

Chap. 9 Virtual Work Principles

9.1 Introduction

- Mechanical work --- scalar product of the force by the displacement through which it acts \rightarrow scalar quantity \rightarrow simpler to manipulate \rightarrow very attractive
- Newton's equilibrium condition: the sum of all forces (regardless of externally applied loads, internal forces, and reaction forces) must vanish
- Analytical mechanics --- powerful tools for complex problems
 - scalar quantities, simpler analysis procedure
 - reaction forces can often be eliminated if the work involved vanishes.
 - systematic development of procedure for approximate solutions (ex: finite element method)
- why still need Newton's formulation?
 - to determine both magnitude and direction of all forces acting within a structure, to estimate failure conditions

- Principle of virtual work (PVW) $\xleftrightarrow{\text{equivalent}}$ Newton's law

9.2 Equilibrium and work fundamentals

9.2.1 static equilibrium conditions

- Newton's 1st law --- every object in a state of uniform motion tends to remain in that state of motion unless an external force is applied to it.
- \rightarrow A particle at rest tends to remain at rest unless the sum of the externally applied force does not vanish.

→ A particle is at rest if and only if the sum of the externally applied forces vanishes.

→ A particle is in static equilibrium

→ " " $\Sigma \underline{F} = 0$ (9.1)

(1) the vector sum of all forces acting on a particle must be zero

(2) the vector polygon must be closed

(3) the component of the vector sum resolved in any coord. system must vanish. $\Sigma \underline{F} = F_1 \underline{i}_1 + F_2 \underline{i}_2 + F_3 \underline{i}_3 \rightarrow F_1 = F_2 = F_3 = 0$

• Newton's 3rd law: if particle A exerts a force on particle B, particle B simultaneously exerts on particle A a force of identical magnitude and opposite direction.

→ Two interacting particles exert on each other forces of equal magnitude, opposite direction, and sharing a common line of action.

• Euler's 1st law

- Fig. 9.1 ... system consisting of N particles

particle i subjected to an external force \underline{F}_i ,

$N-1$ interaction forces \underline{L}_{ij} , $j=1, 2, \dots, N, j \neq i$

Newton's 1st law →

$$\underline{F}_i + \sum_{j=1, j \neq i}^N \underline{L}_{ij} = 0 \quad (9.2)$$

interaction forces ... for rigid body, it will ensure the body shape remain unchanged

elastic body, stress resulting from deformation

planetary system, gravitational pull

- summation of N eqns. for N particles

$$\sum_{i=1}^N \underline{F}_i + \sum_{i=1}^N \sum_{j=1, j \neq i}^N \underline{f}_{ij} = 0$$

- By Newton's 3rd law,

$$\sum_{i=1}^N \sum_{j=1, j \neq i}^N \underline{f}_{ij} = 0 \quad (9.3)$$

Then,
$$\sum_{i=1}^N \underline{F}_i = 0 \quad (9.4)$$

--- Euler's 1st law for a system of particles

necessary condition for " to be in static equilibrium
but not a sufficient condition

• Euler's 2nd law

- taking a vector product of Eq. (9.2) by \underline{r}_i , then summing over all particles

$$\sum_{i=1}^N \underline{r}_i \times \underline{F}_i + \underbrace{\sum_{i=1}^N \sum_{j=1, j \neq i}^N \underline{r}_i \times \underline{f}_{ij}}_{= 0} = 0$$

0 due to Newton's 3rd law (9.5)

Then

$$\sum_{i=1}^N \underline{r}_i \times \underline{F}_i = \sum_{i=1}^N \underline{M}_i = 0 \quad (9.6)$$

--- Euler's 2nd law, both necessary condition for the system of particles
" 1st " to be in static equilibrium, but not a sufficient condition

9.2.2 Concepts of mechanical work

• Definition --- the work done by a force is the scalar product of the force by the displacement of its point of application

① force, displacement collinear: $\underline{F} = F\bar{u}$, $d = d\bar{u}$, $W = Fd$

(+) if the same direction, (-) if the opposite direction

② not collinear : $W = Fd \cos \theta$, θ angle between \bar{u} and \bar{v} .

