chap. 8 Thin-walled beams

- typical aeronautical structures … light-weight, thin-walled, beam-like structures ← complex loading environment

(combined axial, bending, shearing, torsional loads)

- closed or open sections, or a combination of both

   ← profound implications for the structural response (shearing and torsion)

- thin-walled beams … specific geometric nature of the beam will be exploited to simplify the problem's formulation and solution process

  Fig. 8.1 ~ 8.4 … 8.1 : closed section
  8.2 : open section
  8.3 : combination of both
  8.4 : multi-cellular section

8.1 Basic eqn.s for thin-walled beams

C : geometry of the section, along the mid-thickness of the wall

s : length along the contour, orientation along C

$t(s)$: wall thickness

multi-cellular sections … a number of different curves

8.1.1 The thin wall assumption

- wall thickness is assumed to be much smaller than the other representative imensions

- In Fig. 8.1, $\frac{t(s)}{b} \ll 1$, $\frac{t(s)}{h} \ll 1$, or $\frac{t(s)}{\sqrt{b^2+h^2}} \ll 1$ (8.1)

- the thin-walled beam must also be long to enable the beam theory to be a reasonable approximation

$$\frac{\sqrt{b^2+h^2}}{L} \ll 1$$

8.1.2 stress flows

- As in Sec. 5.4.2, 5.5.2, the stress components acting in the plane of the x-s are assumed to be negligible as compared to the others

--> $\sigma_2 \ll \sigma_1$, $\sigma_3 \ll \sigma_1$; $\tau_{23} \ll \tau_{12}$, $\tau_{23} \ll \tau_{13}$

--> only non-vanishing components ... { axial stress σ_1 ; transverse shear stress τ_{12}, τ_{13} }

- it is preferable to use the stress components parallel and normal to C

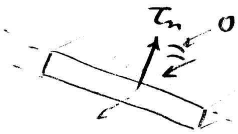
-> τ_s, τ_n (Fig. 8.5), rather than Cartesian components

$$\tau_n = \cos\alpha\, \tau_{12} + \sin\alpha\, \tau_{13} = \tau_{12}\frac{dx_3}{ds} - \tau_{13}\frac{dx_2}{ds} \qquad (8.2a)$$

$$\tau_s = -\sin\alpha\, \tau_{12} + \cos\alpha\, \tau_{13} = \tau_{12}\frac{dx_2}{ds} + \tau_{13}\frac{dx_3}{ds} \qquad (8.2b)$$

$$\cos\alpha = \frac{dx_3}{ds}, \quad \sin\alpha = \frac{-dx_2}{ds}$$

sign convention for s (Fig. 8.5)



- principle of reciprocity of shear stress (Eq. (1.5)) → normal shear stress

{ τ_n must vanish at the two edges of the wall because the outer surface are stress free.
no appreciable magnitude of this ~~st~~ stress component can build up since the wall is very thin }

⇒ { τ_n vanishes through the wall thickness
The only non-vanishing shear stress component : τ_s, tangential shear stress }

- Inverting Eqs. (8.2a), (8.2b), and $\tau_n \simeq 0$,

$$\tau_{12} \simeq \tau_s \frac{dx_2}{ds}, \quad \tau_{13} \simeq \tau_s \frac{dx_3}{ds} \qquad (8.3)$$

- it seems reasonable to assume that τ_s is uniformly distributed across the wall thickness (Fig. 8.6) since the wall is very thin

- concept of "stress flows"

$n(x_1, s) = \sigma_1(x_1, s)t(s)$ "axial stress flow," "axial flow"

$f(x_1, s) = \tau_s(x, s)t(s)$ "shearing " , " " shear flow"

→ only necessary to integrate a stress flow along C, instead of over an area, to compute a force

8.1.3 Stress resultants

- integration over the beam's x-s area → integration along curve C

infinitesimal area of the x-s $dA = t\,ds$

- Axial force ... from Eq. (5.8)

$$N_1(x_1) = \int_A \sigma_1 dA = \int_C \sigma_1 t\, ds = \int_C n\, ds \qquad (8.5)$$

axial flow, Eq. (8.4a)

- bending moments ... from Eq. (5.10)

$$M_2(x_1) = \int_C n x_3\, ds, \qquad M_3(x_1) = -\int_C n x_2\, ds \qquad (8.6)$$

- Shear forces ... from Eq. (5.9)

$$V_2(x_1) = \int_C f \frac{dx_2}{ds} ds, \qquad V_3(x_1) = \int_C f \frac{dx_3}{ds} ds \qquad (8.7)$$

where Eq. (8.7), and shear flow, Eq. (8.4b), are used.

