457.562 Special Issue on River Mechanics (Sediment Transport)

.07 Equation of particle motion 1

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1. Definitions and notation

- D: particle diameter
- ρ_s : sediment density
- ρ : fluid density
- $u_{pi}=(u_p, v_p, w_p)$: Instantaneous particle velocity.
- $u_i=(u, v, w)$: Instantaneous fluid velocity
- V_p : particle volume

2. Equation of particle motion

- The instantaneous equation of momentum of a particle i mmersed in a fluid

$$\rho_s V_p \frac{du_{pi}}{dt} = -\rho_s V_p g_i + \int_S \tau_{ji} n_j \, dS$$

- Stress tensor from N-S

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = +\frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j} + g_i$$

- n_j is denotes an outward unit normal to the surface of the grain.

3. Case: Particle moving through quiescent fluid

- The particle equation has no general solution so we will l ook up the several cases.
- Particle falls through a quiescent fluid.

$$\int_S \tau_{ji} n_j \, dS = B_i + D_i + A_i$$

- Buoyancy force $B_i = -\rho V_p g_i$

- Drag force

$$D_i = -\frac{1}{2} \rho C_D A_D \left| u_p \right| u_{pi}$$

$$A_D = \pi \left(\frac{D}{2} \right)^2$$

$$\left| u_p \right| = \sqrt{u_{pi} u_{pi}}$$

3. Case: Particle moving through quiescent fluid

- When the particle Reynolds number defined as

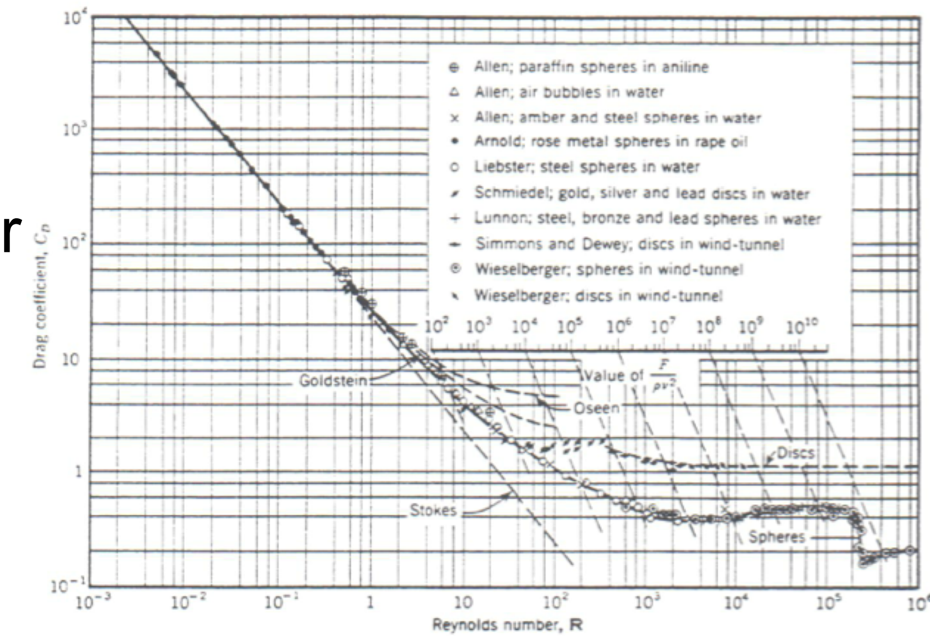
$$R_p = \frac{|u_p| D}{\nu}$$

- For small Reynolds number

$$C_D = \frac{24}{R_p} \qquad D_i = 3\pi\rho\nu D u_{pi}$$

- Larger Reynolds number

$$C_D = 0.4, \qquad D_i = 0.4\pi \frac{D^2}{8} |u_p| u_{pi}$$



3. Case: Particle moving through quiescent fluid

- Finally the added mass term

$$A_i = -\rho C_m V_p \frac{du_{pi}}{dt} \qquad (C_m \sim 0.5)$$

- This added mass term acts as a resistance force.
- If we send it to the left hand side, it plays as like as mass and appear as effective mass.
- More terms
  - Basset force : history force
  - Magnus force : lift force by rotation
- Finally $$(\rho_s + 0.5\rho) V_p \frac{du_{pi}}{dt} = (\rho_s - \rho) V_p g_i - \frac{1}{2} \rho C_D \pi A_p |u_p| u_{pi}$$

