Deformation of Concrete

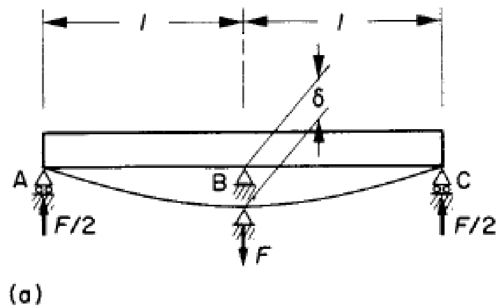
Fall 2010

Dept of Architecture
Seoul National University

10th week

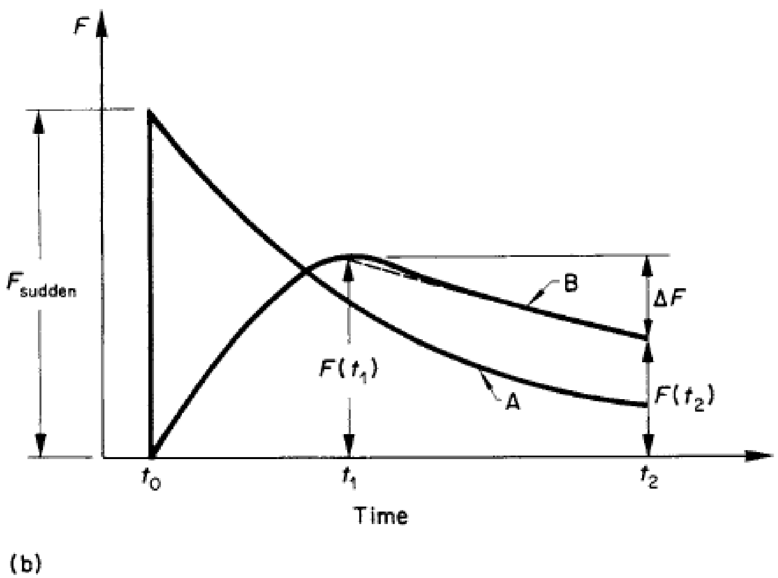
Force method 2.

4.4 Settlement



Idealization of settlement

$$\delta = \frac{FL^3}{48EI}$$



$$\delta = \frac{l^3}{6\bar{E}_c\left(t_1,t_0\right)I_c}F\left(t_1\right)$$

$$\bar{E}_c\left(t_1,t_0\right) = \frac{E_c(t_0)}{1+\chi\varphi\left(t_1,t_0\right)}$$

For proof of Equations (4.7) and (4.8), consider as an example the structure in Fig. 4.6(a). The instantaneous reaction at B due to a sudden settlement δ

$$F_{\text{sudden}} = \left(\frac{6}{l^3} E_c(t_0) I_c\right)\delta \tag{4.12}$$

$$\delta = \left(\frac{l^3}{6\bar{E}_c(t_1, t_0) I_c}\right) F(t_1) \tag{4.13}$$

The term in the large parentheses is the age-adjusted flexibility, or the displacement due to a unit increment of force introduced gradually. $\bar{E}_c(t_1, t_0)$ is the age-adjusted modulus of elasticity of concrete (see Equation (1.31)),

$$\bar{E}_c(t_1, t_0) = \frac{E_c(t_0)}{1 + \chi\varphi(t_1, t_0)} \tag{4.14}$$

$$F(t_1) = F_{\text{sudden}} \frac{1}{1 + \chi\, \varphi(t_1, t_0)} \tag{4.7}$$

$$F(t_2) = F(t_1) \left(1 - \frac{E_c(t_1)}{E_c(t_e)} \frac{\varphi(t_2, t_e) - \varphi(t_1, t_e)}{1 + \chi \varphi(t_2, t_1)}\right) \tag{4.8}$$

Under the effect of the force $F(t_1)$, free creep would increase the deflection by the hypothetical increment:

$$\Delta\delta = \left(\frac{l^3}{6E_c(t_e)I_c}\right) F(t_1)\, [\varphi(t_2, t_e) - \varphi(t_1, t_e)] \tag{4.15}$$

In this equation, $F(t_1)$ is treated as if it were applied in its entire value at the effective time t_e.

In this equation, $F(t_1)$ is treated as if it were applied in its entire value at the effective time t_e.

Because the support settlement does not change during the period t_1 to t_2, an increment of force ΔF must develop such that

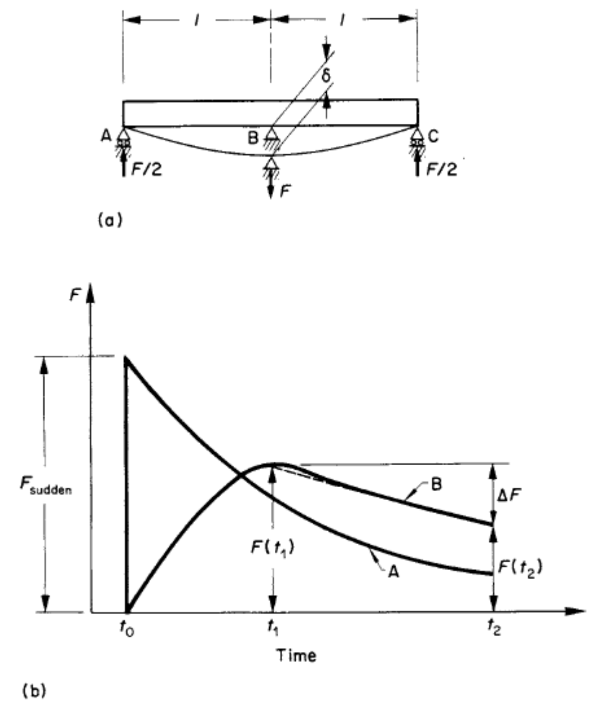
$$\Delta\delta + \left(\frac{l^3}{6\bar{E}_c(t_2, t_1)I_c}\right)\Delta F = 0$$

where

$$\bar{E}_c(t_2, t_1) = \frac{E_c(t_1)}{1 + \chi\varphi(t_2, t_1)}$$

The force at B at time t_2 is

$$F(t_2) = F(t_1) + \Delta F$$



$$F(t_2) = F(t_1)\left(1 - \frac{E_c(t_1)}{E_c(t_e)}\frac{\varphi(t_2, t_e) - \varphi(t_1, t_e)}{1 + \chi\varphi(t_2, t_1)}\right) \quad (4.8)$$

$$F(t_1) = F_{\text{sudden}}\frac{1}{1 + \chi\,\varphi(t_1, t_0)}$$

Example 4.5 Two-span continuous beam: settlement of central support

The continuous concrete beam shown in Fig. 4.6(a) is subjected to a downwards settlement at B. Find the time variation of the force F and the reaction at the central support. Express F in terms of F_{sudden} the value of the instantaneous reaction when the settlement δ is suddenly introduced. Consider two cases:

(a) δ introduced suddenly at $t_0 = 14$ days and maintained constant to $t_2 = 10\,000$ days.

(b) Settlement introduced gradually from zero at $t_0 = 14$ days to a value δ at $t_1 = 104$ days maintained constant thereafter up to $t_2 = 10\,000$ days.

