Deformation of Concrete

Fall 2010

Dept of Architecture
Seoul National University

11th week

Displacement method

5. Displacement Method

- Unit displacement
- Stiffness coefficients
- Degree of freedom

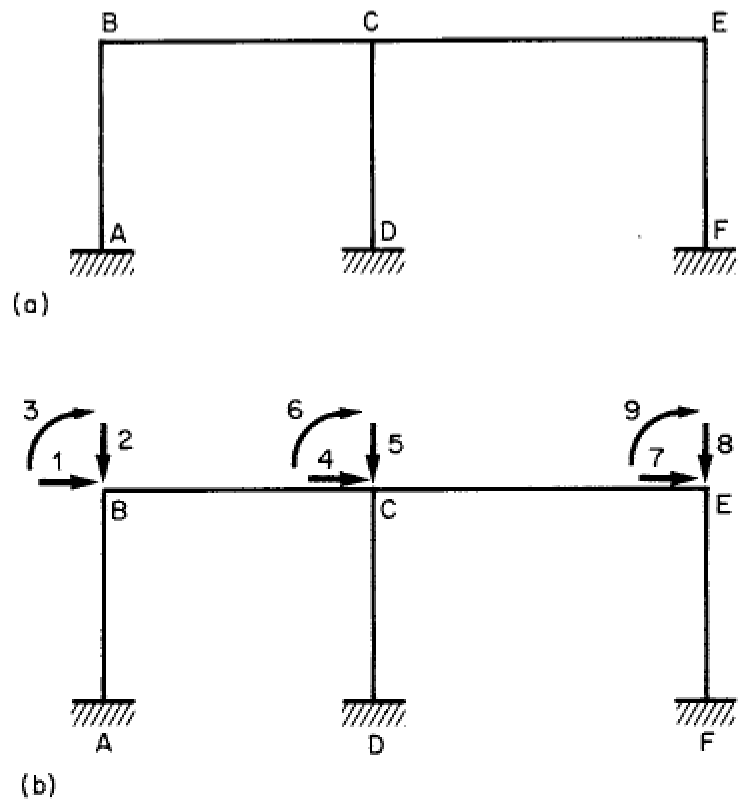


Figure 5.1 Example of a coordinate system (b) employed for the analysis of a plane frame (a) by the displacement method.

Step 1 A coordinate system is established to identify the locations and the positive directions of the joint displacements (Fig. 5.1(b)). The number of coordinates n is equal to the number of possible independent joint displacements (degrees of freedom). There are generally two translations and a rotation at a free (unsupported) joint of a plane frame. The number of unknown displacements may be reduced by ignoring the axial deformations. For example, by considering that the length of the members of the frame in Fig. 5.1(b) remains unchanged, the degrees of freedom are reduced to coordinates 1, 3, 6 and 9.

Step 2 Restraining forces $\{F\}$ are introduced at the n coordinates to prevent the joint displacements. The forces $\{F\}$ are calculated by summing the fixed-end forces for the members meeting at the joints. Also determine $\{A_r\}$, values of the actions with the joints in the restrained position.

Step 3 The structure is now assumed to be deformed such that the displacement at coordinate j, $D_j = 1$, with the displacements prevented at all the other coordinates. The forces $S_{1j}, S_{2j}, \ldots, S_{nj}$ required to hold the frame in this configuration are determined at the n coordinates. This process is repeated for unit values of displacement at each of the coordinates, respectively. Thus a set of $n \times n$ stiffness coefficients is calculated, which forms the stiffness matrix $[S]_{n \times n}$ of the structure; a general element S_{ij} is the force required at coordinate i due to a unit displacement at coordinate j. The values of the actions $[A_u]$ are also determined due to unit values of the displacements; any column j of the matrix $[A_u]$ is composed of the values of the actions at the desired locations due to $D_j = 1$.

Step 4 The displacement $\{D\}$ in the actual (unrestrained) structure is obtained by solving the equilibrium equation:

$$[S]\{D\} = -\{F\} \tag{5.1}$$

The equilibrium Equation (5.1) indicates that the displacements $\{D\}$ must be of such a magnitude that the artificial restraining forces $\{F\}$ are eliminated.

Step 5 Finally, the required values $\{A\}$ of the actions in the actual structure are obtained by adding the values $\{A_r\}$ in the restrained structure (calculated in step 2) to the values caused by the joint displacements. This is expressed by the superposition equation:

$$\{A\}_{m \times 1} = \{A_r\}_{m \times 1} + [A_u]_{m \times n}\{D\}_{n \times 1} \tag{5.2}$$

Step	Force method p 103	Displacement Method
1	Removal of redundant	Identify Dof (unknowns)
2	Displacement at release	Restrained forces are introduced to prevent the joint displacement
3	Apply unit load and flexibility	Apply unit displacement with the displacement prevented at all other coordinate. Stiffness
4	Eliminate inconsistencies by compatibility condition-> redundant	Displacement is obtained by equilibrium -> displacement
5	Released structure + redundant forces	Restrained structure + values caused by the joint displacement

Time-dependent change in fixed-end forces in a homogeneous member

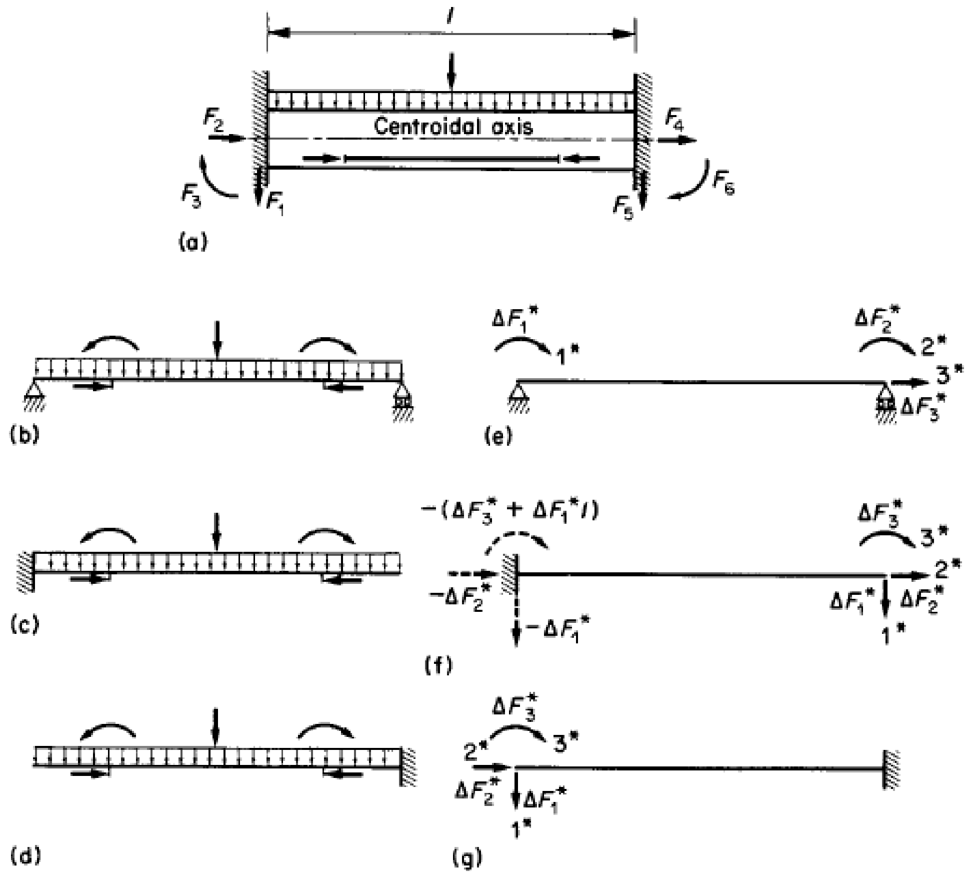


Figure 5.2 Analysis of the time-dependent changes in the end forces of a member caused by fixity introduced after loading: (a) totally fixed beam subjected at time t_0 to a system of forces; (b), (c), (d) statically determinate beams loaded at time t_0, statical system changed to totally fixed beam at time t_1; (e), (f), (g) coordinate systems.

