chap. 7. Torsion

∘ Example of structural components which are designed to carry torsional loads

- power transmission drive shafts … solid or thin-walled circular cross-section
- A/C wing … needs to carry the bending and torsional moments generated by the aerodynamic forces

"bar" rather than "beam"

7.1 Torsion of circular cylinders

∘ Fig. 7.1 … infinitely long, homogeneous, solid or hollow circular cylinder subjected to end torques, Q_1

- 2 types of symmetries

  ① cylindrical symmetry about $\bar{i}_1$

  ② symmetric w.r.t. any plane, P, passing through axis $\bar{i}_1$ (Fig. 7.2)

  … shear stress due to Q_1 must be of constant magnitude along circle C, and tangent to it → loading is antisymm. w.r.t. P

- axial displacement at A and B, u_1^A and u_1^B

$$\left.\begin{array}{l} ① \rightarrow u_1^A = u_1^B \\ ② \rightarrow u_1^A = -u_1^B \end{array}\right\} \rightarrow u_1^A = u_1^B = 0$$

→ axial displacement must vanish.

"the cross-section does not warp out-of plane."

- each x-s "rotates about its own center like a rigid disk."

7.1.1 Kinematic description

- rotation angle Φ_1 … rigid body rotation of each x-s (Fig. 7.3)
- sectional in-plane displacement field

$$u_2(x_1, r, \alpha) = -r\,\Phi_1(x_1)\sin\alpha, \quad u_3(x_1, r, \alpha) = r\,\Phi_1(x_1)\cos\alpha \qquad (7.1)$$

- out-of-plane displacement field

$$u_1(x_1, x_2, x_3) = 0 \qquad (7.2)$$

Eq. (7.1) →

$$\begin{aligned} u_2(\quad " \quad) &= -x_3\,\Phi_1(x_1) \\ u_3(\quad " \quad) &= x_2\,\Phi_1(x_1) \end{aligned} \qquad (7.3)$$

- strain field

$$\epsilon_1 = 0, \quad \epsilon_2 = 0, \quad \epsilon_3 = 0, \qquad \gamma_{23} = 0 \qquad (7.4), (7.5)$$

$$\gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = -x_3 \kappa_1(x_1), \qquad \gamma_{13} = x_2 \kappa_1(x_1) \qquad (7.6)$$

$$\kappa_1(x_1) = \frac{d\Phi_1}{dx_1} \quad \text{"sectional twist rate"} \qquad (7.7)$$

- To visualize the strain field, describe them in the polar coord. (r, α)

→ γ_{r1} and $\gamma_{\alpha 1}$, or simply γ_r and γ_α

- transformation between the Cartesian and the polar strain components

$$\gamma_r = \gamma_{12} \cos\alpha + \gamma_{13} \sin\alpha, \quad \gamma_\alpha = -\gamma_{12} \sin\alpha + \gamma_{13} \cos\alpha \qquad (7.8)$$

Eq. (7.6) → (7.8) : $\gamma_r(x_1, r, \alpha) = 0, \quad \gamma_\alpha(x_1, r, \alpha) = r\kappa_1(x_1) \qquad (7.9)$

circumferential shearing strain (Fig. 7.4)

7.1.2 The stress field

- the only non-vanishing stress components

$$\tau_{12} = -G x_3 \kappa_1(x_1), \qquad \tau_{13} = G x_2 \kappa_1(x_1) \qquad (7.10)$$

using polar coord.,

$$\tau_r(x_1, r, \alpha) = 0, \qquad \tau_\alpha(x_1, r, \alpha) = G r \kappa_1(x_1) \qquad (7.11)$$

radial — circumferential stress components

- distribution of circumferential shear stress (Fig. 7.5)
  1. circumferential direction exists only, radial direction vanishes
  2. varies linearly along the radial direction

7.1.3 Sectional constitutive law

- torque acting on the x-s at a given span-wise location

$$M_1(x_1) = \int_A \tau_\alpha r \, dA \qquad (7.12)$$

Eq. (7.11) → $M_1(x_1) = \int_A G r^2 \kappa_1(x_1) dA = \left[\int_A G r^2 dA\right] \kappa_1(x_1) = H_{11} \kappa_1(x_1) \qquad (7.13)$

"torsional stiffness" $H_{11} = \int_A G r^2 dA$ "for circular x-s only" (7.14)

