Chapter 8 Similitude and Dimensional Analysis

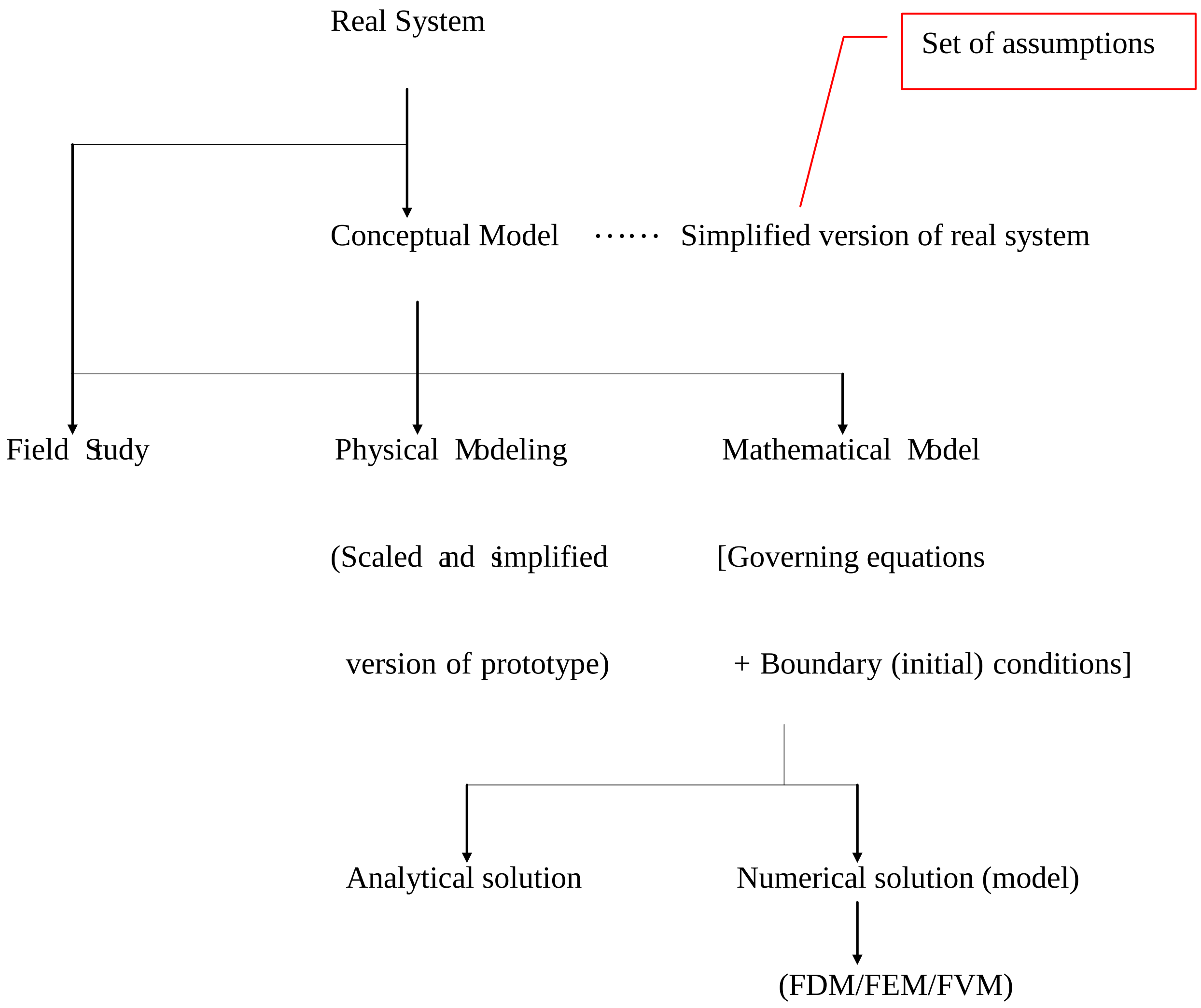
Objectives:

- learn how to begin to interpret fluid flows
- introduce concept of model study for the analysis of the flow phenomena that could not be solved by analytical (theoretical) methods
- introduce laws of similitude which provide a basis for interpretation of model results
- study dimensional analysis to derive equations expressing a physical relationship between quantities

8.0 Introduction

Why we need to model the real system?

Most real fluid flows are complex and can be solved only approximately by analytical methods.



▪ Three dilemmas in planning a set of physical experiments or numerical experiments

1) Number of possible and relevant variables or physical parameters in real system is huge and so the potential number of experiments is beyond our resources.
2) Many real flow situations are either too large or far too small for convenient experiment at their true size.
   → When testing the real thing (prototype) is not feasible, a physical model (scaled version of the prototype) can be constructed and the performance of the prototype simulated in the physical model.
3) The solutions produced by numerical simulations must be verified or numerical models calibrated by use of physical models or measurements in the prototype.

▪ Model study

Physical models have been used for over a hundred years.

Models began to be used to study flow phenomena that could not be solved by analytical (theoretical) methods.

[Ex]

Civil and environmental engineering: models of hydraulic structures, river sections, estuaries and coastal bays and seas

Mechanical engineering: models of pumps and turbine, automobiles

Naval architect: ship models

Aeronautical engineering: model test in wind tunnels

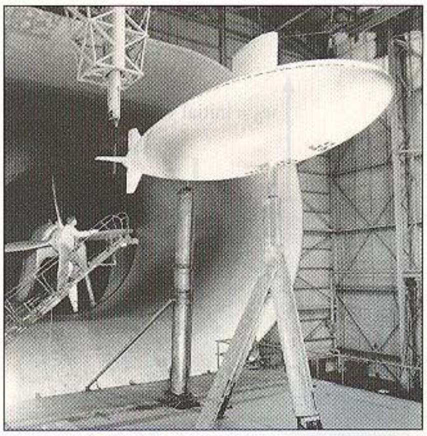


FIGURE 7–21
Similarity can be achieved even when the model fluid is different than the prototype fluid. Here a submarine model is tested in a wind tunnel.
Courtesy NASA Langley Research Center.

▪ Justification for models

1) Economics: A model, being small compared to the prototype, costs little.

2) Practicability: In a model, environmental and flow conditions can be rigorously controlled.