$$\{\Delta D^*\} = \{D^*(t_0)\}[\varphi(t_2, t_0) - \varphi(t_1, t_0)] \tag{5.3}$$

where $\{D^*(t_0)\}$ are the instantaneous displacement at t_0 due to the external loads on the statically determinate system; $\varphi(t_i, t_j)$ is the coefficient for creep at t_i when the age at loading is t_j.

The age-adjusted flexibility matrix is

$$[\bar{f}] = [f][1 + \chi\varphi(t_2, t_1)] \tag{5.4}$$

where $\chi = \chi(t_2, t_1)$ is the aging coefficient (see Section 1.7); $[f]$ is the flexibility matrix of a statically determinate beam (Fig. 5.2(b), (c) or (d)). The modulus of elasticity to be used in the calculation of the elements of $[f]$ is $E_c(t_1)$.

The compatibility Equation (4.5) can now be applied, which is repeated here:

$$[\bar{f}]\{\Delta F^*\} = \{-\Delta D^*\} \tag{5.5}$$

Substitution of Equations (5.3) and (5.4) in Equation (5.5) and solution gives the changes in the three end forces developed during the period t_1 to t_2:

$$\{\Delta F^*\} = \left(\frac{\varphi(t_2, t_0) - \varphi(t_1, t_0)}{1 + \chi\varphi(t_2, t_1)}\right)\{F^*\} \tag{5.6}$$

where

$$\{F^*\} = [f]^{-1}\{-D^*(t_0)\} \tag{5.7}$$

Example 5.1

The cantilever in Fig. 5.3(a) is subjected at age t_0 to a uniformly distributed load q/unit length. At age t_1, end B is made totally fixed. Find the forces at the two ends at a later time t_2. Use the following creep and aging coefficients: $\varphi(t_1, t_0) = 0.9$; $\varphi(t_2, t_0) = 2.6$; $\chi(t_2, t_1) = 0.8$; $\varphi(t_2, t_1) = 2.45$.

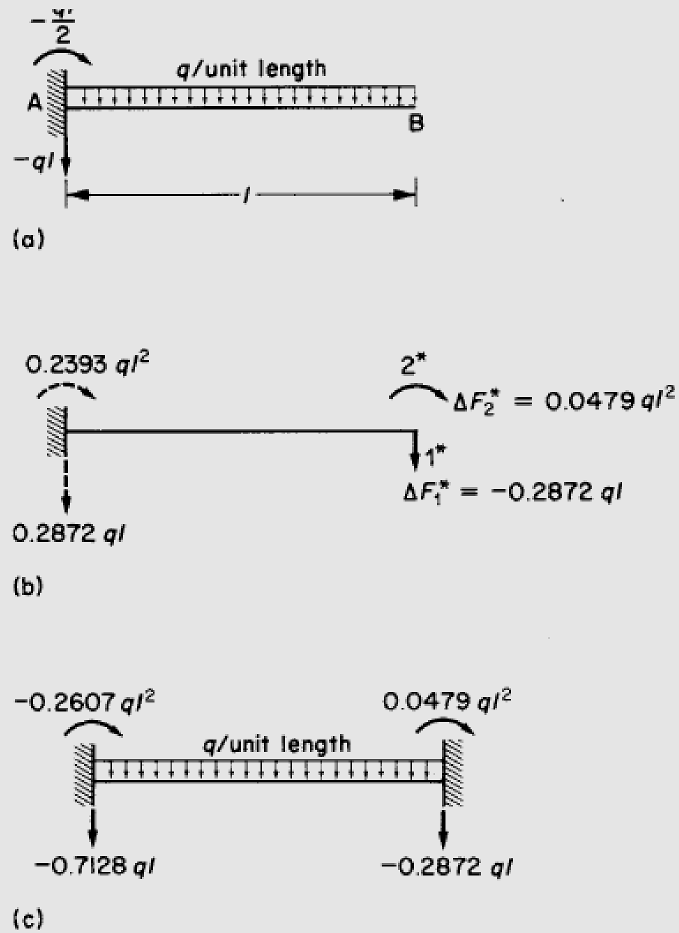


Figure 5.3 Analysis of time-dependent forces in a cantilever transformed into a totally fixed beam after loading (Example 5.1): (a) forces acting at time t_0; (b) changes in end forces between t_1 and t_2; (c) total forces at t_2.

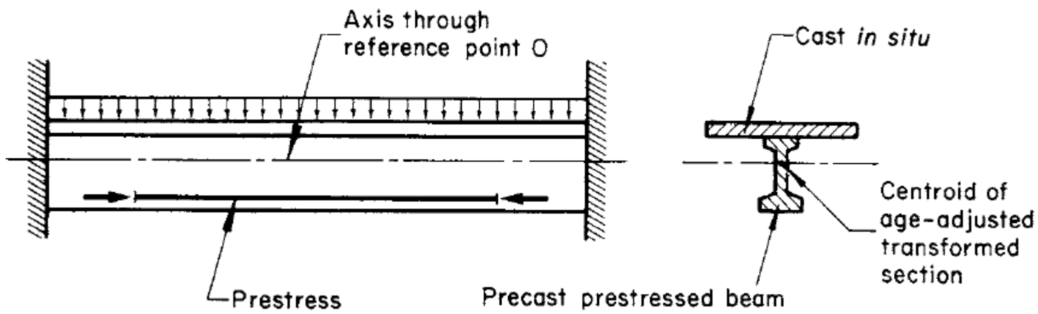
$$\{F^*\} = \begin{Bmatrix} -ql/2 \\ ql^2/12 \end{Bmatrix}$$

Forces developed at end B of the cantilever during the period t_1 to t_2 (Equation (5.6)) are

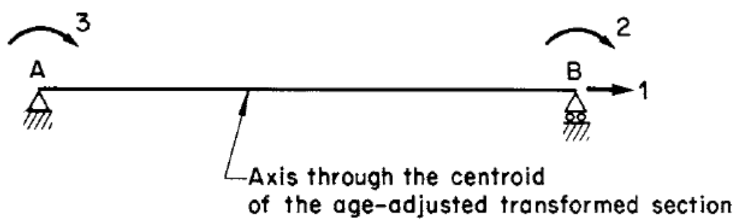
$$\{\Delta F^*\} = \left(\frac{2.6 - 0.9}{1 + 0.8 \times 2.45}\right) \begin{Bmatrix} -ql/2 \\ ql^2/12 \end{Bmatrix} = \begin{Bmatrix} -0.2872ql \\ 0.0479ql^2 \end{Bmatrix}$$

These two forces and their equilibrants at end A are shown in Fig. 5.3(b). Superposition of the forces at the member ends in Fig. 5.3(a) and (b) gives the end forces at time t_2, shown in Fig. 5.3(c).