4. Case: Particle in moving fluid

- Consider the relative motion between the fluid and the p article
- The position of the particle centroid

$$u_{fi}(t) = u_i\left(x_{pi}(t), t\right)$$

- Relative particle velocity

$$u_{ri} = u_{pi} - u_{fi}$$

- Now force balance equation becomes

$$\int_S \tau_{ji} n_j \, dS = B_i + D_i + A_i + L_i$$

- Here L_i is lifting force

4. Case: Particle in moving fluid

- Buoyant force now contains an accelerative effect,

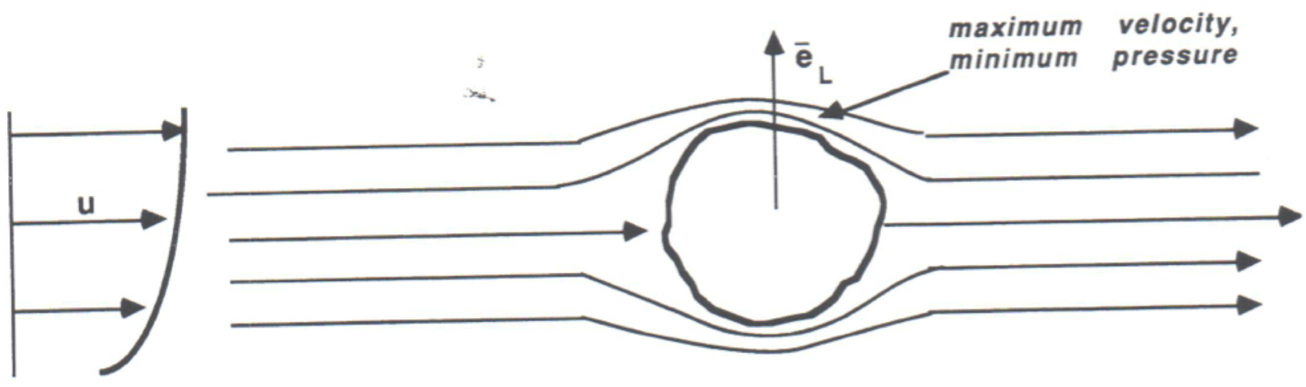
$$B_i = -\rho V_p g_i + \rho V_p \frac{du_{fi}}{dt}$$

- The drag and added mass forces must be expressed in t erms of relative velocity

$$D_i = -\frac{1}{2}\rho C_D A_p |u_r| u_{ri}$$

$$A_i = -\rho C_m V_p \frac{du_{ri}}{dt}$$

4. Case: Particle in moving fluid



- The lift force associated with the filed of shear flow can b e related in dynamic pressure across the grain.
- Net lift force is

$$L_i = \frac{1}{2}\rho C_L A_p \left[\left|u_r\right|^2_{max} - \left|u_r\right|^2_{opp} \right] e_{ri}$$