③ perpendicular : $\cos \theta = \cos \frac{\pi}{2} = 0$, $W = 0$

- "incremental work" --- $dW = \underline{F} \cdot d\underline{r}$
total work --- $W = \int_{\underline{r}_i}^{\underline{r}_f} dW = \int_{\underline{r}_i}^{\underline{r}_f} \underline{F} \cdot d\underline{r}$ (9.7)

- $\underline{F} = F_1 \bar{e}_1 + F_2 \bar{e}_2 + F_3 \bar{e}_3$, $d\underline{r} = dr_1 \bar{e}_1 + dr_2 \bar{e}_2 + dr_3 \bar{e}_3$,

$$dW = \underline{F} \cdot d\underline{r} = F_1 dr_1 + F_2 dr_2 + F_3 dr_3$$

- $d\underline{r} = dr \bar{u}$, $\underline{F} = F_{\parallel} \bar{u} + F_{\perp} \bar{v} \rightarrow dW = (F_{\parallel} \bar{u} + F_{\perp} \bar{v}) \cdot dr \bar{u}$
 $= F_{\parallel} dr$

- superposition --- $\underline{F} = \underline{F}_1 + \underline{F}_2$, $dW = \underline{F} \cdot d\underline{r} = (\underline{F}_1 + \underline{F}_2) \cdot d\underline{r}$
 $= \underline{F}_1 \cdot d\underline{r} + \underline{F}_2 \cdot d\underline{r} = dW_1 + dW_2$

- why is work a quantity of interest for the static analysis?

Ans. : concept of "virtual work" that would be done by a force if it were to displace its point of application a fictitious amount.

9.3 Principle of virtual work

• PVW --- "arbitrary virtual displacement," "arbitrary test", "fictitious"

"arbitrary" --- displacement can be chosen arbitrarily without any restrictions imposed on their magnitudes or orientations

"virtual", "test", "fictitious" --- do not affect the forces acting on the particle

9.3.1 PVW for a single particle

Fig. 9.2 --- particle in static equilibrium under a set of externally applied loads, fictitious displacement of $\underline{\delta}$

- virtual work done

$$W = \left[\sum \underline{F} \right] \cdot \underline{\delta} = 0 \quad (9.8)$$

||
0 due to Newton's 1st law

- assume that one of the externally applied forces, \underline{F}_1 , is an elastic spring force. If for a real, arbitrary displacement, \underline{d} , the spring force will change to become \underline{F}_1' , the sum of externally applied forces, $\sum \underline{F}_i$
 $\rightarrow \sum \underline{F}_i' \neq 0$

- for a virtual or fictitious displacement, do not affect the loads applied to the particle, it remains in static equilibrium, Eq (9.8) holds.

- if Eq. (9.8) is satisfied for all arbitrary virtual displacements, then $\sum \underline{F}_i = 0$, and the particle is in static equilibrium.

◦ Principle 3 (PVW for a particle) --- A particle is in static equilibrium if and only if the virtual work done by the externally applied forces vanishes for all arbitrary virtual displacements.

9.3.2 Kinematically admissible virtual displacements

◦ "arbitrary virtual displacements" --- including those that violate the kinematic constraints of the problem

- "kinematically inadmissible direction", "infeasible direction" --- $\underline{\delta}$ in the track example

$\rightarrow \underline{\delta} = s_2 \bar{i}_2$ kinematically admissible

- reaction forces acts along the kinematically inadmissible direction

◦ Modified version of PVW --- "a particle is in static equilibrium if and only if the virtual work done by the externally applied forces vanishes for all arbitrary kinematically admissible virtual displacements"

\rightarrow constraint (reaction) forces are automatically eliminated

\rightarrow fewer number of eqns

9.3.3 Use of infinitesimal displacements as virtual displacements

- special notation commonly used to denote virtual displacements

$$\underline{\delta} = \delta \underline{u}$$

(9.13)

- virtual work done by a force undergoing virtual displacement $\rightarrow \delta W$

- convenient to use virtual displacements of infinitesimal magnitude
 \rightarrow often simplifies algebraic developments

① Displacement dependent forces

- automatically remain unaltered

② Rigid bodies

- 2 points P, Q of a rigid body \rightarrow must satisfy the rigid body dynamics

$$\underline{v}_P = \underline{v}_Q + \underline{\omega} \times \underline{r}_{QP}$$

$$\rightarrow \frac{d\underline{v}_P}{dt} = \frac{d\underline{v}_Q}{dt} + \frac{d\underline{\omega}}{dt} \times \underline{r}_{QP}$$

$$\rightarrow d\underline{v}_P = d\underline{v}_Q + d\underline{\omega} \times \underline{r}_{QP}$$

