Crystal Micro-Mechanics

Lecture 2 – Classical definition of stress and strain

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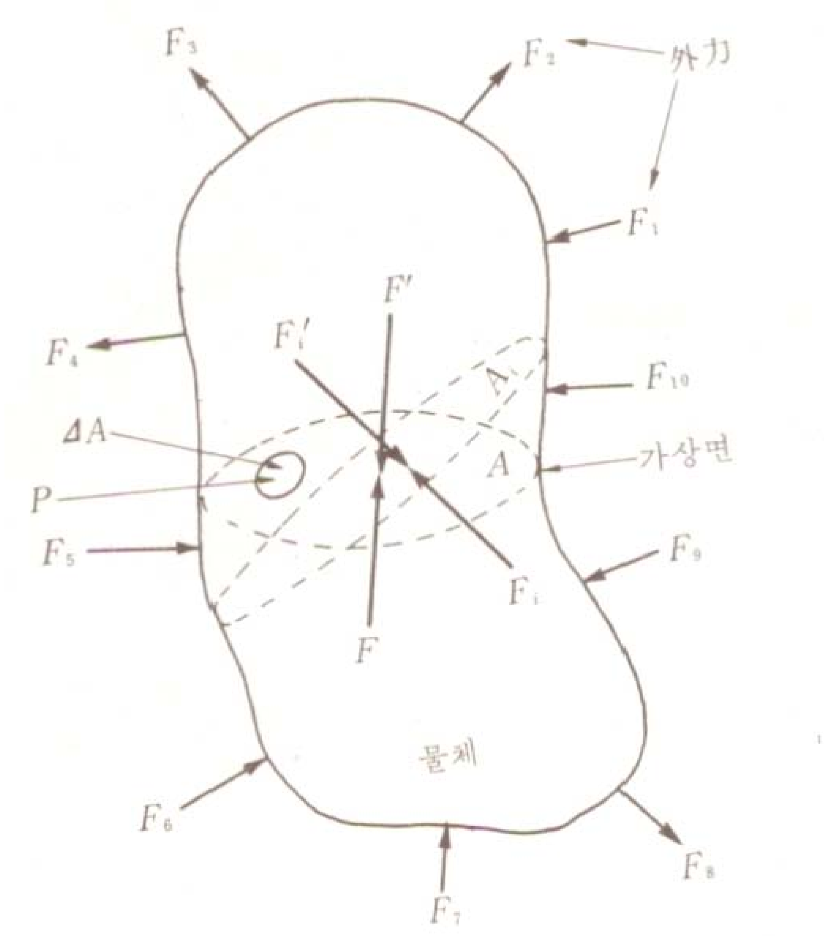
Concept of stress

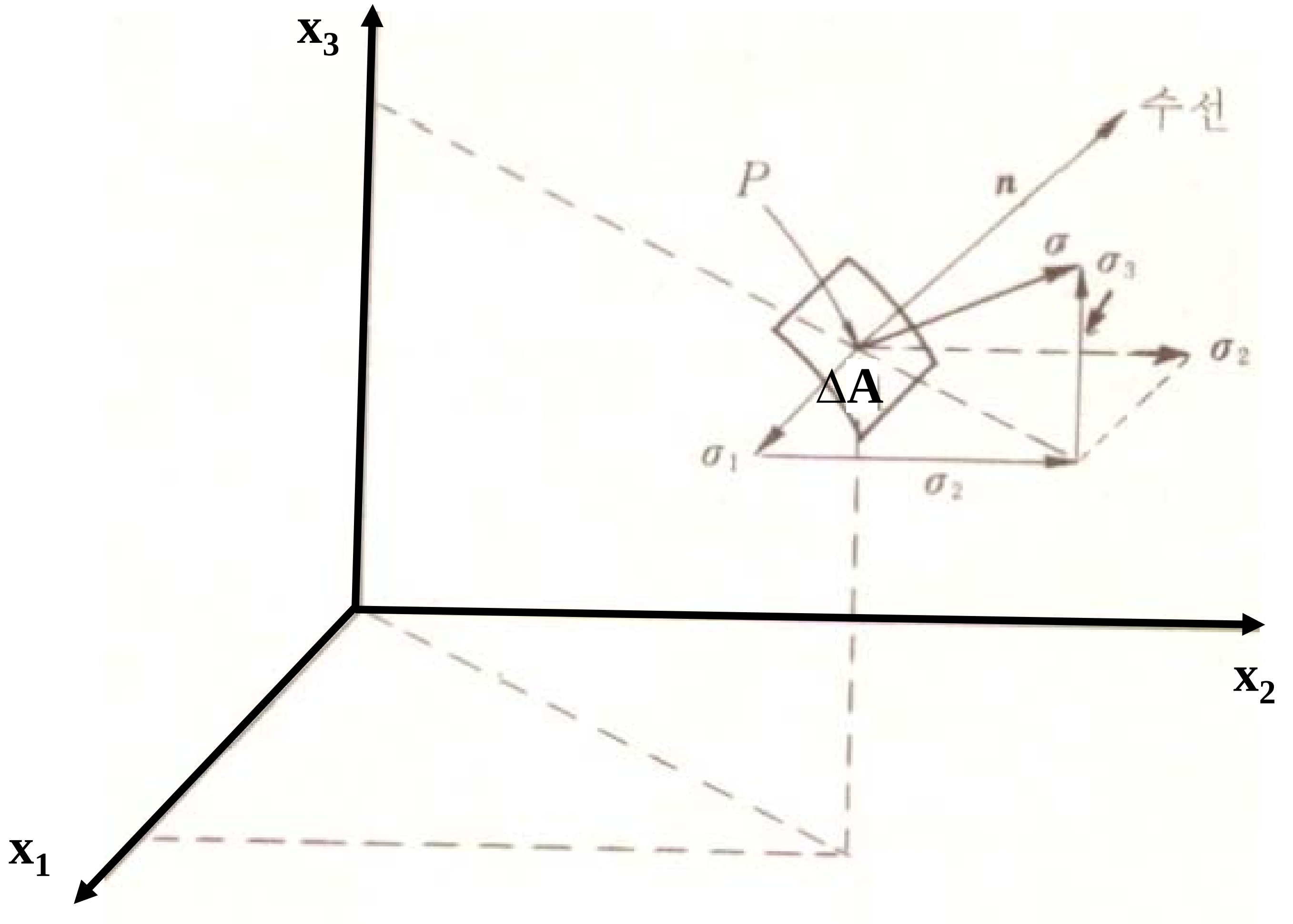


The forces acting within a stressed body are either body forces or surface forces.
We assume an equilibrium state under body forces or surface forces.

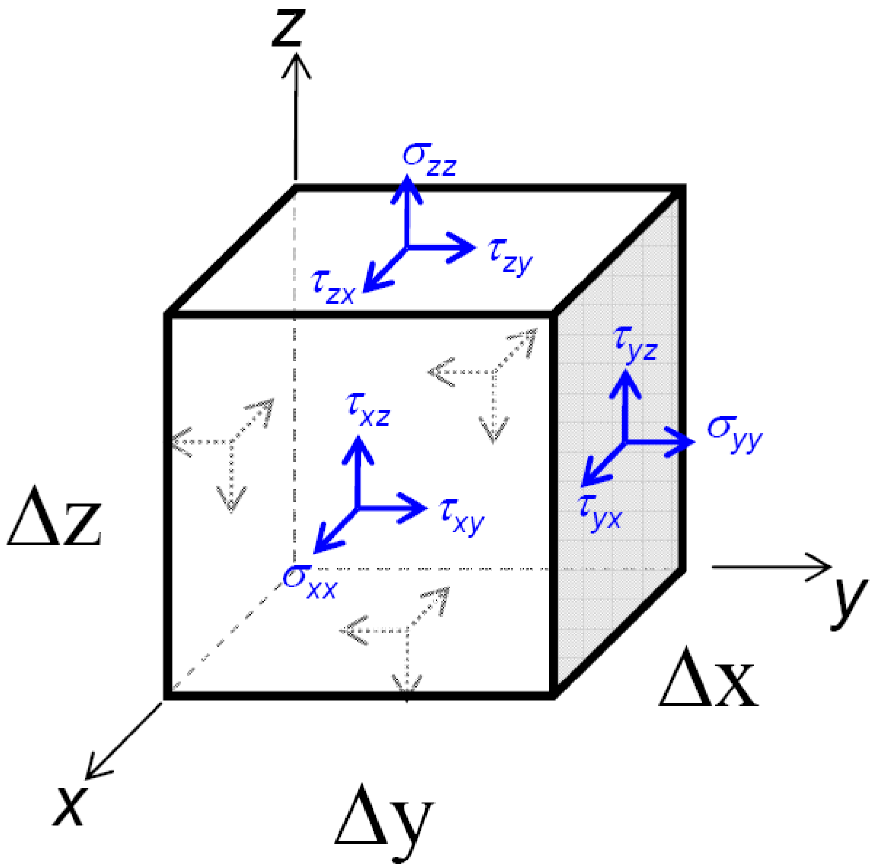
$$\sigma = \lim \frac{\Delta F}{\Delta A} = \frac{dF}{dA}$$

