Crystal Mechanics

Lecture 7 – Plastic Deformation – Basic Principles

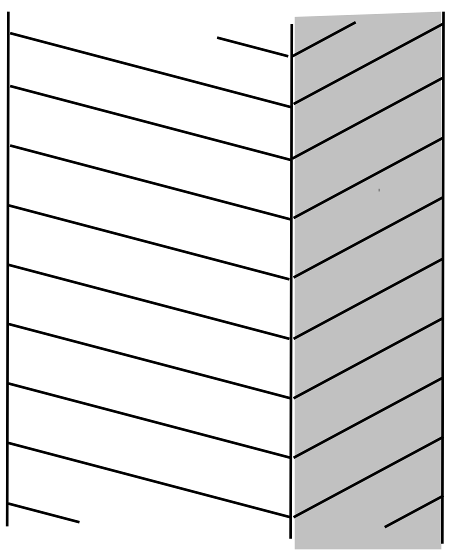
Ref : Texture and Related Phenomena, D. N. Lee, 2006
Continuum Theory of Plasticity, A.S. Khan and S. Huang, 1995
H. Courtney, Mechanical Behavior of Materials, McGraw Hill, 2000
G. E. Dieter, Mechanical Metallurgy, McGraw Hill

Heung Nam Han
Associate Professor
School of Materials Science & Engineering
College of Engineering
Seoul National University
Seoul 151-744, Korea
Tel : +82-2-880-9240
Fax : +82-2-885-9647
email : hnhan@snu.ac.kr

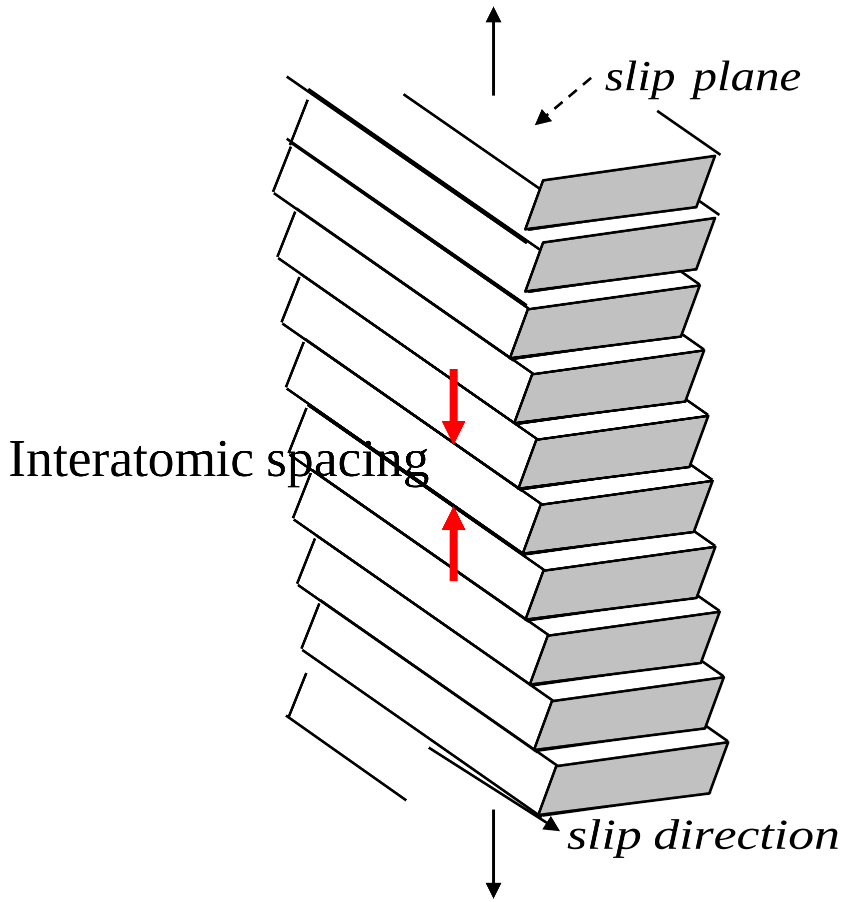
Slip System

Plastic deformation of single crystal in uniaxial tension

Before slip



After slip



Slip System

Metal	Crystal structure	Purity, %	Slip plane	Slip direction	Shear stress, MPa	Reference
Zn	hcp	99.999	(0001)	<1 1-2 0>	0.18	Jillson
Mg	hcp	99.996	(0001)	<1 1-2 0>	0.77	Burke
Cd	hcp	99.95	(0001)	<1 1-2 0>	0.43	Burke
Ti	hcp	99.996	(0001)	<1 1-2 0>	0.58	Schmid
		99.996	(0001)	<1 1-2 0>	0.57	Boas
		99.99	(1010)	<1 1-2 0>	13.7	Churchman
		99.9	(1010)	<1 1-2 0>	90.1	Churchman
Ag	fcc	99.999	{111}	<110>	0.37	daC. Andrade
Al	fcc	99.99	{111}	<110>	0.48	Rosi
Au	fcc	99.97	{111}	<110>	0.73	Rosi
Cu	fcc	99.93	{111}	<110>	1.3	Rosi
Ni	fcc	99.996	{111}	<110>	1.02	Rosi, MCW
		99.99	{111}	<110>	0.91	Sachs
		99.999	{111}	<110>	0.65	Rosi
		99.98	{111}	<110>	0.94	Rosi
		99.8	{111}	<110>	5.7	Rosi
Fe	bcc	99.96	{110}	<111>	27.5	Cox
Mo	bcc		{112}	<111>	49.0	Maddin
			{123}	<111>		
			{110}	<111>		

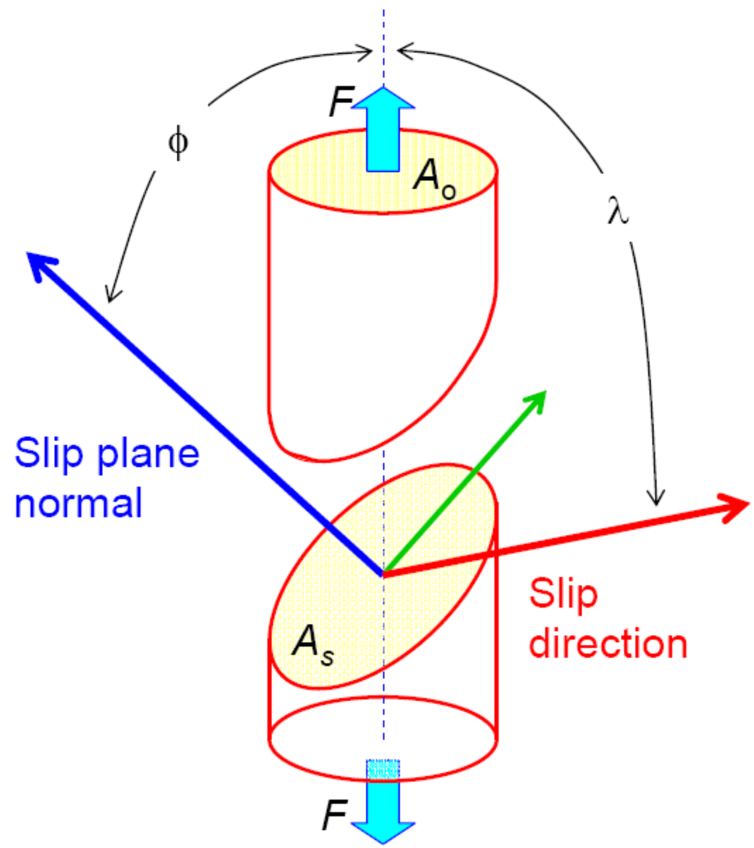
Slip System for Crystal Deformation

Table1.1. Slip systems of some crystal structures

Metals	*Slip Plane*	*Slip Direction*	*Number of Slip Systems*
	Face-Centered Cubic		
Cu, Al, Ni, Ag, Au	$\{111\}$	$\langle 1\bar{1}0 \rangle$	12
	Body-Centered Cubic		
α-Fe, W, Mo	$\{110\}$	$\langle \bar{1}11 \rangle$	12
α-Fe, W	$\{211\}$	$\langle \bar{1}11 \rangle$	12
α-Fe, K	$\{321\}$	$\langle \bar{1}11 \rangle$	24
	Hexagonal Close-Packed		
Cd, Zn, Mg, Ti, Be	$\{0001\}$	$\langle 11\bar{2}0 \rangle$	3
Ti, Mg, Zr	$\{10\bar{1}0\}$	$\langle 11\bar{2}0 \rangle$	3
Ti, Mg	$\{10\bar{1}1\}$	$\langle 11\bar{2}0 \rangle$	6

Schmid's Law

- Plastic deformation is initiated at the critical resolved shear stress (CRSS).
- The CRSS is the stress at which dislocations begin to move.