▪ Laws of similitude

- provide a basis for interpretation of physical and numerical model results and crafting both physical and numerical experiments

▪ Dimensional analysis

- derive equations expressing a physical relationship between quantities

8.1 Similitude and Physical Models

Similitude of flow phenomena not only occurs between a prototype and it modelbut also may exist between various natural phenomena.

There are three basic types of similitude; all three must be obtained if complete similarity is to exist between fluid phenomena.

- Geometrical similarity
- Kinematic similarity
- Dynamic similarity

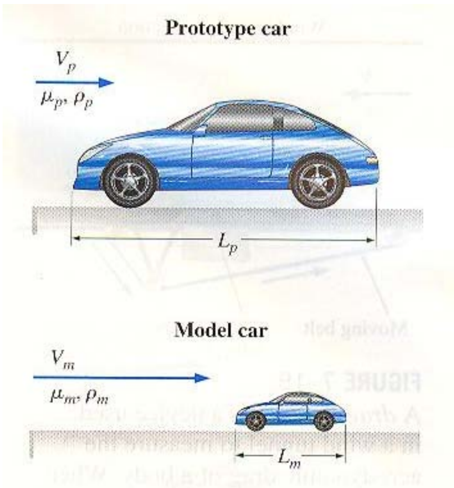
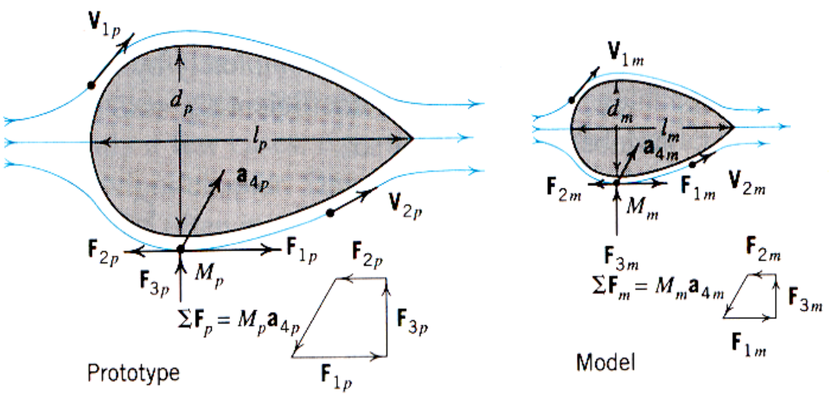


FIGURE 7–17
Geometric similarity between a prototype car of length L_p and a model car of length L_m.

1) Geometrical similarity



~ Flow field and boundary geometry of model and of the prototype have the same shape.

→ The ratios between corresponding lengths in model and prototype are the same.

[Cf] Distorted model

~ not geometrically similar $(l_r > d_r)$

~ The flows are not similar and the models have to be calibrated and adjusted to make them perform properly.

~ used models of rivers, harbor, estuary

~ Numerical models are usually used in their place.

For the characteristic lengths we have

$$d_r = \frac{d_p}{d_m} = \frac{l_p}{l_m} = l_r$$

• Area

$$\frac{A_p}{A_m} = \left(\frac{d_p}{d_m}\right)^2 = \left(\frac{l_p}{l_m}\right)^2$$

• Volume

$$\frac{Vol_p}{Vol_m} = \left(\frac{d_p}{d_m}\right)^3 = \left(\frac{l_p}{l_m}\right)^3$$

2) Kinematic similarity

In addition to the flowfields having the same shape, the ratios of corresponding velocities and accelerations must be the same through the flow.

→ Flows with geometrically similar streamlines are kinematically similar.

$$\frac{\vec{V}_{1p}}{\vec{V}_{1m}} = \frac{\vec{V}_{2p}}{\vec{V}_{2m}}$$

$$\frac{\vec{a}_{3p}}{\vec{a}_{3m}} = \frac{\vec{a}_{4p}}{\vec{a}_{4m}}$$

3) Dynamic similarity

In order to maintain the geometric and kinematic similarity between flowfields, the forces acting on corresponding fluid masses must be related by ratios similar to those for kinematic similarity.

Consider gravity, viscous and pressure forces, and apply Newton's 2nd law

$$\frac{\vec{F}_{1p}}{\vec{F}_{1m}} = \frac{\vec{F}_{2p}}{\vec{F}_{2m}} = \frac{\vec{F}_{3p}}{\vec{F}_{3m}} = \frac{M_p \vec{a}_{4p}}{M_m \vec{a}_{4m}} \tag{8.2}$$

Define inertia force as the product of the mass and the acceleration

$$\vec{F}_I = M\,\vec{a}$$

4) Complete similarity

~ requires simultaneous satisfaction of geometric, kinematic, and dynamic similarity.

→ Kinematically similar flows must be geometrically similar.

→ If the <u>mass distributions</u> in flows are similar, then kinematic similarity (density ratio for the corresponding fluid mass are the same) guarantees complete similarity from Eq. (8.2).

From Fig. 8.1, it is apparent that

$$\vec{F}_{1p} + \vec{F}_{2p} + \vec{F}_{3p} = M_p \, \vec{a}_{4p} \qquad \text{(a)}$$

$$\vec{F}_{1m} + \vec{F}_{2m} + \vec{F}_{3m} = M_m \, \vec{a}_{4m} \qquad \text{(b)}$$

If the ratios between three of the four corresponding terms in Eq.(a) and Eq.(b) are the same, the ratio between the corresponding fourth terms be the same as that the other three. Thus, one of the ratio of Eq.(8.2) is redundant. If the first force ratio is eliminated,

$$\frac{M_p \, \vec{a}_{4p}}{\vec{F}_{2p}} = \frac{M_m \, \vec{a}_{4m}}{\vec{F}_{2m}} \qquad (8.3)$$

$$\frac{M_p \, \vec{a}_{4p}}{\vec{F}_{3p}} = \frac{M_m \, \vec{a}_{4m}}{\vec{F}_{3m}} \qquad (8.4)$$

▪ The scalar magnitude of forces affecting a flow field

Pressure force: $F_p = (\Delta p) A = \Delta p \, l^2$

Inertia force: $F_I = M \, a = \rho l^3 \left(\dfrac{V^2}{l} \right) = \rho \, V^2 \, l^2$

Gravity force: $F_G = M \, g = \rho l^3 g$

Viscosity force: $$F_V = \mu\left(\frac{dv}{dy}\right)A = \mu\left(\frac{V}{l}\right)l^2 = \mu V\, l$$

Elasticity force: $$F_E = EA = E\, l^2$$

Surface tension: $$F_T = \sigma\, l$$

Here l and V are characteristic length and velocity for the system.

