Chap. 10 Energy methods

- 2 virtual work principles

i) PVW ··· entirely equivalent to the equilibrium eqns

however, does not provide any information about the other

2 sets of eqn.s $\begin{cases} \text{strain-displacement relationship} \\ \text{constitutive laws} \end{cases}$

ii) PCVW -- " the strain-displacement relationships

" $\begin{cases} \text{equilibrium eqns} \\ \text{constitutive laws} \end{cases}$

∘ Type of forces

- In virtual work principles, various categories of forces are clearly defined and used.

① internal, external forces

② reaction forces ··· can be eliminated from the formulation since the work they perform vanishes when using kinematically admissible virtual displacements

But, when arbitrary virtual displacements are used, the virtual work does not vanish. → become an integral part of the formulation

∘ Conservative forces

- the work they perform always vanishes for a closed path displacement
- total mechanical energy of the system is preserved
- if the externally applied forces are conservative, they can be derived from a potential → further simplify the calculation of VW
- if the strain energy of an elastic component exists, the corresponding elastic forces can be derived from this strain energy → "

- combination of $\left\{ \begin{array}{l} \text{PVW} \\ \text{strain energy} \\ \text{potential of external forces} \end{array} \right\} \rightarrow$ principle of minimum total potential energy

PVW is always valid

PMTPE is limited to systems involving conservative forces

10.1 Conservative forces

$\underline{r}$: position vector of a particle

$\underline{F}$: force acting the particle, depends only upon the position of the particle, $\underline{F} = \underline{F}(\underline{r})$

- Fig. 10.1 ... two arbitrary paths ACB, ADB

- Definition
- $\underline{F}$ is conservative iff the work it performs along any path joining the same initial and final points is identical.

$$W = \int_{ACB} \underline{F} \cdot d\underline{r} = \int_{ADB} \underline{F} \cdot d\underline{r} \qquad (10.1)$$

- work done along path ADB = (−) · that along BDA
- work over the closed path ACBDA = 0

$$W = \oint_{\text{any path}} \underline{F} \cdot d\underline{r} = \oint_C \underline{F} \cdot d\underline{r} = 0 \qquad (10.2)$$

- Potential of a conservative force
- Stoke's theorem

$$\oint_C \underline{F} \cdot d\underline{r} = \int_A \bar{n} \cdot \nabla \times \underline{F} \, dA = 0 \qquad (10.3)$$

A : area enclosed by curve C

$\bar{n}$: outward normal to area A (Fig. 10.2)

$\rightarrow \nabla \times \underline{F} = 0 \quad \rightarrow \nabla \times \nabla \Phi = 0 \quad (\Phi$: arbitrary scalar fn.)

- sol. of eqn. $\nabla \times \underline{F} = 0$ ...

$$\underline{F} = -\nabla \Phi \qquad (10.4)$$

justified later "potential"

- $$\underline{F} = -\nabla \Phi = -\frac{\partial \Phi}{\partial x_1}\bar{i}_1 - \frac{\partial \Phi}{\partial x_2}\bar{i}_2 - \frac{\partial \Phi}{\partial x_3}\bar{i}_3 \qquad (10.5)$$

- work done by a conservative force

$$W = \int_{\underline{r}_1}^{\underline{r}_2} \underline{F} \cdot d\underline{r} = -\int_{\underline{r}_1}^{\underline{r}_2} \nabla \Phi \cdot d\underline{r}$$

$$= -\int_{\underline{r}_1}^{\underline{r}_2} \left(\frac{\partial \Phi}{\partial x_1} dx_1 + \frac{\partial \Phi}{\partial x_2} dx_2 + \frac{\partial \Phi}{\partial x_3} dx_3 \right) = -\int_{\underline{r}_1}^{\underline{r}_2} d\Phi = \Phi(\underline{r}_1) - \Phi(\underline{r}_2)$$

... depends only on the position of initial / final points

can be evaluated as the difference between the values of the potential fn.

