same control volume previously employed for conservation of mass and work-energy.

For the individual fluid system in the control volume,

$$\sum \vec{F} = m\vec{a} = \frac{d}{dt} m\vec{v} = \frac{d}{dt}\rho\vec{v}dV \qquad \text{(a)}$$

Sum them all

$$\sum F_{ext} = \iiint_{sys} \frac{d}{dt}(\rho\vec{v}dV) = \frac{d}{dt}\iiint_{sys}(\rho\vec{v}dV)$$

Use Reynolds Transport Theorem for steady flow to evaluate RHS

$$\frac{dE}{dt} = \frac{\partial}{\partial t}\left(\cancel{\iiint_{c.v.} i\rho\, dvol}\right) + \oint\oint_{c.s.} i\rho\,\vec{v}\cdot\vec{dA}$$

Steady flow

$$\frac{d}{dt}\iiint_{sys}(\rho\vec{v}dV)=\frac{dE}{dt}=\iint_{c.s.} i\rho\vec{v}\cdot d\vec{A}=\iint_{c.s.out}\rho\vec{v}\left(\vec{v}\cdot d\vec{A}\right)-\iint_{c.s.in}\rho\vec{v}\left(\vec{v}\cdot d\vec{A}\right) \quad \text{(b)}$$

$i=\vec{v}$ for momentum/mass

where E = <u>momentum</u> of fluid system in the control volume

$i=\vec{v}=$ momentum per unit mass

Because the streamlines are straight and parallel at Sections 1 and 2, velocity is constant over the cross sections. The cross-sectional area is normal to the velocity vector over the entire cross section. Thus, integration of terms in Eq. (b) are written as

$$\int_{c.s.out}\vec{v}\left(\rho\vec{v}\cdot dA\right)=\int_{c.s.out}\rho\vec{v}\left(\underbrace{\vec{v}\cdot\vec{n}}_{v}dA\right)=\int_{c.s.out}\rho\vec{v}\,\underbrace{vdA}_{Q}=\rho_2\vec{V}_2Q_2$$

Flux out through Section 2

$$\int_{c.s.in}\vec{v}\left(\rho\vec{v}\cdot d\vec{A}\right)=\int_{c.s.in}\rho\vec{v}\left(\underbrace{\vec{v}\cdot\vec{n}}_{-v}dA\right)=-\rho_1\vec{V}_1Q_1$$

Flux in through Section 1

By Continuity eq: $Q_1\rho_1=Q_2\rho_2=Q\rho$

$\therefore$ R. H. S. of (b) $=Q\rho\left(\vec{V}_2-\vec{V}_1\right)$ (c)

Substitute (c) into (a)

$$\boxed{\sum\vec{F}=Q\rho\left(\vec{V}_2-\vec{V}_1\right)} \quad (6.1)$$

In 2-D flow,

$$\sum F_x=Q\rho\left(V_{2x}-V_{1x}\right) \quad (6.2a)$$

$$\sum F_z=Q\rho\left(V_{2z}-V_{1z}\right) \quad (6.2b)$$

General form in case momentum enters and leaves the control volume at more than one location:

$$\boxed{\sum \vec{F} = \left(\sum Q\rho\vec{v}\right)_{out} - \left(\sum Q\rho\vec{v}\right)_{in}} \qquad (6.3)$$

- The external forces include both normal (pressure) and tangential (shear) forces on the fluid in the control volume, as well as the weight of the fluid inside the control volume at a given time.

▪ Advantages of impulse-momentum principle

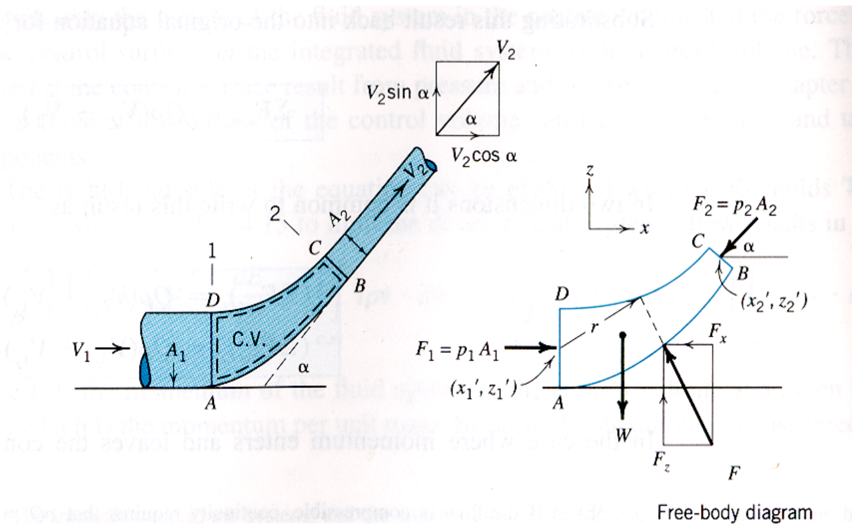
~ Only flow conditions at inlets and exits of the control volume are needed for successful application.

~ Detailed flow processes within the control volume need not be known to apply the principle.

6.2 Pipe Flow Applications

Forces exerted by a flowing fluid on a pipe bend, enlargement, or contraction in a pipeline may be computed by an application of the impulse-momentum principle.

- The reducing pipe bend



Known: flowrate, Q; pressures, p_1, p_2; velocities, V_1, V_2

Find: F (equal & opposite of the force exerted by the fluid on the bend)

= force exerted by the bend on the fluid

• Pressures:

For streamlines essentially straight and parallel at section 1 and 2, the forces F_1, and F_2 result from <u>hydrostatic pressure distributions</u>.