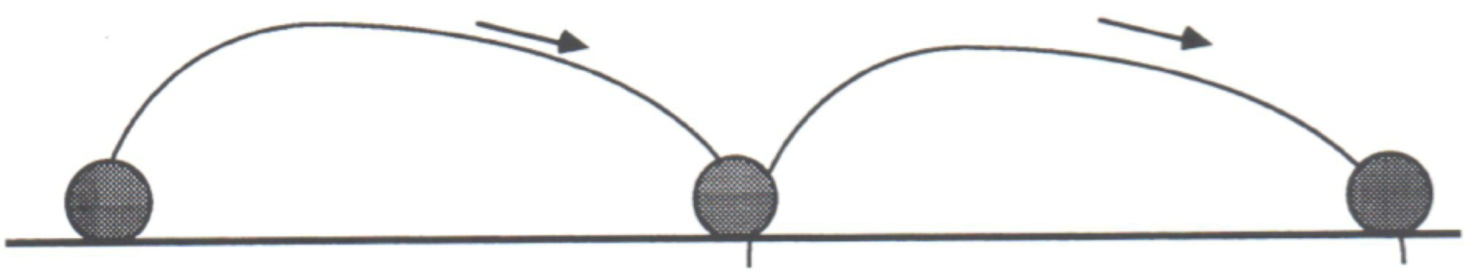
4. Case: Particle in moving fluid

- Now, the general equation of motion of a particle is found to be

$$\rho_s V_p \frac{du_{pi}}{dt} = (\rho_s - \rho) V_p g_i + \rho V_p \frac{du_{fi}}{dt} - \frac{1}{2}\rho C_D A_p |u_r| (u_{pi} - u_{fi})$$
$$- \rho C_m V_p \frac{d}{dt}(u_{pi} - u_{fi}) + \frac{1}{2}\rho C_L A_p \left[|u_r|^2_{max} - |u_r|^2_{opp} \right] e_{ri}$$

5. Collision with the bed (Bedload transportation)

- Sediment particles tend to move in one or two relative distinct modes.
  - The first mode is bedload transport
- In this case, particles tend to roll, slide, or hop along the bed.
- The hopping mode is perhaps the most common
  - It is called ***Saltation.***



5. Collision with the bed (Bedload transportation)

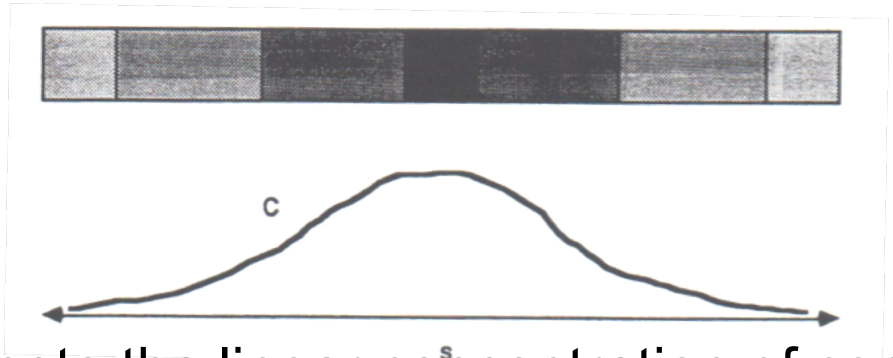
- Saltating particles obtain their forward momentum fron th e flow through the drag term.
- That is the faster flow tends to pull the grain along.
- Under the influence of gravity these grains fall back to th e bed.
- Upon striking the bed, they transfer some of their momen tum to the bed.
- The particles tend to bound back however, as the collisio n is partially elastic.

5. Collision with the bed (Bedload transportation)

- Particle collide with the bed with velocity $u_{pi}\big|_{in}$
  - The subscript "in" denoting "incoming".
- Velocity tangent to the bed surface $u_{pT}\big|_{in}$
- Velocity normal to the bed surface $u_{pN}\big|_{out} = -e \cdot u_{pN}\big|_{in}$
  - "out" denoting "outgoing and e denotes the coefficient of restitution. (0.5~0.75)
- You will have mini-project !!!! (Congrat!)
  - From today during two weeks.