$$W = \Phi(\underline{r}_1) - \Phi(\underline{r}_2) = -\Delta \Phi \qquad (10.6)$$

o Examples of conservative forces

i) gravity force ... $\Phi = mg\,\underline{r} \cdot \bar{i}_3 = mgx_3$

$$\underline{F}_g = -\nabla \Phi = -\partial \Phi / \partial x_3\, \bar{i}_3 = -mg\,\bar{i}_3$$

$$W = \int_{x_{3a}}^{x_{3b}} \underline{F}_g \cdot d\underline{r} = -\int_{x_{3a}}^{x_{3b}} \frac{\partial \Phi}{\partial x_3} dx_3 = \Phi(x_{3a}) - \Phi(x_{3b})$$

ii) restoring force of an elastic spring ...

restoring force $-ku$,

potential $A(u) = \frac{1}{2}ku^2$ ... "strain energy"

elastic force $F_s = -\frac{\partial A}{\partial u} = -ku$

$$W = \int_{u_a}^{u_b} F_s\, du = -\int_{u_a}^{u_b} \frac{\partial A}{\partial u} du = A(u_a) - A(u_b)$$

10.1.1 Potential for internal and external forces

- in PVW, a distinction is made between { internal forces / externally applied loads

- In elastic systems, internal forces { stresses acting in a body / elastic forces in structural components

→ potential of internal forces = "strain energy", "deformation energy", "internal energy" ... A

$$W_I = -\Delta A \quad (10.7)$$

- potential of external forces ... Φ

$$W_E = -\Delta \Phi \quad (10.8)$$

- total potential energy

$$\Pi = A + \Phi \quad (10.9)$$

- total work done by both internal and external forces

$$W = W_I + W_E = -\Delta A - \Delta \Phi = -\Delta \Pi \quad (10.10)$$

... "for conservative systems, the work done by the internal and external forces = negative change in total potential energy"

- adding an arbitrary constant to the potential fn will not alter the work done

10.1.2 Calculation of the potential fn's

- potential of internal forces ... "strain energy", $A = A(\underline{\epsilon})$

it is convenient to select $A(\underline{\epsilon} = 0) = 0$, undeformed or unstrained state

$$W_I = -\Delta A = -[A(\underline{\epsilon}) - A(\underline{\epsilon} = 0)] = -A(\underline{\epsilon})$$

$$A(\underline{\epsilon}) = -W_I \quad (10.11)$$

- it is cumbersome to compute the work done within a solid as the negative product of the internal stress component acting through strains or deformations → alternative approach

- Eq. (9.19), $W_I = -W_E$ →

$$A(\underline{\epsilon}) = W_E \quad (10.12)$$

... if the internal forces in a solid are conservative, the work done by the externally applied forces = strain energy stored in a body

- assumption ... the forces are applied slowly, in a quasi-steady manner
associated kinetic energy is negligible

- potential of the externally applied loads, Φ ... negative of the work done by the external forces acting through the displacements.

N_p forces, P_i, const. magnitude, line of action fixed in space → "dead loads"

$$\Phi = -W_E = -\sum_{i=1}^{N_p} P_i d_i - \sum_{j=1}^{N_Q} Q_j \phi_j \qquad (10.13)$$

- Non-conservative forces

i) aerodynamic force ... lift $\propto$ AOA, non-conservative, cannot be derived from potential

ii) follower force ... const. magnitude, but the orientation of their line of action changes with the rotation of structures

ex) thrust of a rocket jet engine

10.2 Principle of minimum total potential energy

- system represented by N generalized coord. $\underline{q} = \{q_1, q_2, \cdots, q_N\}^T$
- if the system is conservative, strain energy $A = A(\underline{q})$

potential of the externally applied loads $\Phi = \Phi(\underline{q})$

→ infinitesimal increment

$$dA = \frac{\partial A}{\partial q_1} dq_1 + \frac{\partial A}{\partial q_2} dq_2 + \cdots + \frac{\partial A}{\partial q_N} dq_N = \sum_{i=1}^{N} \frac{\partial A}{\partial q_i} dq_i$$

$$d\Phi = \frac{\partial \Phi}{\partial q_1} dq_1 + \frac{\partial \Phi}{\partial q_2} dq_2 + \cdots + \frac{\partial \Phi}{\partial q_N} dq_N = \sum_{i=1}^{N} \frac{\partial \Phi}{\partial q_i} dq_i \qquad (10.14)$$