If <u>mean pressure</u> p_1 and p_2 are large, and the pipe areas are small, then $F_1 = p_1 A_1$ and $F_2 = p_2 A_2$, and assumed to act at the <u>centerline of the pipe</u> instead of the <u>center of pressure</u>.

$\boxed{h_c}$ $\boxed{h_p}$

[Cf] Resultant force

(2.12): $F = \underline{\gamma h_c} A$

$\boxed{p_c = \gamma h_c}$

• Body forces:

= total weight of fluid, W

• Force exerted by the bend on the fluid, F

= resultant of the pressure distribution over the entire interior of the bend between Sections 1&2.

~ distribution is unknown in detail

~ resultant can be predicted by Impulse-momentum Eq.

Now apply Impulse-momentum equation, Eq. (6.2)

(i) *x*-direction:

$$\sum F_x = p_1 A_1 - p_2 A_2 \cos\alpha - F_x \qquad \text{(a)}$$

$$Q\rho\left(V_{2x} - V_{1x}\right) = Q\rho\left(V_2 \cos\alpha - V_1\right) \qquad \text{(b)}$$

Combining the two equations to develop an expression for F_x

$$F_x = p_1 A_1 - p_2 A_2 \cos\alpha + Q\rho(V_1 - V_2 \cos\alpha) \qquad \text{(6.4.a)}$$

(ii) *z*-direction

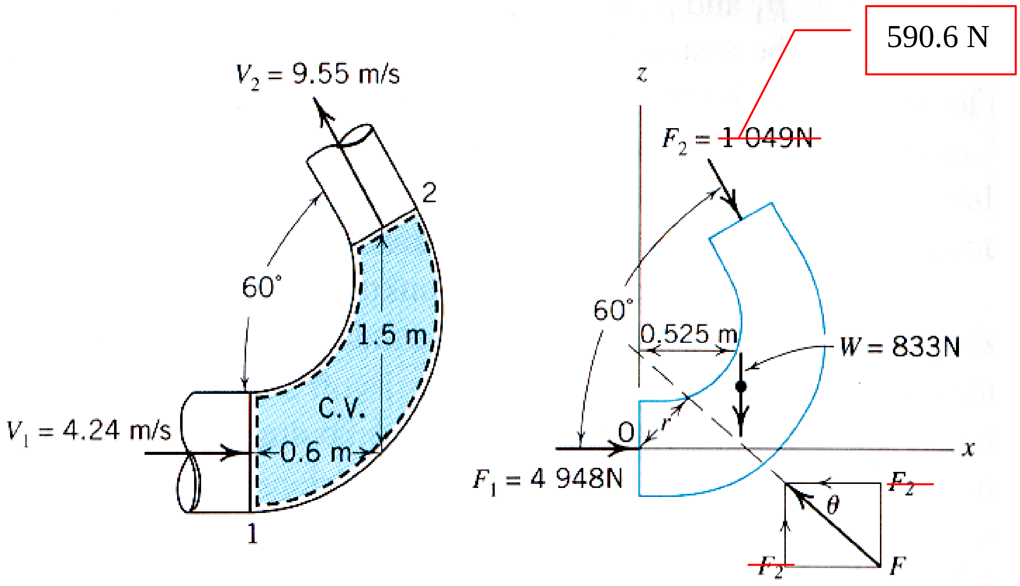
$$\sum F_z = -W - p_2 A_2 \sin\alpha + F_z$$

$$Q\rho\left(V_{2z} - V_{1z}\right) = Q\rho\left(V_2 \sin\alpha - 0\right)$$

$$F_z = W + p_2 A_2 \sin\alpha + Q\rho V_2 \sin\alpha \qquad \text{(6.4.b)}$$

If the bend is relatively sharp, the weight may be negligible, depending on the magnitudes of the pressure and velocities.

[IP 6.1] 300 l/s of water flow through the vertical reducing pipe bend. Calculate the force exerted by the fluid on the bend if the volume of the bend is 0.085 m^3.



Given: $Q = 300 \text{ l/s} = 0.3 \text{ m}^3/\text{s}$; Vol. of bend $= 0.085\ m^3$

$$A_1 = \frac{\pi}{4}(0.3)^2 = 0.071 \text{ m}^2; \quad A_2 = \frac{\pi}{4}(0.2)^2 = 0.031 \text{ m}^2$$

$$p_1 = 70 \text{ kPa} = 70 \times 10^3 \text{ N/m}^2$$

Now, we apply three equation to solve this problem.

1) Continuity Eq.

$$Q = A_1V_1 = A_2V_2 \qquad (4.5)$$

$$V_1 = \frac{0.3}{0.071} = 4.24 \text{ m/s}$$

$$V_2 = \frac{0.3}{0.031} = 9.55 \text{ m/s}$$

2) Bernoulli Eq. between 1 and 2

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2$$

$$\frac{70 \times 10^3}{9{,}800} + \frac{(4.24)^2}{2(9.8)} + 0 = \frac{p_2}{9{,}800} + \frac{(9.55)^2}{2(9.8)} + 1.5$$

$$p_2 = 18.8 \text{ kPa}$$

3) Momentum Eq.

