Crystal Mechanics

Lecture 3 – Strain measures

Ref : Continuum Theory of Plasticity, A.S. Khan and S. Huang, 1995, Chapter 2

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Configurations and displacement

Configuration

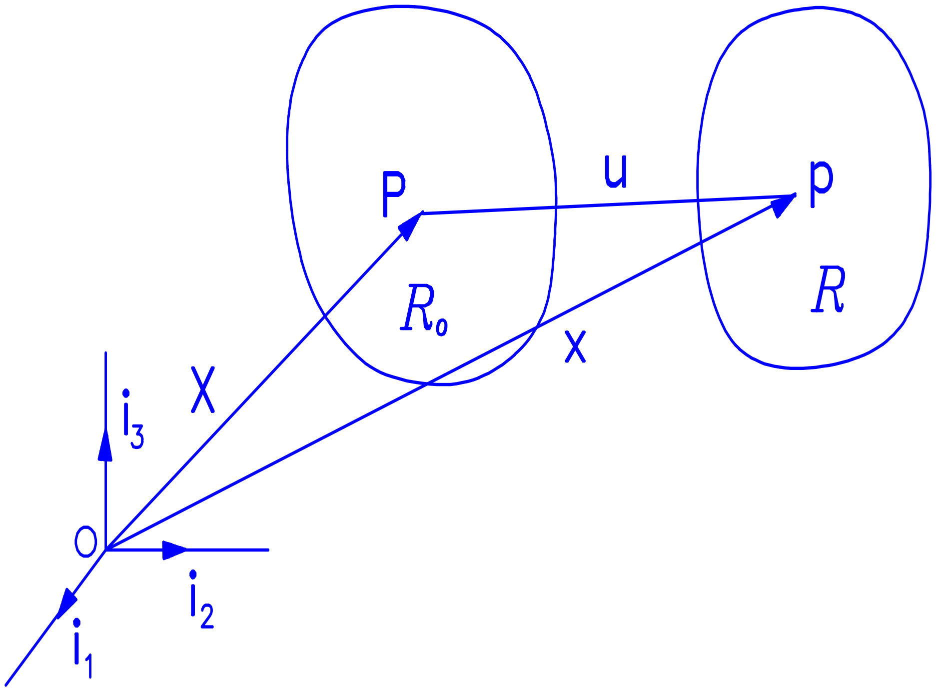
Atom, molecule, particle → Discrete in nature
Material body in large dimension → *"Continuum"*

Configuration and body

Body *B* is a whole set of particle.
The particles in the body *B* are distributed continuously over the region.
Handling variables are classified into two groups *i.e*. time and space.

Configurations and displacement

The origin of our concerned system is the starting point of the every mathematical conceptual manipulation.