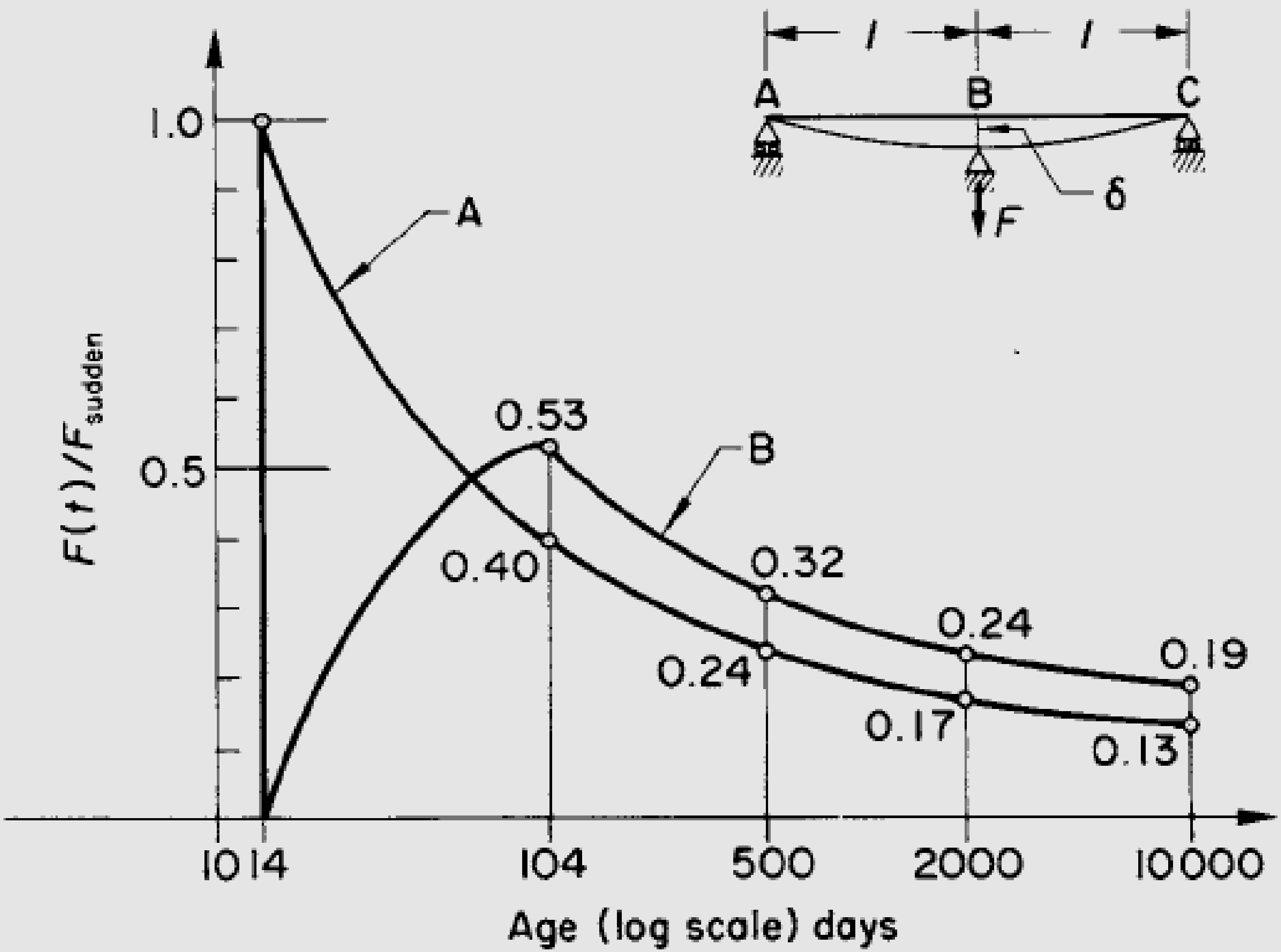


Figure 4.7 Values of the reaction at the central support versus time in a continuous beam subjected to settlement of a support (Example 4.5): A, period of settlement $(t_1 - t_0) = 0$; B, $(t_1 - t_0) = 90$ days.

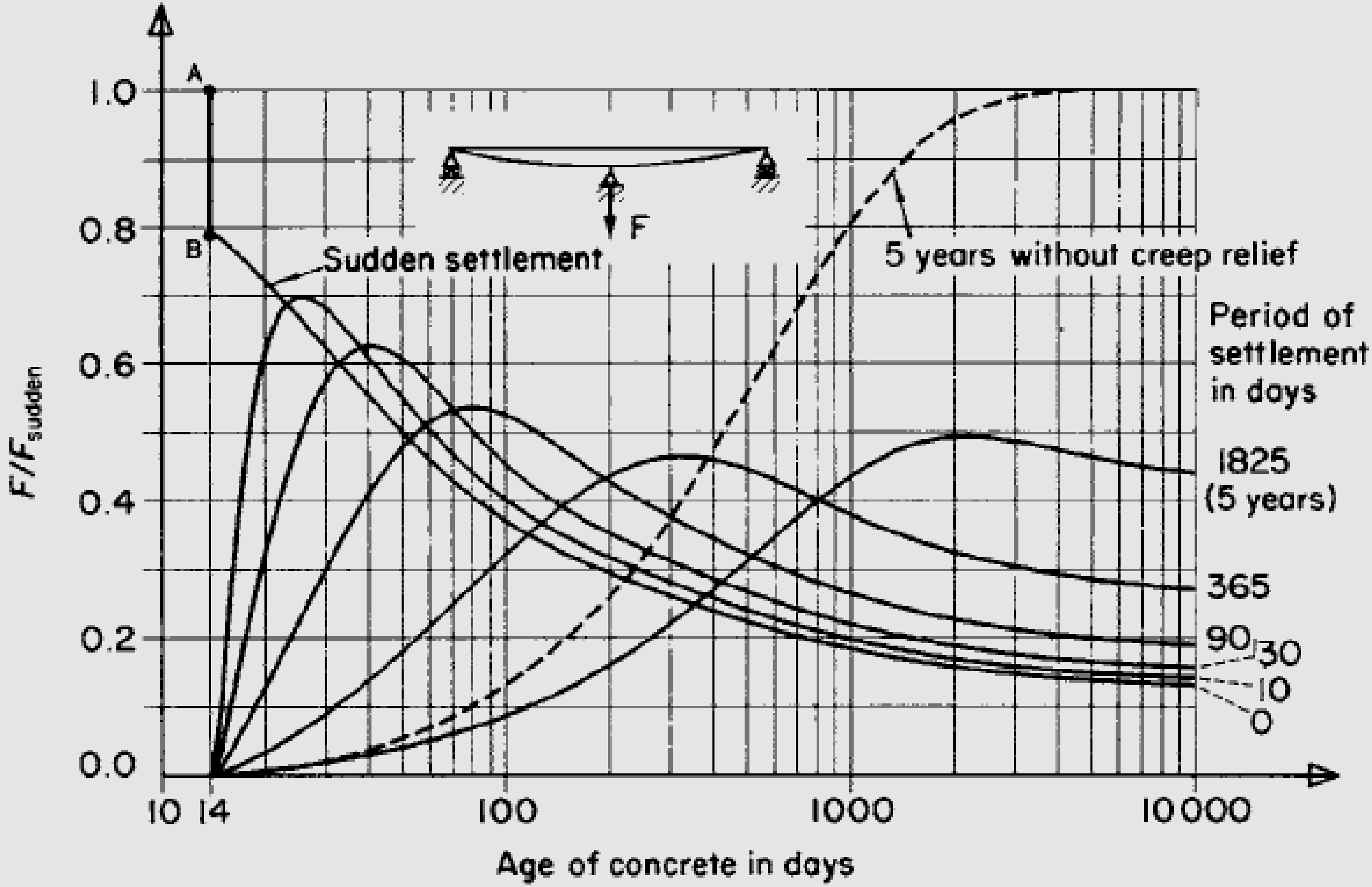


Figure 4.8 Time versus reaction by slow settlement of support occurring in a period of: 0, 10, 30, 365 days or 5 years (Example 4.5).