Resolved Shear Stress

Talyor factor

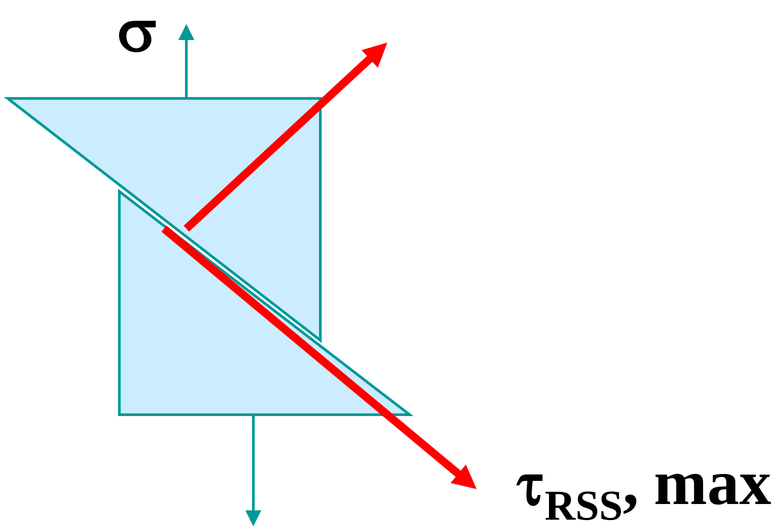
$$\tau_{RSS} = \frac{F}{A_o} \underbrace{\cos\phi \cos\lambda}_{\text{Schmid Factor}} = \sigma / m$$

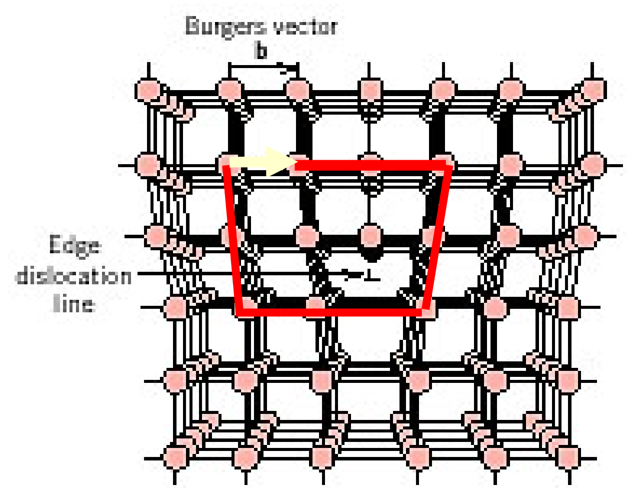
Plastic flow is initiated when τ_{RSS} reaches a critical value, characteristic of the material, called *critical RSS*, when *m* $\tau_{CRSS} = \sigma_{ys}$ *(Schmid law)*.

Schmid factor

MAXIMUM Resolved Shear Stress occurs when $\boldsymbol{\phi = \lambda = 45^o}$ called $\tau_{RSS,max}$. Slip is on the planes 45º from the applied stress.

Then, $\boldsymbol{\tau_{RSS,\,max} = \sigma \cos^2\phi = \sigma /2}$ at $\boldsymbol{\phi = \lambda = 45^o}$.





Slip system , 1/m of which is maximum, operates.

Critical Resolved Shear Stress

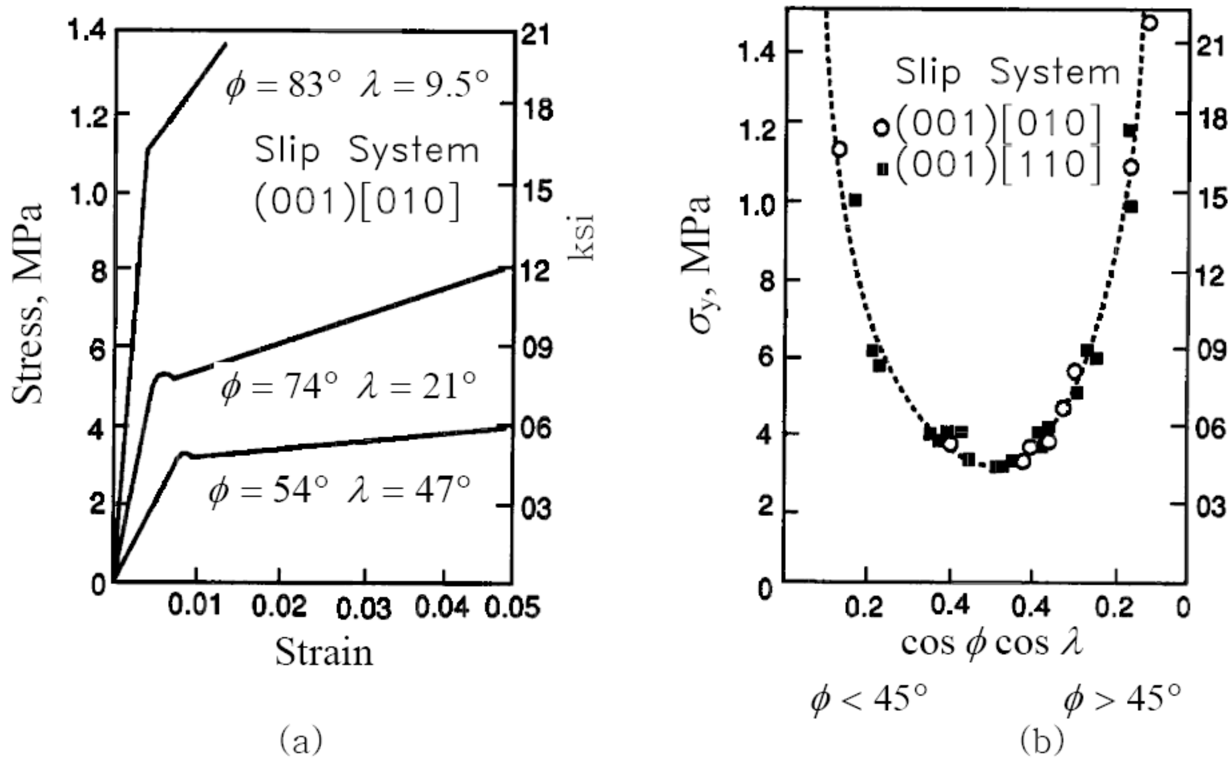


Fig.2.2. Yield behavior of anthracene single crystals. (a) Axial stress-strain curves of crystals having different orientations relative to loading axis; (b) Axial stress vs. Schmid factors. Dotted curve represents Eq.4.2.2 where $\tau_c = 137$ kPa [Robinson, Scott, 1967].

Schmid Factor

$$m = d\gamma / d\varepsilon = \sigma / \tau_{rss}$$

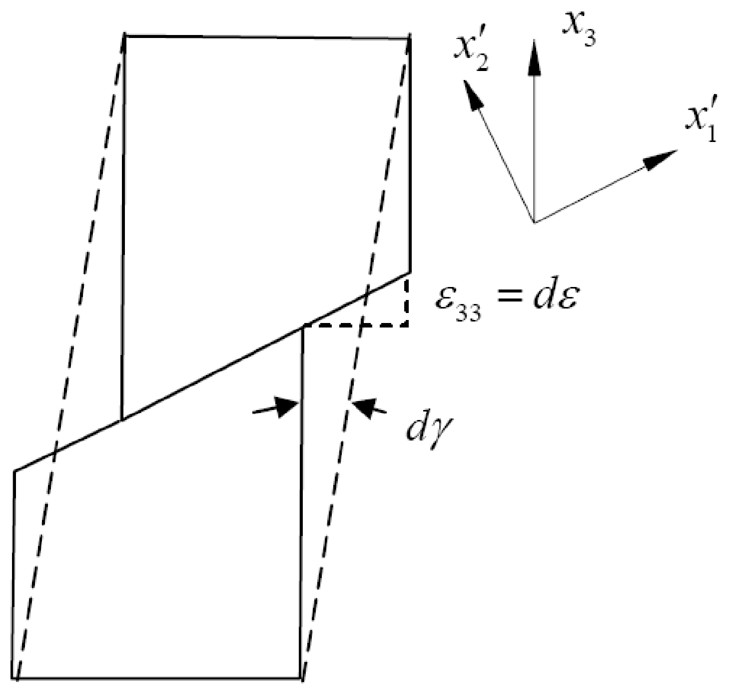


Fig.2.3. Tensile strain $d\varepsilon$ of specimen of unit length subjected to shear strain $d\gamma$.