- it is possible to write

$$\delta \underline{u}_P = \delta \underline{u}_Q + \delta \underline{\psi} \times \underline{r}_{QP}$$

(9.14)

--- field of kinematically admissible virtual displacements for a rigid body

" δ ": virtual, fictitious displacement, leave the forces unchanged, allowed to violate the kinematic constraints

" d ": real, infinitesimal displacement, no requirement for forces, cannot violate the kinematic constraints

- $\underline{\delta\psi}$: vector quantity, but finite rotations are scalar quantity

- virtual displacements of infinitesimal magnitude greatly simplifies the treatment

9.7.4 PVW for a system of particles

• for a particle i ,

$$\delta W_i = \left(\underline{F}_i + \sum_{j=1, j \neq i}^N \underline{f}_{ij} \right) \cdot \delta \underline{u}_i \quad (9.15)$$

- sum of the virtual work - all particles must also vanish.

--- a system of particles is in static equilibrium iff

$$\delta W = \sum_{i=1}^N \left\{ \left[\underline{F}_i + \sum_{j=1, j \neq i}^N \underline{f}_{ij} \right] \cdot \delta \underline{u}_i \right\} \quad (9.16)$$

= 0 for all virtual displacements, $\delta \underline{u}_i$, $i = 1, 2, \dots, N$

→ $3N$ scalar eqns for a system of N particles → $3N$ D.O.F.'s

• External and internal VW

- "internal forces" -- act and reacted within the system

"external" " " -- act on the system but reacted outside the system

$$\delta W_E = \sum_{i=1}^N \underline{F}_i \cdot \delta \underline{u}_i \quad (9.17)$$

$$\delta W_I = \sum_{i=1}^N \left[\sum_{j=1, j \neq i}^N \underline{f}_{ij} \right] \cdot \delta \underline{u}_{ij}$$

$$\text{Eq. (9.16)} \rightarrow \delta W = \delta W_E + \delta W_I = 0 \quad (9.18)$$

Principle 4 (PVW) --- A system of particles is in static equilibrium iff

the sum of the virtual work done by the internal and external forces vanishes for all arbitrary virtual displacements.

- actual displacements : $W = W_E + W_I = 0 \quad (9.19)$

• Euler's law

- virtual displacement of a particle i

$$\delta \underline{u}_i = \delta \underline{u}_0 + \delta \underline{\psi} \times \underline{r}_i \quad (9.20)$$

↑ virtual translation of a rigid body ↑ virtual rotation → 6 independent virtual quantities, far fewer than

$$\delta W = \sum_{i=1}^N \left\{ \left[\underline{F}_i + \sum_{\substack{j=1, \\ j \neq i}}^N \underline{f}_{ij} \right] \cdot (\delta \underline{u}_0 + \delta \underline{\psi} \times \underline{r}_i) \right\}$$

$$= \left(\sum_i \underline{F}_i \right) \cdot \delta \underline{u}_0 + \left(\sum_i \sum_j \underline{f}_{ij} \right) \cdot \delta \underline{u}_0$$

$$+ \sum_i \underline{F}_i \cdot (\delta \underline{\psi} \times \underline{r}_i) + \sum_i \sum_j \underline{f}_{ij} \cdot (\delta \underline{\psi} \times \underline{r}_i)$$

$\left. \begin{array}{l} \underline{a} \cdot (\underline{b} \times \underline{c}) = \\ \underline{b} \cdot (\underline{c} \times \underline{a}) \end{array} \right\}$
 due to Eqs. (7.3), (7.5)

$$= \delta \underline{u}_0 \cdot \left(\sum_i \underline{F}_i \right) + \delta \underline{u}_0 \cdot \left(\sum_i \sum_j \underline{f}_{ij} \right)$$

$$+ \delta \underline{\psi} \cdot \left(\sum_i \underline{r}_i \times \underline{F}_i \right) + \delta \underline{\psi} \cdot \left(\sum_i \sum_j \underline{r}_i \times \underline{f}_{ij} \right)$$

$$= \delta \underline{u}_0 \cdot \left[\sum_i \underline{F}_i \right] + \delta \underline{\psi} \cdot \left[\sum_i \underline{r}_i \times \underline{F}_i \right]$$

!!
0

!!
0

← Euler's 1st, 2nd law
Eqs. (9.4), (9.6)

... necessary but not sufficient cond. for static equilibrium

9.4 Principle of virtual work applied to mechanical systems

• Rigid body ... Eq. (9.20) kinematically admissible virtual displacement field (3-dimensional)

→ 2 vector eqns. Eqs. (9.4) and (9.6), or 6 scalar eqns.