4.5 Accounting for the reinforcement

Analysis of the time-dependent changes in the internal forces in a statically indeterminate structure by Equation (4.5) involves calculation of the displacements of a statically determinate released structure to generate its age-adjusted flexibility matrix $[\bar{f}]$ and the vector $\{\Delta D\}$ of the changes in displacements occurring between two specified instants t_0 and t. By the procedure of analysis presented in Section 2.5, we can determine the changes $\Delta\varepsilon_O$ and $\Delta\psi$ in the axial strain and curvature in a section of a statically determinate structure, taking into account the presence of the reinforcement. The analysis gives the effects of creep, shrinkage and relaxation of steel on the stress and strain distribution, and thus the prestress loss in a prestressed section is automatically accounted for.

$$[\bar{f}]\{\Delta F\} = -\{\Delta D\}$$

from released structures for homogeneous section

We need to calculate $\Delta\varepsilon_0$ and $\Delta\psi$ for accounting of Reba

In step 4 of the analysis, we find the changes in the redundants occurring between t_0 and t by solving the compatibility equations:

$$[\bar{f}]\,\{\Delta F\} = -\{\Delta D\} \tag{4.5}$$

Example 4.6 Three-span precast post-tensioned bridge

A three-span bridge (Fig. 4.9(a)) is made up of precast post-tensioned simple beams for which the cross-section at mid-spans is shown in Fig. 4.9(b). The beams are prestressed at age $\bar{\tau}$, placed in position and made continuous at age t_0 by casting concrete at the joints and by continuous prestress tendons as shown in Fig. 4.9(c). It is required to find the bending moment diagram at time t later than t_0. Assume no cracks are produced at the casting joint and that the joint results in perfect continuity. Also calculate the deflection at time t_0 at the centre of AB and the change in this value during the period t_0 to t.

To simplify the presentation, we shall assume that the difference between $\bar{\tau}$ and t_0 is small and consider that the prestressing, placing the beams in positions and casting of the joints, all occur at age t_0. We shall also ignore the area of the cast *in situ* concrete (hatched area in Fig. 4.9(b)). Other data are: area of concrete section for one beam, A_c = 0.78 m^2 (1200 in^2); moment of inertia about an axis through the centroid of the concrete area, I_c = 0.159 m^4 (382 × 10^3 in^4); dead load of the precast and cast *in situ* concrete (assumed to come into effect at age t_0) = 9.1 kN/m^2 of area of deck; or the dead load per beam = 19.57 kN/m (1.344 kip/ft). A superimposed dead load of 5.0 kN/m^2 (10.75 kN/m per beam (0.737 kip/ft)) is applied shortly after the structure is made continuous. Again, for the sake of simplicity, we shall consider that the superimposed load is applied at t_0 on the continuous structure.

The prestress in each beam is achieved by straight tendons A and parabolic tendons B and C. The prestressing of A and B is applied to simple beams, while C is inserted after placing the beams in position and the cable runs continuous over the whole length of the bridge. Further, we shall consider that cables B and C have identical profiles (Fig. 4.9(d)). The cross-section areas of prestress steel A_{ps} are 430, 1000 and 1000 mm^2 (0.67, 1.55, 1.55 in^2) for tendons A, B and C, respectively; the initial prestress forces are: 500, 1160 and 1160 kN (112, 260 and 260 kip). Consider that these forces exclude friction loss and that the prestress force is constant over the full length of a tendon.

Non-prestressed steel of total area, $A_{ns} = 3750$ mm^2 (5.81 in^2) is distributed over all surfaces of the cross-section; thus, we here assume that A_{ns} has the same centroid as A_c (point O in Fig. 4.9(b)) and that the moment of inertia of the area A_{ns} about an axis through the same centroid is $I_c(A_{ns}/A_c) = 0.764 \times 10^{-3}\ m^4$; this is equivalent to considering that the radius of gyration for A_{ns} is the same as that of A_c.

The material properties are: modulus of elasticity for all reinforcement $E_{ps} = E_{ns} = 200\,GPa$ (29 000 ksi); modulus of elasticity of concrete at age t_0 $E_c(t_0) = 28\,GPa$ (4100 ksi); creep coefficient $\varphi(t, t_0) = 2.6$; aging coefficient $\chi(t, t_0) = 0.8$; free shrinkage during the period $(t - t_0) = \varepsilon_{cs}(t, t_0) = -240 \times 10^{-6}$; reduced relaxation during the same period, $\Delta\bar{\sigma}_{pr} = -90\,MPa$ (-13 ksi).

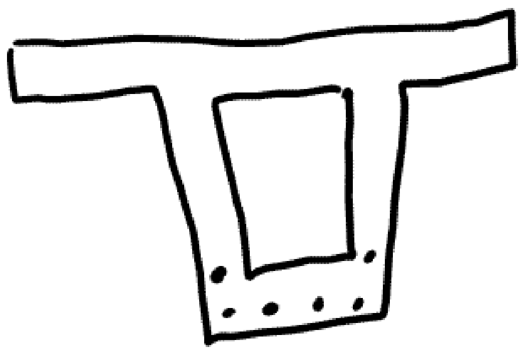
At t_0 the self-weight and the prestress of tendons A and B are applied on simple beams, while tendon C and the superimposed dead load are applied on a continuous beam. The bending moments for the simple and the continuous beams are calculated separately and then superposed; the result is shown in Fig. 4.10(a). Two values of the bending moment are indicated at B, with the larger value being the bending moment in the joint cast *in situ*.

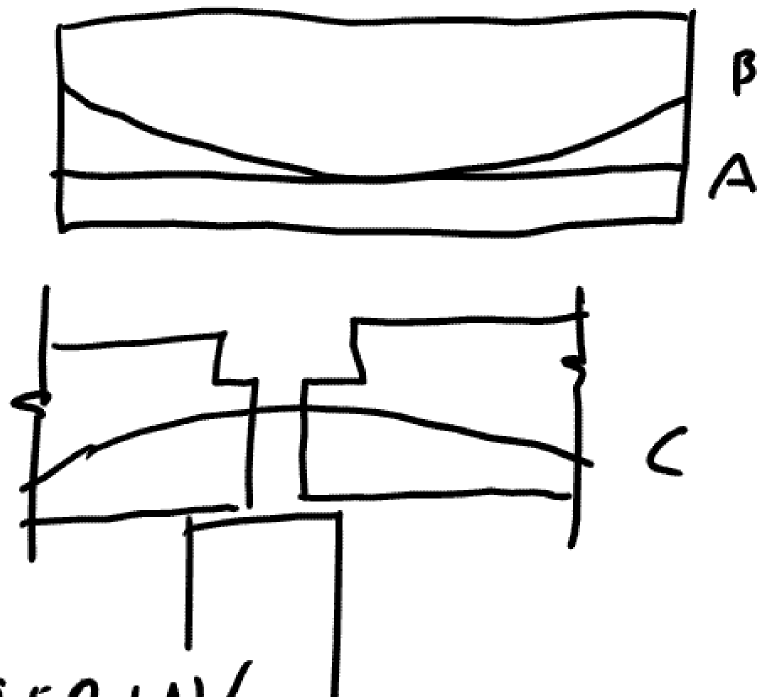
With the axial force and bending moment known at time t_0, the instantaneous axial strain at the reference point O, $\varepsilon_O(t_0)$ and the curvature $\psi(t_0)$ are calculated (by Equation (2.32)) at a number of sections and given in Table 4.1. The reference point O is chosen at the centroid of the concrete and the reference modulus of elasticity used in the calculation of area properties is $E_{ref} = E_c(t_0)$.

The properties of the transformed section at age t_0 in Table 4.1 are calculated for a section composed of A_c plus $(\alpha(t_0)A_{ns})$. The area of prestress steel should have been accounted for in the calculation of the deformations due to the superimposed dead load, but this is ignored here.

The changes in axial strain and in curvature, $\Delta\varepsilon_o$ and $\Delta\psi$ during the period t_0 to t are calculated by Equation (2.40) and the results are given in Table 4.2. These calculations involve the properties of the age-adjusted transformed section which are included in Table 4.1, using as reference modulus $\bar{E}_{ref} = \bar{E}_c(t, t_0) = 9.09\,\text{GPa}$ (1320 ksi) (Equation (1.31)).