Schmid Factor

The work done per unit extension (*e* = 1) is

$$W = m\,\tau_c\,\mathrm{A}$$

The work done is least for the slip system with the smallest *m*, the system that slips preferentially.

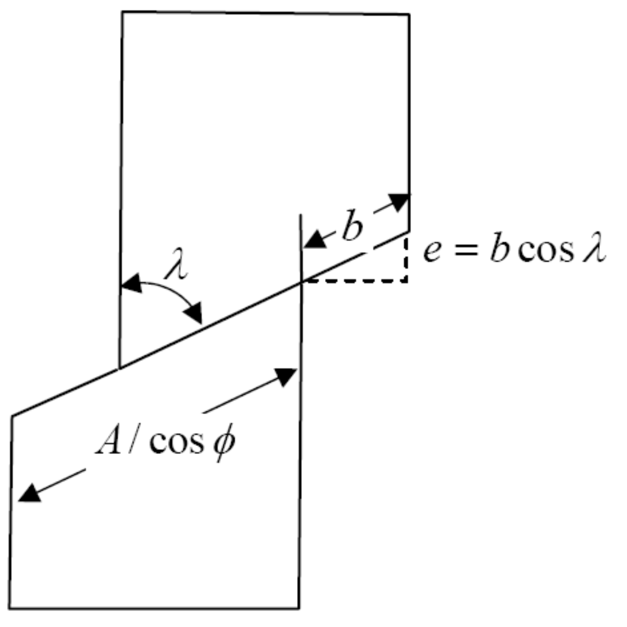


Fig.2.4. Relation between extension e and slip displacement b.

Critical Resolved Shear Stress

Example problem I

Calculate the tensile stress that is applied along the [1-20] axis of a gold crystal to cause slip on the (1-1-1)[0-11] slip system. The critical resolved shear stress is 10 MPa.

Table 4-4 Room-temperature slip systems and critical resolved shear stress for metal single crystals

Metal	Crystal structure	Purity, %	Slip plane	Slip direction	Critical shear stress, MPa	Ref.
Zn	hcp	99.999	(0001)	$[11\bar{2}0]$	0.18	[a]
Mg	hcp	99.996	(0001)	[1120]	0.77	[b]
Cd	hcp	99.996	(0001)	$[11\bar{2}0]$	0.58	[c]
Ti	hcp	99.99	(1010)	$[11\bar{2}0]$	13.7	[d]
		99.9	(1010)	$[11\bar{2}0]$	90.1	[d]
Ag	fcc	99.99	(111)	[110]	0.48	[e]
		99.97	(111)	[110]	0.73	[e]
		99.93	(111)	[110]	1.3	[e]
Cu	fcc	99.999	(111)	[110]	0.65	[e]
		99.98	(111)	[110]	0.94	[e]
Ni	fcc	99.8	(111)	[110]	5.7	[e]
Fe	bcc	99.96	(110) (112) (123)	[111]	27.5	[f]
Mo	bcc	...	(110)	[111]	49.0	[g]

[a] D. C. Jillson, *Trans. AIME*, vol. 188, p. 1129, 1950.
[b] E. C. Burke and W. R. Hibbard, Jr., *Trans. AIME*, vol. 194, p. 295, 1952.
[c] E. Schmid, "International Conference on Physics," vol. 2, Physical Society, London. 1935.
[d] A. T. Churchman, *Proc. R. Soc. London Ser. A*, vol. 226A, p. 216, 1954.
[e] F. D. Rosi, *TRans., AIME*, vol. 200, p. 1009, 1954.
[f] J. J. Cox, R. F. Mehl, and G. T. Horne, *Trans. Am. Soc. Met.*, vol. 49, p. 118, 1957.
[g] R. Maddin and N. K. Chen, *Trans. AIME*, vol. 191, p. 937, 1951.

Example II: FCC Cu with Loading axis [112]

- What is most likely initial slip system?
- If CRSS is 50 MPa, what is the tensile stress at which Cu will start to deform plastically?

Slip Plane **n**	Slip direction **s**	**n·l** $\cos\phi$	**s·l** $\cos\lambda$	Schmidt factor $\cos\phi\ \cos\lambda$	σ (MPa)
(111)	$[\bar{1}10]$ $[\bar{1}01]$ $[0\bar{1}1]$	$2\sqrt{2}/3$	0 $\sqrt{3}/6$ $\sqrt{3}/6$	0 $\sqrt{6}/9$ $\sqrt{6}/9$	Not def. 184 184
$(\bar{1}11)$	$[110]$ $[101]$ $[0\bar{1}1]$	$\sqrt{2}/3$	$\sqrt{3}/3$ $-\sqrt{3}/2$ $\sqrt{3}/6$	$\sqrt{6}/9$ $-\sqrt{6}/6$ $\sqrt{6}/18$	184 -122 367
$(1\bar{1}1)$	$[110]$ $[\bar{1}01]$ $[011]$	$\sqrt{2}/3$	$\sqrt{3}/3$ $-\sqrt{3}/6$ $\sqrt{3}/2$	$\sqrt{6}/9$ $-\sqrt{6}/18$ $\sqrt{6}/6$	184 -367 122
$(11\bar{1})$ $=(\bar{1}\bar{1}1)$	$[\bar{1}10]$ $[101]$ $[011]$	0	0 $\sqrt{3}/2$ $\sqrt{3}/2$	0 0 0	Not def. Not def. Not def.

smallest stress to cause slip (yielding)

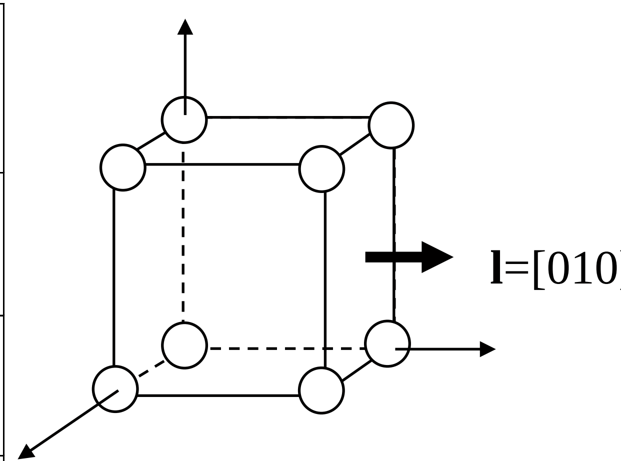
Initial Slip Systems (plane, direction) are then $(\bar{1}11)[101], (1\bar{1}1)[011]$

Example III:

Crystal with simple cubic structure :
slip planes {100} and slip directions <010>

Load is applied along [010].
Determine Schmid factor and what slip occurs.