Position Vector

$$\boldsymbol{X} = X_K \boldsymbol{i}_K$$

$$\boldsymbol{x} = x_k \boldsymbol{i}_k$$

Configurations and displacement

Displacement Vector

$$x = X + u$$

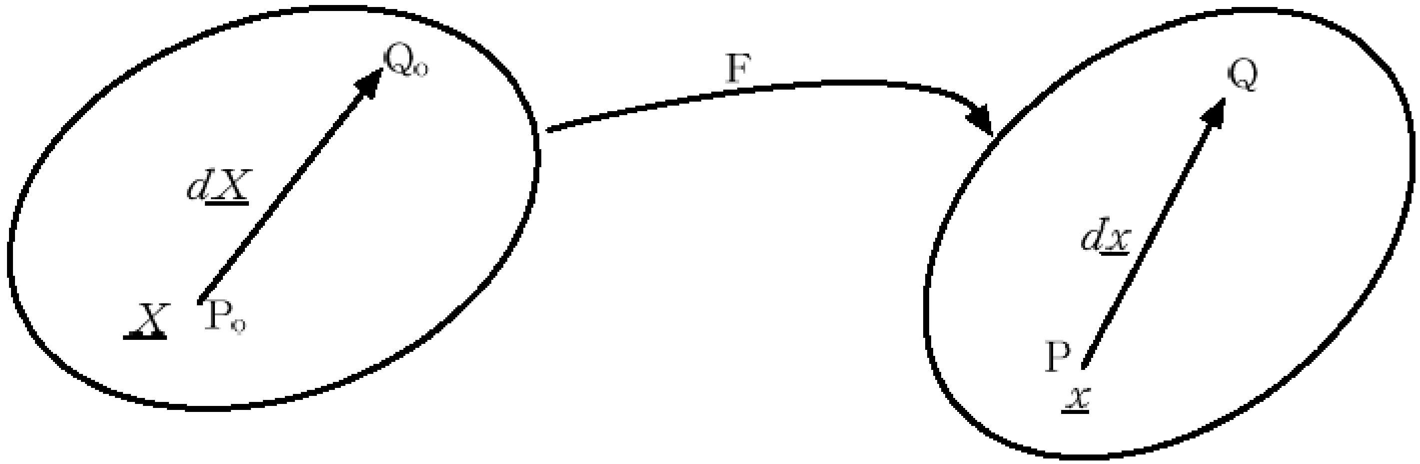
$$u(X, t) = x(X, t) - \boldsymbol{X}$$

in material description (Lagrangian)

$$u(x, t) = x - X(x, t)$$

in spatial description (Eulerian)

Deformation gradient and deformation measure



$$d\boldsymbol{x} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} \cdot d\boldsymbol{X} = \nabla \boldsymbol{x} \cdot d\boldsymbol{X} = \boldsymbol{F} \cdot d\boldsymbol{X}$$

F : Deformation gradient tensor

Deformation gradient in line element

The Physical Meaning of F

1) Diagonal Term : $\dfrac{\partial x_i}{\partial X_i}$: The change of length in i direction.

The degree of stretch along i direction

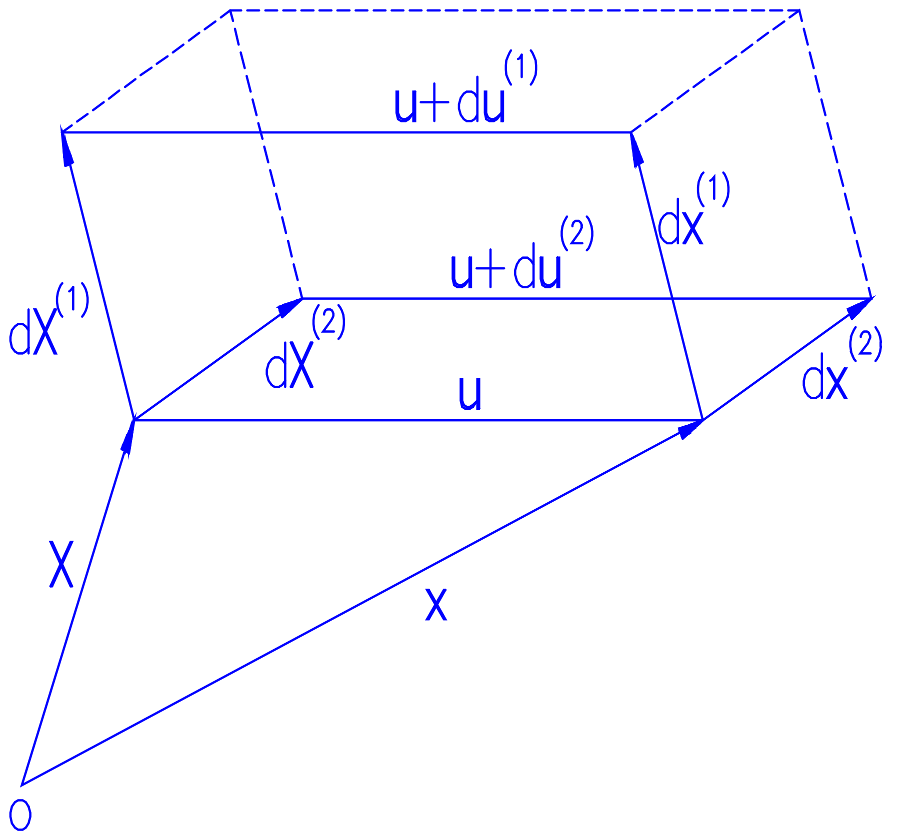
2) Off-diagonal Term : $\dfrac{\partial x_i}{\partial X_j}$: The degree of angle change of i direction in plane j