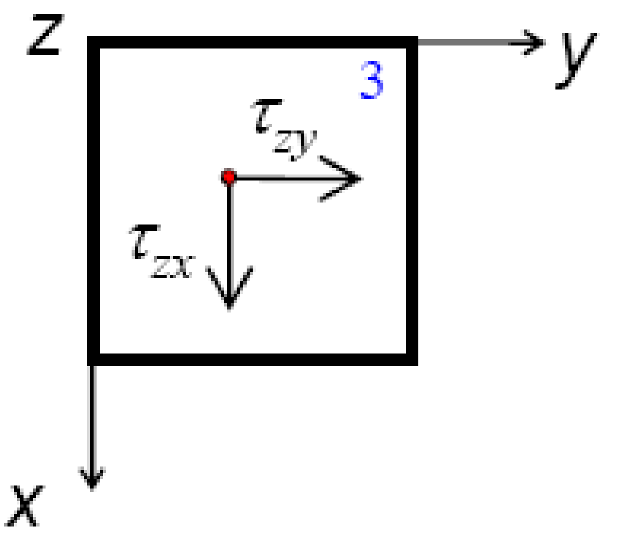
Concept of stress

Decomposition of stress in the direction of coordinate axis



Concept of stress





This construction automatically implies the need for 18 stress components to define the state of stress in the volume element.

Concept of stress

At equilibrium, ΣF = 0 & ΣM = 0

(there can be no net force or torque)

Sum of Forces Parallel To The:

$$x\text{-axis: } \sigma_{xx}A - \sigma_{-x-x}A = 0 \Rightarrow \sigma_{xx} = \sigma_{-x-x}$$

$$y\text{-axis: } \sigma_{yy}A - \sigma_{-y-y}A = 0 \Rightarrow \sigma_{yy} = \sigma_{-y-y}$$

$$z\text{-axis: } \sigma_{zz}A - \sigma_{-z-z}A = 0 \Rightarrow \sigma_{zz} = \sigma_{-z-z}$$

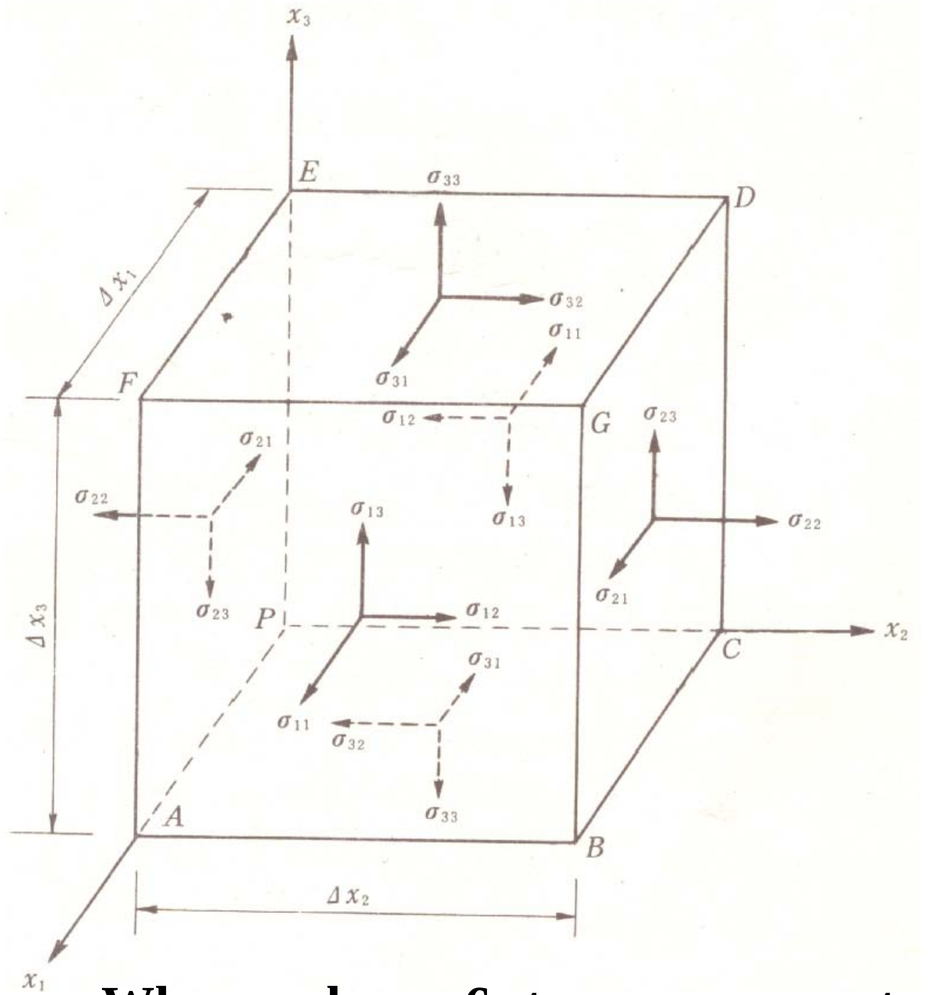
Sum of Moments About The:

$$z\text{-axis: } \left(\tau_{xy}\Delta y\Delta z\right)\Delta x = \left(\tau_{yx}\Delta x\Delta z\right)\Delta y \Rightarrow \tau_{xy} = \tau_{yx}$$

$$y\text{-axis: } \left(\tau_{xz}\Delta z\Delta y\right)\Delta x = \left(\tau_{zx}\Delta x\Delta y\right)\Delta z \Rightarrow \tau_{xz} = \tau_{zx}$$

$$x\text{-axis: } \left(\tau_{zy}\Delta y\Delta x\right)\Delta z = \left(\tau_{yz}\Delta x\Delta z\right)\Delta y \Rightarrow \tau_{zy} = \tau_{yz}$$

Concept of stress



$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

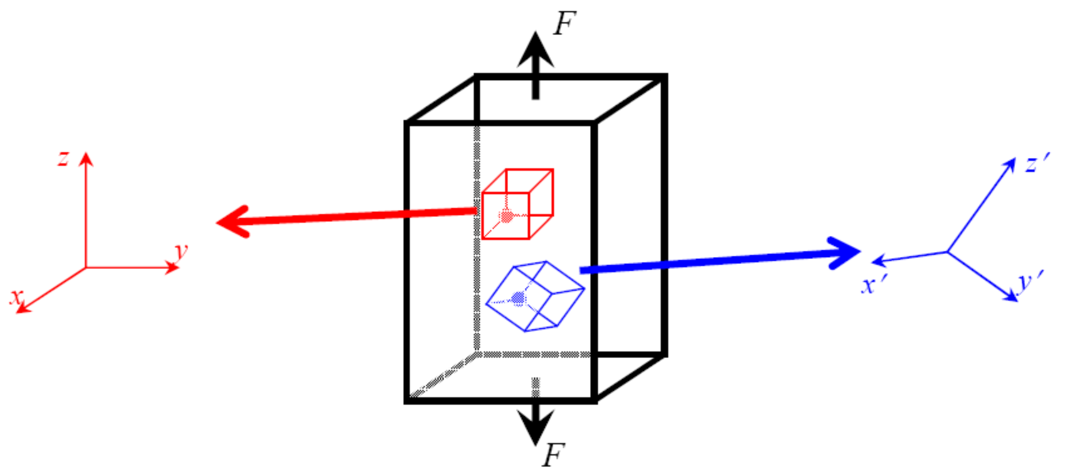
$$\sum M_i = 0$$

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}$$

When we know 6 stress components at a position, we can define the stress tensor at the position in the material.