The released structure and the coordinate system are shown in Fig. 4.10(b). Because of symmetry, the change in displacement ΔD_1 needs to be calculated only at coordinate 1 and can be calculated from the curvature increments $\Delta\psi$ in Table 4.2. The increment in displacement ΔD_1 is equal to the sum of the changes in rotation at B of members BA and BC treated as simple beams. Employing Equations (C.6) and (C.7) gives





$A_c = 0.78 m^2$

$I_c : 0.159 m^2$

Dead Load = 9.1 kN/m² ⇒ 19.57 kN/m

Superimposed D.L. = 10.75 kN/m

	A	B	C
A_{ps}	430	1000	1000
P_i	500	1160	1160

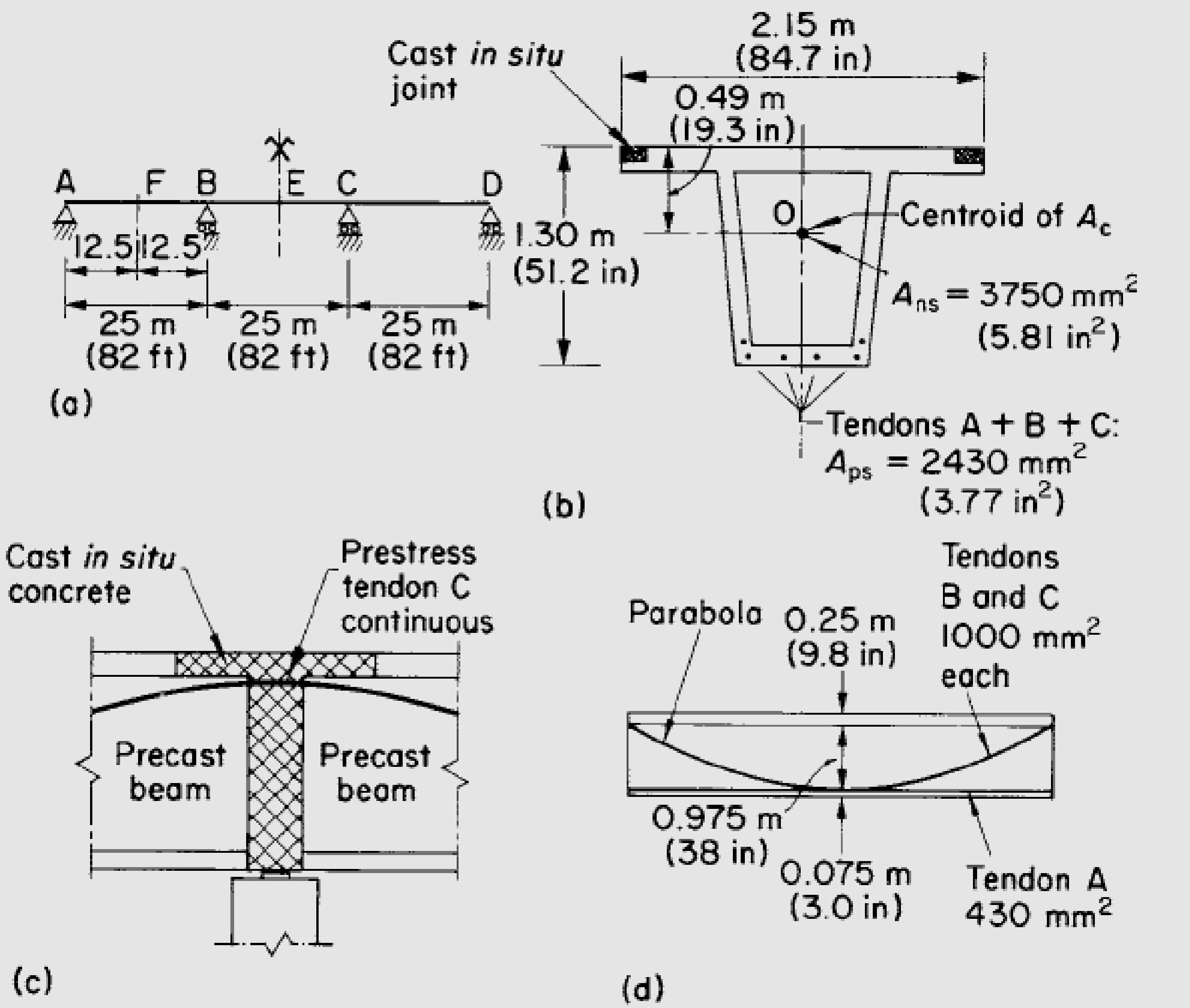
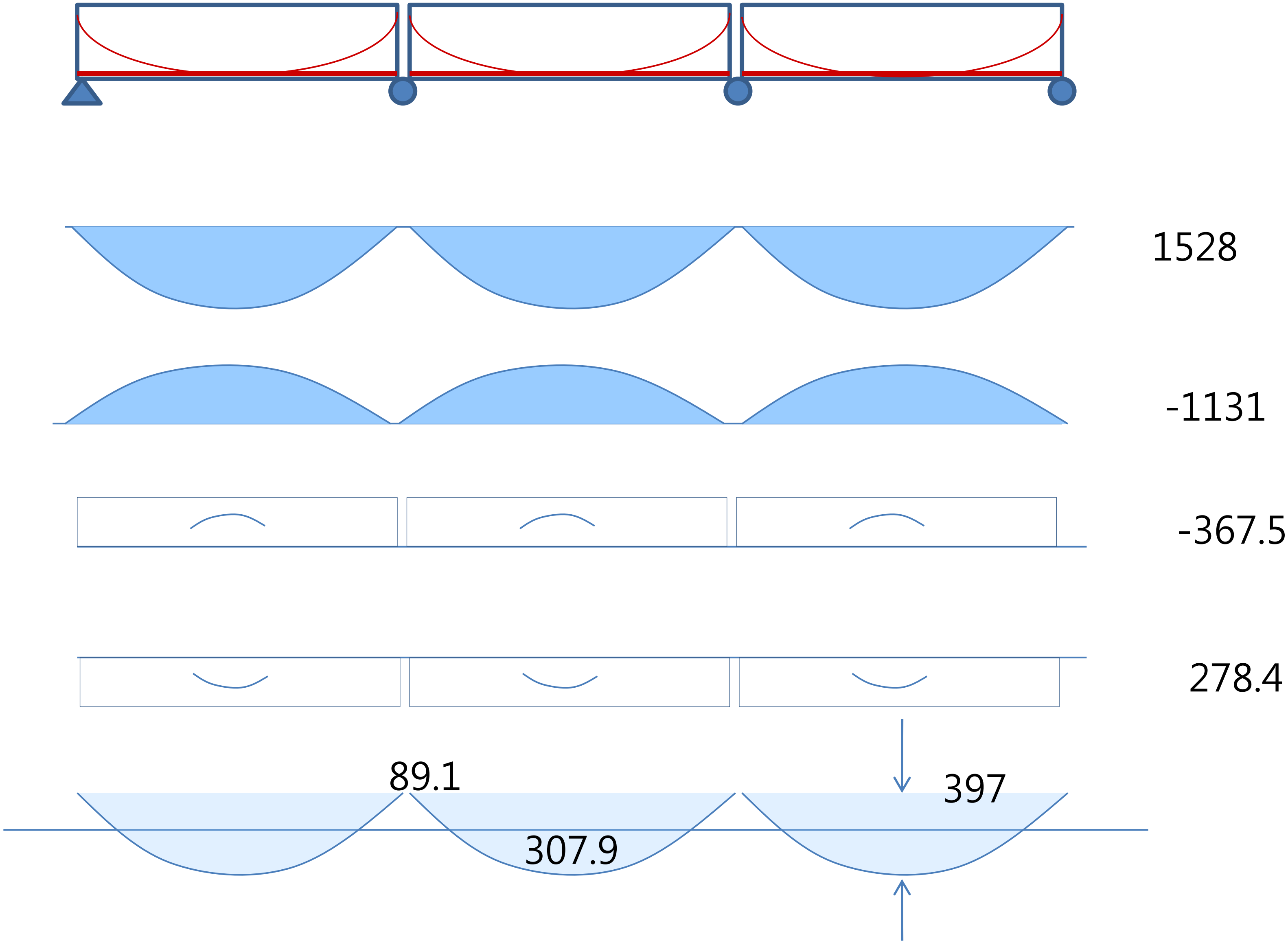
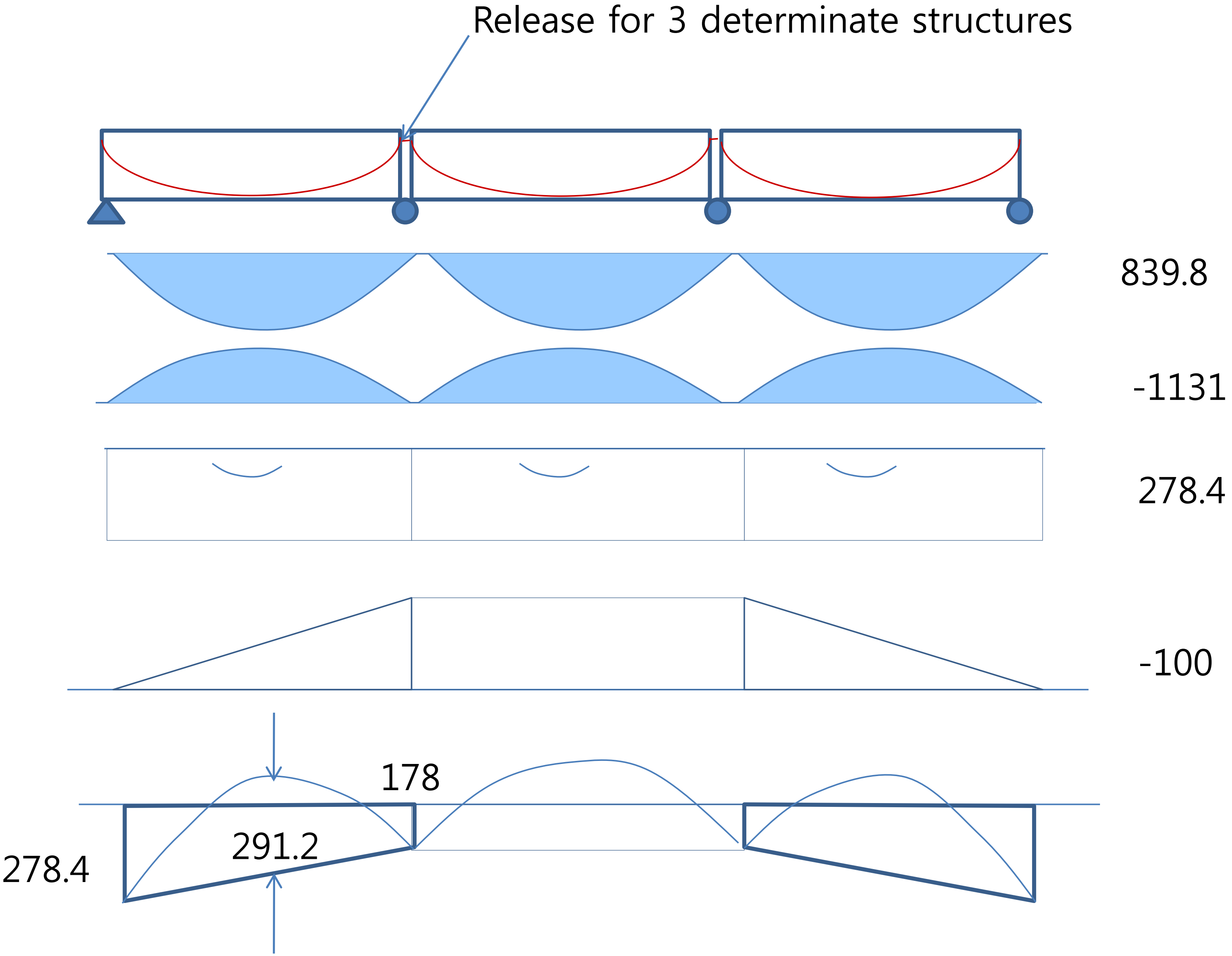
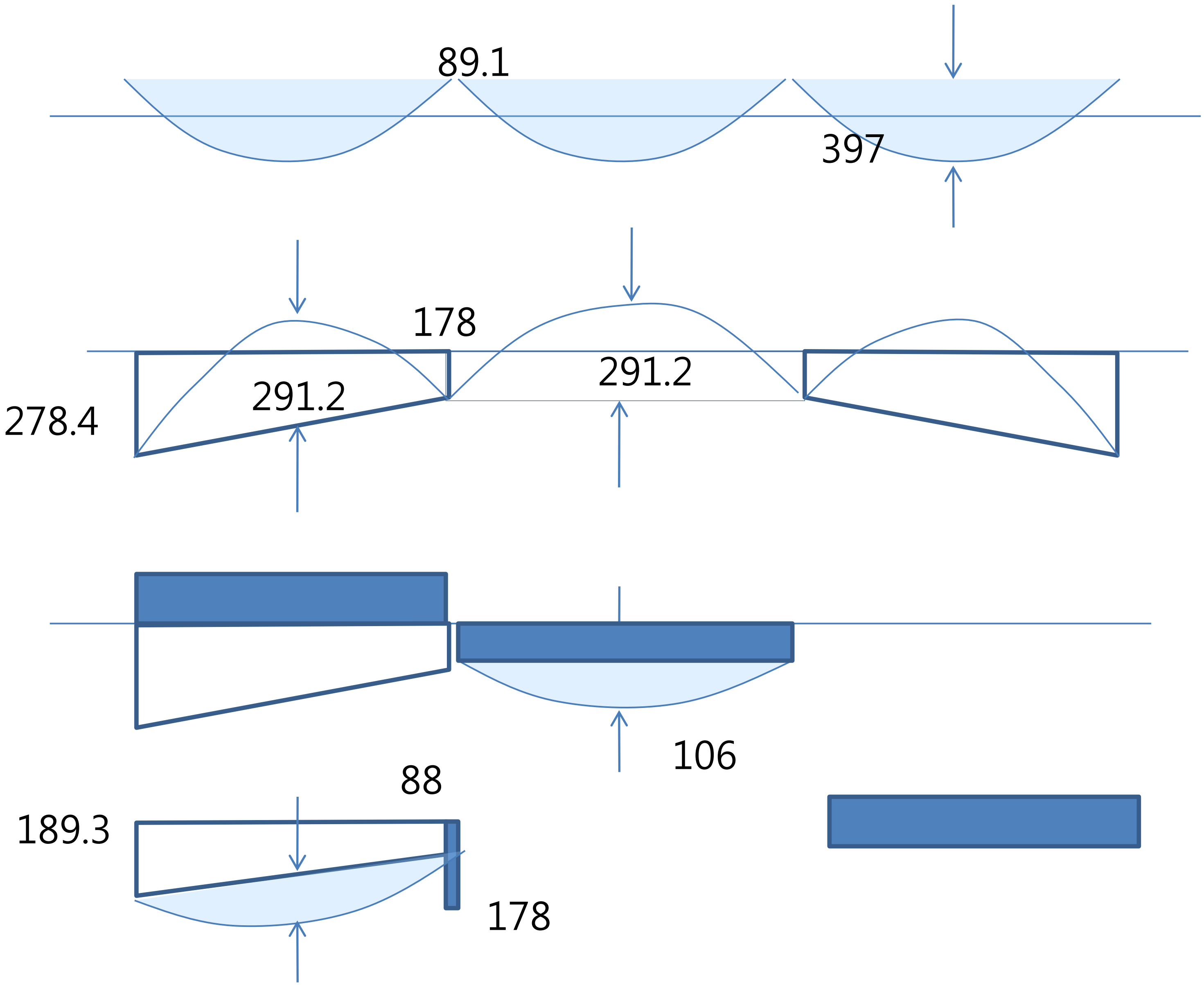


Figure 4.9 Continuous precast bridge of Example 4.6: (a) three-span bridge: (b) cross-section of one beam at mid-span; (c) joint of precast beams at supports B and C; (d) typical prestress tendon profiles in precast beams.