... constitutive → for the torsional behavior of the beam

- If homogeneous material,

$$H_{11} = GJ, \quad \text{where } J = \int_A r^2 dA : \text{"area polar moment"}$$

for circular x-s only

7.1.4 Equilibrium eqn.s

- infinitesimal slice of the cylinder of length dx_1 (Fig. 7.6)
- torsional equilibrium eqn.

$$\frac{dM_1}{dx_1} = -q_1 \tag{7.15}$$

7.1.5 Governing eqn.

- Eq (7.13) → (7.15) and recalling Eq (7.7)

$$\frac{d}{dx_1}\left[H_{11}\frac{d\Phi_1}{dx_1}\right] = -q_1 \tag{7.16}$$

- B.C. ① fixed (clamped) ... $\Phi_1 = 0$

② free (unloaded) ... $M_1 = 0 \rightarrow \kappa_1 = \frac{d\Phi_1}{dx_1} = 0$

③ subjected to a concentrated torque Q_1 ... $M_1 = Q_1 \rightarrow H_{11}\frac{d\Phi_1}{dx_1} = Q_1$

7.1.6 The torsional stiffness

- if homogeneous material

$$H_{11} = G\int_0^{2\pi}\int_0^R r^2 dr d\alpha = \frac{\pi}{2}GR^4 \tag{7.17}$$

- for a circular tube

$$H_{11} = G\int_0^{2\pi}\int_{R_i}^{R_o} r^2 dr d\alpha = \frac{\pi}{2}G(R_o^4 - R_i^4) \tag{7.18}$$

- for a thin-walled circular tube, mean radius $R_m = (R_o+R_i)/2$

$$H_{11} = \frac{\pi}{2}G(R_o^2+R_i^2)(R_o+R_i)(R_o-R_i) \simeq 2\pi G R_m^3 t \tag{7.19}$$

- thin-walled circular tube consisting of N concentric layers (Fig. 7.7)

$$H_{11} = \frac{\pi}{2}\sum_{i=1}^{N} G^{[i]}\left[(R^{[i+1]})^4 - (R^{[i]})^4\right]$$

$$= 2\pi\sum_{i=1}^{N} G^{[i]} t^{[i]}\left(\frac{R^{[i+1]}+R^{[i]}}{2}\right)^3 \tag{7.20}$$

"weighted average" of the shear moduli of the various layers

7.1.7 Measuring the torsional stiffness

- deformation of the test section ... measured by the chevron strain gauge (Fig. 7.8), $\gamma_{12} = e_{+45} - e_{-45}$,
  @ $r = R$, $\kappa_1 = (e_{45} - e_{-45})/R$
  - slope of θ_{3i} vs. κ_{1i} curve → torsional stiffness
    ... valid as long as the cylindrical symmetry is maintained.

7.1.8. The shear stress distribution

- local circumferential stress ... Eq. (7.11) → (7.13)

$$\tau_\alpha = G \frac{M_1(x_1)}{H_{11}} r \tag{7.21}$$

... increases linealy from zero at the center to a max. value at the outer radius (Fig. 7.5)

- concentric layers of distinct material (Fig. 7.7) $\tau_\alpha^{(i)} = G^{(i)} \frac{M_1}{H_{11}} r$
  ... within each layer, still linear distribution, but discontinuities at the interface
- Max. shear stress for homogeneous material

$$\tau_\alpha^{max} = \frac{2M_1(x_1)}{\pi R^3} \tag{7.22}$$

- strength criterion

$$\frac{GR}{H_{11}} |M_1^{max}| \leq \tau_{allow} \tag{7.26}$$

7.1.9. Rational design of cylinders under torsion

- material near the center of the cylinder is not used efficiently since the shear stress becomes small.
  → thin-walled tube is a far more efficient design
- 2 thin-walled tube of the same material, mass per unit span, but different mean radii R_m and R_m'

① torsional stiffness

$$\frac{H_{11}}{H_{11}'} = \frac{(\mu/\rho)GR_m^2}{(\mu/\rho)GR_m'^2} = \left(\frac{R_m}{R_m'}\right)^2 \tag{7.28}$$

② shear stress under the same torque

$$\frac{\tau_{\alpha}}{\tau_{\alpha}'} = \frac{GM_1 R_m / H_{11}}{GM_1 R_m' / H_{11}'} = \frac{R_m H_{11}'}{R_m' H_{11}} = \frac{R_m'}{R_m} \qquad (7.29)$$