Concept of stress

Transformation of Coordinates

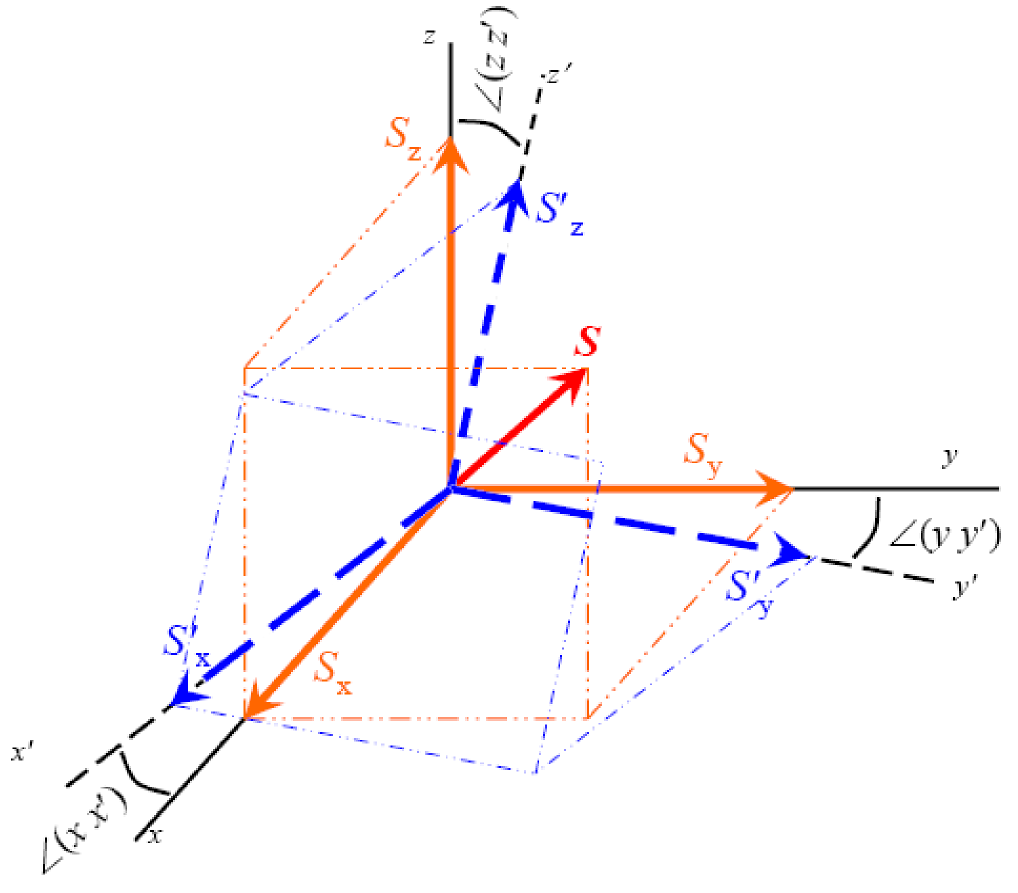


Nearly all solid objects have non-uniform structures (microscopically and macroscopically).

THUS THE STATES OF STRESS GENERALLY VARY FROM POINT TO POINT EVEN THOUGH THE APPLIED FORCES DO NOT CHANGE.

$$\sigma'_{ij} = a_{ik} a_{jl} \sigma_{kl}$$

Transformation of Vector



The vector S can be easily resolved into components that are parallel to any set of reference axes.

Transformation of Vector

The two coordinate systems are related through a series of angles. We define them in terms of direction cosines,

S_i

S_i'		x	y	z
	x'	$\cos(\angle xx')$	$\cos(\angle yx')$	$\cos(\angle zx')$
	y'	$\cos(\angle xy')$	$\cos(\angle yy')$	$\cos(\angle zy')$
	z'	$\cos(\angle xz')$	$\cos(\angle yz')$	$\cos(\angle zz')$

or

S_i

S_i'		x	y	z
	x'	l_x	l_y	l_z
	y'	m_x	m_y	m_z
	z'	n_x	n_y	n_z

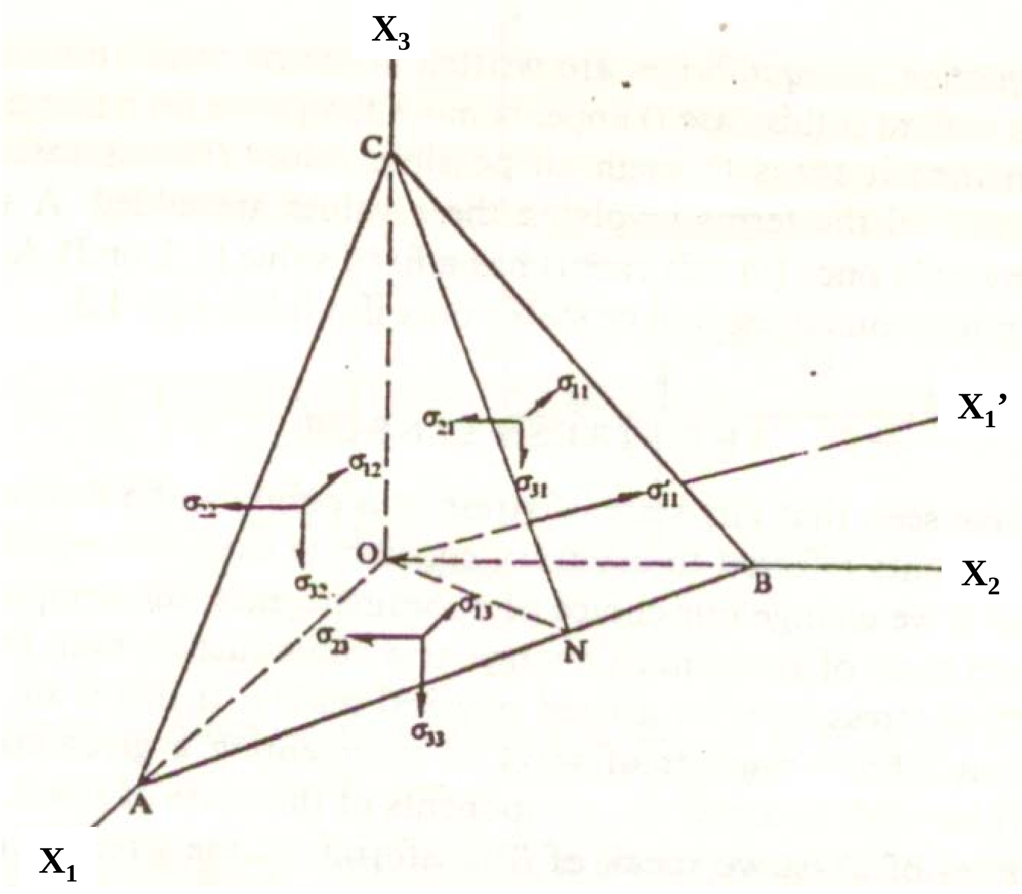
$$\begin{bmatrix} S_x' \\ S_y' \\ S_z' \end{bmatrix} = \begin{bmatrix} l_x & l_y & l_z \\ m_x & m_y & m_z \\ n_x & n_z & n_z \end{bmatrix} \begin{bmatrix} S_x \\ S_y \\ S_z \end{bmatrix}$$

or

$$S_i' = a_{ij} S_j$$

Concept of stress

Principal stresses



Stress components acting on the faces of a tetrahedron OABC

Principal stresses

$$\begin{vmatrix} \sigma_{xx}-\sigma & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy}-\sigma & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz}-\sigma \end{vmatrix} = 0$$