[Re] Other forces

Coriolis force of rotating system $\rightarrow$ Rossby number

Buoyant forces in stratified flow $\rightarrow$ Richardson number

Forces in an oscillating flow $\rightarrow$ Strouhal number

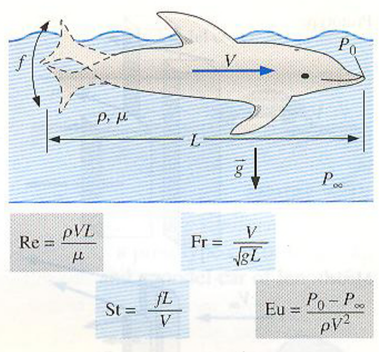


FIGURE 7–15

In a general unsteady fluid flow problem with a free surface, the scaling parameters include a characteristic length L, a characteristic velocity V, a characteristic frequency f, and a reference pressure difference $P_0 - P_\infty$. Nondimensionalization of the differential equations of fluid flow produces four dimensionless parameters: the Reynolds number, Froude number, Strouhal number, and Euler number (see Chap. 10).

▪ Dynamic similarity

To obtain dynamic similarity between two flowfields when all these forces act, all corresponding force ratios must be the same in model and prototype.

(i) $$\left(\frac{F_I}{F_p}\right)_p = \left(\frac{F_I}{F_p}\right)_m = \left(\frac{\rho V^2}{\Delta p}\right)_p = \left(\frac{\rho V^2}{\Delta p}\right)_m \tag{8.5}$$

Define Euler number, $Eu = V\sqrt{\frac{\rho}{2\,\Delta p}}$

$$Eu_p = Eu_m$$

(ii) $$\left(\frac{F_I}{F_V}\right)_p = \left(\frac{F_I}{F_V}\right)_m = \left(\frac{\rho V\, l}{\mu}\right)_p = \left(\frac{\rho V\, l}{\mu}\right)_m \tag{8.6}$$

Define Reynolds number, $Re = \frac{V\, l}{\nu}$

$$\boxed{Re_p = Re_m \quad \rightarrow \text{ Reynolds law}}$$

(iii) $$\left(\frac{F_I}{F_G}\right)_p = \left(\frac{F_I}{F_G}\right)_m = \left(\frac{V^2}{g\, l}\right)_p = \left(\frac{V^2}{g\, l}\right)_m \tag{8.7}$$

Define Froude number, $Fr = \frac{V}{\sqrt{g\, l}}$

$$\boxed{Fr_p = Fr_m \quad \rightarrow \text{ Froude law}}$$

(iv) $$\left(\frac{F_I}{F_E}\right)_p = \left(\frac{F_I}{F_E}\right)_m = \left(\frac{\rho V^2}{E}\right)_p = \left(\frac{\rho V^2}{E}\right)_m \tag{8.8}$$

Define Cauchy number, $Ca = \frac{\rho V^2}{E}$

$$Ca_p = Ca_m$$

[Cf] Define Mach number, $Ma = \sqrt{Ca} = \dfrac{V}{\sqrt{E/\rho}}$

$$Ma_p = Ma_m$$

(v) $\left(\dfrac{F_I}{F_T}\right)_p = \left(\dfrac{F_I}{F_T}\right)_m = \left(\dfrac{\rho l V^2}{\sigma}\right)_p = \left(\dfrac{\rho l V^2}{\sigma}\right)_m$ (8.9)

Define Weber number, $We = \dfrac{\rho l V^2}{\sigma}$

$$We_p = We_m$$

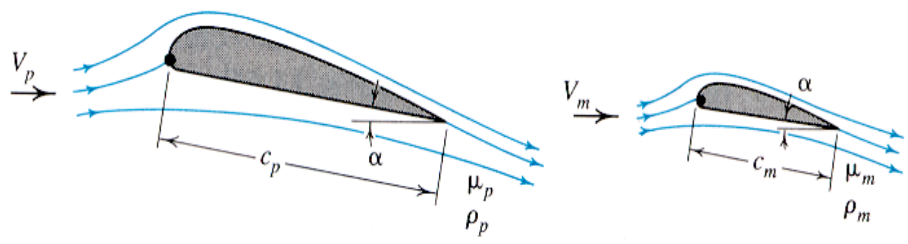
Only four of these equations are independent. → One equation is redundant according to the argument leading to Eq. (8.3) & (8.4). → If four equations are simultaneously satisfied, then dynamic similarity will be ensured and fifth equation will be satisfied.

In most engineering problems (real world), some of the forces above (1) may not act, (2) may be of negligible magnitude, or (3) may oppose other forces in such a way that the effect of both is reduced.

→ In the problem of similitude a good understanding of fluid phenomena is necessary to determine how the problem may be simplified by the elimination of the irrelevant, negligible, or compensating forces.

1. Reynolds similarity

~ Flows in pipe, viscosity-dominant flow



For low-speed submerged body problem, there are no surface tension phenomena, negligible compressibility effects, and gravity does not affect the flowfield.

→ Three of four equations are not relevant to the problem.

→ Dynamic similarity is obtained between model and prototype when the Reynolds numbers (ratio of inertia to viscous forces) are the same.

$$\left(\frac{V\,l}{\nu}\right)_p = Re_p = Re_m = \left(\frac{V\,l}{\nu}\right)_m \tag{8.6}$$

[Re] Reynolds similarity

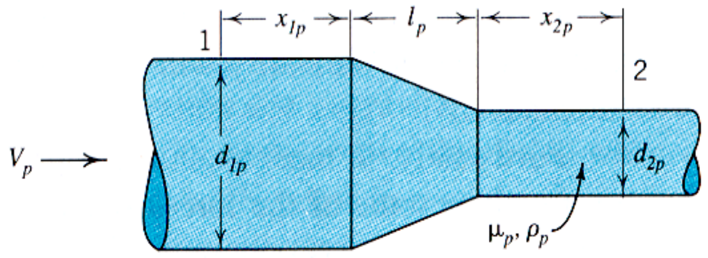
Ratio of any corresponding forces will be the same.

