결정미소역학 (Crystal Mechanics)

Lecture 4 – Stress measures and their examples

Ref : Continuum Theory of Plasticity, A.S. Khan and S. Huang, 1995, Chapter 2

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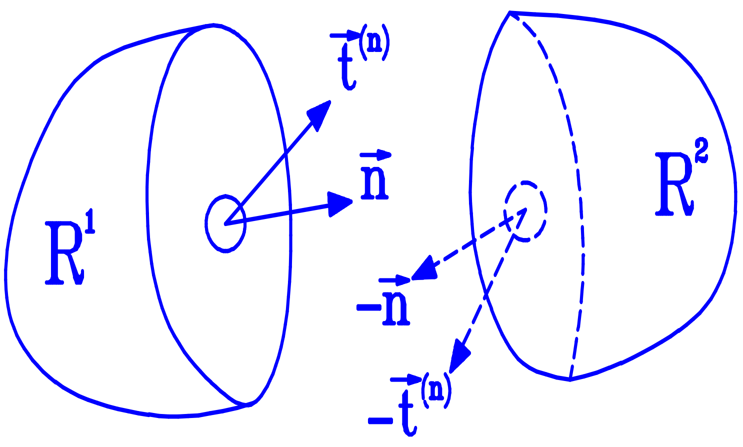
Cauchy stress tensor (at current configuration)

Euler-Cauchy Stress Principle

"Tractions exist across every internal surface element of a body, and in terms of these tractions, the laws of motion apply for any interior region as well as the body as a whole"



$$\mathbf{t}^{(n)} = \lim_{\Delta a \to 0} \frac{\Delta \mathbf{p}}{\Delta a} \quad \Rightarrow \quad \mathbf{t}^{(n)} = \mathbf{n} \cdot \sigma = n_i \sigma_{ij}$$

Cauchy stress tensor

$$\sigma = [\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

(2nd rank order symmetric tensor)

σ_{ij} → i : plane normal and j : direction

$$\sigma_{ij}' = a_{ik} a_{jl} \sigma_{kl}$$

** Defined at current configuration

Stress measures at original configuration

1st Piola-Kirchhoff stress

$$\Sigma^{\mathrm{I}} = \det(\mathbf{F})\mathbf{F}^{-1} \cdot \sigma \quad \rightarrow \text{Non-symmetric}$$

2nd Piola-Kirchhoff stress

$$\Sigma^{II} = \Sigma^{I} \cdot (\mathbf{F}^{-1})^{T} = \det(\mathbf{F})\mathbf{F}^{-1} \cdot \sigma \cdot (\mathbf{F}^{-1})^{T} \quad \rightarrow \text{Symmetric}$$

Relationship

$$\sigma = \frac{1}{\det(\mathbf{F})}\mathbf{F} \cdot \Sigma^{\mathrm{I}}$$

$$\sigma = \frac{1}{\det(\mathbf{F})}\mathbf{F} \cdot \Sigma^{\mathrm{II}} \cdot \mathbf{F}^{\mathrm{T}}$$

$$\Sigma^{\mathrm{I}} = \Sigma^{\mathrm{II}} \cdot \mathbf{F}^{\mathrm{T}}$$

work-conjugate

> **"The choice of stress measure depends on the case of finding strain measure, which is work-conjugate of the selected stress measure"**

$$\int_v \mathrm{tr}(\sigma \cdot \mathrm{D})\, dv = \int_v \mathrm{tr}(\sigma \cdot \mathrm{L})\, dv = \int_v \mathrm{tr}\left(\frac{1}{\det(\boldsymbol{F})}\boldsymbol{F} \cdot \boldsymbol{\Sigma}^{\mathrm{I}} \cdot \boldsymbol{L}\right) dv = \int_V \mathrm{tr}(\Sigma^{\mathrm{I}} \cdot \dot{\boldsymbol{F}})\, dV$$