1528
-1131
-367.5
278.4
89.1
397
307.9


Release for 3 determinate structures
839.8
-1131
278.4
-100
178
291.2
278.4


89.1
397
178
291.2
291.2
278.4
106
88
189.3
178

$$A_{ns} = 3750 \text{ mm}^2$$

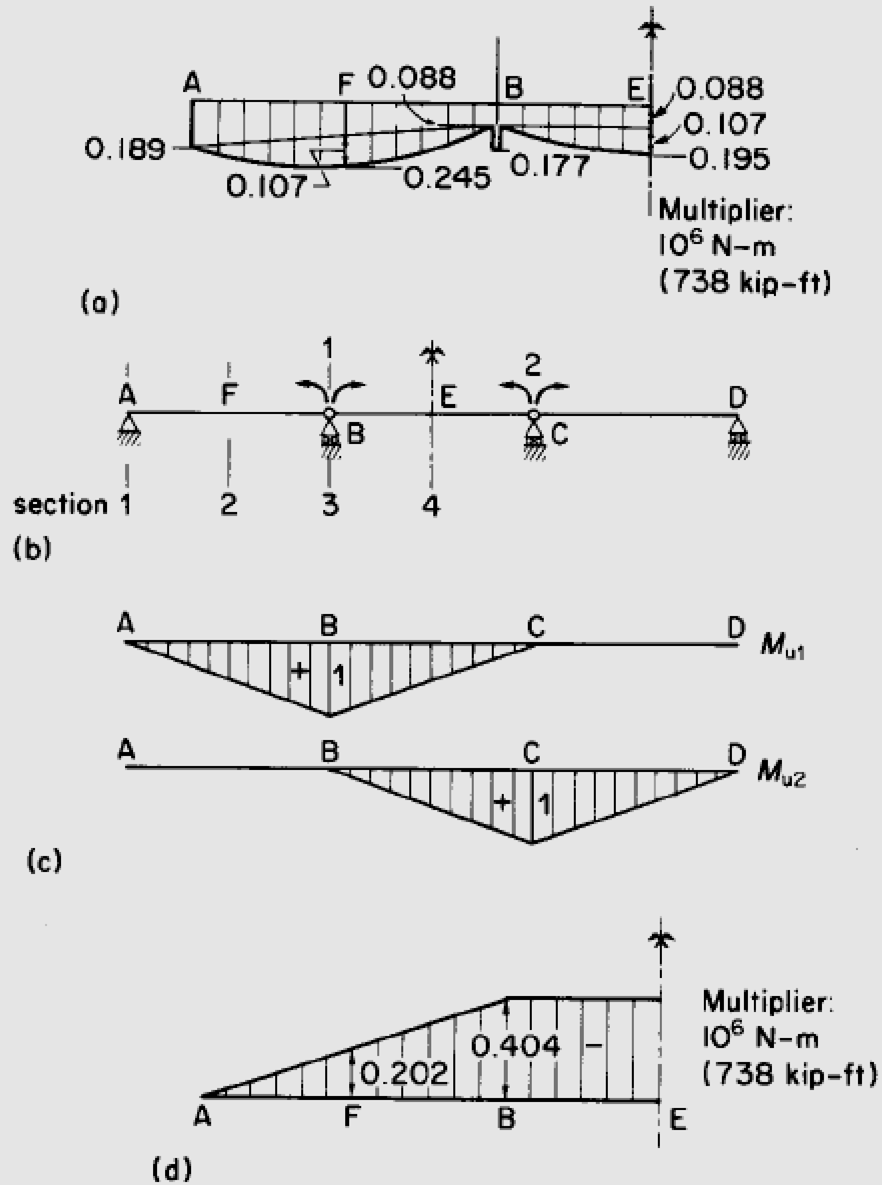
$$E_{rs} = E_{ns} = 200 \text{ GPa}$$

$$\bar{E}_c(t_0) = 28 \text{ GPa}$$

$$\varphi(t, t_0) = 2.6$$

$$\chi(t, t_0) = 0.8$$

$$\Delta\bar{\sigma}_{pr} = -90 \text{ MPa}$$


A
F
0.088
B
E
0.088
0.107
0.189
0.107
0.245
0.177
0.195
Multiplier:
10^6 N-m
(738 kip-ft)
(a)
1
2
A
F
E
D
B
C
section 1
2
3
4
(b)
A
B
C
D
M_{u1}
+
1
A
B
C
D
M_{u2}
+
1
(c)
0.202
0.404
–
Multiplier:
10^6 N-m
(738 kip-ft)
A
F
B
E
(d)

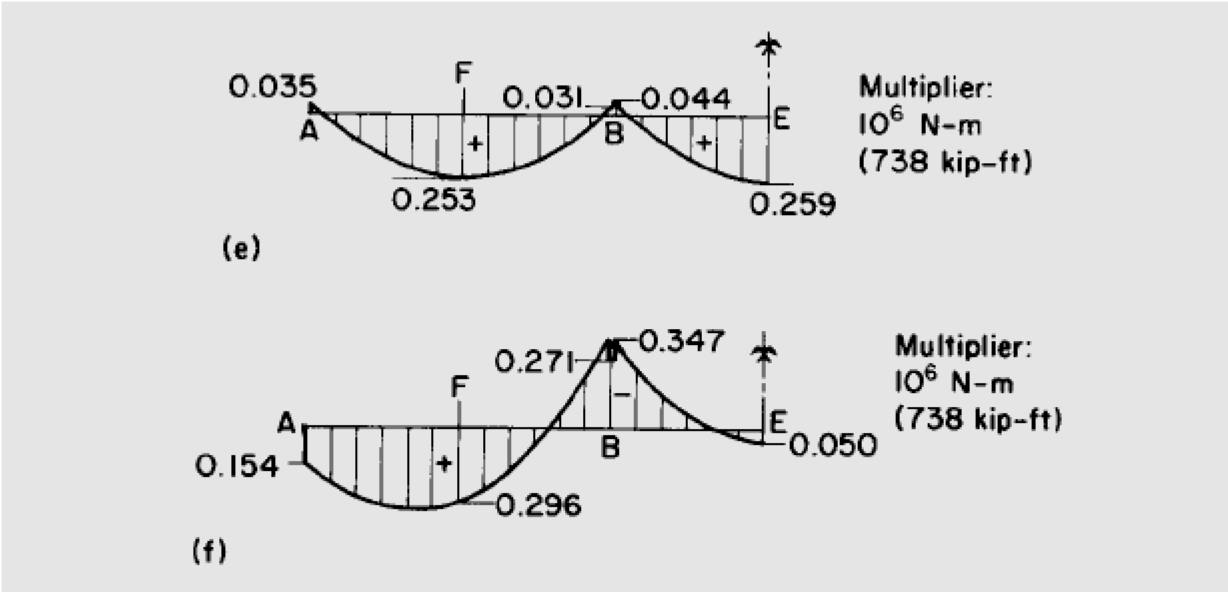
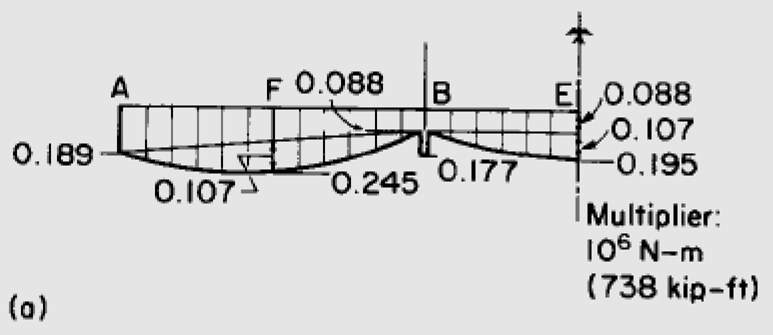