Secular or Characteristic Eq.

$$\sigma^3 - (\sigma_{xx}+\sigma_{yy}+\sigma_{zz})\sigma^2 + (\sigma_{xx}\sigma_{yy}+\sigma_{yy}\sigma_{zz}+\sigma_{xx}\sigma_{zz}-\tau_{xy}^2-\tau_{yz}^2-\tau_{xz}^2)\sigma$$
$$-(\sigma_{xx}\sigma_{yy}\sigma_{zz}+2\tau_{xy}\tau_{yz}\tau_{xz}-\sigma_{xx}\tau_{yz}^2-\sigma_{yy}\tau_{xz}^2-\sigma_{zz}\tau_{xy}^2)=0$$

or

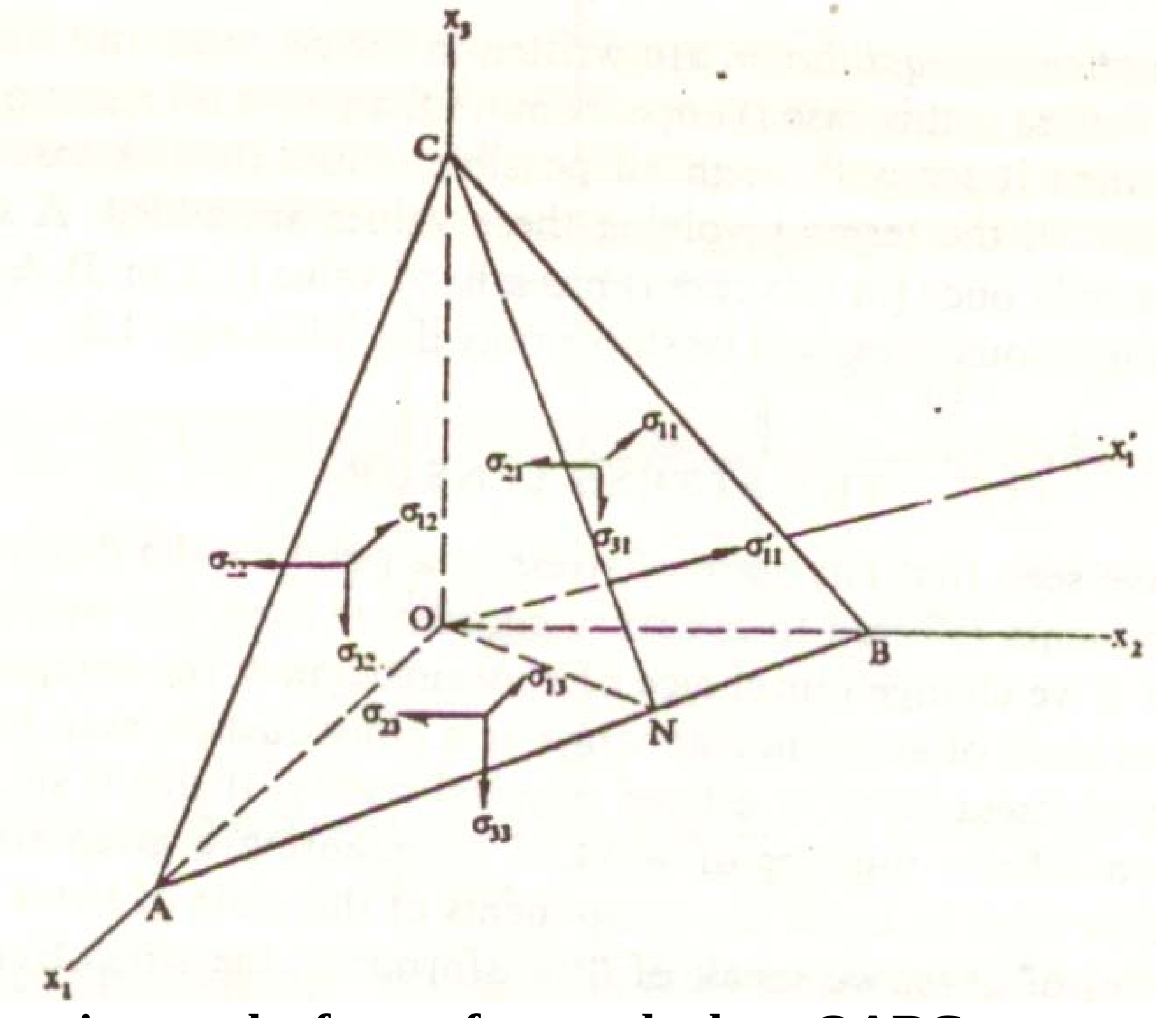
$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

Invariants of stress : I_1, I_2, I_3

Principal stresses

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}$$

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{I} & 0 & 0 \\ 0 & \sigma_{II} & 0 \\ 0 & 0 & \sigma_{III} \end{bmatrix}$$



Stress components acting on the faces of a tetrahedron OABC

Example

Find the principal components and principal axes of the stress

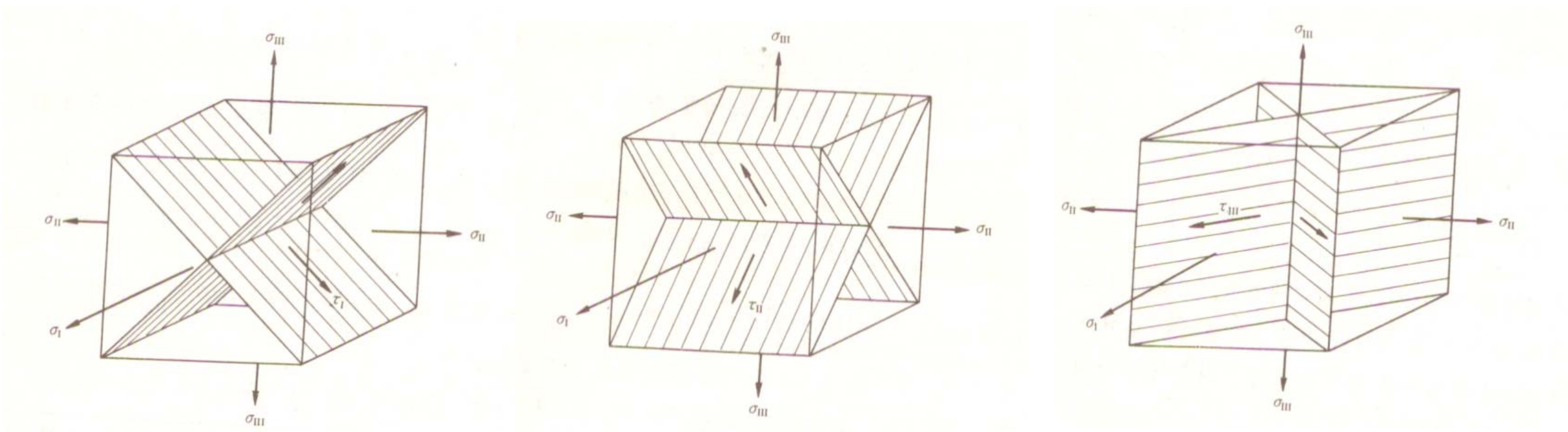
$$\sigma_{ij} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Concept of stress

Maximum shear stress

표 2-1 主剪斷應力面 및 주전단응력

면의 方向餘弦	면 I	면 II	면 III
l_1	0	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$
l_2	$\pm\frac{1}{\sqrt{2}}$	0	$\pm\frac{1}{\sqrt{2}}$
l_3	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$	0
주 전 단 응 력	$\tau_{I}=\frac{\sigma_{II}-\sigma_{III}}{2}$	$\tau_{II}=\frac{\sigma_{I}-\sigma_{III}}{2}$	$\tau_{III}=\frac{\sigma_{I}-\sigma_{II}}{2}$



Concept of stress

HW1 (due 9/13) :

Derive the Table 1 and explain that the maximum shear stress directions bisect the principal stress direction by 45º.