Time-dependent change in the fixed-end forces in a composite member



(a)



(b)

A system of three coordinates is chosen on a statically determinate released structure in Fig. 5.6(b). The analysis involves the solution of the following equation (see Equation (4.5)):

$$[\bar{f}]\ \{\Delta F\} = -\ \{\Delta D\} \tag{5.8}$$

Solution of Equation (5.8) gives

$$\{\Delta F\} = [\bar{f}]^{-1}\,\{-\Delta D\} \tag{5.9}$$

where $[\bar{f}]^{-1}$ is the age-adjusted stiffness corresponding to the coordinate system in Fig. 5.6(b). For a member with constant cross-section,[2]

$$[\bar{f}]^{-1} = \frac{\bar{E}_c}{l}\begin{bmatrix} \bar{A} & 0 & 0 \\ 0 & 4\bar{I} & 2\bar{I} \\ 0 & 2\bar{I} & 4\bar{I} \end{bmatrix} \tag{5.10}$$

$$\{\Delta F\} = \frac{\bar{E}_c}{l}\begin{bmatrix} \bar{A} & 0 & 0 \\ 0 & 4\bar{I} & 2\bar{I} \\ 0 & 2\bar{I} & 4\bar{I} \end{bmatrix}\{-\Delta D\} \tag{5.11}$$

The changes $\{\Delta D\}$ in the displacements of the released structure may be determined by numerical integration or by virtual work using the equation (see Section 3.8):

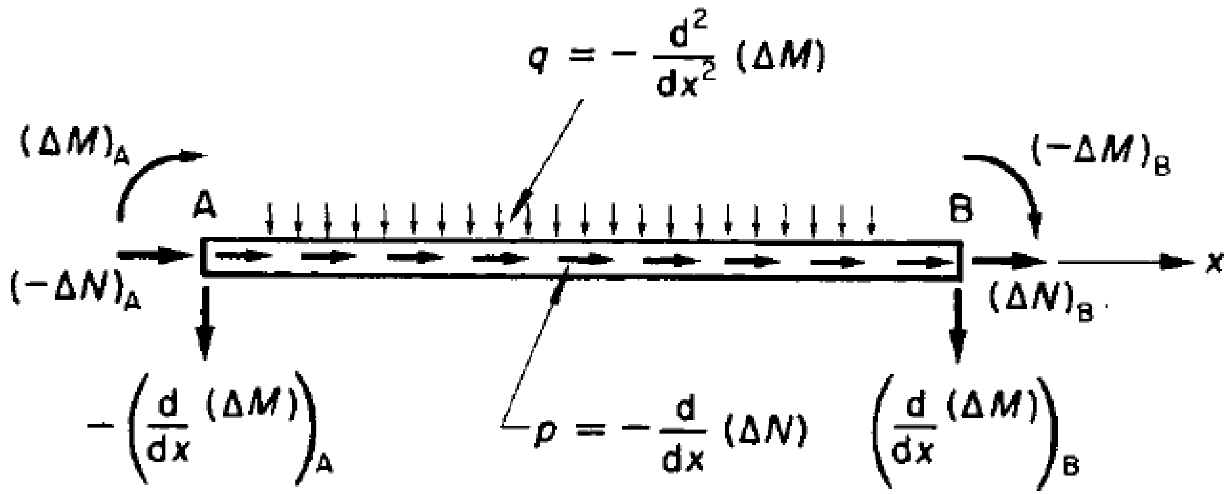
$$\{\Delta D\} = \begin{Bmatrix} \int(\Delta\varepsilon_O) & N_{u1} & \mathrm{d}l \\ \int(\Delta\psi) & M_{u2} & \mathrm{d}l \\ \int(\Delta\psi) & M_{u3} & \mathrm{d}l \end{Bmatrix} \tag{5.12}$$

$\Delta\varepsilon_O$ and $\Delta\psi$ may be calculated by the method presented in Section 2.5 using Equation (2.40) which is rewritten here:

$$\begin{Bmatrix} \Delta\varepsilon_O \\ \Delta\psi \end{Bmatrix} = \frac{1}{\bar{E}_c(\bar{A}\bar{I} - \bar{B}^2)} \begin{bmatrix} \bar{I} & -\bar{B} \\ -\bar{B} & \bar{A} \end{bmatrix} \begin{Bmatrix} -\Delta N \\ -\Delta M \end{Bmatrix} \tag{5.13}$$

where $\{\Delta N, \Delta M\}$ are a normal force at O and a bending moment required to artificially prevent the change in strain in the section during the period t_0 to t. $\bar{B}$ is the first moment of area of the age-adjusted transformed section about an axis through the reference point O.

Artificial restraining forces



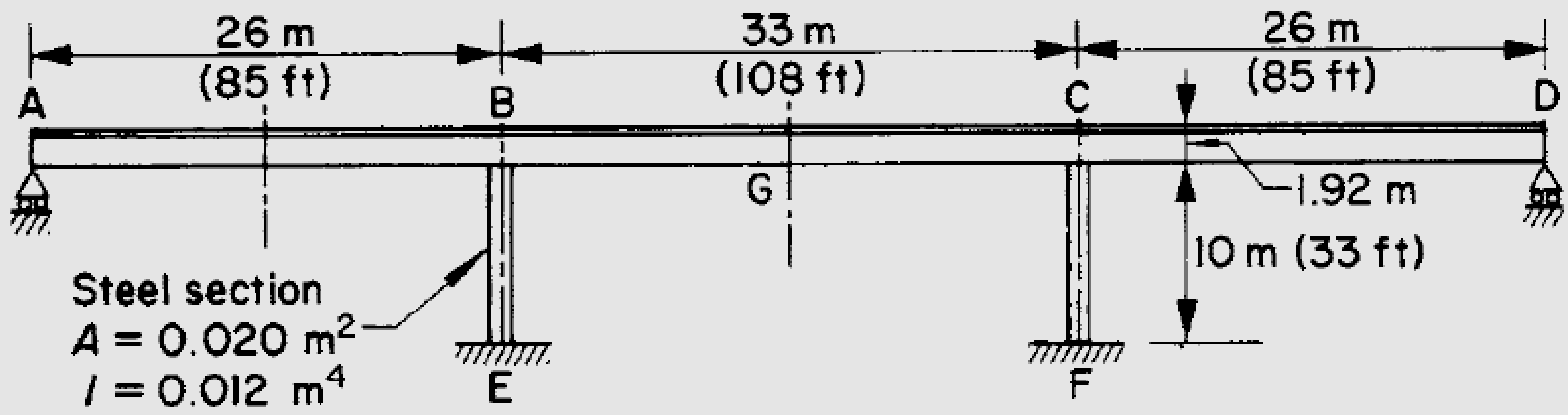
$$p = -\frac{d}{dx}(\Delta N) \tag{5.15}$$