0.035
F
0.031
0.044
A
B
E
+
+
0.253
0.259
Multiplier:
10^6 N-m
(738 kip-ft)
(e)
0.347
0.271
F
A
B
E
+
−
0.154
0.296
0.050
Multiplier:
10^6 N-m
(738 kip-ft)
(f)

Table 4.1 Cross-section properties[1] and calculation of instantaneous axial strain and curvature for a continuous bridge (Example 4.6)

Section number (see Fig. 4.10(b))	Concrete section properties			Transformed section properties at time t_0 $E_{ref} = E_c(t_0) = 28$ GPa			Age-adjusted transformed section properties $E_{ref} = \bar{E}_c(t, t_0) = 9.09$ GPa			Forces applied at time t_0: equivalents of prestress force and dead-load bending moment (Equation (2.31))		Instantaneous axial strain and curvature (Equation(2.32))	
	A_c	B_c	I_c	A	B	I	$\bar{A}$	$\bar{B}$	$\bar{I}$	N	M	$\varepsilon_O(t_0)$	$\psi(t_0)$
1	0.78	0	0.159	0.8068	0	0.1645	0.9160	−0.0036	0.1835	−2.82	0.189	−125	41.0
2	0.78	0	0.159	0.8068	0	0.1645	0.9160	0.0393	0.2047	−2.82	0.245	−125	53.2
3	0.78	0	0.159	0.8068	0	0.1645	0.9160	−0.0036	0.1835	−2.82	0.088	−125	19.1
4	0.78	0	0.159	0.8068	0	0.1645	0.9160	0.0393	0.2047	−2.82	0.195	−125	42.3
Multipliers	m^2	m^3	m^4	m^2	m^3	m^4	m^2	m^3	m^4	10^6 N	10^6 N-m	10^{-6}	10^{-6} m^{-1}

[1] Reference point O is chosen at the common centroid of A_c or A_{rc}.



the change in displacement of the released structure during the time t_0 to t:

$$\Delta D_1 = \frac{25}{6}[0 \quad 2 \quad 1]\begin{Bmatrix} 74.6 \\ 283.4 \\ 25.3 \end{Bmatrix} 10^{-6} + \frac{25}{6}[1 \quad 2 \quad 0]\begin{Bmatrix} 25.3 \\ 261.3 \\ 25.3 \end{Bmatrix} 10^{-6}$$

$$= 4750 \times 10^{-6} \text{ radian.}$$

Use of Equation (C.8) and the curvature values $\psi(t_0)$ from Table 4.1 gives the instantaneous deflection at middle of span AB as

$$\frac{(25)^2}{96}[1 \quad 10 \quad 1]\begin{Bmatrix} 41.0 \\ 53.2 \\ 19.1 \end{Bmatrix} 10^{-6} = 3.85 \times 10^{-3} \text{ m}$$

$$= 3.85\,\text{mm} \qquad (0.152\,\text{in}).$$

(b) Three sections, parabolic variation (Fig. C.1)

$$D_1 = \frac{l}{6}[1\ 4\ 1]\{\varepsilon_O\} \qquad \text{(C.5)}$$

$$D_2 = -\frac{l}{6}[0\ 2\ 1]\{\psi\} \qquad \text{(C.6)}$$

$$D_3 = \frac{l}{6}[1\ 2\ 0]\{\psi\} \qquad \text{(C.7)}$$

$$D_4 = \frac{l^2}{96}[1\ 10\ 1]\{\psi\} \qquad \text{(C.8)}$$

The change in deflection of the released structure during the period t_0 to t (using $\Delta\psi$ values from Table 4.2 and Equation (C.8)) is

$$\frac{(25)^2}{96}[1 \quad 10 \quad 1]\begin{Bmatrix} 74.6 \\ 283.4 \\ 25.3 \end{Bmatrix} 10^{-6} = 19.10 \times 10^{-3}\ \text{m}$$

$$= 19.1\,\text{mm} \qquad (0.752\,\text{in}).$$

For calculation of the age-adjusted flexibility coefficient, apply $F_1 = 1$ at coordinate 1; the diagram of the corresponding bending moment M_{u1} is shown in Fig. 4.10(c). Division of the ordinates of this diagram by $\bar{E}_{ref}\bar{I}_{centroid}$ at sections 1 to 4 gives the curvatures due to $F_1 = 1$. $\bar{I}_{centroid}$ is the moment of inertia of the age-adjusted transformed section about an axis through the centroid:

$$\bar{I}_{centroid} = \bar{I} - \frac{\bar{B}^2}{\bar{A}}$$

$$I = I_o + Ar^2$$

The values of the curvatures due to $F_1 = 1$, calculated in this fashion at the four sections considered, are:

$$\{\psi_{u1}\} = 10^{-9}\ \{0, 0.2710, 0.5995, 0.2710\}\ \text{m}^{-1}/\text{N-m}.$$

The value $\bar{f}_{11}$ is the sum of the rotations just to the left and to the right of section 3, caused by $F_1 = 1$. These rotations can be calculated from the above curvatures, using Equations (C.6) and (C.7), giving

$$\bar{f}_{11} = \frac{25}{6}(2 \times 0.2710 + 1 \times 0.5995)2 \times 10^{-9} = 9.513 \times 10^{-9}\ (\text{N-m})^{-1}.$$

The age-adjusted flexibility coefficient $\bar{f}_{12}$ is the rotation at coordinate 1 due to $F_2 = 1$. Using a similar procedure as above gives

$$\bar{f}_{12} = \frac{25}{6}\ (2 \times 0.2710)\ 10^{-9} = 2.258 \times 10^{-9}\ (\text{N-m})^{-1}.$$

The deflection at the centre of AB due to $F_1 = 1$ (by Equation (C.8))

$$\frac{(25)^2}{96}\ (10 \times 0.2710 + 0.5995)\ 10^{-9} = 21.55 \times 10^{-9}\ \text{m/N-m}.$$

The force $F_2 = 1$ produces no deflection at the centre of AB.

Because of symmetry, the two redundants are equal and can be determined by solving one equation:

$$(\bar{f}_{11} + \bar{f}_{12})\Delta F_1 = -\Delta D_1$$

Thus,

$$\Delta F_1 = \Delta F_2 = \frac{-4750 \times 10^{-6}}{(9.513 + 2.258)10^{-9}} = -0.404 \times 10^6 \text{ N-m.}$$

The statically indeterminate bending moment diagram developed during the period t_0 to t is shown in Fig. 4.10(d).

The change in deflection of the actual structure can now be calculated by the superposition Equation (4.6) which is repeated here:

$$\{\Delta A\} = \{\Delta A_s\} + [\Delta A_u]\{\Delta F\}$$

where $\{\Delta A_s\}$ is the change in deflection of the released structure; $[\Delta A_u]$ are the changes in deflection due to $F_1 = 1$ and due to $F_2 = 1$; $\{\Delta F\}$ are the time-dependent redundant forces. Substitution of the values calculated above gives the change in deflection at the centre of AB during the period t_0 to t:

$$19.10 \times 10^{-3} + 10^{-9}[21.55 \quad 0]\begin{Bmatrix} -0.404 \\ -0.404 \end{Bmatrix} 10^6 = 10.39 \times 10^{-3} \text{ m}$$

$$= 10.39 \text{ mm} \qquad (0.409 \text{ in}).$$

Table 4.2 Changes in axial strain and in curvature of the released structure during the period t_0 to t in Example 4.6