표 2-1 主剪斷應力面 및 주전단응력

면의 方向餘弦	면 I	면 II	면 III
l_1	0	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$
l_2	$\pm\frac{1}{\sqrt{2}}$	0	$\pm\frac{1}{\sqrt{2}}$
l_3	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$	0
주 전 단 응 력	$\tau_{\mathrm{I}}=\frac{\sigma_{\mathrm{II}}-\sigma_{\mathrm{III}}}{2}$	$\tau_{\mathrm{II}}=\frac{\sigma_{\mathrm{I}}-\sigma_{\mathrm{III}}}{2}$	$\tau_{\mathrm{III}}=\frac{\sigma_{\mathrm{I}}-\sigma_{\mathrm{II}}}{2}$

Deviatoric stress

Mean stress :

$$\sigma_m = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} = \frac{\sigma_I + \sigma_{II} + \sigma_{III}}{3} = \frac{\sigma_{ii}}{3} = \frac{I_1}{3} = -P$$

Hydrostatic stress or pressure

Deviatoric stress :

$$\sigma'_{ij} = \begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} & \sigma'_{xz} \\ \sigma'_{xy} & \sigma'_{yy} & \sigma'_{yz} \\ \sigma'_{xz} & \sigma'_{yz} & \sigma'_{zz} \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} - \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}$$

$$\sigma'_{ij} = \sigma_{ij} - \sigma_m \delta_{ij}$$

Concept of stress

Familiar types of stress state

Uniaxial tension

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Uniaxial compression

$$\boldsymbol{\sigma} = \begin{bmatrix} -\sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Pure shear

$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & \sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

by 45^o rotation for ox_3 axis

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & -\sigma & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hydrostatic press

$$\boldsymbol{\sigma} = \begin{bmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{bmatrix}$$

Hydrostatic stress has the same components irrespective of the choice of coordinate axes.

Deviatoric stress

$$\begin{vmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{xy} & \sigma_{zz} \end{vmatrix} = \begin{vmatrix} \sigma_{m} & 0 & 0 \\ 0 & \sigma_{m} & 0 \\ 0 & 0 & \sigma_{m} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} - \sigma_{m} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} - \sigma_{m} & \tau_{yz} \\ \tau_{zx} & \tau_{xy} & \sigma_{zz} - \sigma_{m} \end{vmatrix}$$

Total Stress — Hydrostatic Stress — Stress Deviator

- The stress deviator is what causes distortion of the material element.

Deviatoric stress is a symmetric second rank tensor.

Deviatoric stress

- If we take the determinant of the stress deviator in the same way that we took the determinant of the total stress tensor earlier, we generate a new cubic equation that has three new invariants:

$$\sigma'^3 - J_1\sigma'^2 + J_2\sigma' - J_3 = 0$$

- Two of the new invariants, the invariants of the stress deviator, are of importance:

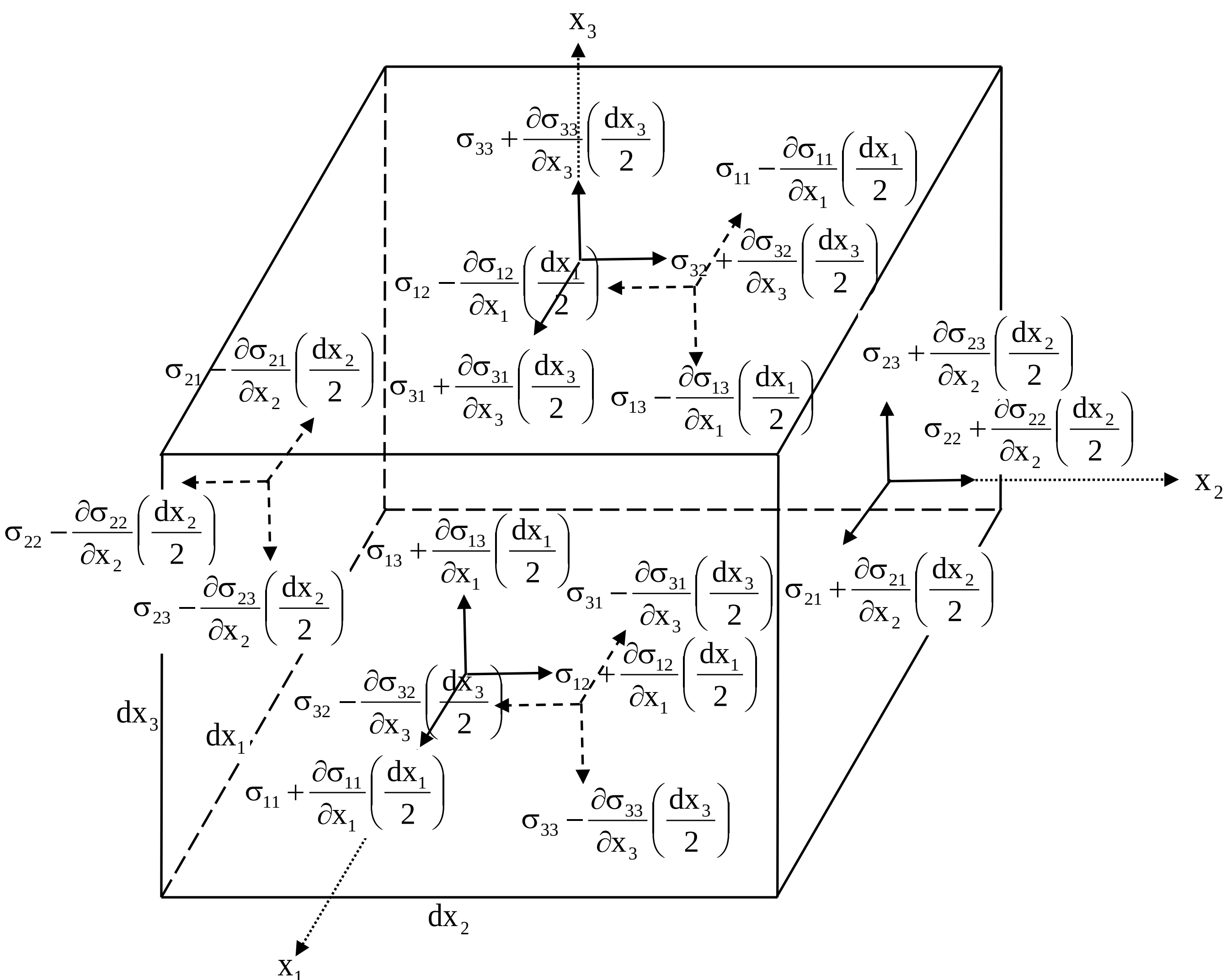
$$J_1 = I_1 - \sigma_m = (\sigma_{xx} - \sigma_M) + (\sigma_{yy} - \sigma_m) + (\sigma_{zz} - \sigma_m) \quad = 0$$

$$J_2 = \frac{1}{6}\left[(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6(\tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2)\right]$$

$$= \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] = 1/2\ \sigma'_{ij}\sigma'_{ij}$$