Example

When $x_1=X_1+aX_2$, $x_2=X_2$, $x_3=X_3$,

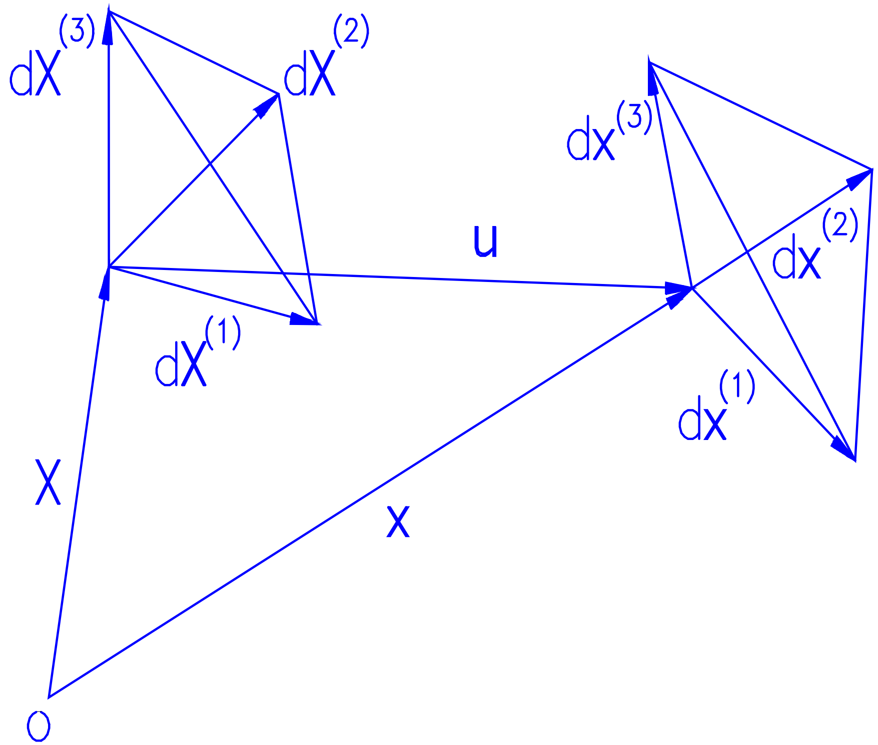
Determine the deformation gradient F.

Deformation gradient in area element



$$d\mathbf{A} = \left[\det(\mathbf{F})\right]^{-1}\mathbf{F}^{T}\cdot d\mathbf{a}$$

Deformation gradient in volume element



$$\frac{\mathbf{d}v}{\mathbf{d}V} = \det(\mathbf{F})$$

Volumetric strain for classical infinitesimal theory

$$\frac{dv - dV}{dV} = \frac{dv}{dV} - 1 \cong tr(u_{i,k}) = u_{i,i}$$

Measures of finite deformation (Cauchy-Green tensor)

$\mathbf{B}^{-1} = (\mathbf{F}^{-1})^T \cdot \mathbf{F}^{-1}$	$\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}$
Cauchy Deformation tensor	Green Deformation tensor
[left Cauchy-Green tensor]	[right Cauchy-Green tensor]
(spatial description)	(material description)

Measures of finite deformation (Strain tensor)

Usually the strain measure should be expected to be zero tensor for rigid body motion.

referential or Lagrangian

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \frac{1}{2}(x_{i,K} x_{i,L} - \delta_{KL}) = E_{KL}$$

spatial or Eulerian

$$\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1}) = \frac{1}{2}(\delta_{ij} - X_{K,i} X_{K,j}) = e_{ij}$$

Measures of finite deformation (Strain tensor)

Green-Lagrangian strain tensor

$$E_{KL} = \frac{1}{2}(u_{K,L} + u_{L,K} + u_{M,K}u_{M,L}) \qquad \mathbf{E} = \frac{1}{2}(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I})$$

Almansi-Eulerian strain tensor

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} - u_{m,i}u_{m,j}) \qquad \mathbf{e} = \frac{1}{2}(\mathbf{I} - (\mathbf{F}^{-1})^T \cdot \mathbf{F}^{-1})$$

Measures of finite deformation (Strain tensor)

Logarithmic (true, natural) strain tensor

referential or Lagrangian

$$\bar{\mathbf{E}} = \frac{1}{2}\ln\mathbf{C} = \ln\mathbf{C}^{1/2}$$

spatial or Eulerian

$$\bar{\mathbf{e}} = -\frac{1}{2}\ln\mathbf{B}^{-1} = \ln(\mathbf{B}^{-1})^{-1/2}$$

For logarithmic tensor, components should be transformed into the principal axis and taken as the eigenvalues.