- Torque about O

$$\underline{M}_0(x_1) = \int_C \underline{r}_p \times \underline{f}\, d\underline{s}$$

$\underline{r}_p = x_2 \bar{i}_2 + x_3 \bar{i}_3$, position vector of point P (Fig. 8.7)

$\underline{ds} = dx_2 \bar{i}_2 + dx_3 \bar{i}_3$, increment in curvilinear coord.

$$\underline{M}_0(x_1) = \int_C (x_2 dx_3 - x_3 dx_2) f \bar{i}_1 = \int_C \left(x_2 \frac{dx_3}{ds} - x_3 \frac{dx_2}{ds}\right) f \bar{i}_1 ds$$

- from Fig. 8.7,

$$r_0 = x_2 \cos\alpha + x_3 \sin\alpha = x_2 \frac{dx_3}{ds} - x_3 \frac{dx_2}{ds} \qquad (8.8)$$

- magnitude of the torque

$$M_{10}(x_1) = \int_C f r_0\, ds \qquad (8.9)$$

... torque = magnitude of the force × perpendicular distance from the point to the line of action of the force

- Torque about an arbitrary point K of the x-s

$$M_{1K}(x_1) = \int_c f\, r_K\, ds \qquad (8.10)$$

r_K ··· perpendicular distance from K to the line of action of the shear flow

8.1.4 Sign conventions

- the sign convention for the torque is independent of the choice of the curvilinear variable, s.

s : counterclockwise, s' : clockwise

$$\rightarrow \quad f'(s') = -f(s), \quad r_o'(s') = -r_o(s) \qquad \text{(Fig 8.9)}$$

However, the resulting torque is unaffected by this choice

$$M_{1o} = \int_c f r_o\, ds = \int_c f' r_o'\, ds'$$

8.1.5 Local equilibrium eqn.

Fig. 8.10 --- a differential element of the thin-walled beam

all the forces acting along axis $\bar{i}_1$ →

$$-n\,ds + \left(n + \frac{\partial n}{\partial x_1} dx_1\right) ds - f\,dx_1 + \left(f + \frac{\partial f}{\partial s} ds\right) dx_1 = 0$$

$$\Rightarrow \quad \frac{\partial n}{\partial x_1} + \frac{\partial f}{\partial s} = 0 \qquad (8.14)$$

--- any change in axial stress flow, n, along the beam axis must be equilibrated by a corresponding change in shear flow, f, along curve C that defines the x-s.

8.2 Bending of thin-walled beams

- Fig 8.11 --- thin-walled beam subjected to axial forces and bending moments

E-B assumptions are applicable for either open or closed x-s.

- Assuming a displacement field in the form of Eq. (6.1)

→ strain field given by Eq. (6.2a) ~ (6.2c)

... axial stresses distribution, from Eq. (6.15)

$$\sigma_1 = E\left[\frac{N_1}{S} - \frac{x_2 H_{23}^c - x_3 H_{33}^c}{\Delta_H} M_2 - \frac{x_2 H_{22}^c - x_3 H_{23}^c}{\Delta H} M_3\right] \quad (8.15)$$

$$\Delta_H = H_{22}^c H_{33}^c - (H_{23}^c)^2$$

Using Eq. (8.4a), axial flow distribution

$$n(x_1, s) = E(s)\, t(s)\left[\frac{N_1(x_1)}{S} - \frac{x_2(s) H_{23}^c - x_3(s) H_{33}^c}{\Delta_H} M_2(x_1) - \frac{x_2(s) H_{22}^c - x_3(s) H_{23}^c}{\Delta H} M_3(x_1)\right] \quad (8.16)$$

8.3 Shearing of thin-walled beams

- bending moments in the thin-walled beams are accompanied by transverse shear force → give rise to shear flow distributions

- evaluated by introducing the axial flow, given by Eq. (8.16) into the local equilibrium eqn., Eq. (8.14)

$$\frac{\partial f}{\partial s} = -Et\left[\frac{1}{S}\frac{dN_1}{ds} - \frac{x_2 H_{23}^c - x_3 H_{33}^c}{\Delta_H}\frac{dM_2}{dx_1} - \frac{x_2 H_{22}^c - x_3 H_{23}^c}{\Delta_H}\frac{dM_3}{dx_1}\right] \quad (8.17)$$

- sectional equilibrium eqns., Eq. (6.16), (1.18), (6.20)

assuming that $p_1, q_2, q_3 = 0$,

$$\frac{\partial f}{\partial s} = -E(s)\, t(s)\left[-\frac{x_2 H_{23}^c - x_3 H_{33}^c}{\Delta_H} V_3 + \frac{x_2 H_{22}^c - x_3 H_{23}^c}{\Delta_H} V_2\right] \quad (8.18)$$