$$J_3 = 1/3\ \sigma'_{ij}\sigma'_{jk}\sigma'_{ki}$$

Equilibrium Equations for Stress

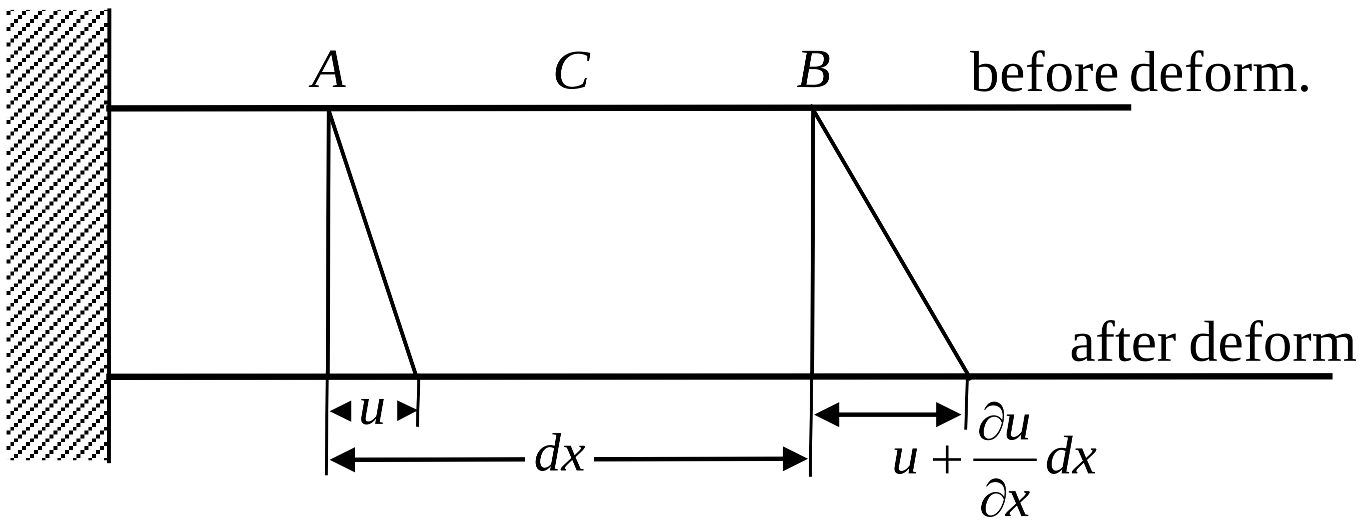


$$\sum F = 0$$

$$\sigma_{ij,j} = 0$$

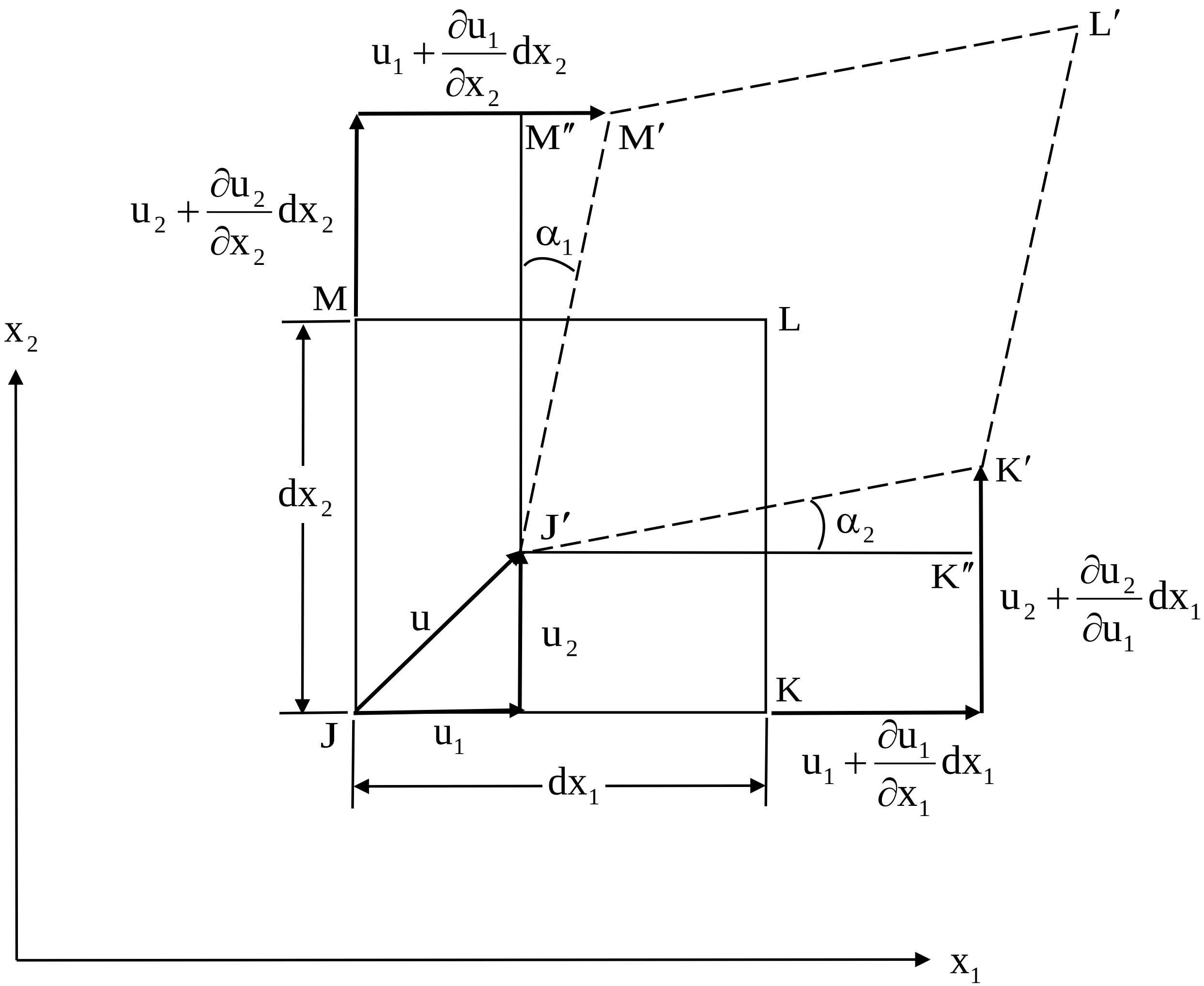
Stress distribution in an infinitesimal element

Concept of strain : classical definition (inifinitesimal strain)