- 2-dimensional or planar mechanism, Eq. (9.20) →

$$\delta \underline{u}_i = \delta \underline{u}_0 + \delta \phi \bar{i}_3 \times \underline{r}_i \quad (9.21)$$

$$\delta \underline{\psi} \rightarrow \delta \phi \bar{i}_3$$

9.4.1 Generalized coord. and forces

• Not convenient to work with Cartesian coord. in many cases

→ will be represented in terms of N "generalized coord."

$$\underline{u} = \underline{u}(\beta_1, \beta_2, \dots, \beta_N)$$

→ virtual displacement

$$\delta \underline{u} = \frac{\partial \underline{u}}{\partial \beta_1} \delta \beta_1 + \frac{\partial \underline{u}}{\partial \beta_2} \delta \beta_2 + \dots + \frac{\partial \underline{u}}{\partial \beta_N} \delta \beta_N$$

- virtual work done by a force \underline{F}

$$\delta W = \underline{F} \cdot \delta \underline{u} = \underbrace{\left(\underline{F} \cdot \frac{\partial \underline{u}}{\partial q_1} \right)}_{\text{generalized force}} \delta q_1 + \left(\underline{F} \cdot \frac{\partial \underline{u}}{\partial q_2} \right) \delta q_2 + \dots + \left(\underline{F} \cdot \frac{\partial \underline{u}}{\partial q_N} \right) \delta q_N \quad (9.22)$$

- then,

$$\delta W = Q_1 \delta q_1 + Q_2 \delta q_2 + \dots + Q_N \delta q_N = \sum_{i=1}^N Q_i \delta q_i \quad (9.23)$$

virtual work = generalized forces \times generalized virtual displacements

- externally applied load or internal force

$$\delta W_I = \sum_{i=1}^N Q_i^I \delta q_i, \quad \delta W_E = \sum_i Q_i^E \delta q_i \quad (9.24)$$

- PVW, Eq. (9.18)

$$\delta W_I + \delta W_E = \sum_i Q_i^I \delta q_i + \sum_i Q_i^E \delta q_i = \sum_i [Q_i^I + Q_i^E] \delta q_i = 0 \quad (9.25)$$

$$\Rightarrow Q_i^I + Q_i^E = 0, \quad i = 1, 2, \dots, N$$

- if arbitrary virtual displacements, reaction forces must be included in Q_i^E

" kinematically admissible " " are eliminated.

o pendulum with a torsional spring (Fig. 9.13)

- virtual work by the gravity load $\delta W_E = -mg \bar{i}_2 \cdot \delta \underline{u}_T$

$$\underline{u}_T = R(\cos \phi \bar{i}_1 + \sin \phi \bar{i}_2), \quad \delta \underline{u}_T = R(-\sin \phi \bar{i}_1 + \cos \phi \bar{i}_2) \delta \phi$$

$$\delta W_E = \underbrace{-mgR \cos \phi}_{Q_\phi^E} \delta \phi = Q_\phi^E \delta \phi$$

- virtual work done by the internal force (restoring moment of the spring)

$$\delta W_I = -k\phi \delta \phi = Q_\phi^I \delta \phi$$

- PVW ... $Q_\phi^I + Q_\phi^E = -mgR \cos \phi - k\phi = 0$

$$\cos \phi \approx 1 \Rightarrow \phi = -mgR/k$$

9.5 P-W applied to truss structures

o truss ... like a simple rectilinear spring of stiffness const. $k = EA/L$

bar slenderness $\rightarrow 100$

9.5.1. Truss structures

• Elongation - displacement equations

- Fig. 9.29, displacement $\underline{\Delta} = \Delta_1 \bar{i}_1 + \Delta_2 \bar{i}_2$

$$e = \text{elongation} \quad (L+e)^2 = (L_1 + \Delta_1)^2 + (L_2 + \Delta_2)^2$$

- Δ_1 , and Δ_2 small compared to the bar's length \rightarrow can be linearized

$$e = \Delta_1 \frac{L_1}{L} + \Delta_2 \frac{L_2}{L} = \Delta_1 \cos \theta + \Delta_2 \sin \theta \quad (9.28)$$

... elongation is the projection of the relative displacement along the bar's direction