- Integration → shear flow distribution arising from V_2, V_3

$$f(s) = c - \int_0^s Et\left[-\frac{x_2 H_{23}^c - x_3 H_{33}^c}{\Delta_H} V_3 + \frac{x_2 H_{22}^c - x_3 H_{23}^c}{\Delta_H} V_2\right] ds \quad (8.19)$$

c : integration constant corresponding to the value at $s = 0$,

the procedure to determine this depends on whether x-s is open or closed.

Since H_{00}^c, V_2, V_3 are fn of x_1 alone,

$$f(s) = c + \frac{Q_3(s) H_{23}^c - Q_2(s) H_{33}^c}{\Delta_H} V_3 - \frac{Q_3(s) H_{22}^c - Q_2(s) H_{23}^c}{\Delta_H} V_2 \quad (8.20)$$

where "stiffness static moments" or "stiffness first moment"

$$Q_2(s) = \int_0^s E\, x_3(s)\, t\, ds, \qquad Q_3(s) = \int_0^s E\, x_2(s)\, t\, ds \quad (8.21)$$

... static moments for the portion of the X-S from $s=0$ to s

8.3.1 Shearing of open sections

- principle of reciprocity of shear stress, Eq. (1.5) → shear flow vanishes at the end points of curve C

Fig. 8.24 ... shear flow must vanish at points A and D since edges AE and DF are stress free

- If the origin of s is chosen to be located at such a stress free edge, the integration constant, c, in Eq. (8.20) must vanish

- Procedure to determine the shear flow distribution over open X-S

① compute the location of the centroid of the X-S, and select a set of centroidal axes, $\bar{i}_1$ and $\bar{i}_2$, and compute the sectional centroidal bending stiffnesses H_{22}^c, H_{33}^c and H_{23}^c. (principal centroidal axes → $H_{23}^c = 0$)

② select suitable curvilinear coord. s to describe the geometry of X-S

③ evaluate the first stiffness moments using Eq. (8.21)

④ $f(s)$ is determined by Eq. (8.20)

8.3.2 Evaluation of stiffness static moments

- homogeneous, thin-walled rectangular strip oriented at an angle α (Fig. 8.2

$$Q_2(s) = \int_0^s E x_3\, t\, ds = E\int_0^s (d_3 + s\sin\alpha)\, t\, ds = E s t \left(d_3 + \frac{s}{2}\sin\alpha\right) \quad (8.22)$$

... Young's modulus × the area st × coord. of the centroid of the local area (i.e., area midpoint)

- similar result of the other stiffness static moment

$$Q_3(s) = E\, st\, (d_2 + \frac{s}{2}\cos\alpha) \qquad (8.23)$$

- Since the strip is made of a homogeneous material, E factors out of integral

$$\rightarrow \quad Q_2(s) = E \underbrace{\int_0^s x_3 t\, ds}_{\text{area static moment}}$$

- thin-walled homogeneous circular arc of radius R (Fig. 8.23)

$$ds = R\, d\theta$$

$$Q_2(s) = \int_0^s E x_3 t\, ds = Et \int_0^\theta (d_3 + R\sin\theta)\, R\, d\theta = Et R^2 (\frac{d_3}{R}\theta + 1 - \cos\theta)$$

$$Q_3(s) = EtR^2 \left[(1 + \frac{d_2}{R})\theta - \sin\theta \right]$$

- stiffness static moment $=$ E $\times$ area $\times$ distance to the area centroid

$$Q = EA\, x_{3c}, \quad Q_3 = EA x_2$$

= "parallel axis theorem" (Sec. 6.8.1), but in this case, only the transport term remains since the static moment about the area centroid itself is zero, by definition.

8.7.3 Shear flow distributions in open sections

Example 8.2 Shear flow continuity conditions

- Two-wall joints --- equilibrium of forces along the beam's axis

$\rightarrow$ $-f_1 + f_2 = 0$, or $f_1 = f_2$ --- the shear flow must be continuous at the junction J.

- Three-wall joint --- $-f_1 - f_2 - f_3 = 0$, or more generally

$$\Sigma f_i = 0 \qquad (8.29)$$

--- the sum of the shear flows converging to a joint must vanish.

8.3.5 Shear center for open sections

- problem is not precisely defined --- whereas the magnitudes of the transverse shear forces are given, their lines of action are not specified.