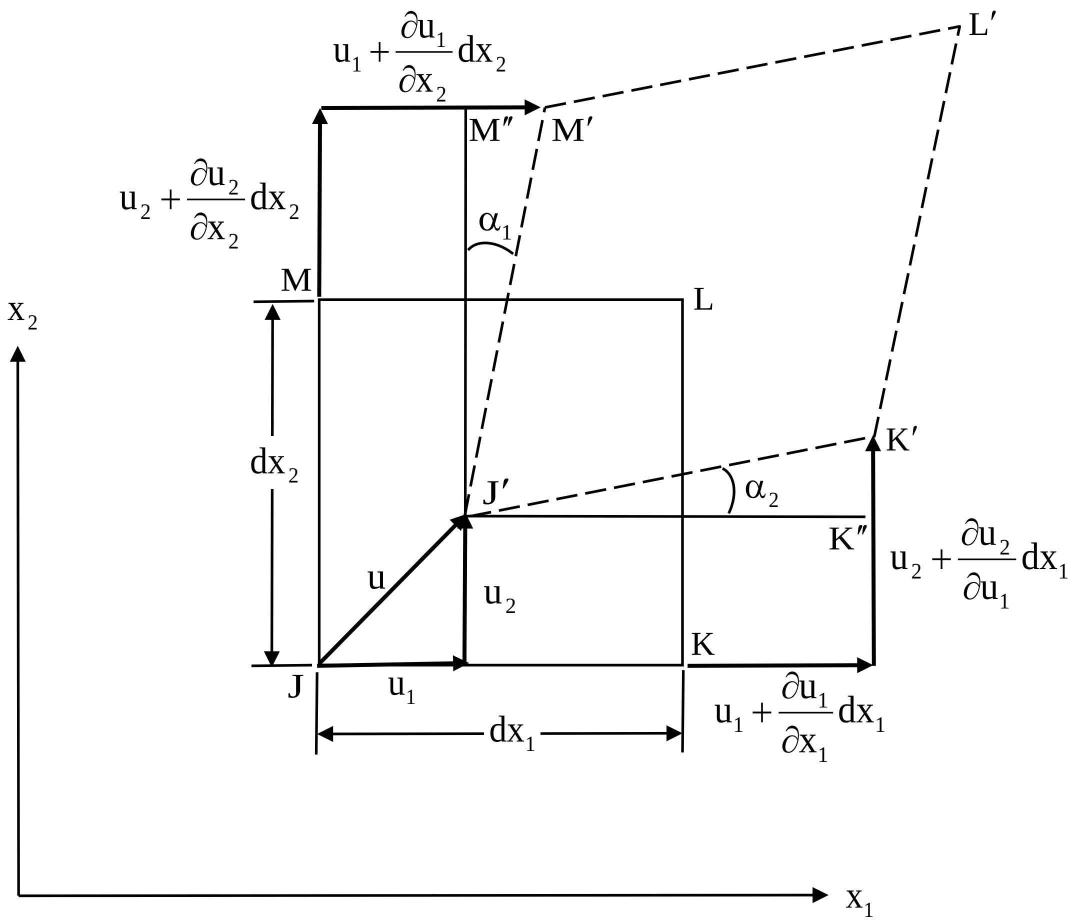
$$\frac{\partial u}{\partial x}$$: **1-dimensional strain**

Concept of strain



2-dimensional strain

Concept of strain



x_1-directional normal

$$e_{11} = \frac{dx_1 + \frac{\partial u_1}{\partial x_1} dx_1 - dx_1}{dx_1} = \frac{\partial u_1}{\partial x_1}$$

x_2-directional normal

$$e_{22} = \frac{dx_2 + \frac{\partial u_2}{\partial x_2} dx_2 - dx_2}{dx_2} = \frac{\partial u_2}{\partial x_2}$$

Shear

$$\gamma_{12} = \alpha_1 + \alpha_2 \cong \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}$$

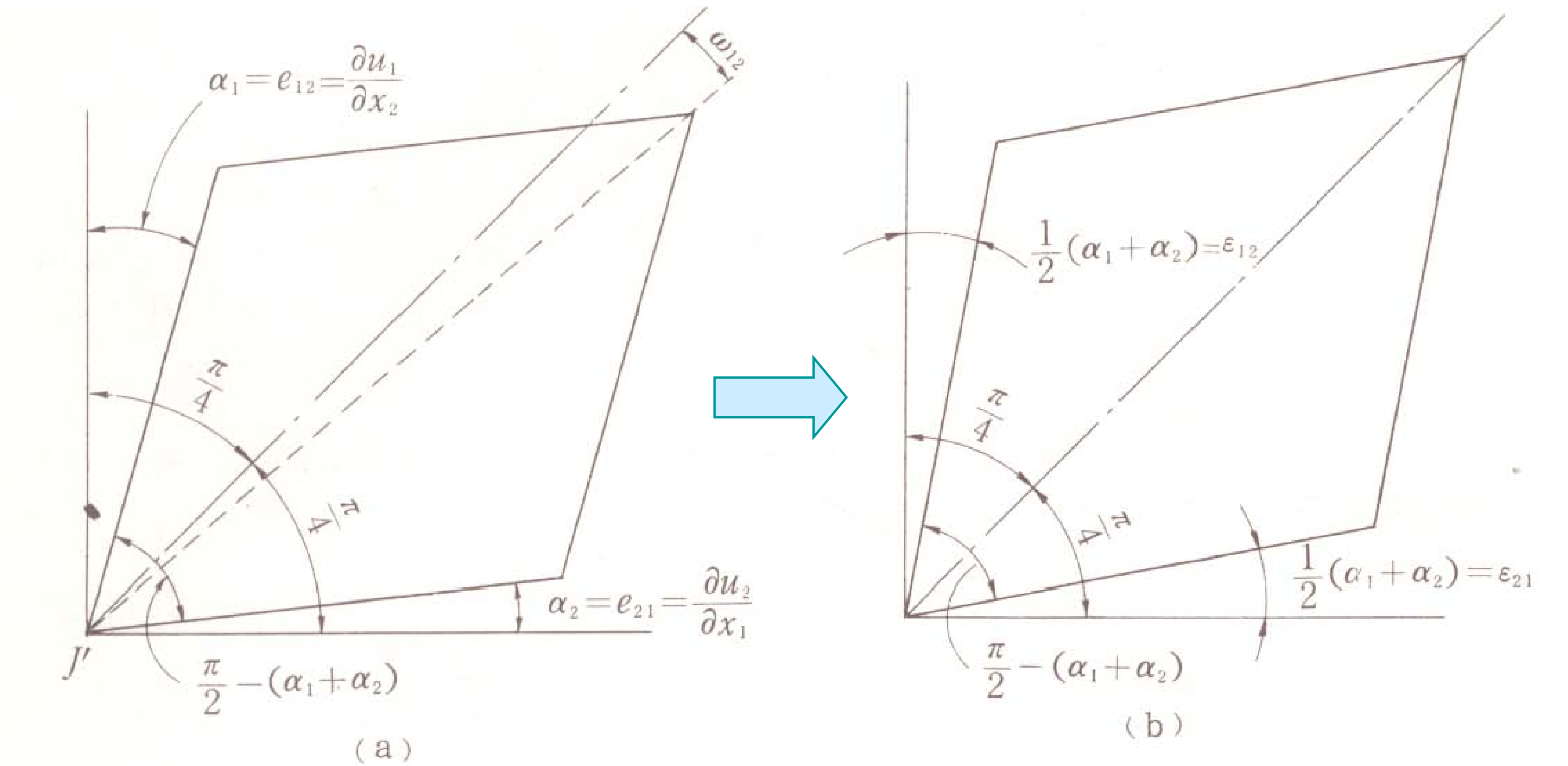
2-dimensional strain

Concept of strain

$$e_{ij} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_1}{\partial x_3} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_2}{\partial x_3} \\ \dfrac{\partial u_3}{\partial x_1} & \dfrac{\partial u_3}{\partial x_2} & \dfrac{\partial u_3}{\partial x_3} \end{bmatrix}$$

3-dimensional strain

Concept of strain



(a) → (b)

Rigid body rotation by ω_{12}

Concept of strain

Strain tensor

$$\varepsilon_{ij}=\begin{bmatrix}\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13}\\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23}\\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right)\\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right)\\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right) & \frac{\partial u_3}{\partial x_3}\end{bmatrix}$$

Spin tensor

$$\varpi_{ij}=\begin{bmatrix}\varpi_{11} & \varpi_{12} & \varpi_{13}\\ \varpi_{21} & \varpi_{22} & \varpi_{23}\\ \varpi_{31} & \varpi_{32} & \varpi_{33}\end{bmatrix}$$

$$=\begin{bmatrix}0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\right)\\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}-\frac{\partial u_3}{\partial x_2}\right)\\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1}-\frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}\right) & 0\end{bmatrix}$$