Consider Drag force, $D = C\rho\,V^2\,l^2$

$$\left(\frac{D}{F_I}\right)_p = \left(\frac{D}{F_I}\right)_m$$

$$\left(\frac{D}{\rho V^2\,l^2}\right)_p = \left(\frac{D}{\rho V^2\,l^2}\right)_m$$

For flow of incompressible fluids through closed passages



Geometric similarity:

$$\left(d_2/d_1\right)_p = \left(d_2/d_1\right)_m$$

$$\left(\frac{l}{d_1}\right)_p = \left(\frac{l}{d_1}\right)_m$$

Assume roughness pattern is similar, surface tension and elastic effect are nonexistent.

Gravity does not affect the flow fields

Accordingly dynamic similarity results when Reynolds similarity, Eq. (8.6) is satisfied.

$$Re_p = Re_m$$

Eq. (8.5) is satisfied automatically.

$$\left(\frac{F_I}{F_P}\right)_p = \left(\frac{F_I}{F_P}\right)_m \quad \rightarrow \quad \left(\frac{p_1 - p_2}{\rho V^2}\right)_p = \left(\frac{p_1 - p_2}{\rho V^2}\right)_m \tag{8.5}$$

◈ **Reynolds similarity**

① Velocity:

$$Re_p = Re_m \qquad \left(\frac{Re_p}{Re_m} = 1 ,\ Re_r = 1 \right)$$

$$\left(\frac{Vd}{\nu} \right)_p = \left(\frac{Vd}{\nu} \right)_m \quad \rightarrow \quad \frac{V_m}{V_p} = \frac{\nu_m}{\nu_p} \frac{1}{\dfrac{d_m}{d_p}} = \frac{\nu_m}{\nu_p} \cdot \frac{d_p}{d_m}$$

$$\text{If} \quad \nu_m = \nu_p \quad \rightarrow \quad \boxed{\frac{V_m}{V_p} = \left(\frac{d_m}{d_p} \right)^{-1}}$$

② Discharge: $Q = VA$

$$\frac{Q_m}{Q_p} = \left(\frac{d_m}{d_p} \right)^2 \frac{V_m}{V_p} = \left(\frac{d_m}{d_p} \right)^2 \frac{\nu_m}{\nu_p} \frac{1}{\dfrac{d_m}{d_p}} = \frac{\nu_m}{\nu_p} \frac{d_m}{d_p}$$

③ Time:

$$\frac{T_m}{T_p} = \frac{\dfrac{l_m}{V_m}}{\dfrac{l_p}{V_p}} = \frac{l_m}{l_p} \frac{1}{\dfrac{V_m}{V_p}} = \frac{l_m}{l_p} \frac{1}{\dfrac{\nu_m}{\nu_p}} \frac{d_m}{d_p} = \frac{\nu_p}{\nu_m} \left(\frac{l_m}{l_p} \right)^2$$

④ Force:

$$\frac{F_m}{F_p} = \frac{\left(m_m l_m / T_m^2 \right)}{\left(m_p l_p / T_p^2 \right)} = \frac{\left(\rho_m l_m^3 l_m / T_m^2 \right)}{\left(\rho_p l_p^3 l_p / T_p^2 \right)} = \left(\frac{\mu_m}{\mu_p} \right)^2 \left(\frac{\rho_p}{\rho_m} \right)$$

⑤ Pressure:

$$\frac{P_m}{P_p} = \left(\frac{\mu_m}{\mu_p} \right)^2 \left(\frac{\rho_p}{\rho_m} \right) \left(\frac{l_p}{l_m} \right)^2$$

[IP 8.1] Water flows in a 75 mm horizontal pipeline at a mean velocity of 3 m/s.

Prototype: Water $0^\circ C$ $\quad \mu_p = 1.781 \times 10^{-3}\ \text{Pa} \cdot \text{s}$

$$\rho_p = 99.8\ \text{kg/m}^3$$

$$\nu_p = \frac{1.781 \times 10^{-3}}{998.8} = 1.78 \times 10^{-6}\ \text{m}^2/\text{s}$$

$$d_p = 75\,\text{mm}, \quad V_p = 3\,\text{m/s}, \quad \Delta p = 14\ \text{kPa}, \quad l_p = 10\,\text{m}$$

Model: Gasoline 20°C $\quad \mu_m = 2.9 \times 10^{-4}\,\text{Pa} \cdot \text{s}$ (Table A 2.1)

$$\rho_m = 0.68 \times 998.8 = 679.2\,\text{kg/m}^3$$

$$\nu_m = 4.27 \times 10^{-7}\,\text{m}^2/\text{s}$$

$$d_m = 25\,\text{mm}$$

[Sol] Use Reynolds similarity; $\text{Re}_p = \text{Re}_m$

$$\frac{V_m}{V_p} = \frac{\nu_m}{\nu_p}\left(\frac{d_m}{d_p}\right)^{-1} = \frac{4.27 \times 10^{-7}}{1.78 \times 10^{-6}} \Big/ \left(\frac{25}{75}\right) = 0.753$$

$$\therefore V_m = 0.753(3) = 2.26\,\text{m/s}$$

$$Eu_p = Eu_m$$

$$\left(\frac{\Delta p}{eV^2}\right)_p = \left(\frac{\Delta p}{eV^2}\right)_m$$

$$\frac{14}{[998.8 \times (3)^2]} = \frac{\Delta p_m}{[679.2 \times (2.26)^2]}$$

$$\therefore \Delta p_m = 5.4\ \text{kPa}$$

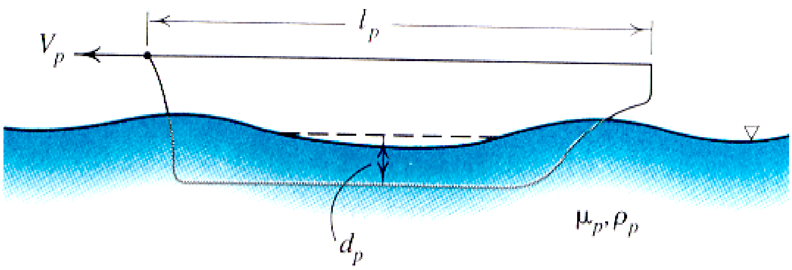
2. Froude similarity

~ Open channel flow, free surface flow, gravity-dominant flow.