$$q = -\frac{d^2}{dx^2}(\Delta M) \tag{5.16}$$

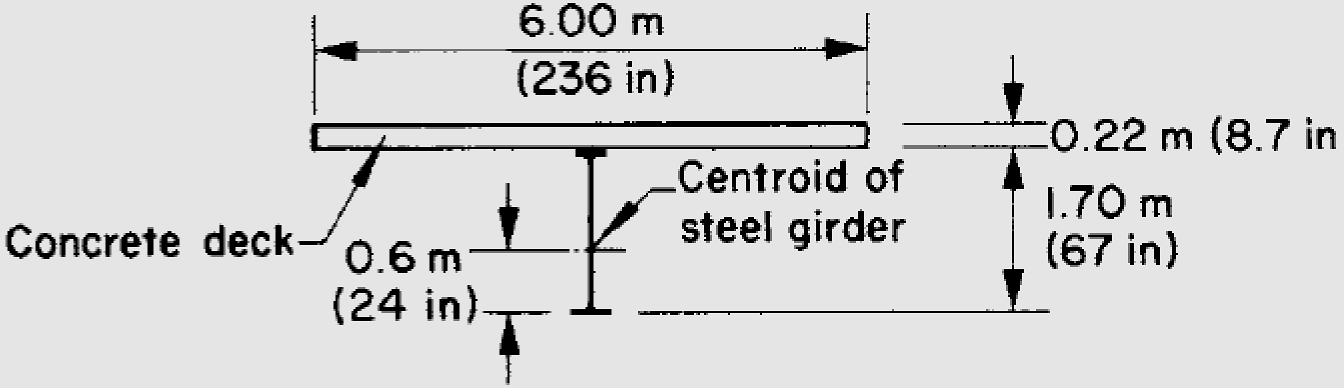
Example 5.2 Steel bridge frame with concrete deck: effects of shrinkage

The bridge frame in Fig. 5.8(a) has a composite section for part AD (Fig. 5.8(b)) and a steel section for the columns BE and CF. It is required to find the changes in the reactions and in the stress distribution in the cross-section at G due to uniform shrinkage of deck slab occurring during a period t_0 to t_1.

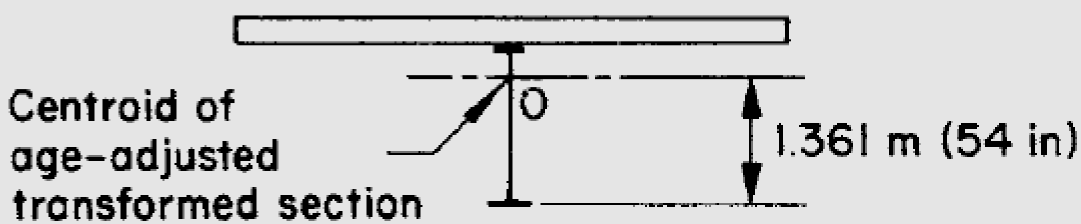
The cross-section properties of members are: for columns BE and CF, area = 20 000 mm^2 (31 in^2) and moment of inertia about an axis through centroid = 0.012 m^4 (29 000 in^4); for part AD, the steel



(a)


6.00 m
(236 in)
0.22 m (8.7 in)
Centroid of
steel girder
1.70 m
(67 in)
Concrete deck
0.6 m
(24 in)

(b)


Centroid of
age-adjusted
transformed section
O
1.361 m (54 in)

(c)

cross-section area = 39 000 mm^2 (60 in^2) and moment of inertia about its centroid = 0.015 m^4 (36 000 in^4).

The material properties are:

$$E_c(t_0) = 30\,\text{GPa}\ (4350\,\text{ksi}) \qquad E_s = 200\,\text{GPa}\ (29\,000\,\text{ksi})$$

$$\varphi(t_1, t_0) = 2.5 \qquad \chi(t_1, t_0) = 0.8 \qquad \varepsilon_{cs}(t_1, t_0) = -270 \times 10^{-6}$$

The following cross-section properties for part AD are needed in the analysis:

Age-adjusted transformed section

$$\bar{E}_c(t_1, t_0) = \frac{30 \times 10^9}{1 + 0.8 \times 2.5} = 10\,\text{GPa} \qquad \bar{\alpha}(t_1, t_0) = \frac{200}{10} = 20.$$

The age-adjusted transformed section is composed of $A_c = 1.32\,\text{m}^2$ plus $\bar{\alpha}A_s = 20 \times 0.039 = 0.780\,\text{m}^2$. A reference point O is chosen at the centroid of the age-adjusted transformed section at 1.361 m above bottom fibre (Fig. 5.8(c)). Using $E_{ref} = \bar{E}_c = 10\,\text{GPa}$, the properties of the age-adjusted transformed section are:

$$\bar{A} = 2.10\,\text{m}^2 \qquad \bar{B} = 0 \qquad \bar{I} = 1.0232\,\text{m}^4.$$

Transformed section at t_0

$$E_c(t_0) = 30\,\text{GPa} \qquad \alpha(t_0) = \frac{200}{30} = 6.667 \qquad E_{\text{ref}} = E_c(t_0)$$

Area and its first and second moment about an axis through the reference point O:

$$A = 1.58\,\text{m}^2 \qquad B = -0.3947\,\text{m}^3 \qquad I = 0.5221\,\text{m}^4.$$

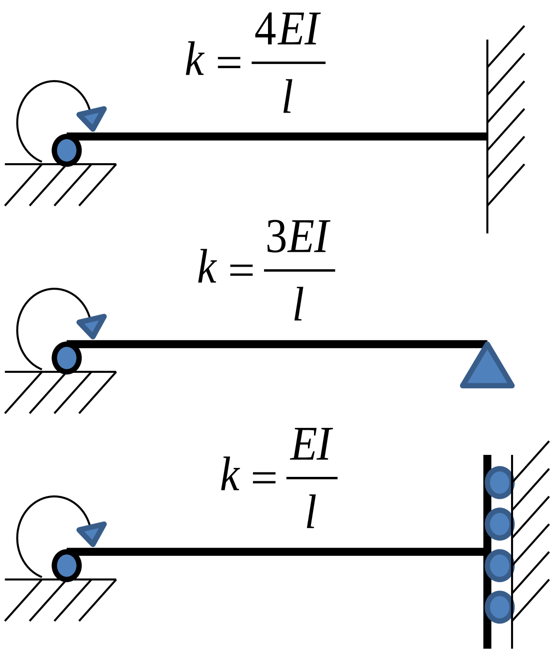
The centroid of this transformed section is 1.611 m above the bottom fibre and moment of inertia about an axis through the centroid is $0.4234\,\text{m}^4$.