Concept of strain

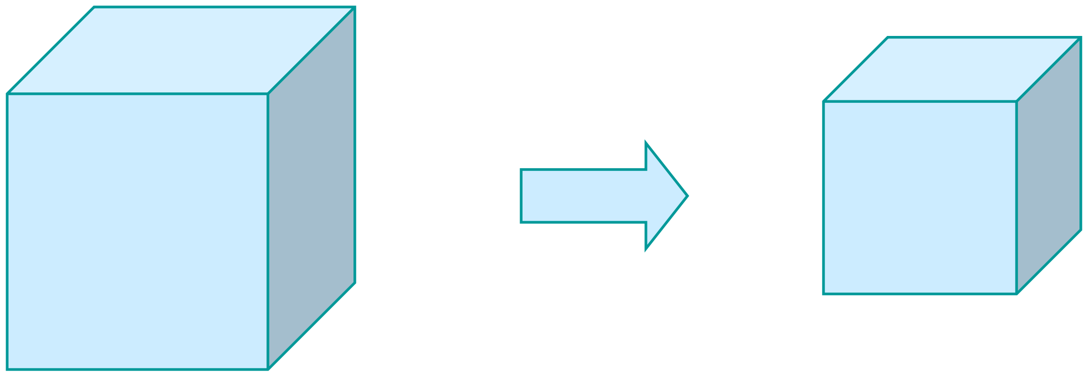
$$e'_{ij} = a_{ki} a_{lj} e_{kl}$$

$$\varepsilon'_{ij} = a_{ki} a_{lj} \varepsilon_{kl}$$

$$\omega'_{ij} = a_{ki} a_{lj} \omega_{kl}$$

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} \Rightarrow \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{\mathrm{I}} & 0 & 0 \\ 0 & \varepsilon_{\mathrm{II}} & 0 \\ 0 & 0 & \varepsilon_{\mathrm{III}} \end{bmatrix}$$

Volumetric Strain



Initial length of side :
dx_1, dx_2, dx_3

Finial length of side :
$(1+\varepsilon_{11})dx_1$, $(1+\varepsilon_{22})dx_2$ $(1+\varepsilon_{33})dx_3$

$$\varepsilon_V = \frac{\Delta V}{V_0} = \frac{V - V_0}{V_0} = \frac{(1+\varepsilon_{11})(1+\varepsilon_{22})(1+\varepsilon_{33})dx_1 dx_2 dx_3 - dx_1 dx_2 dx_3}{dx_1 dx_2 dx_3}$$

$$\varepsilon_V \cong \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = 3\varepsilon_m$$

Deviatoric strain

Mean strain :

$$\varepsilon_m = \frac{\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}}{3} = \frac{\varepsilon_I + \varepsilon_{II} + \varepsilon_{III}}{3} = \frac{\varepsilon_{ii}}{3} = \frac{\varepsilon_V}{3}$$

Deviatoric strain :

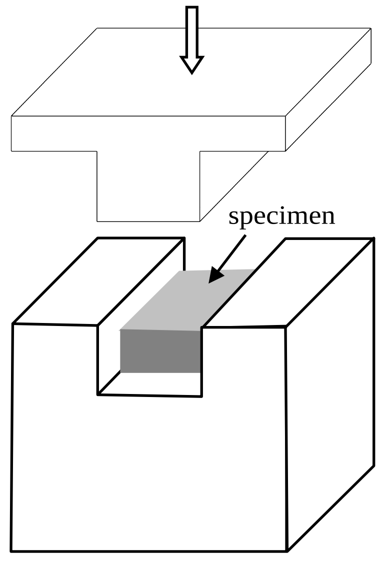
$$\varepsilon'_{ij} = \begin{bmatrix} \varepsilon'_{xx} & \varepsilon'_{xy} & \varepsilon'_{xz} \\ \varepsilon'_{xy} & \varepsilon'_{yy} & \varepsilon'_{yz} \\ \varepsilon'_{xz} & \varepsilon'_{yz} & \varepsilon'_{zz} \end{bmatrix}$$

$$= \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix} - \begin{bmatrix} \varepsilon_m & 0 & 0 \\ 0 & \varepsilon_m & 0 \\ 0 & 0 & \varepsilon_m \end{bmatrix}$$

$$\varepsilon'_{ij} = \varepsilon_{ij} - \varepsilon_m \delta_{ij}$$

Special Types of Strain

Plane strain

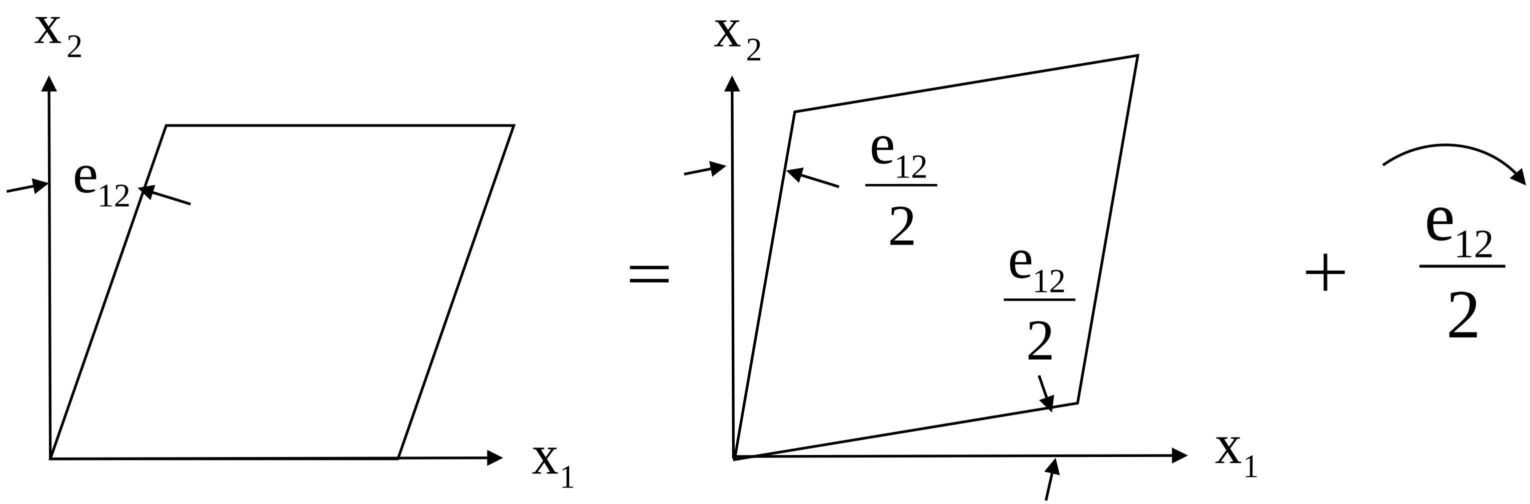


Pure shear

$$\begin{bmatrix} 0 & 0 & \varepsilon_1 \\ 0 & 0 & 0 \\ \varepsilon_1 & 0 & 0 \end{bmatrix}$$

Concept of strain

Simple shear and pure shear



Simple shear

Pure shear

$$e_{12} = (1/2)\,(e_{12} + e_{21}) + (1/2)\,(e_{12} - e_{21})$$

Strain compatibility

General condition

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} = \frac{\partial}{\partial x_1}\left(-\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3}\right)$$

$$\frac{\partial^2 \varepsilon_{22}}{\partial x_3 \partial x_1} = \frac{\partial}{\partial x_2}\left(\frac{\partial \varepsilon_{23}}{\partial x_1} - \frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3}\right)$$

$$\frac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_3}\left(\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} - \frac{\partial \varepsilon_{12}}{\partial x_3}\right)$$

$$2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = \frac{\partial^2 \varepsilon_{11}}{\partial x_2{}^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1{}^2}$$

$$2\frac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3} = \frac{\partial^2 \varepsilon_{22}}{\partial x_3{}^2} + \frac{\partial^2 \varepsilon_{33}}{\partial x_2{}^2}$$

$$2\frac{\partial^2 \varepsilon_{31}}{\partial x_3 \partial x_1} = \frac{\partial^2 \varepsilon_{33}}{\partial x_1{}^2} + \frac{\partial^2 \varepsilon_{11}}{\partial x_3{}^2}$$

Plane strain condition

$$\varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} = 0$$

$$2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = \frac{\partial^2 \varepsilon_{11}}{\partial x_2{}^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1{}^2}$$

Mathematical Theory of Elasticity
I.S. Solkolnikoff, P. 25