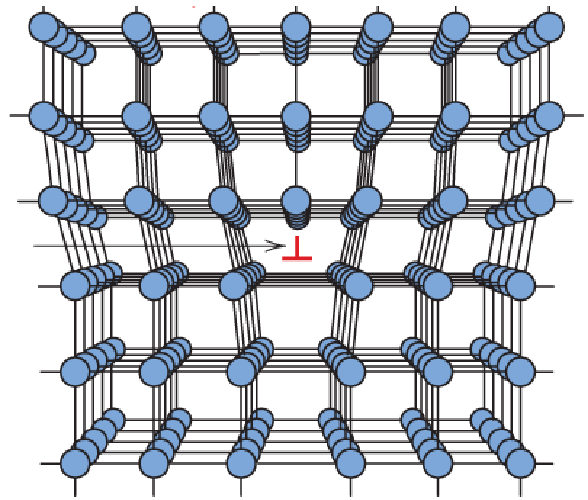
slip plane **n**	ϕ, $\cos\phi$ $\propto$ **l·n**	slip dir. **s**	λ, $\cos\lambda$ $\propto$**l·s**	m $\cos\phi\cos\lambda$
(100)	90^0, 0.0	[010] [001]	0^0, 1.0 90^0, 0.0	0
(010)	0^0, 1.0	[100] [001]	90^0, 0.0 90^0, 0.0	0
(001)	90^0, 0.0	[100] [010]	90^0, 0.0 0^0, 1.0	0

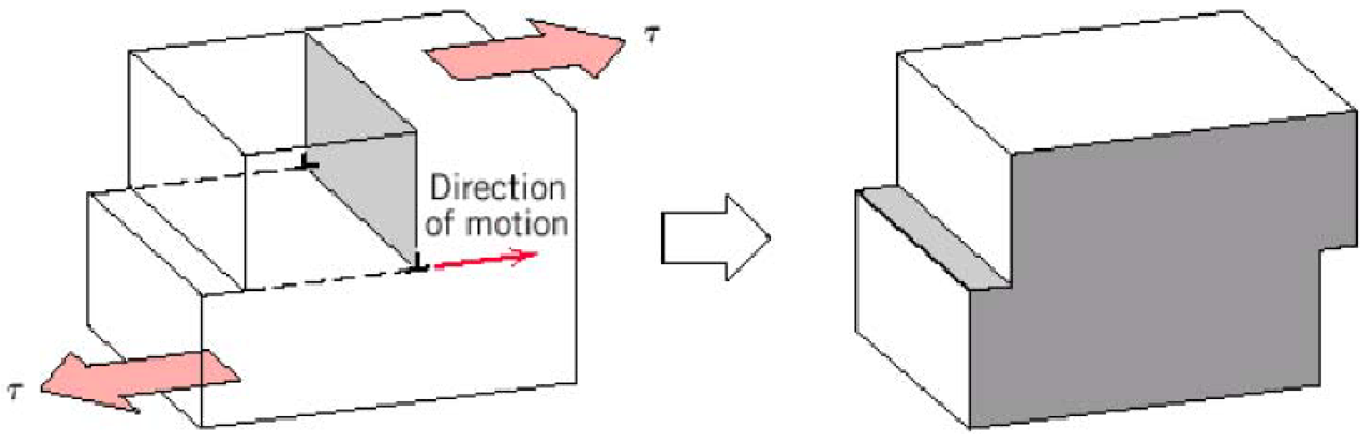


Is there any slip? Why?
If no slip, what must happen finally to material as load is increased?

Line defects (one dimension)

❖ Edge Dislocation

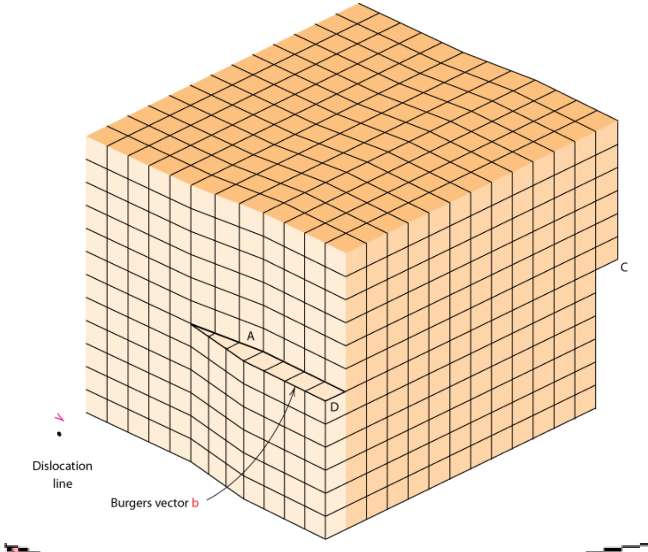


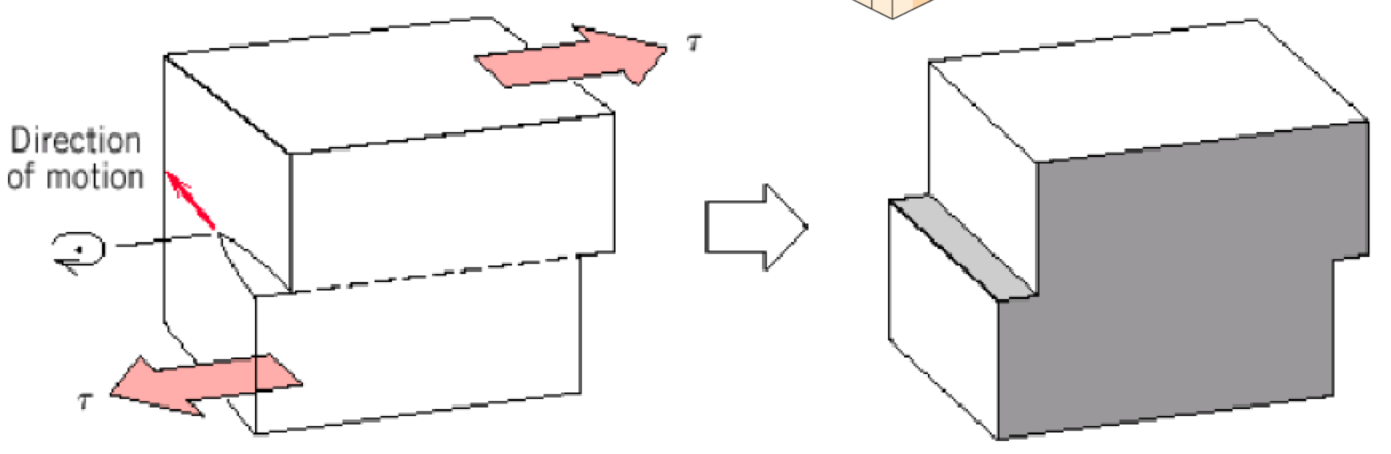


Edge dislocation line moves parallel to applied stress

Line defects (one dimension)

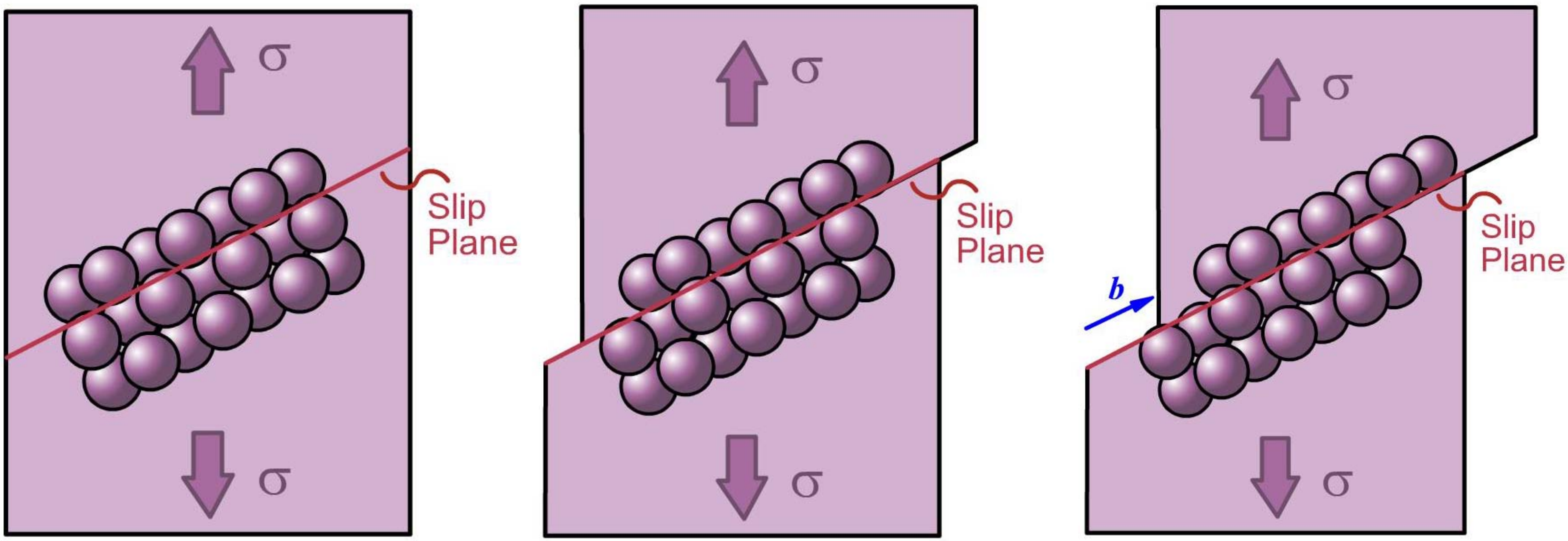
❖ Screw Dislocation





Screw dislocation line moves perpendicular to applied stress

Slip - dislocation



Slip system

- General rule;
  - slip plane: the densest atomic packing
  - slip direction: close-packed atomic direction
- In certain ionic solids, slip can happen in nonclose-packed directions.