Apply Eqs. 6.4a and 6.4b

$$F_x = p_1 A_1 - p_2 A_2 \cos\alpha + Q\rho(V_1 - V_2 \cos\alpha)$$

$$F_z = W + p_2 A_2 \sin\alpha + Q\rho V_2 \sin\alpha$$

$$F_1 = p_1 A_1 = 4948 \text{ N}$$

$$F_2 = p_2 A_2 = 18.8 \times 10^3 \times 0.031 = 590.6 \text{ N}$$

$$W = \gamma(\text{volume}) = 9800 \times 0.085 = 833 \text{ N}$$

$$F_x = 4,948 - (590.6)\cos 120^\circ + (998 \times 0.3)(4.24 - 9.55\cos 120^\circ) = 7,942 \text{ N}$$

$$F_z = 833 + (590.6)\sin 120^\circ + (998 \times 0.3)(9.55\sin 120^\circ - 0) = 3,820 \text{ N}$$

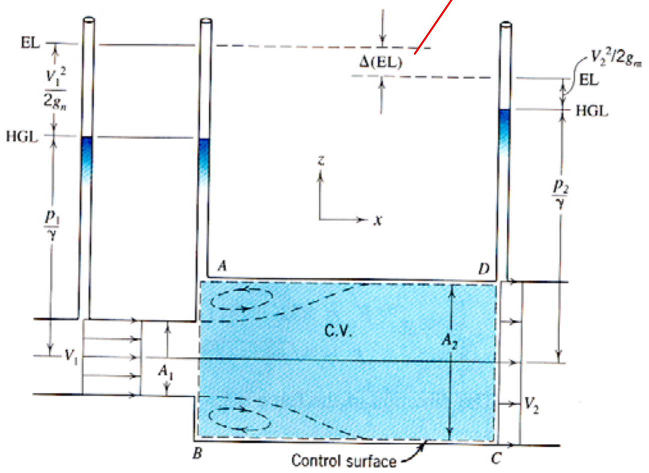
$$F = \sqrt{F_x^2 + F_z^2} = 8,813 \text{ N}$$

$$\theta = \tan^{-1}\frac{F_z}{F_x} = 25.7^\circ$$

• Abrupt enlargement in a closed passage ~ Real fluid flow

The impulse-momentum principle can be employed to predict the fall of the energy line (energy loss due to a rise in the internal energy of the fluid caused by viscous dissipation) at an abrupt axisymmetric enlargement in a passage.

Energy loss



Consider the control surface *ABCD* assuming a one-dimensional flow

i) Continuity

$$Q = A_1 V_1 = A_2 V_2$$

Result from hydrostatic pressure distribution over the area

→ For area *AB* it is an approximation because of the dynamics of eddies in the "dead water" zone.

ii) Momentum

$$\sum F_x = p_1 A_2 - p_2 A_2 = Q\rho (V_2 - V_1)$$

$$(p_1 - p_2)A_2 = \frac{V_2 A_2}{g}\gamma(V_2 - V_1)$$

$$\therefore \ \frac{p_1 - p_2}{\gamma} = \frac{V_2}{g}(V_2 - V_1) \qquad \text{(a)}$$

iii) Bernoulli equation

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + \Delta H$$

$$\frac{p_1 - p_2}{\gamma} = \frac{V_2^2}{2g} - \frac{V_1^2}{2g} + \Delta H \qquad \text{(b)}$$

where $\Delta H =$ Borda-Carnot Head loss

Combine (a) and (b)

$$\frac{V_2(V_2 - V_1)}{g} = \frac{V_2^2}{2g} - \frac{V_1^2}{2g} + \Delta H$$

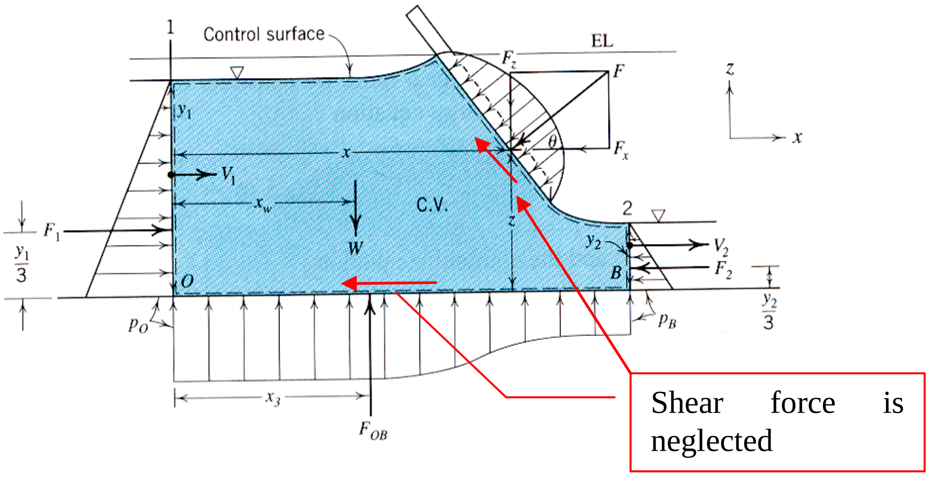
$$\Delta H = \frac{2V_2^2 - 2V_1 V_2}{2g} - \frac{V_2^2}{2g} + \frac{V_1^2}{2g} = \frac{(V_1 - V_2)^2}{2g}$$

6.3 Open Channel Flow Applications

• Applications impulse-momentum principle for Open Channel Flow

- Computation of forces exerted by flowing water on overflow or underflow structures (weirs or gates)
- Hydraulic jump
- Wave propagation

[Case 1] Sluice gate



Consider a control volume that has uniform flow and straight and parallel streamlines at the entrance and exit

Apply first Bernoulli and continuity equations to find values of depths y_1 and y_2 and flowrate per unit width q