For flow field about an object moving on the surface of a liquid such as ship model (William Froude, 1870)

~ Compressibility and surface tension may be ignored.

~ Frictional effects are assumed to be ignored.



$$Fr_p = \left(\frac{V}{\sqrt{g\,l}}\right)_p = Fr_m = \left(\frac{V}{\sqrt{g\,l}}\right)_m \tag{8.7}$$

$$\frac{V_m}{V_p} = \sqrt{\frac{g_m}{g_p}\frac{l_m}{l_p}}$$

[Re] Combined action of gravity and viscosity

For ship hulls, the contribution of wave pattern and frictional action to the drag are the same order.

→ Frictional effects cannot be ignored.

→ This problem requires both Froude similarity and Reynolds similarity.

$$Fr_p = Fr_m = \left(\frac{v}{\sqrt{g\,l}}\right)_p = \left(\frac{v}{\sqrt{g\,l}}\right)_m \quad \rightarrow \quad \frac{V_m}{V_p} = \sqrt{\frac{g_m}{g_p}\frac{l_m}{l_p}} \qquad \text{(a)}$$

$$Re_p = Re_m = \left(\frac{V\,l}{\nu}\right)_p = \left(\frac{V\,l}{\nu}\right)_m \quad \rightarrow \quad \frac{V_m}{V_p} = \frac{\nu_m}{\nu_p}\frac{l_p}{l_m} \qquad \text{(b)}$$

Combine (a) and (b)

$$\sqrt{\frac{g_m}{g_p}\frac{l_m}{l_p}} = \frac{\nu_m}{\nu_p}\frac{l_p}{l_m} \quad \rightarrow \quad \frac{\nu_m}{\nu_p} = \left(\frac{g_m}{g_p}\right)^{0.5}\left(\frac{l_m}{l_p}\right)^{1.5}$$

This requires

(a) A liquid of appropriate viscosity must be found for the model test.

(b) If same liquid is used, then model is as large as prototype.

For $g_m = g_p$

$$\frac{\nu_m}{\nu_p} = \left(\frac{l_m}{l_p}\right)^{1.5} \quad \rightarrow \quad \nu_m = \nu_p \Big/ \left(\frac{l_m}{l_p}\right)^{1.5}$$

If $\dfrac{l_m}{l_p} = \dfrac{1}{10} \quad \rightarrow \quad \nu_m = \dfrac{\nu}{31.6}$

Water: $\mu = 1.0\times 10^{-3}\ \mathrm{Pa\cdot s}$

Hydrogen: $\mu = 0.21 \times 10^{-4}\ \mathrm{Pa \cdot s}$

~ Choose only one equation → Reynolds or Froude

~ Correction (correcting for scale effect) is necessary.

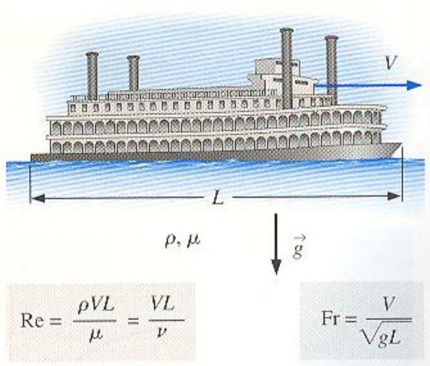


FIGURE 7–42
In many flows involving a liquid with a free surface, both the Reynolds number and Froude number are relevant nondimensional parameters. Since it is not always possible to match both Re and Fr between model and prototype, we are sometimes forced to settle for incomplete similarity.

◈ **Froude law**

① Velocity

$$\boxed{\frac{V_m}{V_p} = \sqrt{\frac{g_m}{g_p}\frac{l_m}{l_p}}}$$

② Time $T = \dfrac{l}{v}$

$$\frac{T_m}{T_p} = \frac{l_m}{l_p}\frac{V_p}{V_m} = \frac{l_m}{l_p}\sqrt{\frac{g_p}{g_m}\frac{l_p}{l_m}} = \sqrt{\frac{g_p}{g_m}\frac{l_m}{l_p}}$$

③ Discharge $Q = vA$

$$\frac{Q_m}{Q_p} = \frac{V_m}{V_p}\left(\frac{l_m}{l_p}\right)^2 = \sqrt{\frac{g_m}{g_p}\frac{l_m}{l_p}}\left(\frac{l_m}{l_p}\right)^2 = \left(\frac{g_m}{g_p}\right)^{0.5}\left(\frac{l_m}{l_p}\right)^{2.5}$$

④ Force

$$\frac{F_m}{F_p} = \left(\frac{\rho_m}{\rho_p}\right)\left(\frac{l_m}{l_p}\right)^3$$

⑤ Pressure

$$\frac{P_m}{P_p} = \left(\frac{\rho_m}{\rho_p}\right)\left(\frac{l_m}{l_p}\right)$$

[IP 8.2] Ship model (Free surface flow)

$$l_p = 120 \text{ m}, \quad l_m = 3 \text{ m} \quad , \quad V_p = 56 \text{ km/h} \quad D_m = 9 \text{ N}$$

Find model velocity and prototype drag.