6. Diffusivity of turbulent (suspended sediment)

- The effect of turbulent is to diffuse, or mix any contaminant such as hear, suspended sediment, momentum.
- Fickian model of diffusion



  – C denote the linear concentration of contaminant. The diffusive discharge Q_d of contaminant, in grams grossing a section as s per unit time, is taken to be given by Fick's law.

$$Q_d = -D_d \frac{dC}{ds}$$

6. Diffusivity of turbulent (suspended sediment)

- D_d denotes a coefficient of diffusivity. Mass balance of contaminant on a strip of length ds requires the following relation to hold:

$$\frac{\partial}{\partial t}\left[\text{mass in strip}\right]=\left[\text{net inflow of mass}\right]$$

$$\frac{\partial}{\partial t}\left[Cds\right]=Q_d\left(s\right)-Q_d\left(s+ds\right)$$

$$\frac{\partial C}{\partial t}=D_d\frac{\partial^2 C}{\partial s^2}$$

- Now the total amount of contaminant in the canal is given by

$$\int_{-\infty}^{\infty} C\,ds$$

6. Diffusivity of turbulent (suspended sediment)

- Two previous equations can be rearranged as

$$\frac{\partial P}{\partial t} = D_d \frac{\partial^2 P}{\partial s^2}$$

$$P = \frac{C}{\int_{-\infty}^{\infty} C\, ds}, \qquad \int_{-\infty}^{\infty} P\, ds = 1$$

- Now suppose we release a patch of contaminant at s=0 and allow it to diffuse in time.
  - Because there is no mean flow, the position of the centroid of the patch should not change.
  - On the other hand the variance of the patch should increas es as diffusion progresses from areas of high concentratio n to low concentration

6. Diffusivity of turbulent (suspended sediment)

- By definition, the s coordinated of the centroid of the diffu sing patch is given by

$$\bar{s} = \int_{-\infty}^{\infty} sP\, ds$$

$$\int_{-\infty}^{\infty} \left(s\frac{\partial P}{\partial t} = sD_d \frac{\partial^2 P}{\partial s^2} \right) ds$$

$$\int_{-\infty}^{\infty} \left(s\frac{\partial P}{\partial t} \right) ds = \int_{-\infty}^{\infty} \left(sD_d \frac{\partial^2 P}{\partial s^2} \right) ds$$

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} (sP)\, ds = \frac{d\bar{s}}{dt}$$

$$\frac{d\bar{s}}{dt} = D_d \int_{-\infty}^{\infty} \left(s\frac{\partial^2 P}{\partial s^2} \right) ds = D_d \left[s\frac{\partial P}{\partial s}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\partial P}{\partial s} ds \right] = 0$$

- Centroid of s does not change with time.

6. Diffusivity of turbulent (suspended sediment)

- Diffusivity $$\sigma^2 = \overline{(s-\bar{s})^2} = \overline{s^2} = \int_{-\infty}^{\infty} s^2 P\, ds$$

$$\int_{-\infty}^{\infty}\left(s^2\frac{\partial P}{\partial t} = s^2 D_d \frac{\partial^2 P}{\partial s^2}\right) ds$$

$$\int_{-\infty}^{\infty}\left(s^2\frac{\partial P}{\partial t}\right) ds = \int_{-\infty}^{\infty}\left(s^2 D_d \frac{\partial^2 P}{\partial s^2}\right) ds$$

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\left(s^2 P\right) ds = \frac{d\sigma^2}{dt}$$

$$\frac{d\sigma^2}{dt} = D_d \int_{-\infty}^{\infty}\left(s^2 \frac{\partial^2 P}{\partial s^2}\right) ds = D_d\left[\left. s^2\frac{\partial P}{\partial s}\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} 2s\frac{\partial P}{\partial s} ds\right]$$

$$\frac{d\sigma^2}{dt} = -D_d \int_{-\infty}^{\infty} 2s\frac{\partial P}{\partial s} ds = 2D_d$$