1) FCC is consistent with the general rule, i.e,

Slip plane	Slip direction	Nonparallel plane		Slip direction per plane		Slip system
$\{111\}$	$<1\bar{1}0>$	4	$\times$	3	$=$	12

2) BCC

Preferable

Slip plane	Slip direction	Nonparallel plane		Slip direction per plane		Slip system
$\{110\}$	$<\bar{1}11>$	6	$\times$	2	$=$	12

Observable

Slip plane	Slip direction	Nonparallel plane		Slip direction per plane		Slip system
$\{112\}$	$<11\bar{1}>$	12	$\times$	1	$=$	12
$\{123\}$	$<11\bar{1}>$	24	$\times$	1	$=$	24

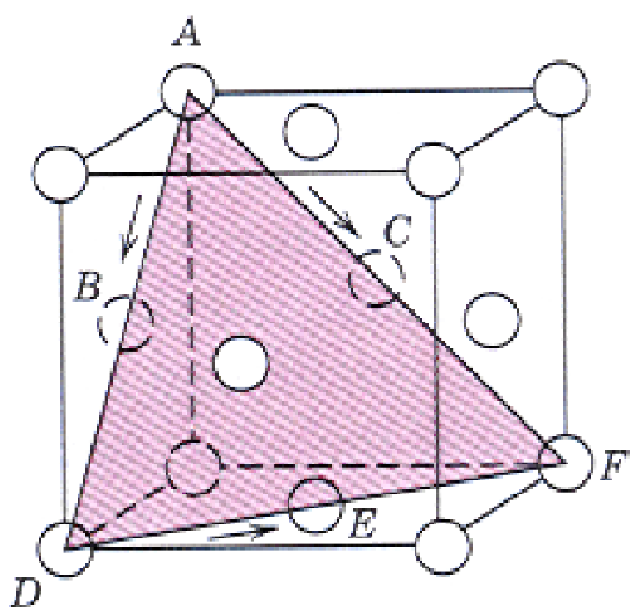
3) HCP

Slip plane	Slip direction	Nonparallel plane		Slip direction per plane		Slip system
$\{0001\}$	$<11\bar{2}0>$	1	$\times$	3	$=$	3
$\{10\bar{1}0\}$	$<11\bar{2}0>$	3	$\times$	1	$=$	3
$\{10\bar{1}1\}$	$<11\bar{2}0>$	6	$\times$	1	$=$	6

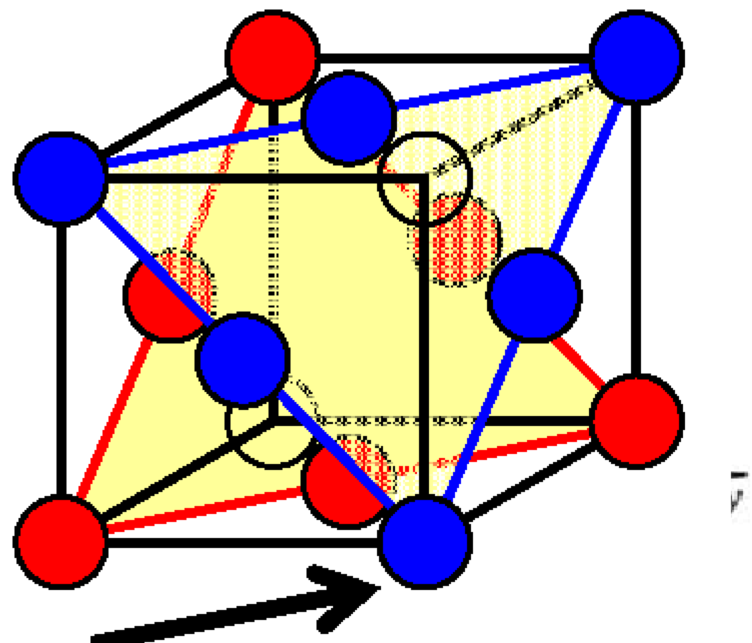
Perfect Dislocation (FCC)

- {111}<1-10> slip system → Burgers vector : a/2<110> → E $\propto 2a^2/4$

- 1/2<110> is a translation vector for the FCC lattice.→ Perfect dislocation



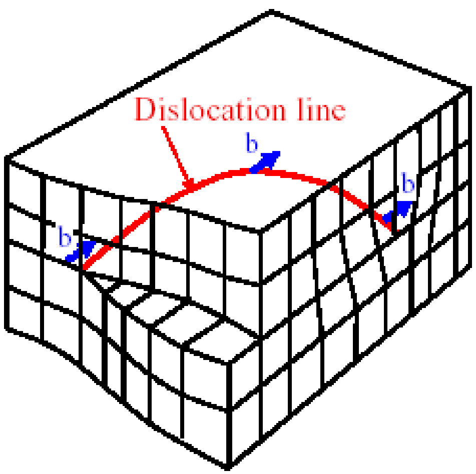
(a)



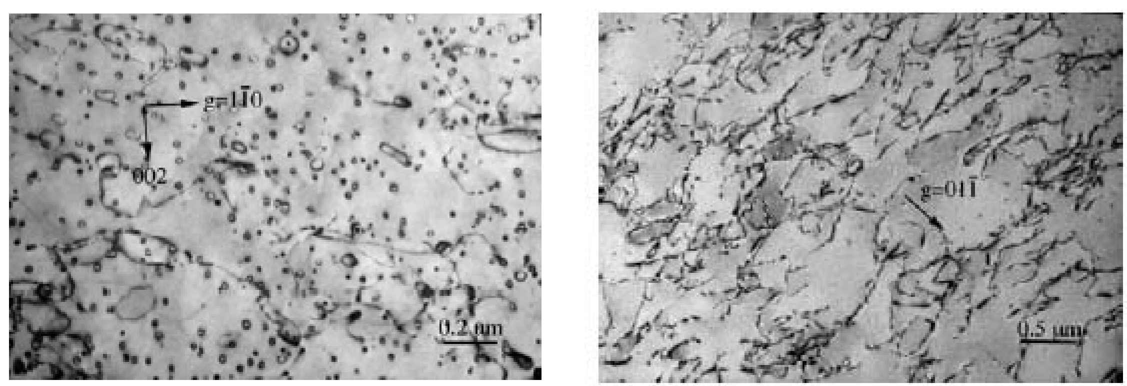
$$b = \frac{a_o\sqrt{2}}{2} = \frac{a_o}{2}\left[10\bar{1}\right]$$

Characteristics of dislocations

Dislocation Characteristic	Type of Dislocation		
	Edge	Screw	**Mixed**
Slip direction	// to *b*	// to *b*	Not // to *b*
Relation between dislocation line and *b*	⊥	//	Not // or ⊥
Direction of line movement relative to *b*	//	⊥	// and ⊥