[Sol]

Use Froude similarity

$$\left(\frac{V}{\sqrt{g\,l}}\right)_p = \left(\frac{V}{\sqrt{g\,l}}\right)_m \tag{8.7}$$

$$l_r = \frac{l_m}{l_p} = \frac{3}{120} = \frac{1}{40}$$

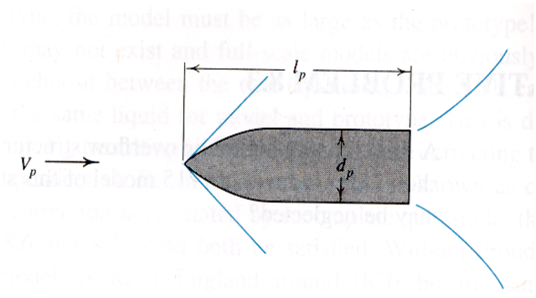
$$V_m = V_p \sqrt{\frac{(g\,l)_m}{(g\,l)_p}} = \frac{56\times10^3}{3600}\left(\frac{3}{120}\right)^{1/2} = 2.46 \text{ m/s}$$

• Drag force ratio

$$\left(\frac{D}{\rho\,V^2\,l^2}\right)_p = \left(\frac{D}{\rho\,V^2\,l^2}\right)_m \tag{8.15}$$

$$D_p = D_m \frac{\left(\rho\,V^2\,l^2\right)_p}{\left(\rho\,V^2\,l^2\right)_m} = 9\times\left(\frac{56\times10^3/3600}{2.46}\right)^2 \times\left(\frac{120}{3}\right)^2 = 575.8 \text{ kN}$$

3. Mach similarity



Similitude in compressible fluid flow

~ gas, air

~ Gravity and surface tension are ignored.

~ Combined action of resistance and elasticity (compressibility)

$$Re_p = Re_m \rightarrow \frac{V_p}{V_m} = \frac{\nu_p}{\nu_m}\frac{l_m}{l_p} \qquad \text{(a)}$$

$$Ma_p = Ma_m = \left(\frac{V}{a}\right)_p = \left(\frac{V}{a}\right)_m \qquad (8.8)$$

$$where\ a = \text{sonic velocity} = \sqrt{\frac{E}{\rho}}$$

$$\frac{V_p}{V_m} = \frac{a_p}{a_m} \qquad \text{(b)}$$

Combine (a) and (b)

$$\frac{l_p}{l_m} = \left(\frac{\nu_p}{\nu_m}\right)\left(\frac{a_m}{a_p}\right)$$

→ gases of appropriate viscosity are available for the model test.

• Velocity

$$\frac{V_m}{V_p} = \frac{a_m}{a_p} = \sqrt{\frac{E_m}{E_p}\frac{\rho_p}{\rho_m}}$$

• Time

$$\frac{T_m}{T_p} = \frac{l_m}{l_p}\frac{V_p}{V_m} = \sqrt{\frac{E_p}{E_m}\frac{\rho_m}{\rho_p}}\ \frac{l_m}{l_p}$$

• Discharge

$$\frac{Q_m}{Q_p} = \left(\frac{l_m}{l_p}\right)^2 \frac{V_p}{V_m} = \sqrt{\frac{E_p}{E_m}\frac{\rho_m}{\rho_p}}\ \left(\frac{l_m}{l_p}\right)^2$$

4. Euler Similarity

~ Modeling of prototype cavitation

~ For cavitation problem, vapor pressure must be included.

[Ex.1] Cavitating hydrofoil model in a water tunnel

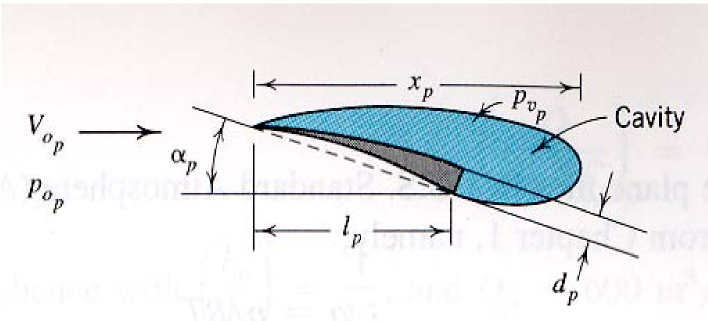


Fig. 8.6 (a)

Here gravity, compressibility, and surface tension are neglected.

Dynamic similitude needs Reynolds similarity and Euler similarity.

$$Re_p = Re_m = \left(\frac{V\,l}{\nu}\right)_p = \left(\frac{V\,l}{\nu}\right)_m$$

$$\sigma_p = \sigma_m = \left(\frac{p_0 - p_v}{\rho V_0^2}\right)_p = \left(\frac{p_0 - p_v}{\rho V_0^2}\right)_m \tag{8.5}$$

$\sigma \quad = \dfrac{p_0 - p_v}{\rho V^2}$ = cavitation number

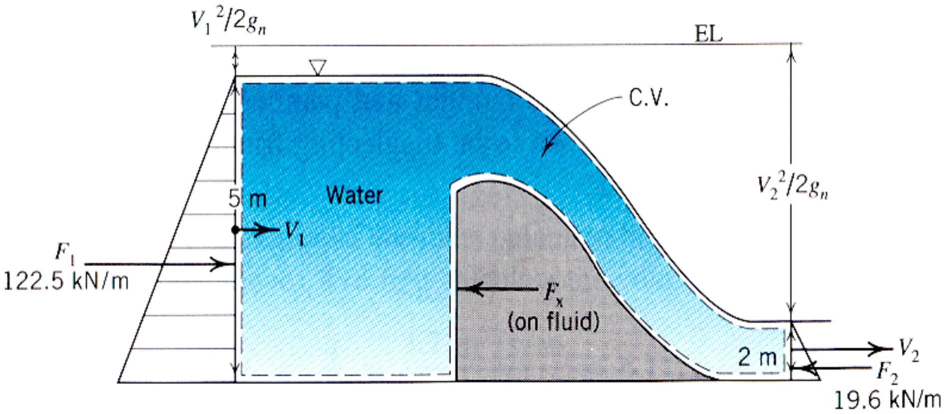
p_0 = absolute pressure

p_v = vapor pressure

~ Virtually impossible to satisfy both equation.

~ Cavitation number must be the same in model and prototype.