Then, apply the impulse-momentum equation to find the force the water exerts on the sluice gate

$$\sum F_x = Q\rho(V_2 - V_1)$$

Discharge per unit width

$$\sum F_x = F_1 - F_2 - F_x = Q\rho(V_{2_x} - V_{1_x}) = q\rho\left(V_2 - V_1\right)$$

where $q = \dfrac{Q}{W} =$ discharge per unit width $= y_1V_1 = y_2V_2$

Assume that the pressure distribution is hydrostatic at sections 1 and 2, replace V with q/y

$$\frac{\gamma y_1^2}{2} - \frac{\gamma y_2^2}{2} - F_x = q^2\rho\left(\frac{1}{y_2} - \frac{1}{y_1}\right) \tag{6.6}$$

[Re] Hydrostatic pressure distribution

$$F_1 = \gamma h_c A = \gamma\frac{y_1}{2}(y_1 \times 1) = \frac{\gamma y_1^2}{2}$$

$$l_p - l_c = \frac{I_c}{l_c A} = \frac{\dfrac{1(y_1)^3}{12}}{\dfrac{y_1}{2}(y_1 \times 1)} = \frac{1}{6}y_1$$

$$C_p = \frac{1}{2}y_1 - \frac{1}{6}y_1 = \frac{1}{3}y_1$$

For ideal fluid (to a good approximation, for a real fluid), the force tnagent to the gate is zero.

→ shear stress is neglected.

→ Hence, the resultant force is normal to the gate.

$$F = F_x / \cos\theta$$

We don't need to apply the impulse-momentum equation in the *z*-direction.

[Re] The impulse-momentum equation in the *z*-direction

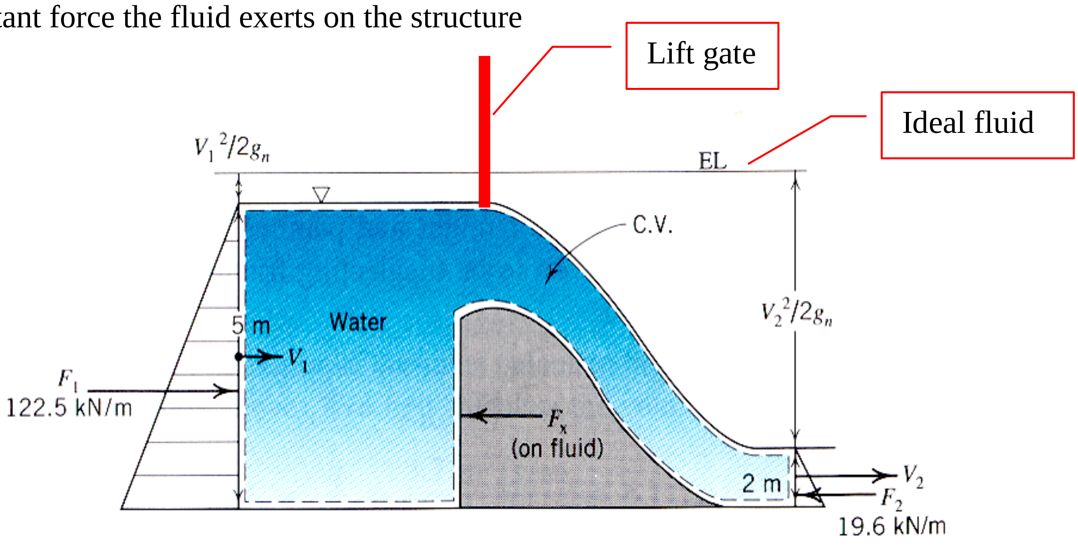
$$\sum F_z = Q\rho(V_{2_z} - V_{1_z})$$

$$\sum F_z = F_{OB} - W - F_z = Q\rho(0 - 0)$$

$$F_z = W - F_{OB}$$

Non-uniform pressure distribution

[IP 6.2] For the two-dimensional overflow structure, calculate the horizontal component of the resultant force the fluid exerts on the structure



• Continuity Eq.

$$q = 5V_1 = 2V_2 \tag{4.7}$$

• Bernoulli's equation between (1) and (2)

$$0 + 5\,\mathrm{m} + \frac{V_1^2}{2g} = 0 + 2\,\mathrm{m} + \frac{V_2^2}{2g} \tag{5.7}$$

Combine two equations

$$V_1 = 3.33\ \mathrm{m/s}$$

$$V_2 = 8.33\ \mathrm{m/s}$$

$$q = 5(3.33) = 16.65\ \mathrm{m^3/s \cdot m}$$

• Hydrostatic pressure principle $\left(\gamma = 9.8 \text{ kN/m}^3\right)$

$$F_1 = \gamma h_c A = \gamma \frac{y}{2} y = 9.8 \frac{(5)^2}{2} = 122.5 \text{ kN/m}$$

$$F_2 = 9.8 \frac{(2)^2}{2} = 19.6 \text{ kN/m}$$

• Impulse-Momentum Eq. ($\rho = 1000 \text{ kg/m}^3$)