Decomposition of deformation gradient

Polar decomposition based on Cauchy theorem

F : non-singular second-order tensor can be **decomposed uniquely into**

$$\mathbf{F} = \mathbf{R} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{R}$$

R : orthogonal rotation tensor
U : right Cauchy tensor (symmetric)
V : left Cauchy tensor (symmetric)

$$\mathbf{B} = \mathbf{F} \cdot \mathbf{F}^{\mathrm{T}} = \mathbf{V} \cdot \mathbf{R} \cdot \mathbf{R}^{\mathrm{T}} \cdot \mathbf{V}^{\mathrm{T}} = \mathbf{V}^{2}$$

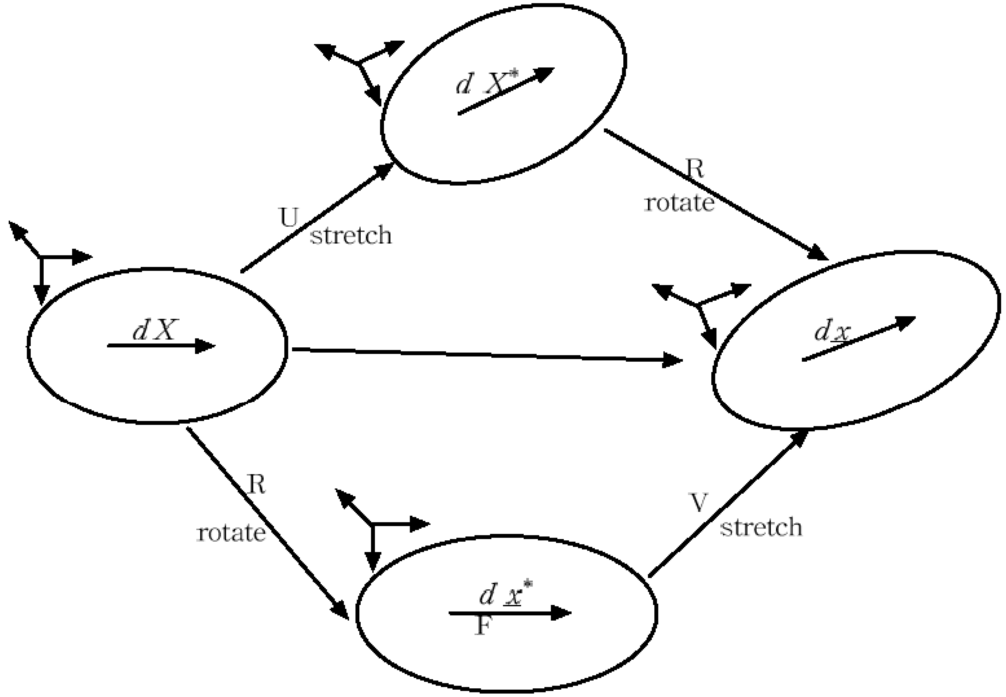
$$\mathbf{C} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} = \mathbf{U}^{\mathrm{T}} \cdot \mathbf{R}^{\mathrm{T}} \cdot \mathbf{R} \cdot \mathbf{V} = \mathbf{U}^{2}$$

Decomposition of deformation gradient

$$d\mathbf{x} = \mathbf{F}\cdot dX$$

$$= \mathbf{R}\cdot\mathbf{U}\cdot d\mathbf{x} = \mathbf{I}\cdot\mathbf{R}\cdot\mathbf{U}\cdot d\mathbf{x} = \mathbf{R}\cdot\mathbf{I}\cdot\mathbf{U}\cdot d\mathbf{x} = \mathbf{R}\cdot\mathbf{U}\cdot\mathbf{I}\cdot d\mathbf{x}$$

$$= \mathbf{V}\cdot\mathbf{R}\cdot d\mathbf{x} = \mathbf{I}\cdot\mathbf{V}\cdot\mathbf{R}\cdot d\mathbf{x} = \mathbf{V}\cdot\mathbf{I}\cdot\mathbf{R}\cdot d\mathbf{x} = \mathbf{V}\cdot\mathbf{R}\cdot\mathbf{I}\cdot d\mathbf{x}$$



Velocity and Acceleration

Velocity : rate of change of displacement

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\mathbf{x}(\mathbf{X}, t + \Delta t) - \mathbf{x}(\mathbf{X}, t)}{\Delta t} = \left(\frac{\partial \mathbf{x}}{\partial t} \right)_X$$

Acceleration : rate of change of velocity

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\mathbf{v}(\mathbf{X}, t + \Delta t) - \mathbf{v}(\mathbf{X}, t)}{\Delta t} = \left(\frac{\partial \mathbf{v}}{\partial t} \right)_X$$