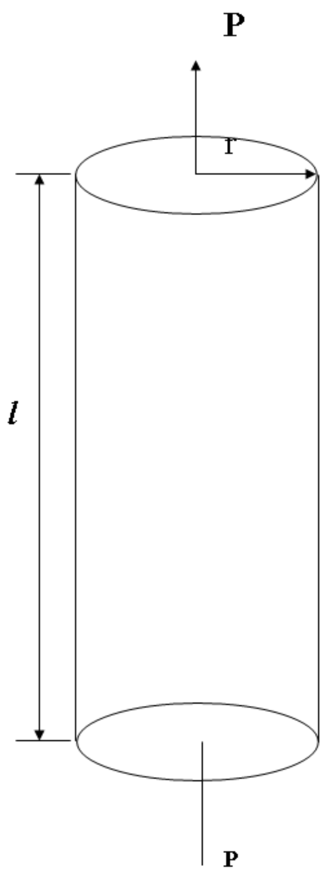
$$\mathrm{tr}(\Sigma^{\mathrm{I}} \cdot \dot{\boldsymbol{F}}) = \mathrm{tr}(\Sigma^{\mathrm{II}} \cdot \boldsymbol{F}^{\mathrm{T}} \cdot \dot{\boldsymbol{F}}) = \mathrm{tr}\left(\Sigma^{\mathrm{II}} \cdot \boldsymbol{F}^{\mathrm{T}} \cdot \frac{\boldsymbol{L} + \boldsymbol{L}^{\mathrm{T}}}{2} \cdot \boldsymbol{F}\right) = \mathrm{tr}(\Sigma^{\mathrm{II}} \cdot \dot{\boldsymbol{E}})$$

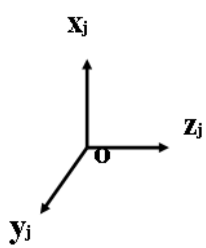
Stress power

$$\sigma : \mathbf{D} = \mathbf{\Sigma}^{\mathbf{I}} : \dot{\mathbf{F}} = \mathbf{\Sigma}^{\mathbf{II}} : \dot{\mathbf{E}}$$

Cauchy stress $\sigma \leftrightarrow$	Deformation Rate Tensor	$\mathbf{D}$
1st P-K stress $\Sigma^{\mathrm{I}} \leftrightarrow$	Deformation Gradient Rate	$\dot{\mathbf{F}}$
2nd P-K stress $\Sigma^{\mathrm{II}} \leftrightarrow$	Green-Lagrangian Strain Rate	$\dot{\mathbf{E}}$

Examples
- Uniaxial Tension or Compression



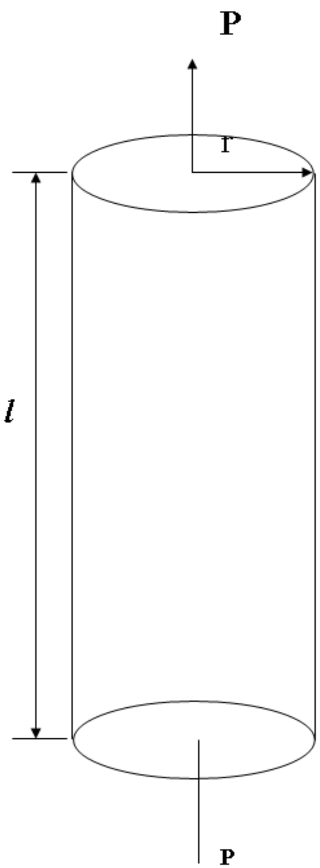


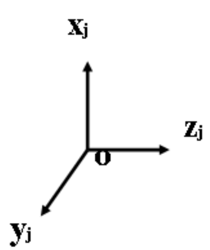
$$x = \lambda_1 X$$
$$y = \lambda_2 Y$$
$$z = \lambda_2 Z$$

$$\lambda_1 = \frac{l}{L}$$
$$\lambda_2 = \frac{r}{R}$$

R, F, U, V, C, B, E, e, L, W, D, σ, Σ^{I}, Σ^{II} ?