- VW done by the internal forces $\delta W_I = -\delta A(\underline{q})$

external " $\delta W_E = -\delta \Phi(\underline{q})$

$$\delta W_I = -\delta A = -\sum_{i=1}^{N} \frac{\partial A}{\partial q_i} \delta q_i$$

$$\delta W_E = -\delta \Phi = -\sum_{i=1}^{N} \frac{\partial \Phi}{\partial q_i} \delta q_i \qquad (10.15)$$

- comparing Eq. (9.24) and (10.15),

$$Q_i^I = -\frac{\partial A}{\partial q_i}, \quad Q_i^E = -\frac{\partial \Phi}{\partial q_i} \qquad (10.16)$$

- PVW : $Q_i^I + Q_i^E = 0$, by introducing Eq. (10.16)

$$-\frac{\partial A}{\partial q_i} - \frac{\partial \Phi}{\partial q_i} = \frac{\partial (A+\Phi)}{\partial q_i} = \frac{\partial \Pi}{\partial q_i} = 0 \qquad (10.17)$$

total potential

$$\delta W = -\delta \Pi$$

- Principle 4 : a system is in static equilibrium iff the sum of the VW done by the internal and external forces vanishes for all arbitrary virtual displacements, $\rightarrow \delta W = -\delta \Pi = 0$

$$\rightarrow \delta \Pi = 0 \qquad (10.18)$$

$$\delta \Pi = \sum_{i=1}^{N} \left[\frac{\partial \Pi}{\partial q_i} \right] \delta q_i = 0 \qquad (10.19)$$

$\frac{\partial \Pi}{\partial q_i} = 0 \rightarrow$ Eq. (10.17)

- Principle 8 : A conservative system is in equilibrium iff virtual changes in the total PE vanish for all virtual displacements.
  "Principle of stationary TPE"

- Kinematically admissible virtual displacements are used $\rightarrow$ reaction forces are eliminated from the formulation.

  Arbitrary virtual displacements " $\rightarrow$ reaction forces must be treated as externally applied loads

- Graphical illustration of Principle 8 (Fig. 10.3)
  ... TPE is stationary at points A, B and C.

- increments in TPE by Taylor series

$$d\Pi \simeq \sum_{i=1}^{N} \frac{\partial \Pi}{\partial q_i} dq_i + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial^2 \Pi}{\partial q_i \partial q_j} dq_i dq_j$$

in the neighborhood of static equilibrium,

$$d\pi \simeq \sum_{i=1}^{N}\sum_{j=1}^{N} \underbrace{\frac{\partial^2 \pi}{\partial q_i \partial q_j} dq_i dq_j} \qquad (10\text{-}20)$$

① $\sqcup > 0$ for all dq_i → TPE is minimum at equilibrium

→ "stable" (A) ... TPE cannot increase without an external source of E

② $\sqcup = 0$ → "neutrally stable" (B)

③ $\sqcup < 0$ → "unstable" (C) ... released PE is converted to KE, leading to spontaneous motion of the system

∘ Principle 9 --- A conservative system is in a "stable" state of equilibrium iff the TPE is a min. w.r.t. changes in the generalized coord.

10.2.1 Non-conservative external forces

- if the externally applied loads are not conservative

$$\delta W = \delta W_I + \delta W_E = -\delta A + \delta W_E^{nc} = 0$$

→ Principle 10 --- A system is in equilibrium iff virtual changes in the strain energy equal the VW done by the externally applied loads for all arbitrary virtual displacements

- if externally applied forces are a mixture of $\begin{cases}\text{conservative} \\ \text{non-conservative}\end{cases}$ forces

$$\delta W_E = \delta W_E^{c} + \delta W_E^{nc}$$

$$\delta(A + \Phi) = \underbrace{\delta W_E^{nc}}_{}$$

↖ VW done by the non-conservative forces

10.3 Strain energy in springs

- strain energy ... function of deformation of the structure

$$A = A(\underline{\epsilon})$$

deformation field → function of $\begin{cases}\text{the displacement field} \\ \text{generalized coord.}\end{cases}$

spring $\begin{cases} \text{rectilinear spring} \\ \text{torsional / rotational spring} \end{cases}$

10.3.1 Rectilinear springs

- 2 primary lumped properties $\begin{cases} \text{stiffness constant} \\ \text{un-stretched length} : u_0 \end{cases}$