Material Derivatives

Time derivatives with Lagrangian coordinate

→ Material derivatives

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\mathbf{v}(\mathbf{X}, t+\Delta t) - \mathbf{v}(\mathbf{X}, t)}{\Delta t} = \left(\frac{\partial \mathbf{v}(x(X,t),t)}{\partial t} \right)_{\mathbf{X}}$$

$$= \left(\frac{\partial\, \mathbf{v}(x,t)}{\partial t} \right)_{x} + \frac{\partial\, \mathbf{v}(x,t)}{\partial x} \cdot \left(\frac{\partial x}{\partial t} \right)_{\mathbf{X}} = \left(\frac{\partial\, \mathbf{v}(x,t)}{\partial t} \right)_{x} + \mathbf{v} \cdot \frac{\partial\, \mathbf{v}}{\partial x}$$

$$= \left(\frac{\partial\, \mathbf{v}}{\partial \mathrm{t}} \right)_{\mathbf{x}} + \mathbf{v} \cdot \mathrm{grad}\ \mathbf{v} = \frac{\mathrm{D}\mathbf{v}}{\mathrm{Dt}}$$

$$\frac{\mathrm{D}}{\mathrm{Dt}} = \left(\frac{\partial}{\partial \mathrm{t}} \right)_{\mathbf{x}} + \mathbf{v} \cdot \mathrm{grad}$$

Velocity gradient

F : Total deformation of material related with **X** and **x**

Rate of change in area, line, volume can expressed by what?
Answer is velocity gradient **L.**

,

$$d\mathbf{v} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \cdot d\mathbf{x} = \text{grad}\,\mathbf{v} \cdot d\mathbf{x} = \mathbf{L} \cdot d\mathbf{x}$$

$$\mathbf{L} = \text{grad}\,\mathbf{v} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \qquad L_{ij} = v_{i,j} = \dot{x}_{i,j}$$

$$\dot{\mathbf{F}} = \mathbf{L} \cdot \mathbf{F}$$

Velocity gradient

Material derivatives of line, area, volume in initial configuration

$$\frac{\mathrm{D}}{\mathrm{Dt}}(\mathrm{dX}) = \frac{\mathrm{D}}{\mathrm{Dt}}(\mathrm{d}\,\mathbf{A}) = \frac{\mathrm{D}}{\mathrm{Dt}}(\mathrm{d}\mathbf{V}) = 0$$

,

Material derivatives of line, area, volume in current configuration

$$\frac{\mathrm{D}}{\mathrm{Dt}}(\mathrm{d}\,\mathbf{x}) = \mathbf{L}\cdot \mathrm{d}\,\mathbf{x}$$

$$\frac{\mathrm{D}}{\mathrm{Dt}}(\mathrm{d}\,\mathbf{a}) = [\mathrm{tr}(\mathbf{L})\,\mathbf{I} - (\mathbf{L})^{\mathrm{T}}]\cdot \mathrm{d}\,\mathbf{a}$$

$$\frac{\mathrm{D}}{\mathrm{Dt}}(\mathrm{d}\,\mathbf{v}) = \mathrm{tr}(\mathbf{L})\mathrm{d}\,\mathbf{v}$$

Deformation rate and spin tensors

$$\mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \mathbf{D} + \mathbf{W}$$

$$\mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^{\mathrm{T}}\right) \quad \text{Deformation Rate Tensor}$$

,

$$\mathbf{W} = \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^{\mathrm{T}}\right) \quad \text{Spin Rate Tensor}$$

$$\mathrm{tr}(\mathbf{L}) = \mathrm{tr}(\mathbf{D})$$

$$\mathrm{tr}(\mathbf{W}) = 0$$

Material derivatives of strain tensors

$$\dot{\mathbf{C}} = 2\mathbf{F}^{\mathrm{T}} \cdot \mathbf{D} \cdot \mathbf{F}$$

$$\dot{\mathbf{B}} = \mathbf{B} \cdot \mathbf{L}^{\mathbf{T}} + \mathbf{L} \cdot \mathbf{B}$$

,

$$\dot{\mathbf{E}} = \frac{1}{2}\dot{\mathbf{C}} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{D} \cdot \mathbf{F}$$

$$\dot{\mathbf{e}} = -\frac{1}{2}\dot{\mathbf{B}}^{-1} = \frac{1}{2}\left(\mathbf{L}^{\mathrm{T}} \cdot \mathbf{B}^{-1} + \mathbf{B}^{-1} \cdot \mathbf{L}\right) = \mathbf{D} - \left(\mathbf{e} \cdot \mathbf{L} + \mathbf{L}^{\mathrm{T}} \cdot \mathbf{e}\right)$$

Homework