→ it is not possible to verify the torque equilibrium of the x-s.

- Definition of the shear center
  - Fig. 8.30 -- subjected to horizontal and vertical shear forces V_2, V_3 with lines of action passing through K, (x_{2K}, x_{3K}), no external torque applied, $M_{1K} = 0$
  - 3 equipollence conditions

① integration of the horizontal component of the shear flow over the x-s must equal the applied horizontal shear force → $\int_c f \left(\frac{dx_2}{ds}\right) ds = V_2$

--- will be satisfied since it simply corresponds to the def. of shear force, Eq. (8.7a)

② " vertical " vertical → $\int_c f \left(\frac{dx_3}{ds}\right) ds = V_3$, identical to Eq. (8.7b)

③ torque generated by the distributed shear flow is equivalent to the externally applied torque, about the same point

--- does require the line of action of the applied shear forces about point K, the torque, $M_{1K} = \int_c f r_K \, ds$ (Eq. (8.10))

torque generated by the external forces w.r.t. point K = 0,

$M_{1K} = 0 + 0 \cdot V_2 + 0 \cdot V_3 = 0$

⇒ $M_{1K} = \int_c f r_K \, ds = 0$

--- point K cannot be an arbitrary point; its coords must satisfy the torque equipollence condition

$$M_{1K} = \int_c f r_K \, ds = 0 \qquad (8.39)$$

"definition of the shear center location"

- Alternative definition
  - perpendicular distance from an arbitrary point A to the line of action

$$r_a = r_0 - x_{2a}\frac{dx_3}{ds} + x_{3a}\frac{dx_2}{ds}$$

(x_{2a}, x_{3a}) : coord. of point A

- subtracting this eqn. from Eq. (8.11)

$$r_K = r_a - (x_{3K} - x_{2a})\frac{dx_3}{ds} + (x_{3K} - x_{2K})\frac{dx_2}{ds}$$

- substituting into the torque equipollence condition. Eq. (8.39)

$$\int_c f r_a ds - (x_{2K} - x_{2a})\left[\int_c f \frac{dx_3}{ds} ds\right] + (x_{3K} - \ _{3a})\left[\int_c f \frac{dx_2}{ds} ds\right]$$

$$= \int_c f r_a ds - (x_{2K} - x_{2a}) V_3 + (x_{3K} - x_{3a}) V_2 = 0$$

- torque generated about point A by the shear flow distribution $M_{1a} = \int_c f r_a$

$$M_{1a} = \int_c f r_a ds = (x_{2K} - x_{2a}) V_3 - (x_{3K} - x_{3a}) V_2 \qquad (8.40)$$

... moment at A due to force and moment resultants at point K

$$M_{1a} = M_{1K} + (x_{2K} - x_{2a}) V_3 - (x_{3K} - x_{3a}) V_2$$

$\quad\quad M_{1K} = 0$ by Eq. (8.39)

Eq. (8.39), (8.40) ... torque generated by the shear flow distribution associated with transverse shear force must vanish w.r.t. the shear center

- Summary
  - " a beam bends without twisting if and only if the transverse shear loads are applied at the shear center."

    " if the transverse loads are not applied at the shear center, the beam will both bend and twist."
  - If the x-s features a plane of symmetry, the shear center must lie in that plane of symmetry.

Example 8.8 shear center for an angle section

.. lines of actions of two resultant of the shear flow distributions, R_1 and R_2, will intersect at point K → produces no torque about this point → must then be the shear center

8.3.7 Shearing of closed sections

- same governing eqn., Eq. (8.19), still applies, but no boundary condition is readily available to integrate this eqn.
  - exception --- axis of symmetry, Fig. 8.34
    if V_3 acts in the plane of symm., $(\bar{i}_1, \bar{i}_3)$ → mirror image of shear flow distribution
  - point A --- joint equilibrium condition, Eq. (8.29) → $f_1 + f_2 = 0$
    symmetry condition → $f_1 = f_2$
    → $f_1 = f_2 = 0$
    shear flow vanishes at A and similarly B
- Fig. 8.35 --- 1st step: beam is cut along its axis at an arbitrary point
  → "auxiliary problem", shear flow distribution $f_0(s)$
  - $f_0(s)$ creates a shear strain γ_s (Fig. 8.36) → infinitesimal axial displacement du_1

$$du_1 = \gamma_s \, ds = \frac{\tau_s}{G} ds = \frac{f_0(s)}{Gt} ds \qquad (8.43)$$