- force applied to the spring : F, force in the spring : F_s

constitutive behavior : $F = F(\Delta)$, $\Delta = u - u_0$: extension

$F(\Delta = 0) = F(u = u_0) = 0$

∘ Linearly elastic spring

- relationship between an applied load and the resulting extension is linear ($F = k\Delta$) → spring is linear

- k : stiness constant, unt : force / length, N/m

- strain energy in the spring (10.21)

$$A = W_E = \int_{u_0}^{u} F\,du = \int_{u_0}^{u} k\Delta\,du = \int_0^{\Delta} k\Delta\,d\Delta = \frac{1}{2}k\Delta^2 = \frac{1}{2}F\Delta$$

: positive-definite fn. of Δ, i.e., $A > 0$ for any (+) or (−) Δ

vanishes only when $\Delta = 0$

- internal force in the spring $F_s = -\frac{\partial A}{\partial u} = -k\Delta$

(−) : force in the spring opposes the externally applied force

- constitutive law : straight line in the force vs. extension plot (Fig. 10.5)

strain energy (A) : shaded area under the curve

- complementary strain energy (A′), stress energy : shaded area to the left of the straight line. "force energy"

$$A' = \int_0^F (u - u_0)\,dF = \int_0^F \Delta\,dF = \int_0^F \frac{F}{k}\,dF = \frac{1}{2}\frac{F^2}{k} = \frac{1}{2}F\Delta \quad (10.22)$$

$$A' = \frac{1}{2}\frac{F^2}{k} = \frac{1}{2}F\Delta = \frac{1}{2}k\Delta^2 = A$$

Fig. 10.5 $\quad A = A' = \frac{1}{2}F\Delta$, $\quad A + A' = F\Delta$ $\quad$ (10.23)

– Nonlinearly elastic springs

- metals (aluminum, copper) ... slight amount of nonlinearly elastic behavior prior to yield point

  elastomers ... quite obvious nonlinearly elastic behavior

- analytical models, the simplest form

$$F = F_0 \tanh\left(\frac{\Delta}{u_0}\right) \qquad (10.24)$$

F_0 : ref. force, u_0 : ref. displacement

- Fig. 10.6 ... aluminum, no sharp transition from linear to nonlinear behavior

$$k = \frac{\partial F}{\partial \Delta} = \frac{F_0}{u_0}\,\mathrm{sech}^2\left(\frac{\Delta}{u_0}\right) = k_0\,\mathrm{sech}^2\left(\frac{\Delta}{u_0}\right)$$

$k_0 : = \frac{F_0}{u_0}$, stiffness of the spring at zero elongation

- strain energy

$$A = \int_0^\Delta F\,du = F_0 u_0 \int_0^{\bar{\Delta}} \tanh\bar{\Delta}\, d\bar{\Delta} = F_0 u_0 \ln(\cosh\bar{\Delta})$$

complementary strain energy

$$A' = \int_0^F \Delta\, dF = F_0 u_0 \int_0^{\bar{F}} \mathrm{arctanh}(\bar{F})\, d\bar{F} = u_0 F_0\left(\bar{F}\,\mathrm{arctanh}\bar{F} + \ln\sqrt{1-\bar{F}^2}\right)$$

- In contrast to the linearly elastic spring, $A \neq A'$, however, $A + A' = F\Delta$

- elastic force in the spring

$$F = \frac{\partial A}{\partial \Delta} = \frac{1}{u_0}\frac{\partial}{\partial\bar{\Delta}}\left[F_0 u_0 \ln(\cosh\bar{\Delta})\right] = F_0 \tanh\left(\frac{\Delta}{u_0}\right) \qquad (10.25)$$

- Fig. 10.7, upper ... strain energy or potential

  middle ... force-extension relationship

  → "softening spring", decreasing stiffness at higher extensions

10.3.2 Torsional springs

- angular motion, θ, under the action of an externally applied torque, M (Fig. 10.9)
- linearly elastic torsional spring : $M = k\theta$

  k : unit ... N·m/rad , N·m/deg

10.3.3 Bars

- strain energy

$$A = \frac{1}{2} k e^2 = \frac{1}{2} \frac{EA}{L} e^2 \qquad (10.29)$$

  e : bar elongation

10.4 Strain energy in beams

10.4.1 Beam under axial loads

- beam subjected only to axial loads (Fig. 5.6)

  infinitesimal slice, left face displacement $\bar{u}_1$

  right " $\bar{u}_1 + \left(\frac{d\bar{u}_1}{dx_1}\right) dx_1$

- left face, axial force N_1, displacement from 0 to $\bar{u}_1$, work :