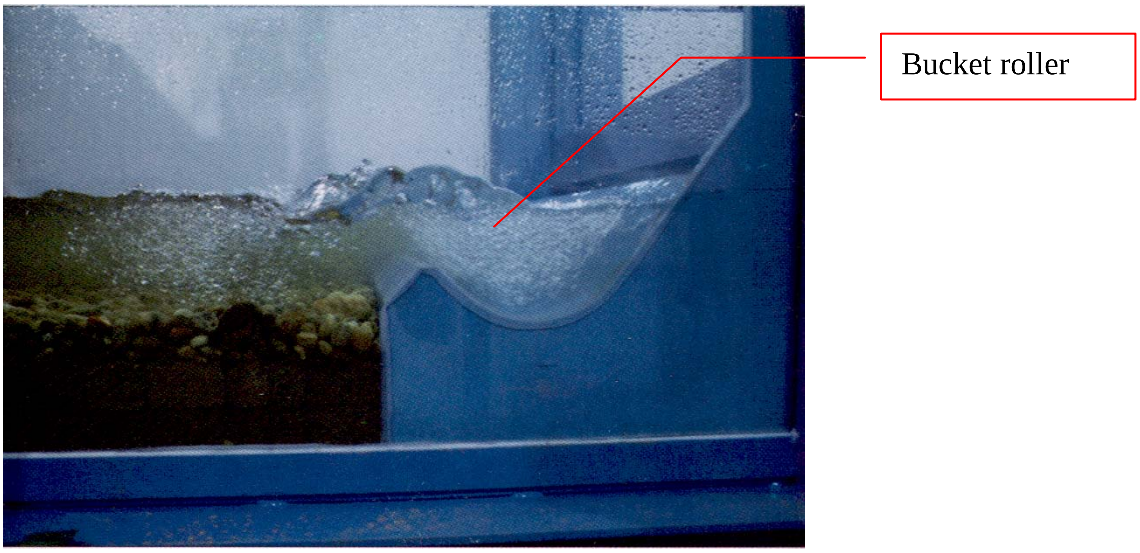
$$\sum F_x = 122{,}500 - F_x - 19{,}600 = (1000 \times 16.65)(8.33 - 3.33)$$

$$F_x = 19.65 \text{ kN/m}$$

[Cf] What is the force if the gate is closed?

Jamshil submerged weir (Seo, 1999)

Jamshil submerged weir with gate opened ($Q = 200\ m^3/s$)



Jamshil submerged weir Model Test; Q = 200 m^3/s (Seo, 1999)

Jamshil submerged weir Model Test (Q = 5,000 m^3/s)

[Case 2] Hydraulic Jump

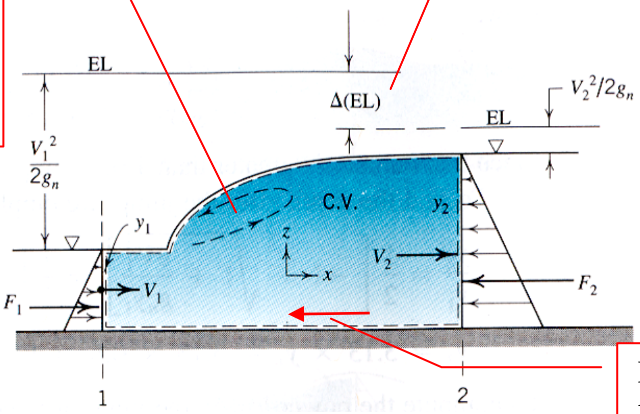
When liquid at high velocity discharges into a zone of lower velocity, a rather abrupt rise (a standing wave) occurs in water surface and is accompanied by violent turbulence, eddying, air entrainment, surface undulation.

→ such as a wave is known as a hydraulic jump

→ large head loss (energy dissipation)

Head loss due to hydraulic jump

turbulence, eddying, air entrainment, surface undulation in the hydraulic jump



Neglect shear force

Apply impulse-momentum equation to find the relation between the depths for a given flowrate

Construct a control volume enclosing the hydraulic jump between two sections 1 and 2 where the streamlines are straight and parallel

$$\sum F_x = F_1 - F_2 = \frac{\gamma y_1^2}{2} - \frac{\gamma y_2^2}{2} = q\rho(V_2 - V_1)$$

where $q =$ flowrate per unit width

Substitute the continuity relations

$$V_1 = \frac{q}{y_1};\; V_2 = \frac{q}{y_2}$$

Rearrange (divide by γ)

$$\frac{q^2}{gy_1} + \frac{y_1^2}{2} = \frac{q^2}{gy_2} + \frac{y_2^2}{2}$$

Solve for y_2/y_1

$$\frac{y_2}{y_1} = \frac{1}{2}\left[-1 + \sqrt{1 + \frac{8q^2}{gy_1^3}}\right] = \frac{1}{2}\left[-1 + \sqrt{1 + \frac{8V_1^2}{gy_1}}\right] \tag{6.7}$$

Set $Fr_1 = \frac{V_1}{\sqrt{gy_1}}$

$$Fr_2 = \frac{V_2}{\sqrt{gy_2}}$$

William Froude (1810~1879)

where $Fr =$ Froude number $= \dfrac{Inertia\ Force}{Gravity\ Force}$

$$= \frac{V}{\sqrt{gy}}$$

Then, we have

$$\frac{y_2}{y_1} = \frac{1}{2}\left[-1+\sqrt{1+8Fr_1^2}\right]$$ Jump Equation

(a) $Fr_1 = 1$: critical flow

$$\rightarrow \frac{y_2}{y_1} = \frac{1}{2}\left[-1+\sqrt{1+8}\right] = 1$$ $y_1 = y_2 \rightarrow$ No jump

(b) $Fr_1 > 1$: super-critical flow

$$\rightarrow \frac{y_2}{y_1} > 1$$ $y_2 > y_1 \quad \rightarrow$ Hydraulic jump

(c) $Fr_1 < 1$: sub-critical flow

$$\rightarrow \frac{y_2}{y_1} < 1$$ $y_2 < y_1 \quad \rightarrow$ physically impossible

($\because$ rise of energy line through the jump is impossible)

Conclusion: For a hydraulic jump to occur, the upstream conditions must be such that

$$V_1^2 / g y_1 > 1.$$

[IP 6.3] p. 199 ; Water flows in a horizontal open channel.

$$y_1 = 0.6 \text{ m}$$

$$q = 3.7 \text{ m}^3/\text{s} \cdot \text{m}$$

Find y_2 , and power dissipated in hydraulic jump.