... inversely proportional to the mean radius

- large mean radius --- high H_{11}, lower max τ

But, in practice, limits ← "torsional buckling"

7.2 Torsion combined with axial force and bending moments

- What is the proper strength criterion to be used when both axial and shear stresses are acting simultaneously?

i) Propeller shaft under torsion and thrust

- torque M_1 and thrust N_1 →

$$\tau = \frac{2M_1}{\pi R^3}, \quad \sigma = \frac{N_1}{\pi R^2} \qquad (7.30)$$

- Tresca's criterion, Eq. (2.31) → most stringent condition among 3

$$\left(\frac{N_1}{\pi R^2 \sigma_y}\right)^2 + 16\left(\frac{M_1}{\pi R^3 \sigma_y}\right)^2 = 1 \quad \text{--- ellipse in Fig. 7.}$$

- von Mises' criterion, Eq. (2.36)

$$\left(\frac{N_1}{\pi R^2 \sigma_y}\right)^2 + \left(\frac{M_1}{\pi R^3 \sigma_y}\right)^2 \le 1 \quad \text{--- ''}$$

ii) Shaft under torsion and bending

- bending moment M_3 and torque M_1 →

$$\sigma = \frac{4M_3 r}{\pi R^4}, \quad \tau = \frac{2M_1 r}{\pi R^3} \qquad (7.31)$$

- Tresca's criterion

$$16\left(\frac{M_3}{\pi R^3 \sigma_y}\right)^2 + 16\left(\frac{M_1}{\pi R^3 \sigma_y}\right)^2 = 1 \quad \text{--- Fig. 7.11}$$

- von Mises' criterion

$$16\left(\frac{M_3}{\pi R^3 \sigma_y}\right)^2 + 12\left(\frac{M_1}{\pi R^3 \sigma_y}\right)^2 \le 1 \quad \text{--- ''}$$

7.3 Torsion of bars with arbitrary cross-sections

7.3.1 Introduction

- circular symmetry of the problem is not maintained any more.
  - Fig. 7.14 ... at any point along the edge of the bar's section, the shear stress must be tangent to the edge. → $\tau_{13} = 0$.
    But, non-zero τ_{13} is required from the circular symmetry
  - fewer symmetries than the circular x-s has.
  - Fig. 7.15 -- symm. w.r.t planes $(\bar{i}_1, \bar{i}_2)$ and $(\bar{i}_1, \bar{i}_3)$, but no circular symm.
    torsional loading and the resulting sol. -- anti symm.
    w.r.t. $(\bar{i}_1, \bar{i}_2)$ → $u_1^A = -u_1^B$, $u_1^C = -u_1^D$
    " $(\bar{i}_1, \bar{i}_3)$ → $u_1^A = -u_1^D$, $u_1^B = -u_1^C$ } → x-s will warp out-of-plane

7.3.2 Saint-Venant's sol.

i) Kinematic description

- each x-s rotates like a rigid body, and warps out-of-plane
  → assumed displacement field

$$u_1(x_1, x_2, x_3) = \Psi(x_2, x_3)\,\kappa_1(x_1)$$
$$u_2(\quad " \quad) = -x_3\,\Phi_1(x_1), \quad u_3(\quad " \quad) = x_2\,\Phi_1(x_1) \tag{7.32}$$

$\Psi(x_2, x_3)$: unknown warping function, will be determined by enforcing equilibrium eqns for the resulting stress field

ii) The strain field

- Eq. (7.32) → Eqs. (1.63) and (1.71)

$$\epsilon_1 = \Psi(x_2, x_3)\frac{d\kappa_1}{dx_1} = 0 \quad \leftarrow \text{due to "uniform torsion"}$$
$$\epsilon_2 = 0, \quad \epsilon_3 = 0, \quad \gamma_{23} = 0 \tag{7.33}$$
$$\gamma_{12} = \left(\frac{\partial \Psi}{\partial x_2} - x_3\right)\kappa_1, \quad \gamma_{13} = \left(\frac{\partial \Psi}{\partial x_3} + x_2\right)\kappa_1$$

iii) The stress field

$$\sigma_1 = 0, \quad \sigma_2 = 0, \quad \sigma_3 = 0, \quad \tau_{23} = 0 \qquad (7.34)$$