Examples
- Uniaxial Tension or Compression





$$x = \lambda_1 X$$
$$y = \lambda_2 Y$$
$$z = \lambda_2 Z$$

$$\lambda_1 = \frac{l}{L}$$

$$\lambda_2 = \frac{r}{R}$$

$$\mathbf{R} = \mathbf{I}, \quad \mathbf{F} = \mathbf{U} = \mathbf{V} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \quad \mathbf{F}^{-1} = \begin{bmatrix} \lambda_1^{-1} & 0 & 0 \\ 0 & \lambda_2^{-1} & 0 \\ 0 & 0 & \lambda_2^{-1} \end{bmatrix}$$

$$\mathbf{C} = \mathbf{U}^2 = \mathbf{F}^T \cdot \mathbf{F} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_2^2 \end{bmatrix},$$

Examples
- Uniaxial Tension or Compression

$$\mathbf{B}^{-1} = \mathbf{V}^{-2} = \left(\mathbf{F}^{-1}\right)^{T}\left(\mathbf{F}^{-1}\right) = \begin{bmatrix} \frac{1}{\lambda_1^2} & 0 & 0 \\ 0 & \frac{1}{\lambda_2^2} & 0 \\ 0 & 0 & \frac{1}{\lambda_2^2} \end{bmatrix}$$

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \frac{1}{2}\begin{bmatrix} \lambda_1^2 - 1 & 0 & 0 \\ 0 & \lambda_2^2 - 1 & 0 \\ 0 & 0 & \lambda_2^2 - 1 \end{bmatrix}$$

$$\mathbf{e} = \frac{1}{2}\left(\mathbf{I} - \mathbf{B}^{-1}\right) = \frac{1}{2}\begin{bmatrix} 1 - \frac{1}{\lambda_1^2} & 0 & 0 \\ 0 & 1 - \frac{1}{\lambda_2^2} & 0 \\ 0 & 0 & 1 - \frac{1}{\lambda_2^2} \end{bmatrix}$$

Examples
- Uniaxial Tension or Compression

$$\bar{\mathbf{e}} = \ln \mathbf{V} = \begin{bmatrix} \ln \lambda_1 & 0 & 0 \\ 0 & \ln \lambda_2 & 0 \\ 0 & 0 & \ln \lambda_2 \end{bmatrix} = \begin{bmatrix} \ln \frac{l}{L} & 0 & 0 \\ 0 & \ln \frac{r}{R} & 0 \\ 0 & 0 & \ln \frac{r}{R} \end{bmatrix}$$

$$\mathbf{L} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} = \begin{bmatrix} \dot{\lambda}_1 & 0 & 0 \\ 0 & \dot{\lambda}_2 & 0 \\ 0 & 0 & \dot{\lambda}_2 \end{bmatrix} \begin{bmatrix} \lambda_1^{-1} & 0 & 0 \\ 0 & \lambda_2^{-1} & 0 \\ 0 & 0 & \lambda_2^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\dot{\lambda}_1}{\lambda_1} & 0 & 0 \\ 0 & \frac{\dot{\lambda}_2}{\lambda_2} & 0 \\ 0 & 0 & \frac{\dot{\lambda}_2}{\lambda_2} \end{bmatrix} = \begin{bmatrix} \frac{\dot{l}}{l} & 0 & 0 \\ 0 & \frac{\dot{r}}{r} & 0 \\ 0 & 0 & \frac{\dot{r}}{r} \end{bmatrix}$$

$$\mathbf{L} = \mathbf{D}, \qquad \mathbf{W} = \mathbf{0}$$

Examples
- Uniaxial Tension or Compression

$$\sigma = \begin{bmatrix} \frac{P}{\pi r^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{P}{\lambda_2^2 \pi R^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{P}{\lambda_2^2 A_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sum^{I} = \det(\mathbf{F})\mathbf{F}^{-1} \cdot \sigma = \lambda_1 \lambda_2^2 \begin{bmatrix} \frac{1}{\lambda_1} & 0 & 0 \\ 0 & \frac{1}{\lambda_2} & 0 \\ 0 & 0 & \frac{1}{\lambda_2} \end{bmatrix} \begin{bmatrix} \frac{P}{\lambda_2^2 A_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{P}{A_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sum^{II} = \det(\mathbf{F})\mathbf{F}^{-1} \cdot \sigma \cdot \left(\mathbf{F}^{-1}\right)^{T} = \begin{bmatrix} \frac{P}{\lambda_1 A_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$