[Sol]

(i) Continuity

$$q = y_1 V_1 = y_2 V_2$$

$$V_1 = \frac{3.7}{0.6} = 6.17 \text{ m/s}$$

$$Fr_1 = \frac{V_1}{\sqrt{g y_1}} = \frac{6.17}{\sqrt{9.8(0.6)}} = 2.54 > 1 \rightarrow \text{ hydraulic jump occurs}$$

(ii) Jump Eq.

$$y_2 = \frac{y_1}{2}\left[-1 + \sqrt{1 + 8 Fr_1^2}\right]$$

$$= \frac{0.6}{2}\left[-1 + \sqrt{1 + 8(2.54)^2}\right]$$

$$= 1.88 \text{ m}$$

$$V_2 = \frac{3.7}{1.88} = 1.97\ m/s$$

(iii) Bernoulli Eq. (Work-Energy Eq.)

$$y_1 + \frac{V_1^2}{2g} = y_2 + \frac{V_2^2}{2g} + \Delta E$$

$$0.6 + \frac{(6.17)^2}{2(9.8)} = 1.88 + \frac{(1.97)^2}{2(9.8)} + \Delta E$$

$$\therefore \Delta E = 0.46 \text{ m}$$

$$\text{Power} = \gamma Q \Delta E = 9800(3.7)(0.46) = 16.7 \text{ kW/meter of width}$$

→ The hydraulic jump is excellent energy dissipator (used in the spillway).

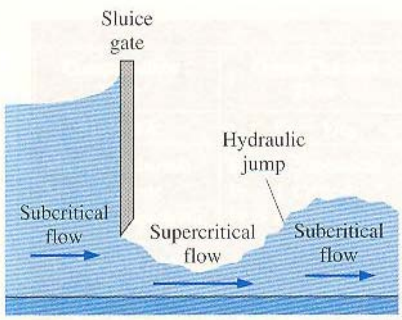
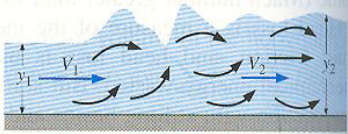
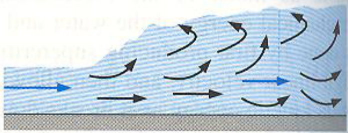
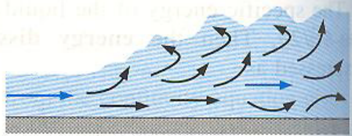
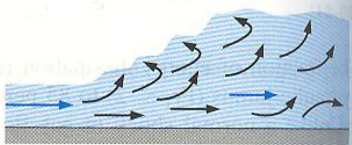
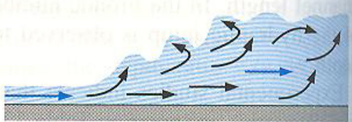


FIGURE 13–10
Supercritical flow through a sluice gate.

TABLE 13–4

Classification of hydraulic jumps

Source: U.S. Bureau of Reclamation (1955).

Upstream Fr_1	Depth Ratio y_2/y_1	Fraction of Energy Dissipation	Description	Surface Profile
<1	1	0	*Impossible jump.* Would violate the second law of thermodynamics.	
1–1.7	1–2	<5%	*Undular jump* (or *standing wave*). Small rise in surface level. Low energy dissipation. Surface rollers develop near Fr = 1.7.	
1.7–2.5	2–3.1	5–15%	*Weak jump.* Surface rising smoothly, with small rollers. Low energy dissipation.	
2.5–4.5	3.1–5.9	15–45%	*Oscillating jump.* Pulsations caused by entering jets at the bottom generate large waves that can travel for miles and damage earth banks. Should be avoided in the design of stilling basins.	
4.5–9	5.9–12	45–70%	*Steady jump.* Stable, well-balanced, and insensitive to downstream conditions. Intense eddy motion and high level of energy dissipation within the jump. Recommended range for design.	
>9	>12	70–85%	*Strong jump.* Rough and intermittent. Very effective energy dissipation, but may be uneconomical compared to other designs.	

Pulsating jump

ΔE/E ~ 85%

[Case 3] Wave Propagation

The velocity (celerity) of small gravity waves in a body of water can be calculated by the impulse-momentum equation.

•small gravity waves

~ appears as a small localized rise in the liquid surface which propagate at a velocity *a*

~ extends over the full depth of the flow

[Cf] small surface disturbance (ripple)

~ liquid movement is restricted to a region near the surface

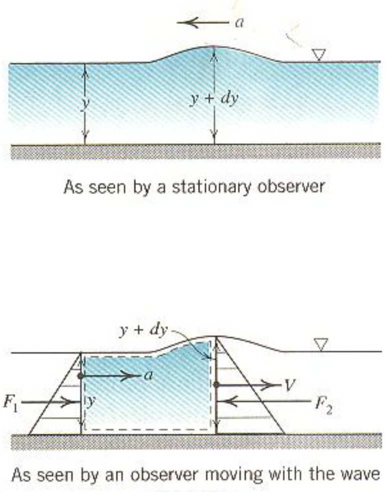


Fig. 6.6

For the steady flow, assign the velocity under the wave as *a'*

From continuity

$$ay = a^{'}\left(y + dy\right)$$

From impulse-momentum

$$\frac{\gamma y^2}{2} - \frac{\gamma\left(y + dy\right)^2}{2} = \left(ay\right)\rho\left(a^{'} - a\right) \tag{6.2a}$$

Combining these two equations gives

$$a^2 = g\left(y + dy\right)$$

Letting *dy* approach zero results in

$$\boxed{a = \sqrt{gy}} \tag{6.8}$$

→ The celerity of the samll gravity wave depends only on the depth of flow.