Concrete deck slab Area, first and second moment of the concrete deck slab alone about an axis though the reference point O:

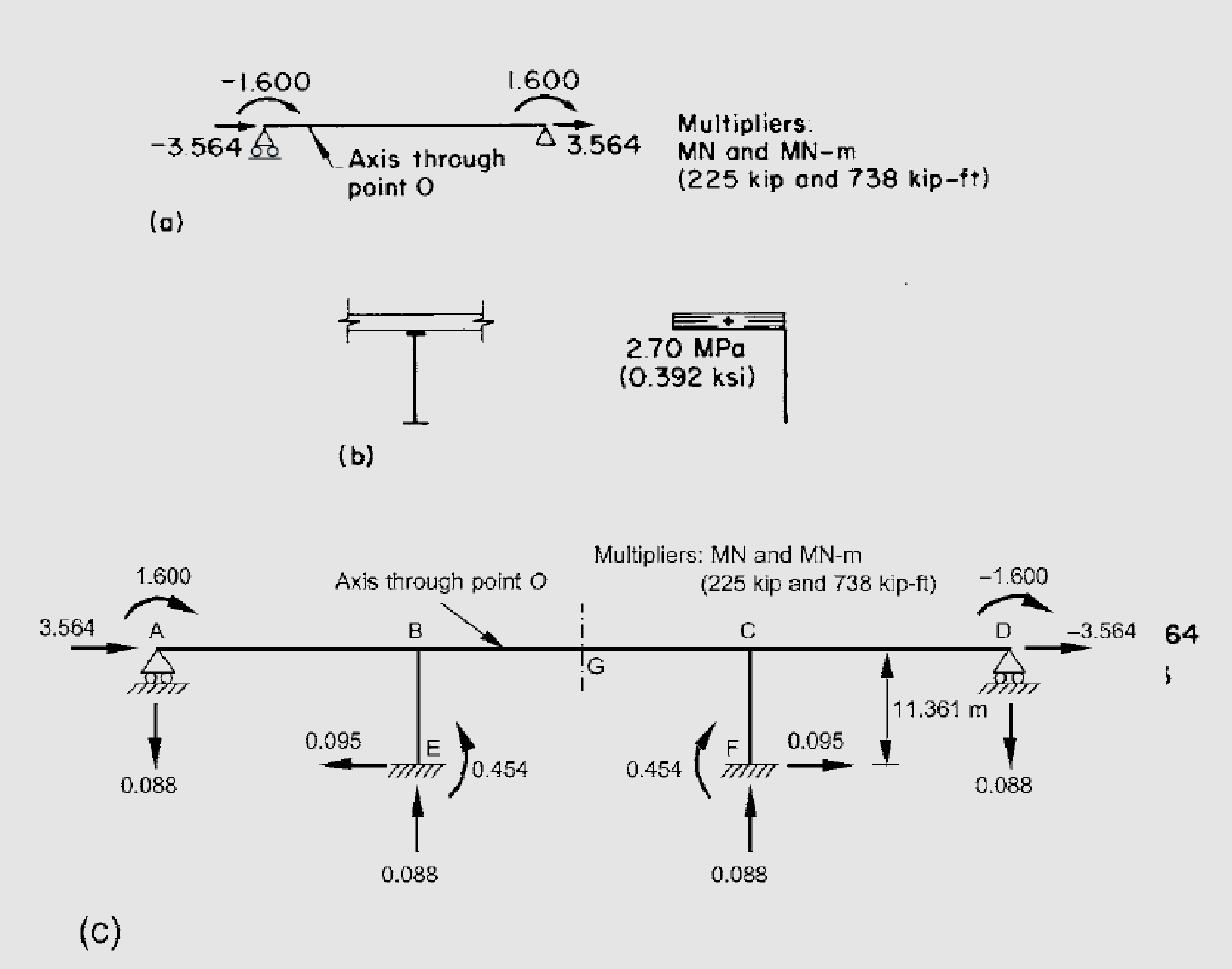
$$A_c = 1.32\,\text{m}^2 \qquad B_c = -0.5927\,\text{m}^3 \qquad I_c = 0.2714\,\text{m}^4.$$

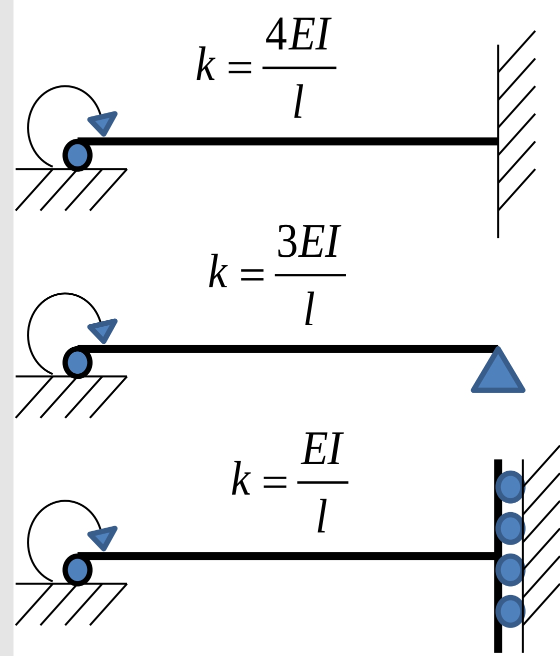
The resultant of stresses if shrinkage were restrained at all sections of AD (Equation (2.43));

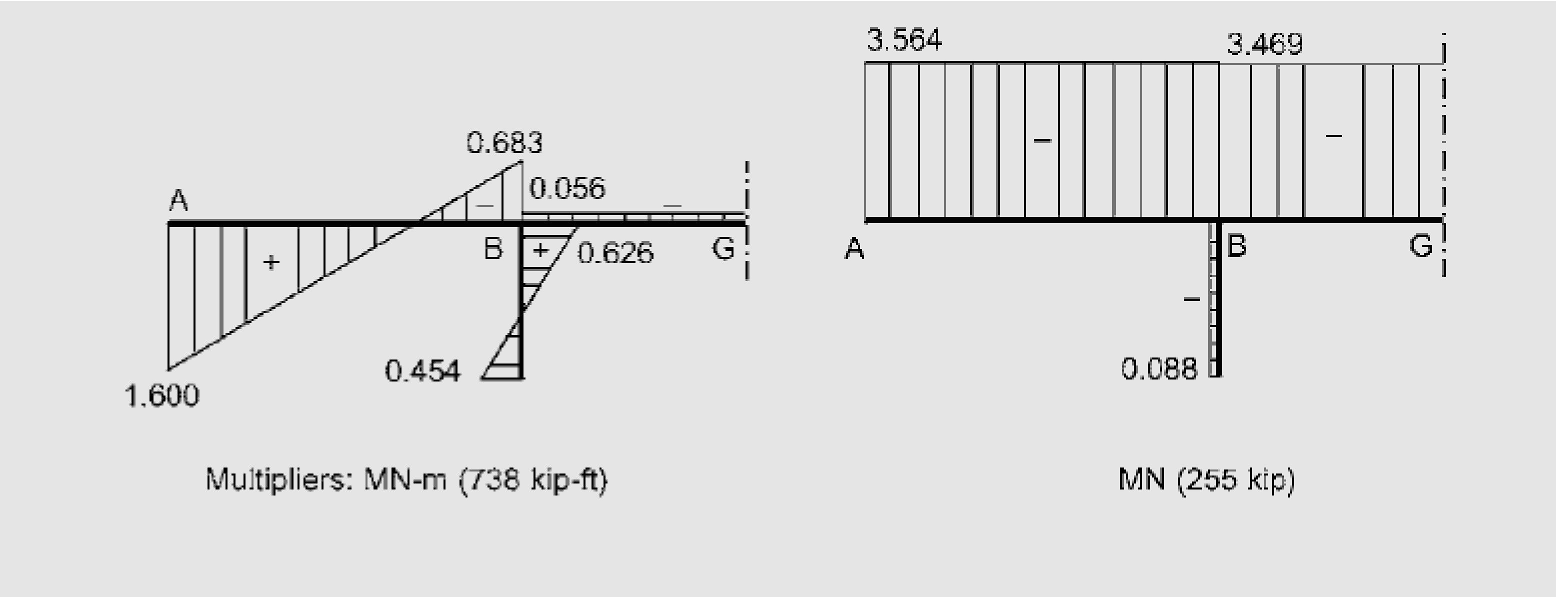
$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix} = -10 \times 10^{9}(-270 \times 10^{-6}) \begin{Bmatrix} 1.32 \\ -0.5927 \end{Bmatrix} = \begin{Bmatrix} 3.564 \times 10^{6} \text{ N} \\ -1.600 \times 10^{6} \text{ N-m} \end{Bmatrix}$$