• Internal virtual work for a bar

- Fig. 9.30 ... general planar truss member.

virtual work done by the root and tip forces

$$\delta W = \underline{F}^r \cdot \delta \underline{u}^r + \underline{F}^t \cdot \delta \underline{u}^t = \underline{F} \bar{b} \cdot (\delta \underline{u}^t - \delta \underline{u}^r)$$

virtual work by the internal forces

$$\delta W_I = -\underline{F}^r \cdot \delta \underline{u}^r - \underline{F}^t \cdot \delta \underline{u}^t = -\underline{F} \bar{b} \cdot (\delta \underline{u}^t - \delta \underline{u}^r)$$

$$\text{virtual elongation } \delta e = \bar{b} \cdot (\delta \underline{u}^t - \delta \underline{u}^r)$$

$$\text{Then, Eq. (9.28)} \rightarrow \delta W_I = -\underline{F} \delta e$$

$$\delta e = (\sin \theta \bar{i}_1 + \cos \theta \bar{i}_2) \cdot (\delta u_1^t \bar{i}_1 + \delta u_2^t \bar{i}_2 - \delta u_1^r \bar{i}_1 - \delta u_2^r \bar{i}_2)$$

$$= (\delta u_1^t - \delta u_1^r) \sin \theta + (\delta u_2^t - \delta u_2^r) \cos \theta$$

9.5.2 Solution using Newton's law

• Fig. 9.31 ... five-bar planar truss

Newton's law \rightarrow equilibrium conditions at 4 joints A, B, C, D

\rightarrow total 8 scalar eqns ("method of joints")

$$P_A - F_{AD} = 0, \quad H_A + F_{AB} = 0$$

(9.31)

⋮

isostatic problem, 5 bar axial force + 3 reaction forces

9.5.3 Solution using kinematically admissible virtual displacements

Eq. (9.31) → 5 corresponding to equilibrium in an unconstrained direction, multiplied by virtual displacements (kinematically admissible)

$$[P_A - F_{AD}] \delta u_1^A + [P_B - F_{BC} - F_{BD} \sin \theta] \delta u_1^B + [-F_{AB} - F_{BD} \cos \theta] \delta u_2^B + [F_{BC}] \delta u_1^C + [P_C - F_{CD}] \delta u_2^C = 0 \quad (9.32)$$

- Regrouping

$$\underbrace{P_A \delta u_1^A + P_B \delta u_1^B + P_C \delta u_2^C}_{\delta W_E} - \underbrace{F_{AB} \delta u_2^B - F_{AD} \delta u_1^A - F_{BC} (\delta u_1^B - \delta u_1^C) - F_{BD} (\delta u_1^B \sin \theta + \delta u_2^B \cos \theta) - F_{CD} \delta u_2^C}_{\delta W_I} = 0 \quad (9.33)$$

$$\delta W_I = -F_{AB} \delta e_{AB} - F_{AD} \delta e_{AD} - F_{BC} \delta e_{BC} - F_{BD} \delta e_{BD} - F_{CD} \delta e_{CD} \quad (9.35)$$

$$\Rightarrow \delta W = \delta W_E + \delta W_I = 0 \quad (9.36)$$

Principle 5 (PVW) A structure is in static equilibrium iff the sum of the internal and external virtual work vanishes for all kinematically admissible virtual displacements.

9.5.4 Solution using arbitrary virtual displacements

Eq. (9.31) → 8 equilibrium multiplied by a virtual displacement

$$[P_A - F_{AD}] \delta u_1^A + [H_A + F_{AB}] \delta u_2^A + [P_B - F_{BC} - F_{BD} \sin \theta] \delta u_1^B + \dots \quad (9.37)$$

- Regrouping

$$\underbrace{P_A \delta u_1^A + P_B \delta u_1^B + P_C \delta u_2^C + H_A \delta u_2^A + V_D \delta u_1^D + H_D \delta u_2^D}_{\delta W_E} + \dots \quad (9.38)$$

$$\underbrace{\dots}_{\delta W_I}$$

Principle 6 (PVW) A structure is in static equilibrium iff the sum of the internal and external work vanishes for all virtual displacements.