6.4 The Angular Impulse-Momentum Principle

The angular impulse-momentum equation can be developed using moments of the force and momentum vectors

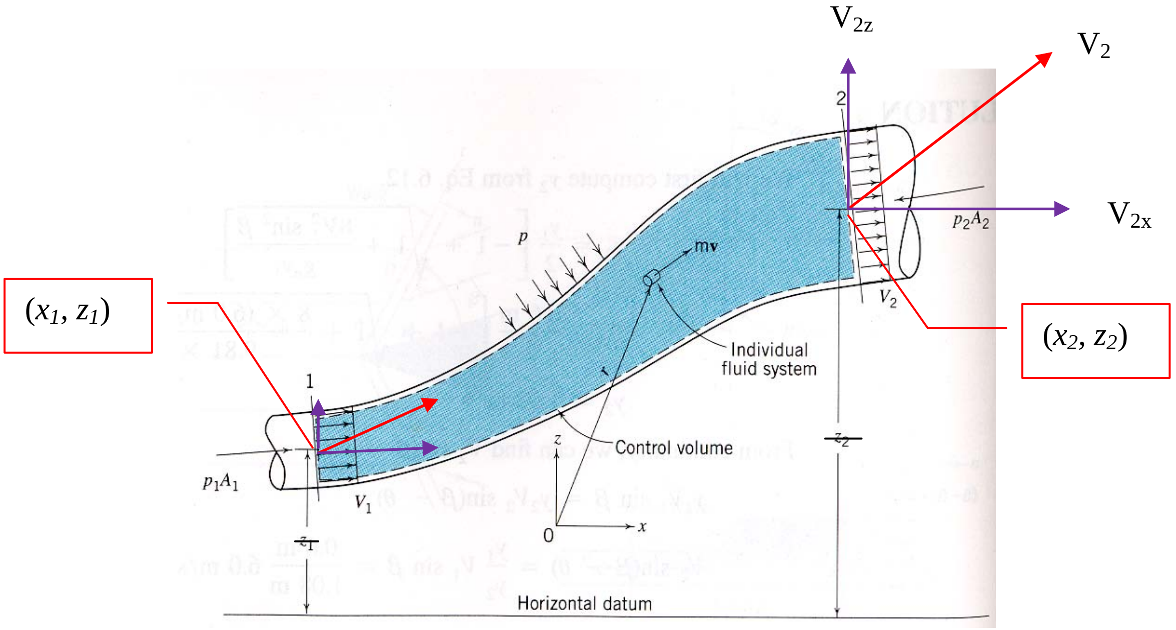


Fig. 6.8

Take a moment of forces and momentum vectors for the small individual fluid system about *0*

$$\sum \vec{r}\times\vec{F} = \frac{d}{dt}(\vec{r}\times m\vec{v}) = \frac{d}{dt}(\vec{r}\times\rho\, dVol\,\vec{v})$$

Sum this for control volume

$$\sum \vec{r}\times\overrightarrow{F_{ext}} = \frac{d}{dt}\iiint_{sys}(\vec{r}\times\vec{v})\rho d\,Vol. \qquad \text{(a)}$$

Use Reynolds Transport Theorem to evaluate the integral

$$i = \vec{r} \times \vec{v}$$

$$\frac{dE}{dt} = \frac{d}{dt}\iiint_{sys} (\vec{r} \times \vec{v})\rho d\,Vol. = \iint_{C.S.} i\rho\vec{v} \cdot \overrightarrow{dA}$$

$$= \iint_{C.S.out} (\vec{r} \times \vec{v})\rho\vec{v} \cdot \overrightarrow{dA} + \iint_{C.S.in} (\vec{r} \times \vec{v})\rho\vec{v} \cdot \overrightarrow{dA} \qquad \text{(b)}$$

where $E =$ moment of momentum of fluid system

$i = \vec{r} \times \vec{v}$ = moment of momentum per unit mass

Restrict to control volume where the fluid enters and leaves at sections where the streamlines are straight and parallel and with the velocity normal to the cross-sectional area

$$\frac{d}{dt}\iiint_{sys} (\vec{r} \times \vec{v})\rho d\,Vol. = \iint_{C.S.out} (\vec{r} \times \vec{v})\rho dQ - \iint_{C.S.in} (\vec{r} \times \vec{v})\rho dQ$$

Because velocity is uniform over the flow cross sections

$$\frac{d}{dt}\iiint_{sys} (\vec{r} \times \vec{v})\rho dVol. = Q\rho(\overrightarrow{r_{out}} \times \overrightarrow{V_{out}}) - Q\rho(\overrightarrow{r_{in}} \times \overrightarrow{V_{in}})$$

$$= Q\rho\left[(\vec{r} \times \vec{V})_{out} - (\vec{r} \times \vec{V})_{in}\right] \qquad \text{(c)}$$

where $\vec{r} =$ position vector from the moment center to the centroid of entering or leaving flow cross section of the control volume

Substitute (c) into (a)

$$\sum(\vec{r}\times\overrightarrow{F_{ext}})=\sum\overrightarrow{M_0}=Q\rho\left[(\vec{r}\times\vec{V})_{out}-(\vec{r}\times\vec{V})_{in}\right] \tag{6.13}$$

In 2-D flow,

$$\boxed{\sum M_0=Q\rho(r_2V_{2t}-r_1V_{1t})} \tag{6.14}$$

where V_t = component of velocity normal to the moment arm *r*.