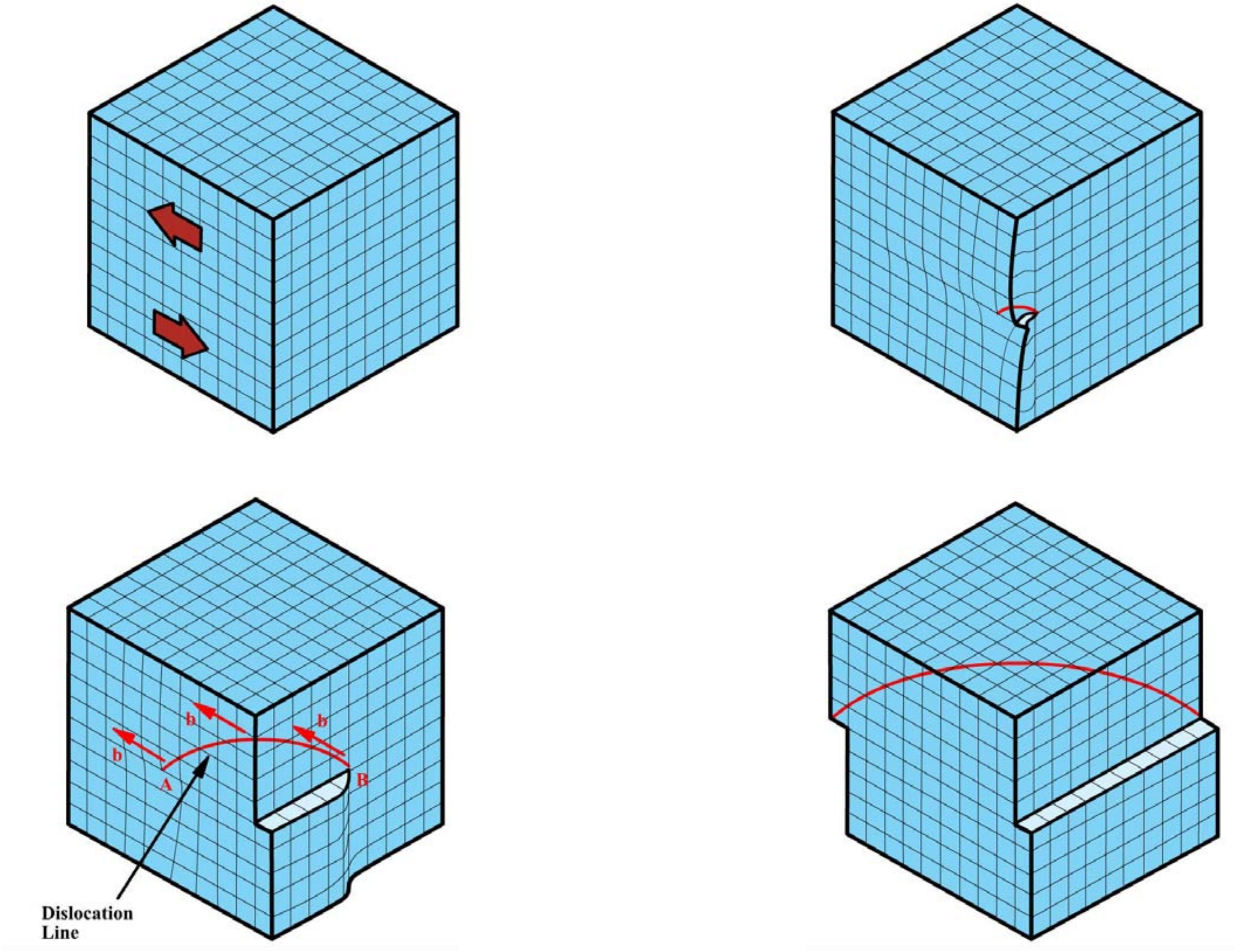
Mixed dislocation



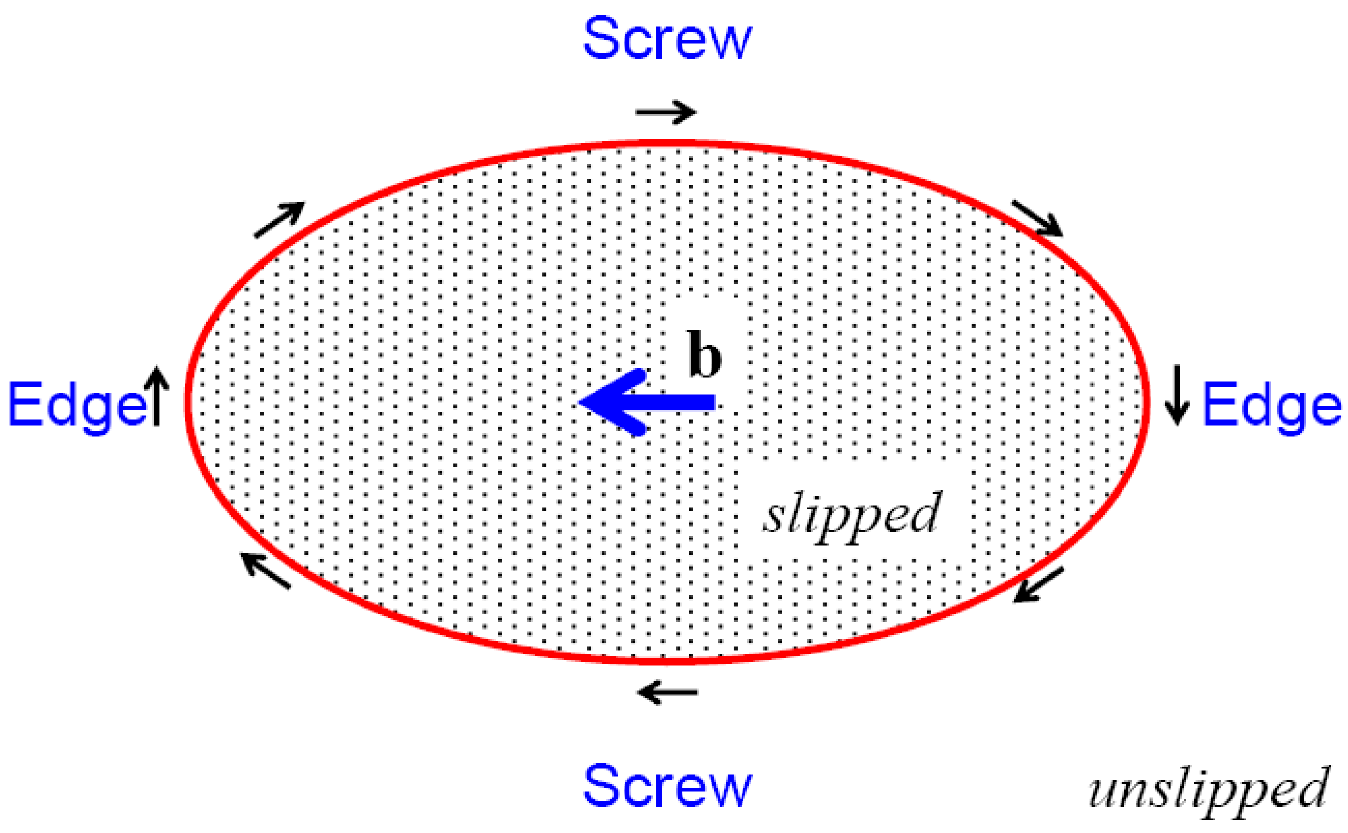
Dislocations imaged in NiAl-0.5Zr single crystals deformed at elevated temperatures.

Most dislocations are curved.