$$\tau_{12} = G\kappa_1\left(\frac{\partial \Psi}{\partial x_2} - x_3\right), \quad \tau_{13} = G\kappa_1\left(\frac{\partial \Psi}{\partial x_3} + x_2\right)$$

iv) Equilibrium eqn.s

- stress field must satisfy the general equilibrium eqn.s, Eq. (1.4) at all points of the section. Neglecting body forces, the remaining eqn. is

$$\frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3} = 0 \qquad (7.35)$$

- Eq. (7.34c) → (7.35) :

$$\frac{\partial^2 \Psi}{\partial x_2^2} + \frac{\partial^2 \Psi}{\partial x_3^2} = 0 \qquad (7.36)$$

... the warping fn must satisfy the PDE at all points of the X-S.

- B.C. ... satisfaction of the equilibrium eqns along the outer contour of the sec (Fig. 7.16)

- along C, according to Fig. 7.14, $\tau_n = 0$ (7.37)

τ_s does not necessarily vanish.

in terms of Cartesian components,

$$\tau_n = \tau_{12}\sin\beta + \tau_{13}\cos\beta = \tau_{12}\left(\frac{dx_3}{ds}\right) + \tau_{13}\left(-\frac{dx_2}{ds}\right) = 0 \qquad (7.3$$

- Eq. (7.34c) → (7.38) :

$$\left(\frac{\partial \Psi}{\partial x_2} - x_3\right)\frac{dx_3}{ds} - \left(\frac{\partial \Psi}{\partial x_3} + x_2\right)\frac{dx_2}{ds} = 0 \qquad (7.39$$

Eq. (7.36) ... Laplace's eqn.

" (7.39) ... rather complicated B.C.

v) Prandtl's stress function

- Alternative formulation leading to simpler B.C. ... stress function, $\phi(x_2, x_3)$

$$\tau_{12} = \frac{\partial \phi}{\partial x_3}, \quad \tau_{13} = -\frac{\partial \phi}{\partial x_2} \qquad (7.41)$$

... automatically satisfies the local equilibrium eqn., Eq. (7.35)

- by comparing Eq. (7.34c) and (7.41)

$$\tau_{12} = G\kappa_1\left(\frac{\partial \Psi}{\partial x_2} - x_3\right) = \frac{\partial \phi}{\partial x_3}, \quad \tau_{13} = G\kappa_1\left(\frac{\partial \Psi}{\partial x_3} + x_2\right) = -\frac{\partial \phi}{\partial x_2} \quad (7.4$$

$$\underbrace{\frac{\partial}{\partial x_3}[\ ] \qquad\qquad \frac{\partial}{\partial x_2}[\ ]}_{(-)}$$

$$\frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} = -2G\kappa_1 \quad (7.43)$$

- B.C. … from Eqs. (7.38), (7.41)

$$\tau_n = \frac{\partial \phi}{\partial x_3}\frac{dx_3}{ds} + \frac{\partial \phi}{\partial x_2}\frac{dx_2}{ds} = \frac{d\phi}{ds} = 0 \quad (7.44$$

const. value may be chosen to vanish

… constant ϕ along C

Eq. (7.43) … Poisson's eqn.

" (7.44) … much simpler B.C.

vi) Sectional equilibrium

- Global equilibrium of the section
  - resultant shear forces

$$V_2 = \int_A \tau_{12}\, dA = \int_{x_2}\int_{x_3} \frac{\partial \phi}{\partial x_3}\, dx_2\, dx_3 = \int_{x_2}\left[\int_{x_3} \frac{\partial \phi}{\partial x_3}\, dx_3\right] dx_2 = 0$$

$$V_3 = \ldots = 0$$

… no shear forces are applied.

  - total torque acting on the section

$$M_1 = \int_A (x_2\tau_{13} - x_3\tau_{12})\, dA = \int_A \left(-x_2\frac{\partial \phi}{\partial x_2} - x_3\frac{\partial \phi}{\partial x_3}\right) dA \quad (7.46$$

Integrating by parts

$$M_1 = 2\int_A \phi\, dA - \int_{x_3} \cancel{[x_2\phi]_{x_2}}\, dx_3 - \int_{x_2} \cancel{[x_3\phi]_{x_3}}\, dx_2 \quad (7.47$$

… applied torque = 2 X "volume" under the stress function,

only valid for solid x-s bounded by a single curve.

otherwise, we Eq. (7.46)

- Summary --- bar of arbitrary x-s subjected to uniform torsion
  - stress distribution -- $\begin{cases} \text{warping fn (Eq. (7.40))} \\ \text{stress fn (Eq. (7.45))} \end{cases}$
  - stress field -- Eqs. (7.34c) or (7.41)

    --- exact sol. although the displacement field is assumed as in Eq. (7.