  $-\frac{1}{2} N_1 \bar{u}_1$, (−) due to that displacement and force are counted positive in opposite directions

- right face, work : $\frac{1}{2} N_1 \left[\bar{u}_1 + \left(\frac{d\bar{u}_1}{dx_1}\right) dx_1\right]$
- total work : $\frac{1}{2} N_1 \left(\frac{d\bar{u}_1}{dx_1}\right) dx_1 = \frac{1}{2} N_1 \bar{\epsilon}_1 dx_1$
- external work : $dW_E = \frac{1}{2} N_1 \bar{\epsilon}_1 dx_1 = \frac{1}{2} S \bar{\epsilon}_1^2 dx_1 \qquad (10.33)$

$$a(\bar{\epsilon}_1) = \frac{1}{2} S \bar{\epsilon}_1^2 \qquad (10.34)$$

  : "strain energy density function"

  ... potential of the axial force, $N_1 = -\frac{\partial a(\bar{\epsilon}_1)}{\partial \bar{\epsilon}_1} = -S\bar{\epsilon}_1$

  internal force in the beam

- total strain energy by the axial force distribution

$$A(\bar{\epsilon}) = \int_0^L a(\bar{\epsilon}_1)\, dx_1 = \frac{1}{2}\int_0^L S\,\bar{\epsilon}_1^2\, dx_1 \qquad (10.35)$$

- in term of the axial force

"total stress E"

$$A(\bar{\epsilon}) = \int_0^L \frac{N_1^2}{2S}\, dx_1 = A'(N_1) \quad \text{"complementary E"} \qquad (10.36)$$

$$a'(N_1) = \frac{N_1^2}{2S}$$: "stress energy density function"

"complementary strain energy density"

10.4.2 Beam under transverse loads

- beams subjected to transverse loads (Fig 5.14)
- left face rotation : $\frac{d\bar{u}_2}{dx_1}$

right " $\frac{d\bar{u}_2}{dx_1} + \left(\frac{d^2\bar{u}_2}{dx_1^2}\right)dx_1$

- work by bending moment M_3 at left face : $-\frac{1}{2}M_3\frac{d\bar{u}_2}{dx_1}$

(−) due to that rotation and moment are counted positive in opposite directions

" right " : $\frac{1}{2}M_3\left[\frac{d\bar{u}_2}{dx_1} + \left(\frac{d^2\bar{u}_2}{dx_1^2}\right)dx_1\right]$

- total work : $\frac{1}{2}M_3\left(\frac{d^2\bar{u}_2}{dx_1^2}\right)dx_1 = \frac{1}{2}M_3\,\kappa_3\, dx_1$

sectional curvature

- external work : $dW_E = \frac{1}{2}M_3\kappa_3\, dx_1 = \frac{1}{2}H_{33}^c\kappa_3^2\, dx_1 \qquad (10.37)$

$$a(\kappa_3) = \frac{1}{2}H_{33}^c\kappa_3^2 \quad : \text{"strain energy density fn"} \qquad (10.38)$$

... potential of the bending moment $M_3 = -\frac{\partial a(\kappa_3)}{\partial\kappa_3} = -H_{33}^c\kappa_3$

internal moment in the beam

- total strain E by the bending moment distribution

$$A(\kappa_3) = \int_0^L a(\kappa_3)\, dx_1 = \frac{1}{2}\int_0^L H_{33}^c\kappa_3^2\, dx_1 \qquad (10.39)$$

or

$$A(u_2(x_1)) = \frac{1}{2}\int_0^L H_{33}^c\left(\frac{d^2\bar{u}_2}{dx_1^2}\right)^2 dx_1 \qquad (10.40)$$

or

$$A(M_3) = \int_0^L \frac{M_3^2}{2H_{33}^c}\, dx_1 = A'(M_3) \qquad (10.41)$$

$$a'(M_3) = \frac{1}{2}\frac{M_3^2}{H_{33}^c} \quad \text{"stress E density fn"}$$