Motion of Mixed Dislocations

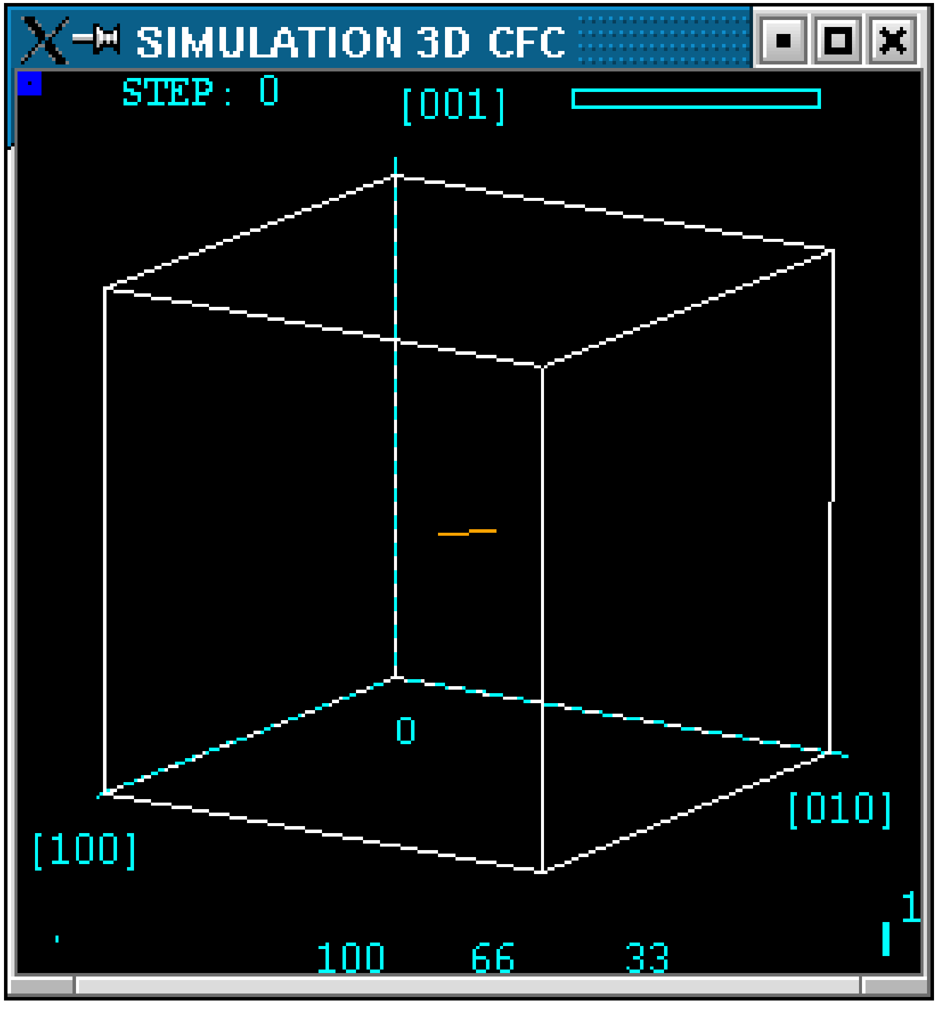


Dislocations move via slip



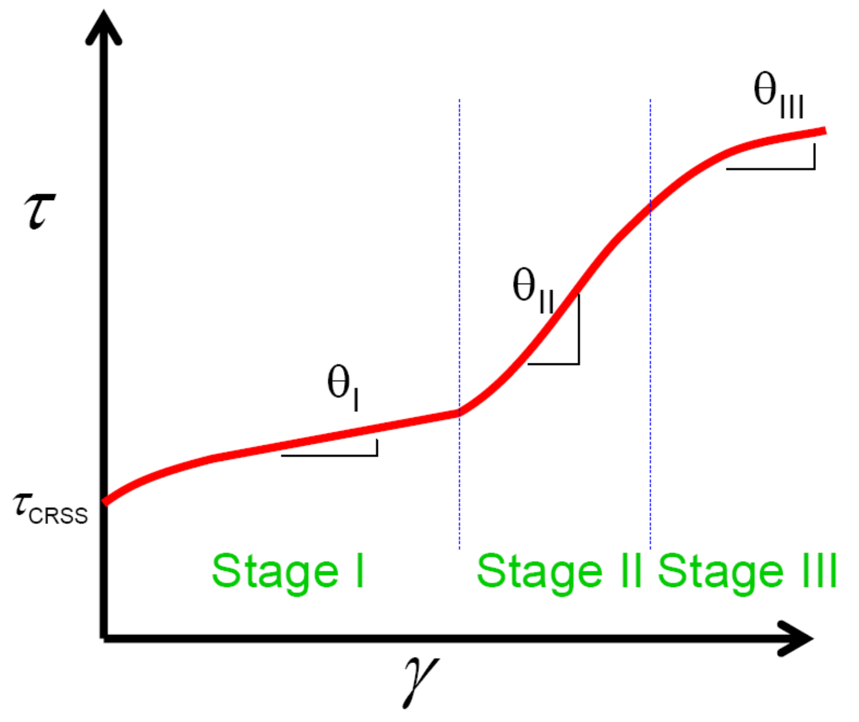
Schematic representation of a *dislocation loop*

Multiplication of Dislocations (Frank-Read Source)



Intersection of Dislocations

Deformation of single crystals



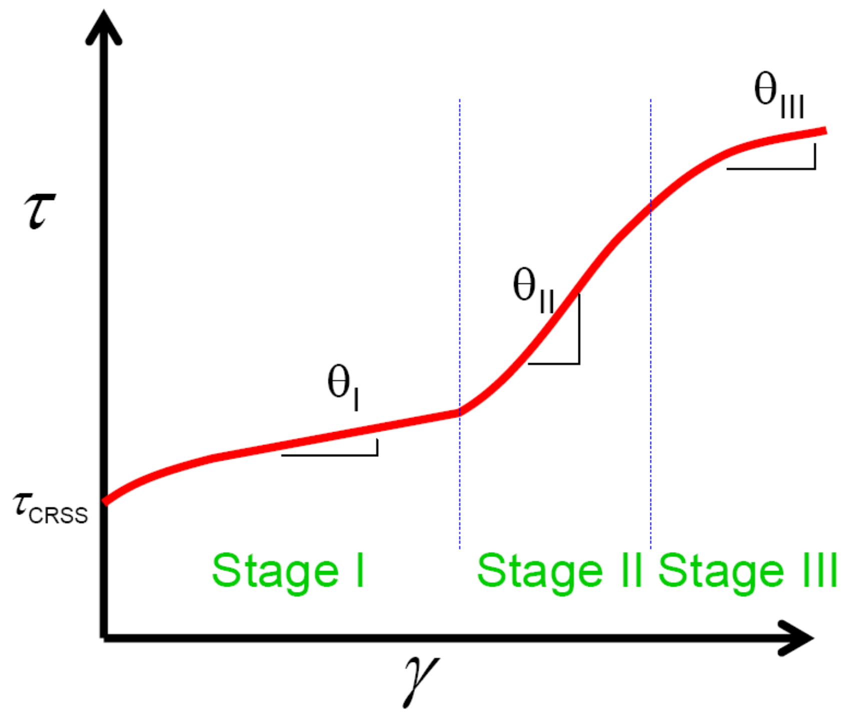
Stage I:

•After yielding, the shear stress for plastic deformation is essentially constant. There is little or no work hardening.

•This is typical when there is a single slip system operative. Dislocations do not interact much with each other. "Easy glide"

•Active slip system is one with maximum Schmid factor.

Deformation of single crystals



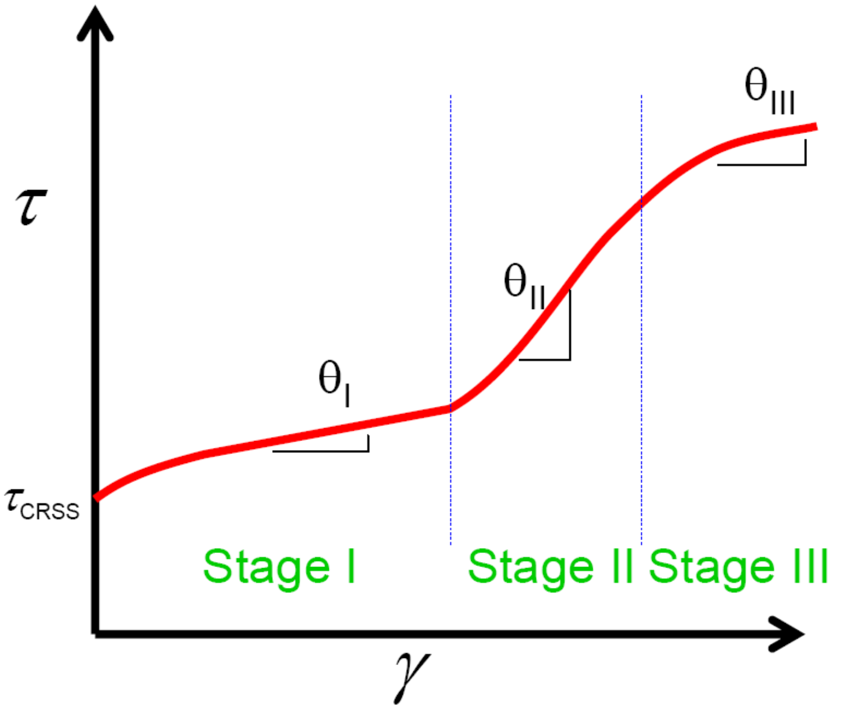
Stage II:

•The shear stress needed to continue plastic deformation begins to increase in an almost linear fashion. There is extensive work hardening ($\theta \cong G/300$).

•This stage begins when slip is initiated on multiple slip systems.

•Work hardening is due to interactions between dislocations moving on intersecting slip planes.