7.3.3. Saint-Venant's sol. for a retangular x-s

- 2 sol. $\begin{cases} \text{approximate sol. based on the co-location approach} \\ \text{exact} \quad \text{"} \quad \text{Fourier series expansion} \end{cases}$

i) Approximate sol.

- rectangular x-s of width a, height b (Fig. 7.21)
  - assumed stress function $\phi(\eta,\zeta) = C_0(\eta^2 - \frac{1}{4})(\zeta^2 - \frac{1}{4})$

$$\eta = \frac{x_2}{a}, \quad \zeta = \frac{x_3}{b}$$

--> $\phi(\eta = \pm\frac{1}{2}, \zeta) = 0, \quad \phi(\eta, \zeta = \pm\frac{1}{2}) = 0 \rightarrow \phi = 0$ along the edge

--> PDE, Eq. (7.43) ... $2C_0(\zeta^2 - \frac{1}{4})\frac{1}{a^2} + 2C_0(\eta^2 - \frac{1}{4})\frac{1}{b^2} = -2G\kappa_1$

--- assumed sol. does not satisfy PDE.

- Approx. sol. --- "co-location method", satisfy PDE only at a specific points of the x-s.
  - PDE will be satisfied at the center, $(\eta,\zeta) = (0,0)$

$$-\frac{C_0}{2a^2} - \frac{C_0}{2b^2} = -2G\kappa_1, \quad C_0 = \frac{4G\kappa_1 a^2 b^2}{a^2+b^2}$$

then, $\phi(\eta,\zeta) = \frac{4a^2b^2G\kappa_1}{a^2+b^2}(\eta^2 - \frac{1}{4})(\zeta^2 - \frac{1}{4})$

$M_1 = 2\int_A \phi \, dA$, torsional stiffness H_{11},

shear stress field τ_{12}, τ_{13}

ii) Open form exact sol. using a Fourier series

- Fourier series expansion of the stress function

$$\phi(\eta,\zeta) = \sum_{i=odd}^{\infty}\sum_{j=odd}^{\infty} C_{ij}\cos i\pi\eta \cos j\pi\zeta$$

- satisfaction of B.C., Eq. (7.45b) ... when i, j = odd, $\phi = 0$.
  Thus only odd values of i, j are included.
- governing PDE, Eq. (7.43)

$$\sum_{i=odd}^{\infty}\sum_{j=odd}^{\infty} C_{ij}\left[\left(\frac{i\pi}{a}\right)^2+\left(\frac{j\pi}{b}\right)^2\right]\cos i\pi\eta \cos j\pi\zeta = 2G\kappa_1$$

- By using the orthogonality properties of cosine function.

$$\sum_{i=odd}^{\infty}\sum_{j=odd}^{\infty} C_{ij}\left[\left(\frac{i\pi}{a}\right)^2+\left(\frac{j\pi}{b}\right)^2\right]\left[\int_{-\frac{1}{2}}^{\frac{1}{2}}\cos m\pi\eta \cos i\pi\eta \, d\eta\right]\cdot$$

$$\left[\int_{-\frac{1}{2}}^{\frac{1}{2}}\cos n\pi\zeta \cos j\pi\zeta \, d\zeta\right] = -2G\kappa_1\left[\int_{-\frac{1}{2}}^{\frac{1}{2}}\cos m\pi\eta \, d\eta\right]\left[\int_{-\frac{1}{2}}^{\frac{1}{2}}\cos n\pi\zeta \, d\zeta\right]$$

- The bracket integrals vanish when $m \neq i$ or $n \neq j$. The remaining terms

$$C_{mn}\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]\frac{1}{4} = \frac{8}{mn\pi^2}(-1)^{\frac{m-1}{2}}(-1)^{\frac{n-1}{2}}G\kappa_1$$

- Then,

$$\phi(\eta,\zeta) = \frac{32G\kappa_1}{\pi^2}\sum_{i=odd}^{\infty}\sum_{j=odd}^{\infty}\frac{(-1)^{\frac{i+j-2}{2}}}{\left[\left(\frac{i\pi}{a}\right)^2+\left(\frac{j\pi}{b}\right)^2\right]}\cos i\pi\eta \cos j\pi\zeta \quad (7.5$$

- externally applied torque M_1
- torsional stiffness H_{11}
- shear stress field

... Although it is a doubly infinite series, it converges rapidly (1, 2 terms) → Figs 7.22, 7.23

iii) Comparison of sol.