[I.P.8.3] Model of hydraulic overflow structure → spillway model



$$Q_p = 600 \text{ m}^3/\text{s}$$

$$l_r = \frac{l_m}{l_p} = \frac{1}{15}$$

[Sol]

Since gravity is dominant, use Froude similarity.

$$\frac{Q_m}{Q_p} = \left(\frac{g_m}{g_p}\right)^{0.5} \left(\frac{l_m}{l_p}\right)^{2.5}$$

$$Q_m = Q_p \left(\frac{l_m}{l_p}\right)^{2.5} = 600 \left(\frac{1}{15}\right)^{2.5}$$

$$= 0.69 \text{ m}^3/\text{s} = 690 \text{ l/s}$$

8.2 Dimensional Analysis

Dimensional analysis

~ mathematics of the dimensions of quantities

~ is closely related to laws of similitude

~ based on Fourier′s principle of dimensional homogeneity (1882)

→ An equation expressing a physical relationship between quantities must be dimensionally homogeneous.

→ The dimensions of each side of equation must be the same.

~ cannot produce analytical solutions to physical problems.

~ powerful tool in formulating problems which defy analytical solution and <u>must be solved experimentally</u>.

~ It points the way toward a maximum of information from a <u>minimum of experiment</u> by the formation of dimensionless groups, some of which are identical with the force ratios developed with the laws of similitude.

▪ Four basic dimension

~ directly relevant to fluid mechanics

~ independent fundamental dimensions

length, L

mass, M or force, F

time, t

thermodynamic temperature T

Newton′s 2nd law

$$F = M\,a = \frac{M\,L}{t^2}$$

~ There are only independent fundamental dimensions.

(1) Rayleigh method

Suppose that power, P, derived from hydraulic turbine is dependent on Q , γ , E_T

Suppose that the relation between these four variables is unknown but it is known that these are the only variables involved in the problem.

$$P = f\left(\,Q\,,\gamma\,,E_T\,\right) \qquad \text{(a)}$$

Q = flow rate

γ = specific weight of the fluid

E_T = unit mechanical energy by unit weight of fluid (Fluid system → turbine)

Principle of dimensional homogeneity

→ Quantities involved cannot be added or subtracted since their dimensions are different.

Eq. (a) should be a combination of <u>products of power</u> of the quantities.

$$P = C\,Q^a\,\gamma^b\,E_T^c \qquad \text{(b)}$$

where C = dimensionless constant ~ cannot be obtained by dimensional methods

a, b, c = unknown exponents

Eq. (b) can be written dimensionally as

$$\left(\text{Dimensions of } P\right)=\left(\text{Dimensions of } Q\right)^a\left(\text{Dimensions of } \gamma\right)^b\left(\text{Dimensions of } E_T\right)^c$$

$$\frac{ML^2}{t^3}=\left(\frac{L^3}{t}\right)^a\left(\frac{M}{L^2t^2}\right)^b(L)^c \qquad \text{(c)}$$

Using the principle of dimensional homogeneity, the exponent of each of the fundamental dimensions is the same on each side of the equation.

$$M:\ 1=b$$

$$L:\ 2=3a-2b+c$$

$$t:\ -3=-a-2b$$

Solving for a, b, and c yields

$$a=1,\ b=1,\ c=1$$

Resubstituting these values Eq. (b) gives

$$P=C\ Q\ \gamma\ E_T \qquad \text{(d)}$$

$C=$ dimensionless constant that can be obtained from

① a physical analysis of the problem

② an experimental measurement of P, Q, γ, E_T

Rayleigh method ~ early development of a dimensional analysis

(2) Buckingham theorem

~ generalized method to find useful dimensionless groups of variables to describe process (E. Buckingham, 1915)

▪ Buckingham′s Π - theorem

1. n variables are functions of each other

→ Then k equations of their exponents $(a, b, c, \cdots)$ can be written.

k = largest number of variables among n variables which cannot be combined into a dimensionless group

[Ex]

Drag force $D \sim f\left(l, V, \rho, \mu, g\right)$ on ship

2. In most cases, k is equal to the number m of independent dimensions (M, L, t)

$$k \le m$$

3. Application of dimensional analysis allows expression of the functional relationship in terms of $(n-k)$ distinct dimensionless groups.

[Ex] $n = 6,\ k = m = 3 \ \rightarrow\ n - k = 3$ groups

$$\pi_1 = \frac{D}{\rho l^2 V^2}$$

$$\pi_2 = R_e = \frac{\rho V l}{\mu}$$

$$\pi_3 = F_r = \frac{V}{\sqrt{gl}}$$

[Ex] Drag on a ship

$$f\left(D, l, \rho, \mu, V, g\right) = 0$$

Three basic variables = repeating variables

$$\begin{array}{ccc} V, & l, & \rho \\ \vdots & \vdots & \vdots \\ t, & L, & M \end{array}$$

Other variables D, μ, g appear only in the unique group describing the ratio of inertia force to force related to the variable.

• Procedure:

1. Find the largest number of variables which do not form a dimensionless Π - group.

For drag problem, No. of independent dimensions is $m = 3$ and $\underline{V, \rho \text{ and } l}$ cannot be formed into a Π - group, so $k = m = 3$

2. Determine the number of Π - groups to be formed: $n = 6,\ k = m = 3$

$\therefore$ No. of Π - group = $n - k = 3$

3. Combine sequentially the variables that cannot be formed into a dimensionless group, with each of the remaining variables to form the requisite Π - groups.

$$\Pi_1 = f_1\left(D, \rho, V, l\right)$$

$$\Pi_2 = f_2\left(\mu, \rho, V, l\right)$$

$$\Pi_3 = f_3\left(g, \rho, V, l\right)$$

4. Determine the detailed form of the dimensionless groups using principle of dimensional homogeneity.

i) Π_1

$$\Pi_1 = D^a \rho^b V^c l^d \qquad \text{(a)}$$

Since Π_1 is dimensionless, writing Eq. (a) dimensionally

$$M^0 L^0 t^0 = \left(\frac{ML}{t^2}\right)^a \left(\frac{M}{L^3}\right)^b \left(\frac{L}{t}\right)^c (L)^d \qquad \text{(b)}$$

The following equations in the exponents of the dimensions are obtained

$$M:\ 0 = a + b$$

$$L:\ 0 = a - 3b + c + d$$

$$t:\ 0 = -2a - c$$

Solving these equations in terms of *a* gives

$$b = -a, c = -2a,\ d = -2a$$

$$\Pi_1 = D^a \rho^{-a} V^{-2a} l^{-2a} = \left(\frac{D}{\rho l^2 V^2}\right)^a$$

The exponent may be taken as any convenient number other than zero.