In rectangular components, assuming *V* is directed with positive components in both *x* and *z*-direction, and with the moment center at the origin of the *x*-*z* coordinate system, for clockwise positive moments,

$$\boxed{\sum M_0=Q\rho\left[(z_2V_{2x}-x_2V_{2z})-(z_1V_{1x}-x_1V_{1z})\right]} \tag{6.15}$$

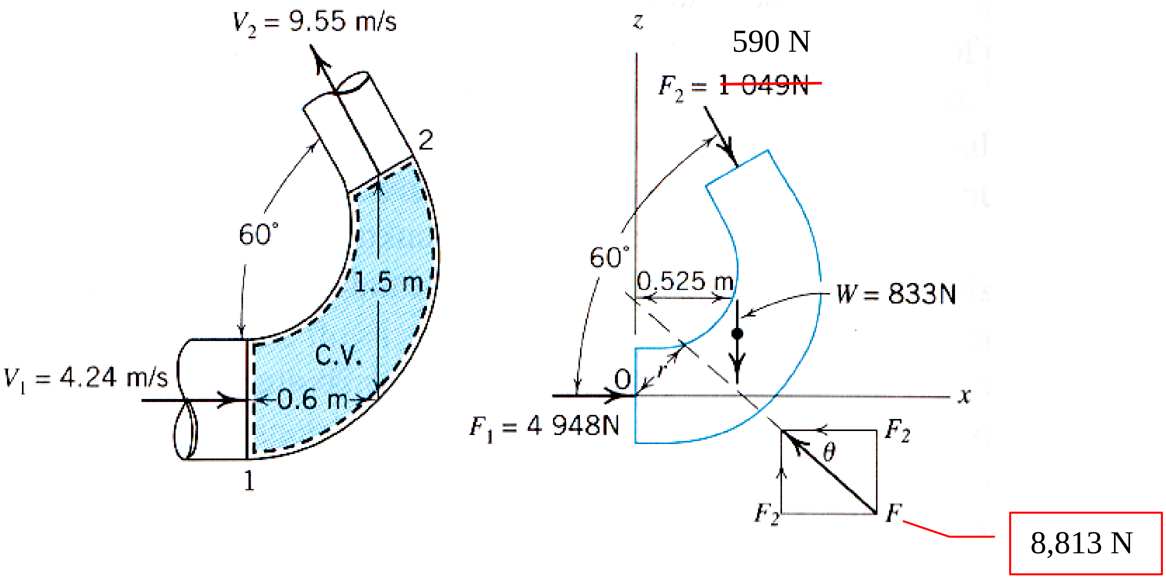
where $x_1,\ z_1=$ coordinates of centroid of the entering cross section

$x_2,\ z_2=$ coordinates of centroid of the leaving cross section

For the fluid that enter or leave the control volume at more than one cross-section,

$$\sum M_0=(\sum Q\rho rV_t)_{out}-(\sum Q\rho rV_t)_{in} \tag{6.16}$$

[IP 6.6] Compute the location of the resultant force exerted by the water on the pipe bend.



Assume that center of gravity of the fluid is 0.525 m to the right of section 1, and the forces F_1 and F_2 act at the centroid of the sections rather than at the center of pressure.

Take moments about the center of section 1

$$\sum M_0 = Q\rho\left[(z_2 v_{2x} - x_2 v_{2x}) - (z_1 v_{1x} - x_1 v_{1x})\right]$$

For this case, $x_1 = 0,\ z_1 = 0,\ x_2 = 0.6,\ z_2 = 1.5$

$$\sum T = -r(8{,}813) + 0.525(833) + 1.5(590\cos 60^\circ) - 0.6(-590\sin 60^\circ)$$

$$= (0.3\times 998)\left[1.5(-9.55\cdot\cos 60^\circ) - 0.6(9.55\cdot\sin 60^\circ)\right]$$

$$\therefore \quad r = 0.59\ \text{m}$$

[Re] Torque for rotating system

$$\vec{T} = \sum(\vec{r} \times \vec{F}) = \frac{d}{dt}(\vec{r} \times m\vec{v}_c)$$

Where $\vec{T} =$ torque

$\vec{T}\, dt =$ torque impulse

$\vec{r} \times m\overrightarrow{v_c} =$ angular momentum (moment of momentum)

$\vec{r} =$ radius vector from the origin *0* to the point of application of a force

[Re] Vector product (cross product)

$$\vec{V} = \vec{F} \times \vec{G}$$

-Magnitude:

$$\left|\vec{V}\right| = \left|\vec{F}\right|\left|\vec{G}\right| \sin\phi$$

-Direction: perpendicular to the plane of $\vec{F}$ and $\vec{G}$ (right-hand rule)

If $\vec{F}, \vec{G}$ are in the plane of x and y, then the $\vec{V}$ is in the z plane.

Homework Assignment # 6

Due: 1 week from today

Prob. 6.1

Prob. 6.6

Prob. 6.14

Prob. 6.16

Prob. 6.30

Prob. 6.34

Prob. 6.36

Prob. 6.40

Prob. 6.55

Prob. 6.60