  - total relative axial displacement at the cut, u_0

$$u_0 = \int_c \frac{f_0(s)}{Gt} ds$$

  -- last step: f_c is applied to eliminate the relative axial displacement, thereby returning the section to its original, closed state (f_c : "closing shear flow")
  ... total shear flow $f(s) = f_0(s) + f_c(s)$

$$u_t = \int_c \frac{f_0(s) + f_c}{Gt} ds = 0 \qquad (8.44)$$

  ... displacement compatibility eqn. for the closed section

$$f_c = - \frac{\int_c \frac{f_0(s)}{Gt} ds}{\int_c \frac{1}{Gt} ds} \qquad (8.45)$$

- summary of the procedure

① $f_0(s)$ for an auxiliary problem

② $f_c(s)$ by Eq. (A.41)

③ $f(s) = f_0(s) + f_c(s)$

A.3.8 Shearing of multi-cellular sections

- Fig. A.39 ... a typical wing section with 2 closed cells
  - → procedure similar to that used for a single closed section must be developed
- One cut per cell is required.
- shear flow distribution in the resulting open sections is evaluated using the procedure in Sec. A.3.1 → $f_0(s_1)$, $f_0(s_2)$, $f_0(s_3)$ along C_1, C_2, C_3
- closing shear flows are applied at each cut : f_{c1} and f_{c2}
- Then, shear flow distribution : $f_0(s_1) + f_{c1}$, $f_0(s_2) + f_{c2}$, $f_0(s_3) + (f_{c1} + f_{c2})$, along C_1, C_2, C_3

  2 unknown f_{c1}, f_{c2} will be evaluated by enforcing the displacement compatibility condition for each cell.
- front cell (clockwise (+)) :

$$u_{t1} = \int_{C_1} \frac{f_0(s_1) + f_{c1}}{Gt}\, ds_1 + \int_{C_3} \frac{f_0(s_3) + (f_{c1} + f_{c2})}{Gt}\, ds_3 = 0$$

- aft cell (counterclockwise (+)) :

$$u_{t2} = \int_{C_2} \frac{f_0(s_2) + f_{c2}}{Gt}\, ds_2 + \int_{C_3} \frac{f_0(s_3) + (f_{c1} + f_{c2})}{Gt}\, ds_3 = 0$$

- can be recast as a set of 2 linear eqn.s for f_{c1} and f_{c2}

$$\begin{cases} \left[\int_{C_1+C_3} \frac{1}{Gt}\, ds\right] f_{c1} + \left[\int_{C_3} \frac{1}{Gt}\, ds\right] f_{c2} = -\int_{C_1+C_3} \frac{f_0(s)}{Gt}\, ds ; \\ \left[\int_{C_3} \frac{1}{Gt}\, ds\right] f_{c1} + \left[\int_{C_2+C_3} \frac{1}{Gt}\, ds\right] f_{c2} = -\int_{C_2+C_3} \frac{f_0(s)}{Gt}\, ds \end{cases}$$

• Extension to multi-cellular section with N closed cells

① open section by N cuts, one per cell → shear flow distribution in open section by the procedure in Sec. 8.7.1

② closing shear flows are applied at each cut and displacement compatibility conditions are imposed → N simultaneous eqn.s

③ total shear flow distribution is found by adding the closing shear flow to that for the open section

8.4 The shear center

• Chap. 6 ... assumption that transverse loads are applied in "such a way that the beam will bend without twisting"

→ more precise statement: the lines of action of all transverse loads pass through the shear center

- If the shear forces are not applied at the shear center, the beam will undergo both bending and twisting.

8.4.1 Calculation of the shear center location

• involves two linearly independent loading cases

① $(\cdot)^{[2]}$, unit shear force $V_2^{[2]} = 1$, no shear force along $\bar{i}_3$, $V_3^{[2]} = 0$

→ shear flow $f^{[2]}(s)$

② $(\cdot)^{[3]}$, $V_3^{[3]} = 1$, $V_2^{[3]} = 0$ → $f^{[3]}(s)$

- from Eq. (8.7), shear forces equipollent to $f^{[2]}(s)$

$$V_2^{[2]} = \int_c f^{[2]} \frac{dx_2}{ds} ds = 1, \quad V_3^{[2]} = \int_c f^{[2]} \frac{dx_3}{ds} ds = 0 \qquad (8.51)$$

- shear center location K (x_{2K}, x_{3K}) : Eq (8.10) →

$$M_{1K} = \int_c f^{[2]} r_K ds = \int_c f^{[2]} \left(r_0 - x_{2K}\frac{dx_3}{ds} + x_{3K}\frac{dx_2}{ds}\right) ds$$

r_K : distance from K to the tangent to contour C, Eq. (8.11)