7. Limiting equation for fine particles.

- Consider in the limiting case of very fine particles.
- The lift force can be neglected because the size of the p article is much smaller than any characteristic lengths
- The drag law can be take to be stokes form

$$D_i = -3\pi\rho\nu D\left(u_{pi} - u_{fi}\right)$$

$$\rho V_p \frac{du_{pi}}{dt} = \left(\rho_s - \rho\right) V_p g_i + \rho V_p \frac{du_{fi}}{dt} - \frac{1}{2}\rho C_D A_p \left|u_r\right|\left(u_{pi} - u_{fi}\right)$$

$$-\rho C_m V_p \frac{d}{dt}\left(u_{pi} - u_{fi}\right) + \frac{1}{2}\rho C_L A_p \left[\left|u_r\right|_{max}^2 - \left|u_r\right|_{opp}^2\right] e_{ri}$$

$$\Rightarrow V_p \frac{d}{dt}\left[\left(\rho_s + C_m \rho\right) u_{pi} - \rho\left(1 + C_m\right) u_{fi}\right] = \left(\rho_s - \rho\right) V_p g_i - 3\pi\rho\nu D\left(u_{pi} - u_{fi}\right)$$

7. Limiting equation for fine particles.

- If the particles are very small and if the time scale of fluctuation of the flow is large compared to the response time of the particle (e.g., characteristic eddy size is much larger than particle size), then the acceleration term can be dropped.
- The remaining terms are gravity and drag; the particle is taken to respond instantaneously to changing flow.
- The resulting relation is usefully expressed in a coordinate system with z oriented upward vertically such that

$$g_i = -(0,0,g) = -g\delta_{3i}$$

$$0 = -\frac{4}{3}\pi\left(\frac{D}{2}\right)^3 Rg\delta_{3i} - 3\pi\nu D\left(u_{pi} - u_{fi}\right)$$

7. Limiting equation for fine particles.

- At this point, consider flow to be in quiescent water

$$u_{fi}=0 \quad Then \;\; u_{pi}=(0,0,-v_s)$$

- Then

$$0=-\frac{4}{3}\pi\left(\frac{D}{2}\right)^3 Rg+3\pi\nu D v_s$$

$$v_s=\frac{1}{18}\frac{RgD^2}{\nu} \qquad (stokes' \; falling \; velocity)$$

- Now put this one into original force balance equation, the n

$$0=-\frac{4}{3}\pi\left(\frac{D}{2}\right)^3 Rg\delta_{3i}-3\pi\nu D\left(u_{pi}-u_{fi}\right)$$

$$u_{pi}-u_{fi}=-\frac{4}{3}\frac{\pi D^3 Rg\delta_{3i}}{8\cdot 3\pi\nu D}=-\frac{1}{18}\frac{RgD^2}{\nu}\delta_{3i}=v_s\delta_{3i}$$

$$\boxed{u_{pi}=u_{fi}-v_s\delta_{3i}}$$

8. Taylor's theory of diffusion by continuous movement

- Assuming that the particle motion is neutrally buoyant (so fine as to exactly follow the fluid motion).
- In this case $u_{pi} = u_{fi}$
- By the way $u_{pi} = \dfrac{dx_{pi}}{dt}$ $\quad (where\ x_{pi}\ is\ particle\ position)$
- Then

$$\frac{dx_{pi}}{dt} = u_i\left(x_{pj}(t),t\right) \quad (where\ u_i\ is\ Eulerian\ fluid\ velocity\ field)$$

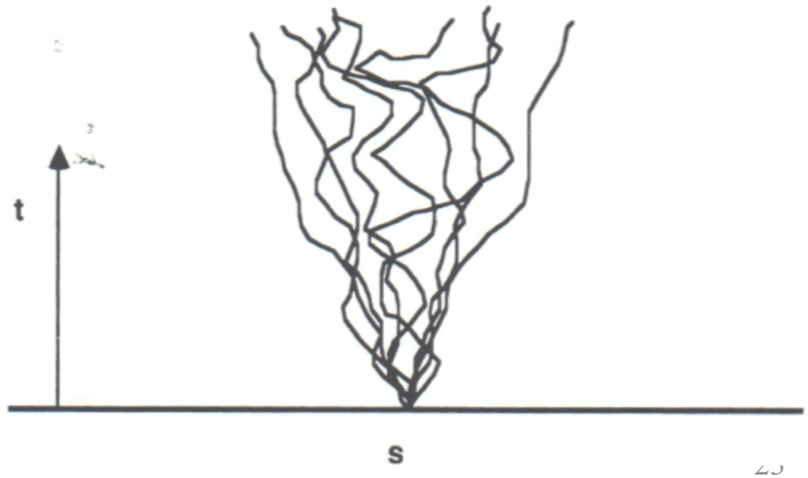
- One dimensional form is

$$\frac{dS}{dt} = U(t) \quad (where\ x_i \to S(t), u_i(x_{pj}(t),t) \to u(S(t),t) = U(t))$$

by G.I. Taylor.