10.4.3 Beam under torsional loads

- circular cylindrical beam subjected to torsion
- rotation of the left face : ϕ_1
- " right " : $\phi_1 + \left(\frac{d\phi_1}{dx_1}\right) dx_1$
- work by the torque M_1 at the left face : $-\frac{1}{2} M_1 \phi_1$

  (−) due to that rotation and torque are counted positive in opposite dir.
- " right " : $\frac{1}{2} M_1 \left[\phi_1 + \left(\frac{d\phi_1}{dx_1}\right) dx_1\right]$
- total work : $\frac{1}{2} M_1 \left(\frac{d\phi_1}{dx_1}\right) dx_1 = \frac{1}{2} M_1 \kappa_1 dx_1$

  ↑ sectional twist rate
- external work $dW_E = \frac{1}{2} M_1 \kappa_1 dx_1 = \frac{1}{2} H_{11} \kappa_1^2 dx_1$ (10.42)

  $a(\kappa_1) = \frac{1}{2} H_{11} \kappa_1^2$: "strain E density fn" (10.43)

  ... potential of the torque, $M_1 = -\frac{\partial a(\kappa_1)}{\partial \kappa_1} = -H_{11} \kappa_1$

  ↑ internal torque in the beam
- total strain energy by the torque distribution

$$A(\kappa_1) = \int_0^L a(\kappa_1) dx_1 = \frac{1}{2} \int_0^L H_{11} \kappa_1^2 dx_1 \quad (10.44)$$

or

$$A(M_1) = \int_0^L \frac{M_1^2}{2H_{11}} dx_1 = A'(M_1) \quad \text{"total complementary stain E stored"} \quad (10.45)$$

$$a'(M_1) = \frac{M_1^2}{2H_{11}} \quad : \text{"stress E density fn"}$$

10.4.4 Relationship with VW

- internal VW by a bending moment M_3 : $dW_I = -M_3 \kappa_3 dx_1$, Eq. (9.69)

$$dW_E = -dW_I = M_3 \kappa_3 dx_1$$

However, in Sec. 10.4, strain E stored in beam is

$$dW_E = \frac{1}{2} M_3 \kappa_3 dx_1$$

↑ $\frac{1}{2}$ factor difference

- internal VW : bending moment is assumed to remain constant while undergoing a curvature

$$dW_E = \left[\int_0^{\kappa_3} M_3\, d\kappa_3\right] dx_1 = \left[M_3 \int_0^{\kappa_3} d\kappa_3\right] dx_1 = M_3 \kappa_3\, dx_1$$

- strain E stored in beam : bending moment is assumed grow in proportion to the curvature

$$dW_E = \left[\int_0^{\kappa_3} M_3\, d\kappa_3\right] dx_1 = \left[\int_0^{\kappa_3} k \kappa_3\, d\kappa_3\right] dx_1 = \frac{1}{2} k \kappa_3^2\, dx_1$$

$$= \frac{1}{2} M_3 \kappa_3\, dx_1$$

- same reasoning for torsion

internal, external VW : $dW_E = -dW_I = M_1 \kappa_1\, dx_1$

strain E : $dW_E = \frac{1}{2} H_{11} \kappa_1^2\, dx_1$

difference

- when computing VW and CVW : virtual displacements do not affect the forces or stresses in the system

strain E stored in the structure : internal forces and moments increase in proportion to the deformation

10.5 Strain energy in solids

10.5.1 3-D solid

- Sec. 9.7.3, work done by the constant, external stress

$$W_E = \int_V \underline{\sigma}^T \underline{\epsilon}\, dV \quad \text{Eq. (9.76)}$$

- Then, if the stresses increase in proportion to the deformations

$$W_E = \frac{1}{2} \int_V \underline{\sigma}^T \underline{\epsilon}\, dV \qquad (10.46)$$

- Hooke's law, $\underline{\sigma} = \underline{\underline{C}}\, \underline{\epsilon}$ (2.13)

$$\underline{\underline{C}} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \qquad (2.14)$$

$$W_E = \frac{1}{2} \int_V \frac{E}{(1+\nu)(1-2\nu)} \Big[(1-\nu)(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2) + 2\nu(\epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_3)$$

$$+ \frac{1-2\nu}{2}(\gamma_{23}^2 + \gamma_{31}^2 + \gamma_{12}^2) \Big] dV = \int_V a(\epsilon)\, dV = A(\underline{\epsilon})$$