$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix}_{shrinkage} = -\sum_{i=1}^{m} \left\{ \bar{E}_c \begin{bmatrix} A_c & B_c \\ B_c & I_c \end{bmatrix} \begin{Bmatrix} \varepsilon_{cs} \\ 0 \end{Bmatrix} \right\}_i$$


−1.600
1.600
−3.564
3.564
Axis through point O
Multipliers:
MN and MN−m
(225 kip and 738 kip−ft)
(a)
2.70 MPa
(0.392 ksi)
(b)
Multipliers: MN and MN-m
(225 kip and 738 kip-ft)
Axis through point O
1.600
−1.600
3.564
−3.564
A
B
C
D
G
E
F
11.361 m
0.095
0.095
0.454
0.454
0.088
0.088
0.088
0.088
(c)


$k = \frac{4EI}{l}$
$k = \frac{3EI}{l}$
$k = \frac{EI}{l}$



0.683
0.056
A
B
0.626
G
1.600
0.454
Multipliers: MN-m (738 kip-ft)
3.564
3.469
A
B
G
0.088
MN (255 kip)

(d)

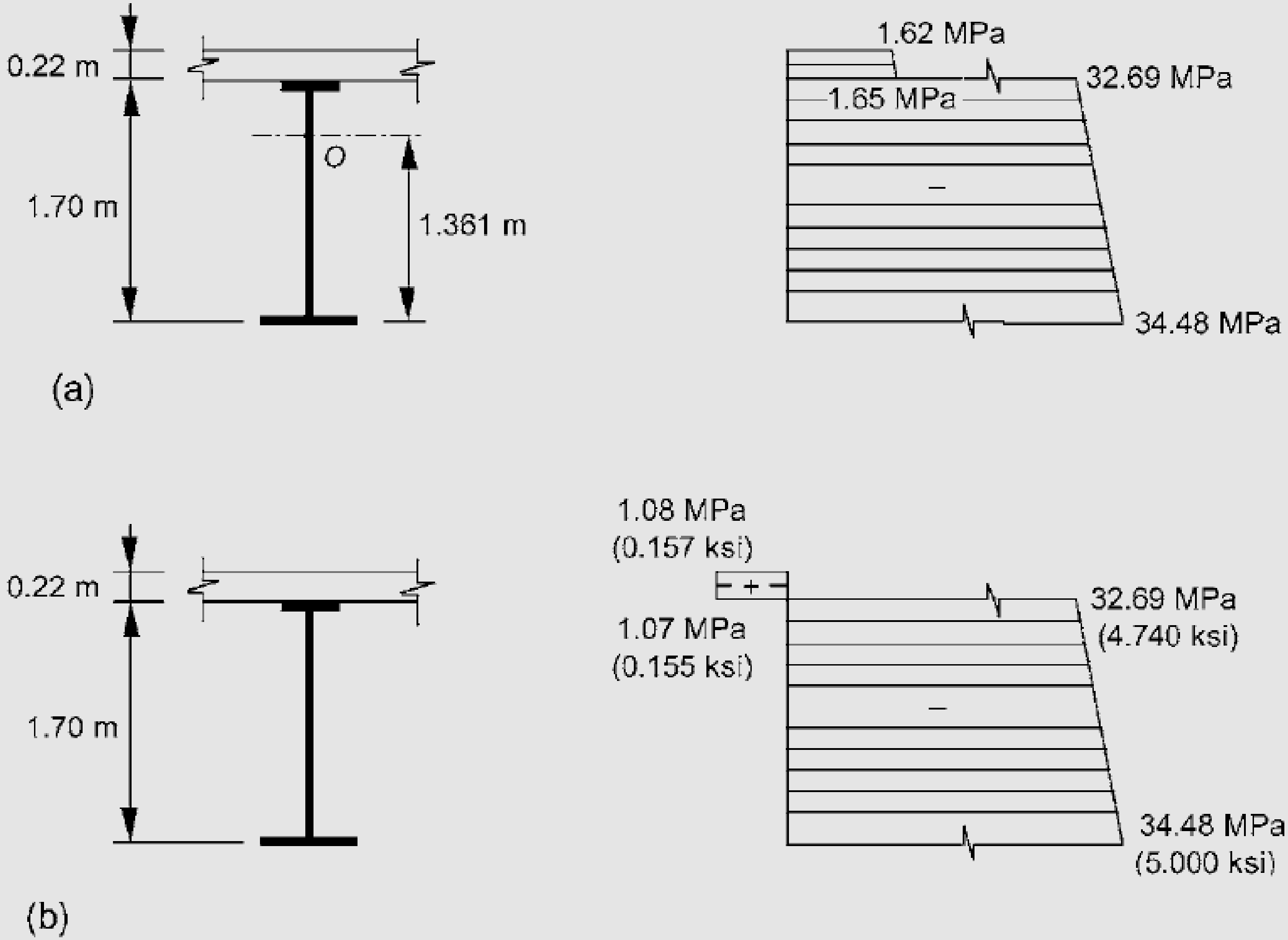
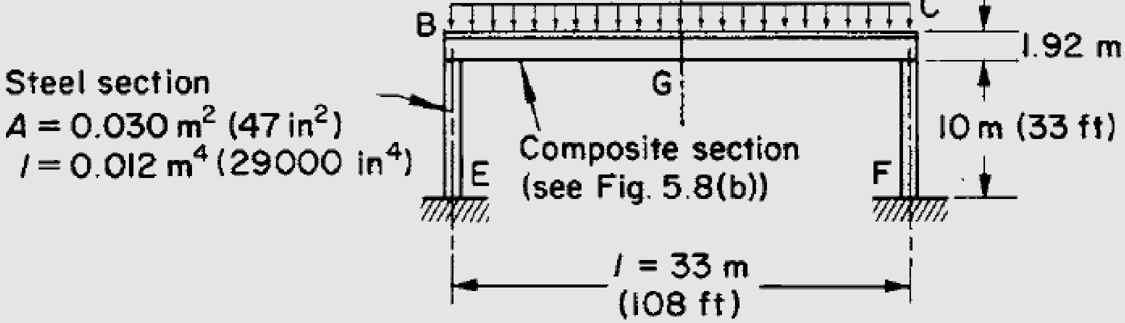


Figure 5.10 Analysis of stresses at section G due to shrinkage in a composite continuous frame of Example 5.2; (a) stress distribution due to $N = -3.469$ MN at O and $M = -0.056$ MN-m applied on age-adjusted transformed section; (b) total stress due to shrinkage (superposition of Figs 5.9(b) and 5.10(a)).