$\overline{H}_{11}$ ... Fig. 7.24, non-dimensional shear stress ... Figs. 7.25, 7.26

... large discrepancies, approximate sol. not good enoug

7.4 Torsion of a thin rectangular cross-section

- Fig. 7.28 ... $t \ll b$, assume that both stress function and associated shear stress distributions will be nearly const. along $\bar{i}_3$

$$\rightarrow \frac{\partial \phi}{\partial x_3} \simeq 0$$

- Governing eqn., Eq. (7.43) $\rightarrow$ $\dfrac{d^2\phi}{dx_2^2} = -2G\kappa_1$ (7.56)

$$\phi(x_2) = -G\kappa_1 x_2^2 + C_1 x_2 + C_2$$

- B.C., Eq. (7.45b) ... $\phi(x_2 = \pm \frac{t}{2}) = 0 \rightarrow C_1 = 0,\ C_2 = \frac{1}{4} G\kappa_1 t^2$

$$\rightarrow \phi(x_2) = -G\kappa_1 \left(x_2^2 - \frac{t^2}{4} \right) \quad (7.57)$$

- resulting torque

$$M_1 = 2\int_A \phi\, dA = -2G\kappa_1 \int_{-\frac{t}{2}}^{\frac{t}{2}} \left(x_2^2 - \frac{t^2}{4}\right) b\, dx_2 = \frac{1}{3} G\kappa_1 b t^3$$

- torsional stiffness

$$H_{11} = \frac{M_1}{\kappa_1} = \frac{1}{3} G b t^3 \quad (7.58)$$

- shear stress distribution

$$\tau_{12} = \frac{\partial \phi}{\partial x_3} = 0, \quad \tau_{13} = -\frac{\partial \phi}{\partial x_2} = 2G\kappa_1 x_2 = \frac{6M_1}{bt^3} x_2 \quad (7.59)$$

R.H.S. of Fig. 7.28

- warping function ... Eq. (7.57) $\rightarrow$ (7.42)

$$\frac{\partial \Psi}{\partial x_2} = \frac{1}{G\kappa_1}\frac{\partial \phi}{\partial x_3} + x_3 = x_3, \quad \frac{\partial \Psi}{\partial x_3} = -\frac{1}{G\kappa_1}\frac{\partial \phi}{\partial x_2} - x_2 = x_2$$

$$\Psi = x_3 x_2 + f(x_3) \qquad \Psi = x_2 x_3 + g(x_2)$$

$$\Psi = x_2 x_3$$

- axial displacement

$$u_1(x_2, x_3) = \Psi(x_2, x_3)\,\kappa_1 = \kappa_1 x_2 x_3 \quad (7.60)$$

... anti-symm. w.r.t. both $\bar{i}_2$ and $\bar{i}_3$

7.5 Torsion of thin-walled open sections

- Gradient of the stress function will vanish along the local tangent to the section's thin wall; corresponding shear stress will be linear through the wall thickness (Fig. 7.30)
  - torsional stiffness ← Eq. (7.58) ⋯ $H_{11} = G \frac{lt^3}{3}$ (7.61)
  - shear stress ⋯ tangential shear stress, τ_s, only nonvanishing component, vary linearly from 0 at the middle to max. (+) and (−) at edges

$$\tau_s^{max} = G t \kappa_1 \quad (7.62)$$

- More general thin-walled open section ⋯ multiple curved and straight sections (Fig. 7.31)
  - torsional stiffness ⋯ sum of those corresponding to the individual segments

$$H_{11} = \sum_i H_{11}^{(i)} = \frac{1}{3} \sum_i G_i l_i t_i^3 \quad (7.64)$$

  - Max. shear stress

$$\tau_s^{max} = G t_{max} \frac{M_1}{H_{11}} \quad (7.65)$$

  - warping ⋯ more complex, described in Chap. 8