9.6 Principle of complementary virtual work

Fig. 9.33 --- basic eqns of linear elasticity (Chap. 1)

→ 3 groups $\left\{ \begin{array}{l} \text{equilibrium eqns} \\ \text{strain-displacement relationships} \\ \text{constitutive laws} \end{array} \right.$

- strain compatibility eqns --- do not form an independent set of eqns and are not required to solve elasticity problems

- However, it is a over-determined problem since 6 strain components are expressed in terms of 3 displacement components only

⇒ Solution of any elasticity problem requires 3 groups of basic eqns (Fig. 9.33)

→ PVW alone does not provide enough information to solve the problems

⇒ PCVW will augment equilibrium eqns and constitutive laws to derive complete solutions, entirely equivalent to the compatibility eqns

9.6.1 Compatibility eqns for a planar truss

o Compatibility conditions

- Fig. 9.34 --- 2-bar truss, arbitrary elongations e_A, e_C

configuration of the truss compatible with these elongations is easily found

→ intersection of 2 circles (of radii $L_A + e_A, L_C + e_C$) → O'

- Fig. 9.35 --- 3-bar truss, again arbitrary elongations e_A, e_C

but configuration of bar B is now uniquely defined, since it must join B and O'

$e_B = L_B' - L_B$, 3 elongations are no longer independent

- same conclusion can be reached by the elongation-displacement relationship instead of the geometric reasoning

elongation --- projection of displacement vector along bar's direction, Eq (9.27)

$$e_A = u_1 \cos \theta + u_2 \sin \theta, \quad e_C = u_1 \cos \theta - u_2 \sin \theta \quad (9.43)$$

- for a 2-bar truss, final configuration is uniquely determined if the 2 displacement components, u_1 and u_2 , are given.

- 3-bar truss (Fig. 9.35)

$$e_A = u_1 \cos \theta + u_2 \sin \theta, \quad e_B = u_1, \quad e_C = u_1 \cos \theta - u_2 \sin \theta \quad (9.44)$$

→ it is not possible to express the 2 displacement components in terms of 3 elongations. Because 3 elongations form an over-determined set for the 2 unknown displacement components

- however, it is possible to eliminate 2 displacement components to obtain the compatibility eqn.

$$e_A - 2e_B \cos \theta + e_C = 0 \quad (9.45)$$

... 3 elongations in terms of 2 displacement components → 1 compatibility condition

- 2-bar truss ... isostatic, order of redundancy, # of compatibility eqn. = 0

3 " ... hyperstatic, # of compatibility eqn. = order of redundancy of the hyperstatic problem

- 3-bar truss ... 3 force components, 2 equilibrium eqn.s → hyperstatic of degree 1

3 elongations, 2 displacement components → 1 compatibility eqn.s

9.6.2 PCVW for trusses

o 3-bar truss under applied load

- Fig. 9.36 --- assumed to undergo compatible deformations so that the 3 bar elongations satisfy the elongation-displacement relationship, Eq. (9.44)

$$\delta W' = - [e_A - u_1 \cos \theta - u_2 \sin \theta] \delta F_A - [e_B - u_1] \delta F_B - [e_C - u_1 \cos \theta + u_2 \sin \theta] \delta F_C = 0 \quad (9.46)$$

"complementary VW" "virtual forces"

$$\delta W' = - e_A \delta F_A - e_B \delta F_B - e_C \delta F_C + u_1 (\delta F_A \cos \theta + \delta F_B + \delta F_C \cos \theta) + u_2 \sin \theta (\delta F_A - \delta F_C) = 0 \quad (9.47)$$

- free body diagram \rightarrow equilibrium eqns

$$F_A \cos \theta + F_B + F_C \cos \theta = P, \quad F_A - F_C = 0$$

--- A set of forces that satisfies these equilibrium eqns is said to be "statically admissible"

- "statically admissible virtual forces"

$$\begin{cases} \delta F_A \cos \theta + \delta F_B + \delta F_C \cos \theta = 0 \\ \delta F_A - \delta F_C = 0 \end{cases} \quad (9.48)$$

--- do not include the externally applied loads since $\delta P = 0$, geometry of the system is given $\rightarrow \delta \theta = 0$

- Eq. (9.47) becomes much simpler due to Eq. (9.48)

$$\delta W' = - e_A \delta F_A - e_B \delta F_B - e_C \delta F_C = 0 \quad (9.49)$$

for all statically admissible virtual forces

- Eq. (9.49) --- "internal complementary VW"

$$\delta W_I' = - e_A \delta F_A - e_B \delta F_B - e_C \delta F_C = - \sum_{i=1}^{N_b} e_i \delta F_i \quad (9.50)$$