Section number (see Fig. 4.10(b))	*Calculation of restraining forces*								Changes in axial strain and in curvature (Equation (2.40))	
	Creep (Equation (2.42))		Shrinkage (Equation (2.43))		Relaxation (Equation (2.44))		Total restraining forces (Equation (2.41))			
	ΔN	ΔM	ΔN	ΔM	ΔN	ΔM	ΔN	ΔM	$\Delta\varepsilon_O$	$\Delta\psi$
1	2.304	−0.1541	1.702	0	−0.219	0.0148	3.787	−0.1393	−455	74.6
2	2.304	−0.1999	1.702	0	−0.219	−0.1607	3.787	−0.3606	−467	283.4
3	2.304	−0.0718	1.702	0	−0.219	0.0148	3.787	−0.0570	−455	25.3
4	2.304	−0.1590	1.702	0	−0.219	−0.1607	3.787	−0.3197	−466	261.3
Multipliers	10^6 N	10^6 N-m	10^6 N	10^6 N-m	10^6 N	10^6 N-m	10^6 N	10^6 N-m	10^{-6}	10^{-6} m^{-1}

$$\begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix} = \begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix}_{creep} + \begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix}_{shrinkage} + \begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix}_{relaxation}$$

$$\begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix}_{creep} = -\sum_{i=1}^{m}\left\{\bar{E}_c\begin{bmatrix}A_c & B_c\\ B_c & I_c\end{bmatrix}\varphi\begin{Bmatrix}\varepsilon_o(t_0)\\ \psi(t_0)\end{Bmatrix}\right\}_i$$

$$\begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix}_{shrinkage} = -\sum_{i=1}^{m}\left\{\bar{E}_c\begin{bmatrix}A_c & B_c\\ B_c & I_c\end{bmatrix}\begin{Bmatrix}\varepsilon_{cs}\\ 0\end{Bmatrix}\right\}_i$$

$$\bar{E}_c(t,t_0) = \frac{E_c(t_0)}{1+\chi\varphi(t,t_0)}$$

$$\begin{Bmatrix}\Delta N\\ \Delta M\end{Bmatrix}_{prestressing} = \sum\begin{Bmatrix}A_{ps}\Delta\bar{\sigma}_{pr}\\ A_{ps}y_{ps}\Delta\bar{\sigma}_{pr}\end{Bmatrix}_i$$

The restraining forces

$$\begin{Bmatrix} \Delta\varepsilon_o \\ \Delta\psi \end{Bmatrix} = \frac{1}{\bar{E}_c\left(\bar{A}\bar{I} - \bar{B}^2\right)} \begin{bmatrix} \bar{I} & -\bar{B} \\ -\bar{B} & \bar{A} \end{bmatrix} \begin{Bmatrix} -\Delta N \\ -\Delta M \end{Bmatrix}$$

The forces $\{F\}_{i+\frac{1}{2}}$ and the displacement $\{D\}_{i+\frac{1}{2}}$ at the end of any interval i may be expressed as the sum of incremental forces $\{\Delta F\}_j$ and displacements, $\{\Delta D\}_j$ occurring at the middle of the intervals $j = 1, 2, \ldots, i$. Thus,

$$\{F\}_{i+\frac{1}{2}} = \sum_{j=1}^{i} \{\Delta F\}_j \tag{4.21}$$

$$\{D\}_{i+\frac{1}{2}} = \sum_{j=1}^{i} \{\Delta D\}_j \tag{4.22}$$

The compatibility Equation (4.20) applied at the end of the ith interval may be written in the form:

$$\sum_{j=1}^{i} \{[f(t_{i+\frac{1}{2}}, t_j)]\{\Delta F\}_j\} = -\sum_{j=1}^{i} \{\Delta D\}_j \tag{4.23}$$

$$[f(t_{i+\frac{1}{2}}, t_i)]\ \{\Delta F\}_i = -\ \{D\}_{i+\frac{1}{2}} - \sum_{j=1}^{i-1} \{[f(t_{i+\frac{1}{2}}, t_j)]\{\Delta F\}_j\} \tag{4.24}$$

$$f_{\text{instantaneous}} = \frac{l^3}{6E_c I_c} \tag{4.25}$$

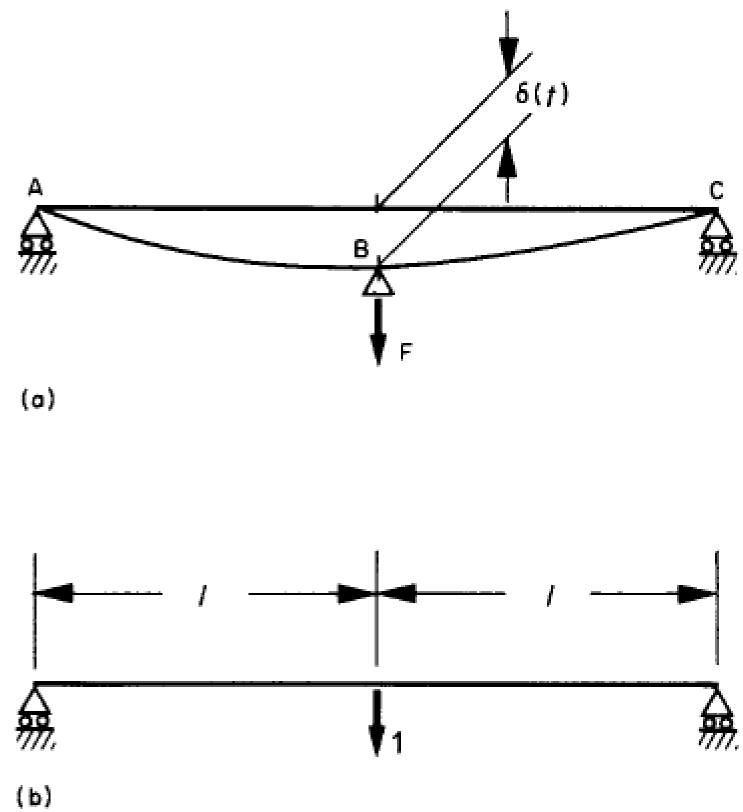


Figure 4.12 Reaction due to settlement of support of a continuous beam by a step-by-step procedure employing Equation (4.24): (a) continuous beam; (b) statically determinate released structure.

$$f(t_{i+\frac{1}{2}}, t_j) = C\frac{1}{E_c(t_j)}[1+\varphi(t_{i+\frac{1}{2}}, t_j)] \tag{4.26}$$

where C is a constant independent of time related to the geometry of the structure

$$C = \frac{l^3}{6I_c}. \tag{4.27}$$

At the end of any interval i

$$D_{i+\frac{1}{2}} = -\delta(t_{i+\frac{1}{2}}). \tag{4.28}$$

The minus sign is included in this equation because it represents a displacement caused by the redundant force F (rather than eliminated by it).

Substitution of Equations (4.26) and (4.28) into Equation (4.24) gives

$$f(t_{i+\frac{1}{2}}, t_i)(\Delta F)_i = \delta(t_{i+\frac{1}{2}}) - C\sum_{j=1}^{i-1}\left(\frac{1+\varphi(t_{i+\frac{1}{2}}, t_j)}{E_c(t_j)}(\Delta F)_j\right). \tag{4.29}$$

The magnitude of the reaction at the central support at the end of the ith interval is

$$F(t_{i+\frac{1}{2}}) = F(t_{i-\frac{1}{2}}) + (\Delta F)_i. \tag{4.30}$$

Solving Equation (4.29) for $(\Delta F)_i$ and substitution in Equation (4.30) gives

$$F(t_{i+\frac{1}{2}}) = F(t_{i-\frac{1}{2}})$$
$$+ [f(t_{i+\frac{1}{2}}, t_i)]^{-1}\left[\delta(t_{i+\frac{1}{2}}) - C\sum_{j=1}^{i-1}\left(\frac{1+\varphi(t_{i+\frac{1}{2}}, t_j)}{E_c(t_j)}(\Delta F)_j\right)\right] \tag{4.31}$$

Equation (4.31) has been used to derive the graphs in Fig. 4.8 (see Example 4.5).