If a = 1, then

$$\Pi_1 = \frac{D}{\rho l^2 V^2} \qquad \text{(c)}$$

ii) Π_2

$$\Pi_2 = \mu^a \rho^b V^c l^d$$

$$M^0 L^0 t^0 = \left(\frac{M}{Lt}\right)^a \left(\frac{M}{L^3}\right)^b \left(\frac{L}{t}\right)^c (L)^d$$

$$M \ : \ 0 = a + b$$

$$L \ : \ 0 = -a - 3b + c + d$$

$$t \ : \ 0 = -a - c$$

Solving these equations in terms of *a* gives

$$b = -a, \ c = -a, \ d = -a$$

$$\Pi_2 = \mu^a \rho^{-a} V^{-a} l^{-a} = \left(\frac{\mu}{\rho l V}\right)^a$$

If *a* = -1, then

$$\Pi_2 = \frac{V l \rho}{\mu} = \text{Re} \qquad \text{(d)}$$

iii) Π_3

$$\Pi_3 = g^a l^b \rho^c V^d$$

$$M^0 L^0 t^0 = \left(\frac{L}{t^2}\right)^a L^b \left(\frac{M}{L^3}\right)^c \left(\frac{L}{t}\right)^d$$

$$M \;:\; 0 = c$$

$$L \;:\; 0 = a + b - 3c + d$$

$$t \;:\; 0 = -2a - d$$

Solving these equations in terms of *a* gives

$$b = a\,,\;\; c = 0\,,\;\; d = -2a$$

$$\Pi_3 = g^a l^a V^{-2a} = \left(\frac{g\,l}{V^2}\right)^a \qquad \text{(e)}$$

If *a* = -1/2, then

$$\Pi_3 = \frac{V}{\sqrt{g\,l}} = \mathrm{Fr}$$

Combining these three equations gives

$$f'\left(\frac{D}{\rho\, l^2 V^2}\,,\; \mathrm{Re},\; \mathrm{Fr}\right) = 0$$

$$\frac{D}{\rho\, l^2 V^2} = f''\left(\,\mathrm{Re},\; \mathrm{Fr}\right)$$

Dimensional analysis

~ no clue to the functional relationship among $D/\rho l^2 V^2$, Re and Fr

~ arrange the numerous original variables into a relation between a smaller number of dimensionless groups of variables.

~ indicate how test results should be processed for concise presentation

[Problem 8.48] Head loss in a pipe flow

$$f\left(h_L, D, l, \rho, \mu, V, g\right) = 0$$

Pipe diameter

Repeating variables: l, ρ, V

$$\Pi_1 = f_1\left(h_L, l, \rho, V\right)$$

$$\Pi_2 = f_1\left(D, l, \rho, V\right)$$

$$\Pi_3 = f_3\left(\mu, l, \rho, V\right)$$

$$\Pi_4 = f_4\left(g, l, \rho, V\right)$$

(i) $\Pi_1 = h_L^a l^b \rho^c V^d$

$$M^0 L^0 t^0 = L^a L^b \left(\frac{M}{L^3}\right)^c \left(\frac{L}{t}\right)^d$$

$$M : 0 = c$$

$$L : 0 = a + b - 3c + d$$

$$t : 0 = -d \qquad\qquad b = -a$$

$$\therefore \quad \Pi_1 = \left(\frac{h_L}{l}\right)^a$$

$$If \quad a = 1 : \Pi_1 = \frac{h_L}{l}$$

(ii) $\Pi_2 = D^a l^b \rho^c V^d$

$$M^0 L^0 t^0 = L^a L^b \left(\frac{M}{L^3}\right)^c \left(\frac{L}{t}\right)^d$$

$M \ : 0 = c$ ①

$L \ : 0 = a + b - 3c + d$ ②

$t \ : 0 = -d$ ③

② : $0 = a + b \qquad b = -a$

$$\therefore \quad \Pi_2 = \left(\frac{D}{l}\right)^a$$

$$If \ \ a = 1 \qquad : \ \Pi_2 = \frac{D}{l}$$

(iii) $\Pi_3 = \mu^a l^b \rho^c V^d$

$$M^0 L^0 t^0 = \left(\frac{M}{LT}\right)^a L^b \left(\frac{M}{L^3}\right)^c \left(\frac{L}{t}\right)^d$$

$M \ : 0 = a + c$ ① $\rightarrow \ c = d \ \rightarrow \ c = -a$

$L \ : 0 = -a + b - 3c + d$ ②

$t \ : 0 = -a - d \rightarrow d = -a$ ③

② $d + b - 3d + d = 0 \ b = d \ \rightarrow \ b = -a$

$$\therefore \quad \Pi_3 = \mu^a l^{-a} \rho^{-a} V^{-a}$$

$$If \quad a = -1 \qquad \therefore \quad \Pi_3 = \frac{l\rho V}{\mu} = \mathrm{Re}$$

(iv) $\Pi_4 = g^a l^b \rho^c V^d$

$$M^0 L^0 t^0 = \left(\frac{L}{t^2}\right)^a L^b \left(\frac{M}{L^3}\right)^c \left(\frac{L}{t}\right)^d$$

$M \,:\, 0 = c$ ①

$L \,:\, 0 = a + b - 3c + d$ ②

$t \,:\, 0 = -2a - d$ ③

③ $d = -2a$

② $0 = a + b - 0 - 2a \rightarrow \quad b = a$

$$\Pi_4 = g^a l^a V^{-2a} = \left(\frac{g^{l}}{V^2}\right)^a$$

$$If \quad a = -\frac{1}{2} \quad : \quad \Pi_4 = \frac{V}{\sqrt{g\,l}} = \mathrm{Fr}$$

$$f\left(\frac{h_L}{l}, \frac{l}{D}, \mathrm{Re}, \mathrm{Fr}\right) = 0$$

$$\frac{h_L}{l} = f'\left(\frac{l}{D}, \mathrm{Re}, \mathrm{Fr}\right)$$

Homework Assignment # 8

Due: 1 week from today

Prob. 8.6

Prob. 8.10

Prob. 8.14

Prob. 8.20

Prob. 8.24

Prob. 8.30

Prob. 8.56

Prob. 8.59