8. Taylor's theory of diffusion by continuous movement

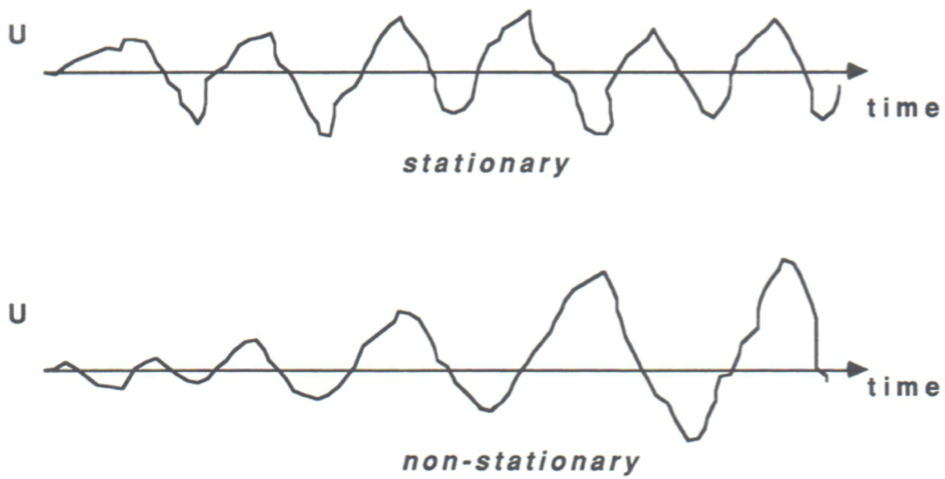
- Taylor envisioned releasing many particles from the point s=0 in a turbulent fluid with no mean flow. Averaging over all the particle displacements should result in a mean dis placement of zero.
- It can be expected, however, that the spread or variance, of particle position should increase in time due to the dis persive nature of turbulence.



8. Taylor's theory of diffusion by continuous movement

- Averaging

$$\overline{\left(\frac{dS}{dt}=U(t)\right)} \Rightarrow \bar{S}=0 \;\; if \;\; \bar{U}=0 \;\; (see\ figure)$$



- Let us consider what a time record of U might look like. It should consist entirely of turbulent fluctuations about me an.

8. Taylor's theory of diffusion by continuous movement

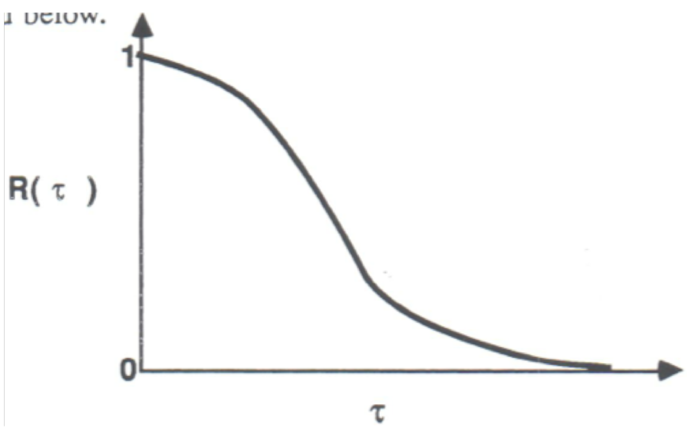
- The amplitude of these fluctuations is measured by the r oot-mean-squared velocity U', where by definition

$$U' = \left(\overline{U^2}\right)^{1/2}$$

- Correlation function can be similarly defined for U:

$$R(t,\tau) = \frac{\overline{U(t)U(t+\tau)}}{U'^2} \quad (where\ \tau\ is\ time\ lag)$$

$$R(t,\tau) \rightarrow 0\ as\ \tau \rightarrow \infty$$



8. Taylor's theory of diffusion by continuous movement

- If the turbulence has the same statistical characteristics everywhere, it is said to be *stationary*. Under this conditi on, the correlation function has exactly the same form re gardless of the time t at which it is measured. That is

$$R(t,\tau)=R(\tau)$$

- This correlation function can be used to define a Lagrang ian (following a fluid particle) integral time scale of turbul ent fluctuation T_L:

$$T_L=\int_0^\infty R(\tau)\,d\tau$$

8. Taylor's theory of diffusion by continuous movement

- Now we compute the variance of the dispersion particles :

$$\sigma^2 = \overline{\left(S-\overline{S}\right)} = \overline{S^2}$$

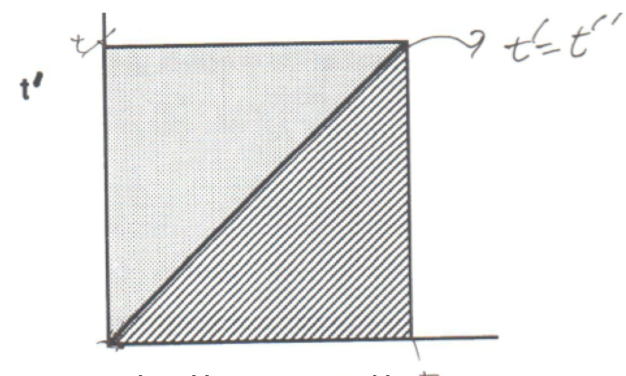
$$\frac{dS}{dt} = U(t) \;\Rightarrow\; S = \int_0^t U(t')\,dt'$$

- With above two equations

$$\sigma^2 = \overline{\int_0^t U(t')\,dt' \int_0^t U(t'')\,dt''}$$

$$= \int_0^t \int_0^t \overline{U(t')U(t'')}\,dt'\,dt''$$

$$= 2\int_0^t \int_0^{t'} \overline{U(t')U(t'')}\,dt'\,dt''$$



If $f(t',t'') = f(t'',t')$ then

$$\int_0^t \int_0^t f(t',t'')\,dt'\,dt''$$

$$= 2\int_0^t \int_0^{t'} f(t',t'')\,dt'\,dt''$$

8. Taylor's theory of diffusion by continuous movement

- Let's $t'' = t' + \tau_1$. In the integral in t'', t' can be taken as constant, so $dt'' = d\tau_1$.

$$\sigma^2 = 2\int_0^t \int_{-t'}^0 \overline{U(t')U(t'+\tau_1)}\,dt'\,dt = 2U'^2 \int_0^t dt' \int_{-t'}^0 R(\tau)\,d\tau_1$$

- Now let $\tau_1 = -\tau$ and use the property $R(\tau) = R(-\tau)$ to obtain

$$\sigma^2 = 2U'^2 \int_0^t dt' \int_0^{t'} R(\tau)\,d\tau_1$$

$$= 2U'^2 \left[t\int_0^t R(\tau)\,d\tau - \int_0^t \tau R(\tau)\,d\tau \right]$$

8. Taylor's theory of diffusion by continuous movement

$$\sigma^2 = 2U'^2\left[t\int_0^t R(\tau)d\tau - \int_0^t \tau R(\tau)d\tau\right]$$

- The first term for the long time converges to tT_L, thence i ncreasing linearly in time. The second term converges t o a constant, and thus becomes negligible to the first ter m for long time.
- Finally, the characteristics of the turbulent can be related to the diffusivity

$$\sigma^2 \cong 2U'^2 T_L t \qquad D_d \cong \frac{1}{2}\frac{d\sigma^2}{dt} = U'^2 T_L$$