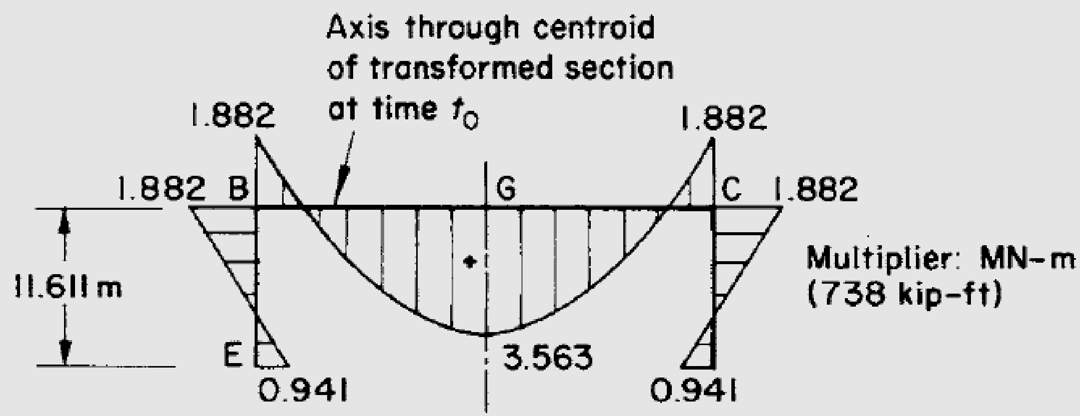
Example 5.3 Composite frame: effects of creep

The frame in Fig. 5.11(a) has a composite cross-section for part BC and a steel section for the columns BE and CF. The dimensions of the cross-sections and the properties of the materials are the same as for member BC in Example 5.2; see Fig. 5.8. The properties of the cross-sections of the columns BE and CF are given in Fig. 5.11(a). At time t_0, a uniformly distributed downward load of intensity $q = 40\,\text{kN/m}$ is applied on BC and sustained to a later time t_1. It is required to find the change in the bending moment due to creep during the period t_0 to t_1. Use the same creep and aging coefficients as in Example 5.2. Also find the stress distribution and the deflection at section G at time t_1.

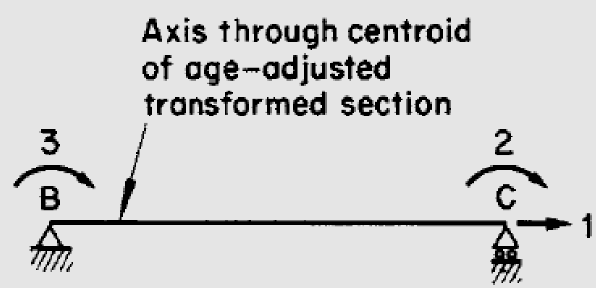
The properties of the cross-section for member BC are the same as for part AD of the frame of Example 5.2, and thus this part of the calculation is not discussed here.


B
C
1.92 m
Steel section
$A = 0.030\ m^2\ (47\ in^2)$
$I = 0.012\ m^4\ (29000\ in^4)$
G
Composite section
(see Fig. 5.8(b))
10 m (33 ft)
E
F
l = 33 m
(108 ft)

(a)


Axis through centroid
of transformed section
at time t_0
1.882
1.882
1.882 B
G
C 1.882
11.611 m
Multiplier: MN-m
(738 kip-ft)
E
3.563
0.941
0.941

(b)


Axis through centroid
of age-adjusted
transformed section
3
2
B
C
1

(c)

$$\{\Delta F\} = \frac{10 \times 10^9}{33} \begin{bmatrix} 2.10 & 0 & 0 \\ 0 & 4(1.0232) & 2(1.0232) \\ 0 & 2(1.0232) & 4(1.0232) \end{bmatrix} \begin{Bmatrix} 1691 \\ 807 \\ -807 \end{Bmatrix} 10^{-6}$$

$$= \begin{Bmatrix} 1.0761\,\text{MN} \\ 0.5004\,\text{MN-m} \\ -0.5004\,\text{MN-m} \end{Bmatrix}$$

Table 5.1 Instantaneous axial strain and curvature at t_0, immediately after application of the load q (Example 5.3, Fig. 5.11)

Member	Section	Properties of transformed[1] section at age t_0 ($E_{ref} = E_c(t_0) = 30$ GPa)			Internal forces introduced at t_0		Axial strain and curvature at t_0 (Equation (2.32))		Properties of concrete area			Deflection at G (Equation (C.8))
		A	B	I	N	M	$\varepsilon_O(t_0)$	$\psi(t_0)$	A_c	B_c	I_c	$D(t_0)$
	B	1.58	−0.3947	0.5221	−0.2431	−1.821	−42.1	−148.1	1.32	−0.5927	0.2714	
BC	G	1.58	−0.3947	0.5221	−0.2431	3.624	64.9	280.5	1.32	−0.5927	0.2714	28.46
	C	1.58	−0.3947	0.5221	−0.2431	−1.821	−42.1	−148.1	1.32	−0.5927	0.2714	
Multiplier		m^2	m^3	m^4	10^6 N	10^6 N-m	10^{-6}	10^{-6} m^{-1}	m^2	m^3	m^4	10^{-3} m

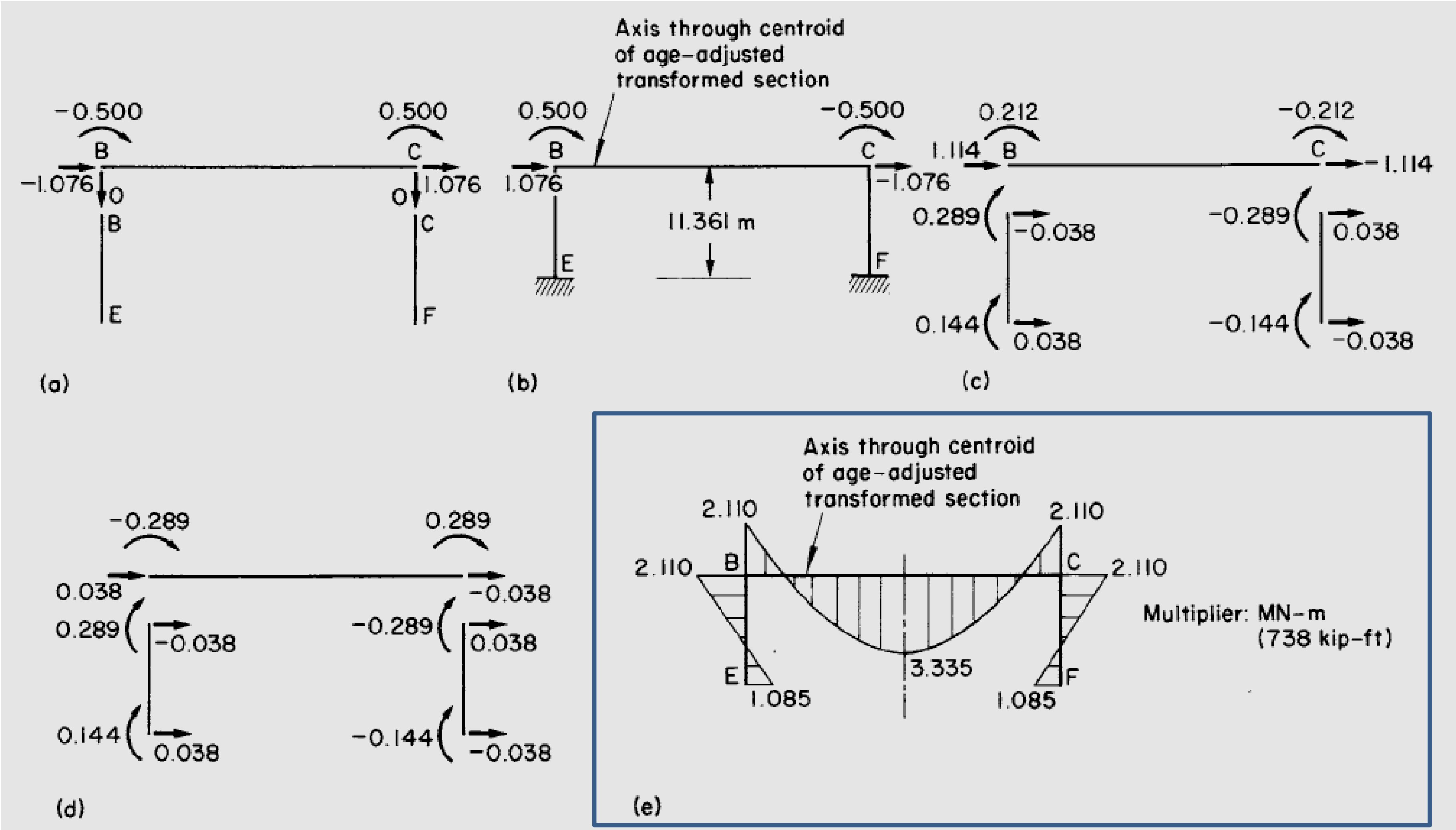
[1] The reference point O is at the centroid of age-adjusted transformed section (Fig. 5.8(c))

Table 5.2 Changes in axial strain and in curvature and corresponding elongation and end rotations of the released structure in Fig. 5.11(c)

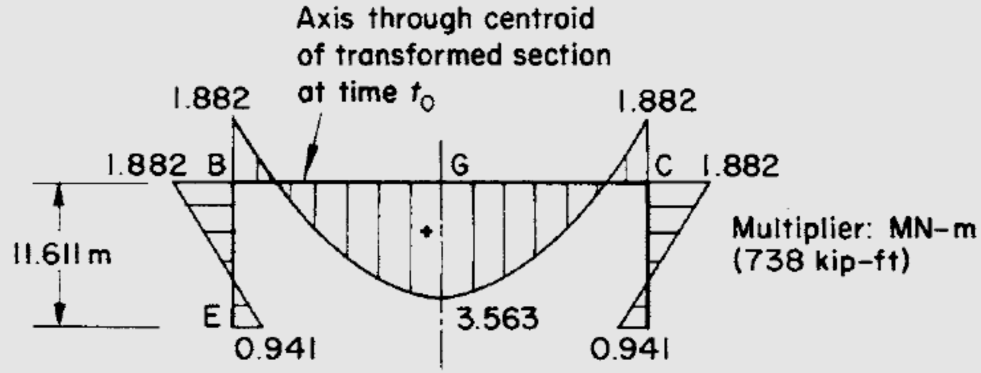
		Internal forces to restrain creep (Equation (2.42))		*Properties of age-adjusted transformed section ($E_{ref} = \bar{E}_c(t_1, t_0) = 10$ GPa)*			*Changes in axial strain and in curvature (Equation (2.40))*		*Changes in displacements at the coordinates in Fig. 5.11(c) (Equations (C.5–7))*			*Change in deflection at G (Equation (C.8))*
Member	*Section*	ΔN	ΔM	$\bar{A}$	$\bar{B}$	$\bar{I}$	$\Delta\varepsilon_O$	$\Delta\psi$	ΔD_1	ΔD_2	ΔD_3	ΔD
	B	−0.8052	0.3810	2.10	0	1.0232	38.3	−37.2				
BC	G	2.015	−0.9415	2.10	0	1.0232	−96.0	92.0	−1691	−807	807	9.59
	C	−0.8052	0.3810	2.10	0	1.0232	38.3	−37.2				
Multipliers		10^6 N	10^6 N-m	m^2	m^3	m^4	10^{-6}	10^{-6} m^{-1}	10^{-6} m	10^{-6} radian	10^{-6} radian	10^{-3} m

$$\{\Delta F\} = \frac{10 \times 10^9}{33}\begin{bmatrix} 2.10 & 0 & 0 \\ 0 & 4(1.0232) & 2(1.0232) \\ 0 & 2(1.0232) & 4(1.0232) \end{bmatrix}\begin{Bmatrix} 1691 \\ 807 \\ -807 \end{Bmatrix}10^{-6}$$

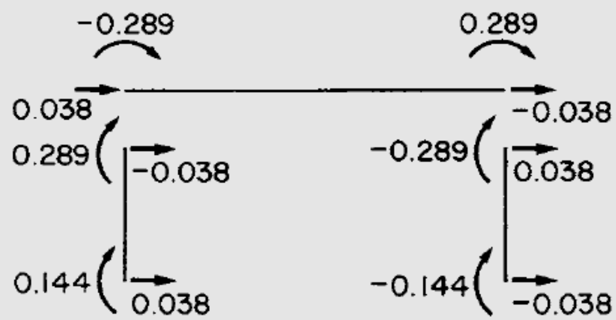
$$= \begin{Bmatrix} 1.0761\ \text{MN} \\ 0.5004\ \text{MN-m} \\ -0.5004\ \text{MN-m} \end{Bmatrix}$$



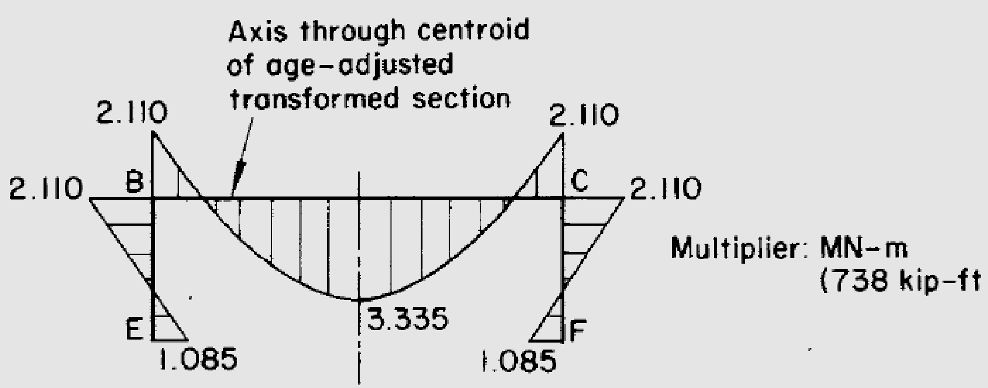
Total moment= b) of Fig 511+d) of 5.12


Axis through centroid
of transformed section
at time t_0
1.882
1.882
1.882
B
G
C
1.882
11.611 m
Multiplier: MN-m
(738 kip-ft)
E
3.563
0.941
0.941

(b)


−0.289
0.289
0.038
−0.038
0.289
−0.038
−0.289
0.038
0.144
0.038
−0.144
−0.038

(d)


Axis through centroid
of age-adjusted
transformed section
2.110
2.110
2.110
B
C
2.110
Multiplier: MN-m
(738 kip-ft)
3.335
E
F
1.085
1.085