Deformation of single crystals

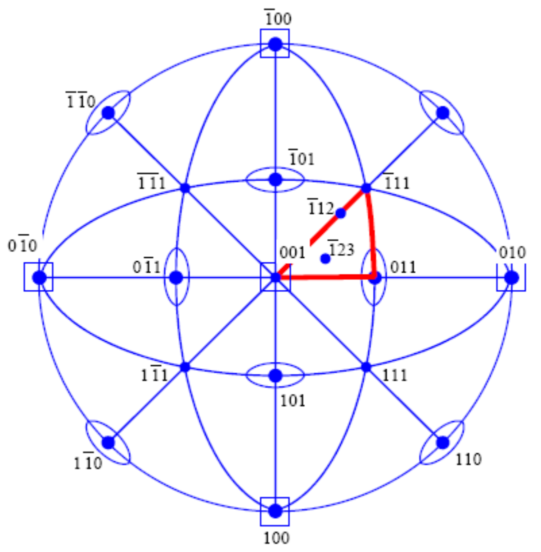


Stage III:

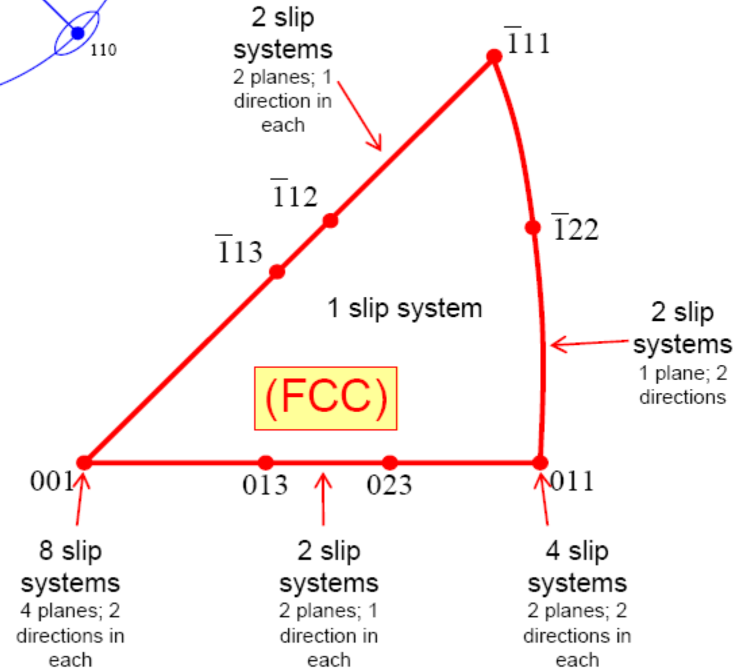
•There is a decreasing rate of work hardening.

•This decrease is due to an increase in the degree of cross slip resulting in a parabolic shape to the curve.

[001] stereographic projection of cubic crystal



Standard Triangle

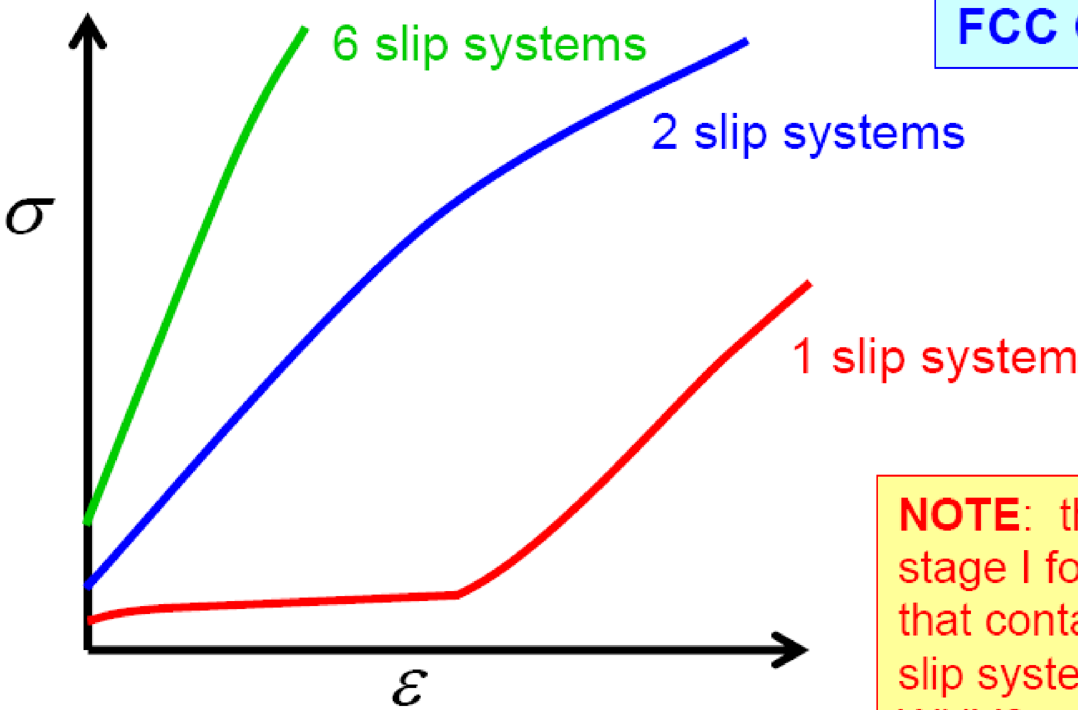


Influence of stress axis orientation

The stress axis orientation plays a **major role** in the stress-strain behavior of a single crystal

More slip systems means a "harder"material.

FCC Crystal



NOTE: there is no stage I for crystals that contain multiple slip systems? WHY?

Work/Strain Hardening

Work/Strain Hardening

$$\tau_{\max} = \frac{Gb}{L} \quad \Leftarrow \quad \bar{L} = \frac{\text{constant}}{\sqrt{\rho_\perp}} \cong \frac{\alpha}{\sqrt{\rho_\perp}}$$

$$\Downarrow$$

$$\tau = \tau_o + \alpha G b \sqrt{\rho_\perp}$$

where:

$\tau_o =$ intrinsic flow strength for $\perp$ free material

$\alpha =$ constant (0.2 for FCC, 0.4 for BCC)

Work/Strain Hardening

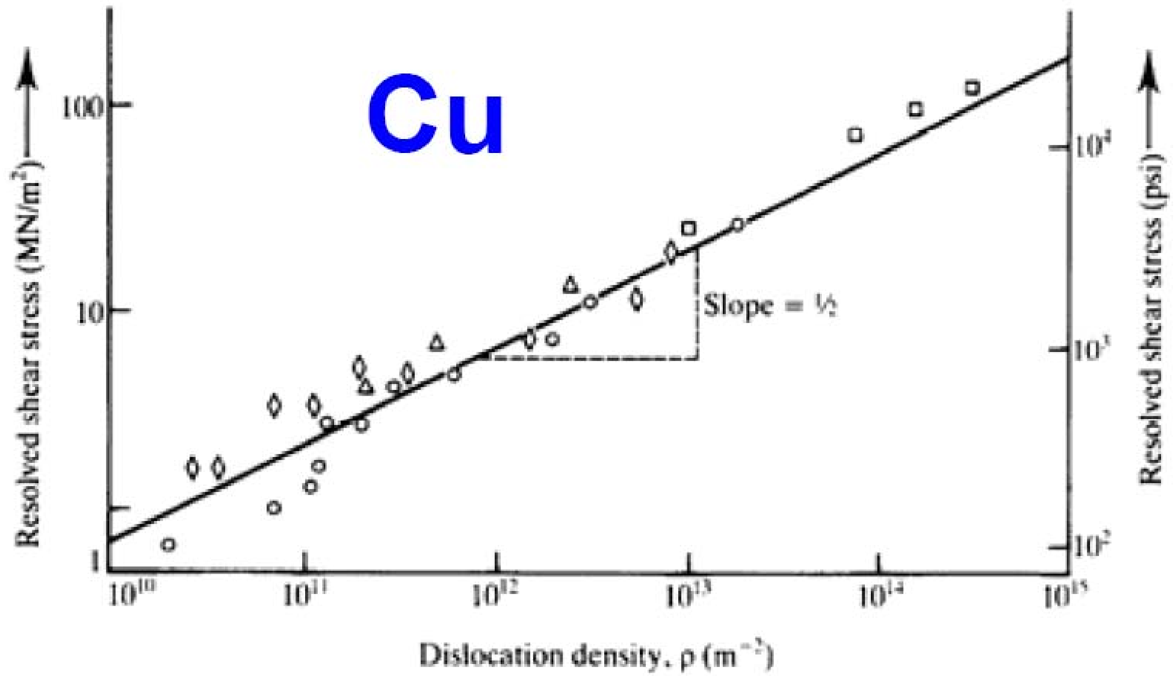
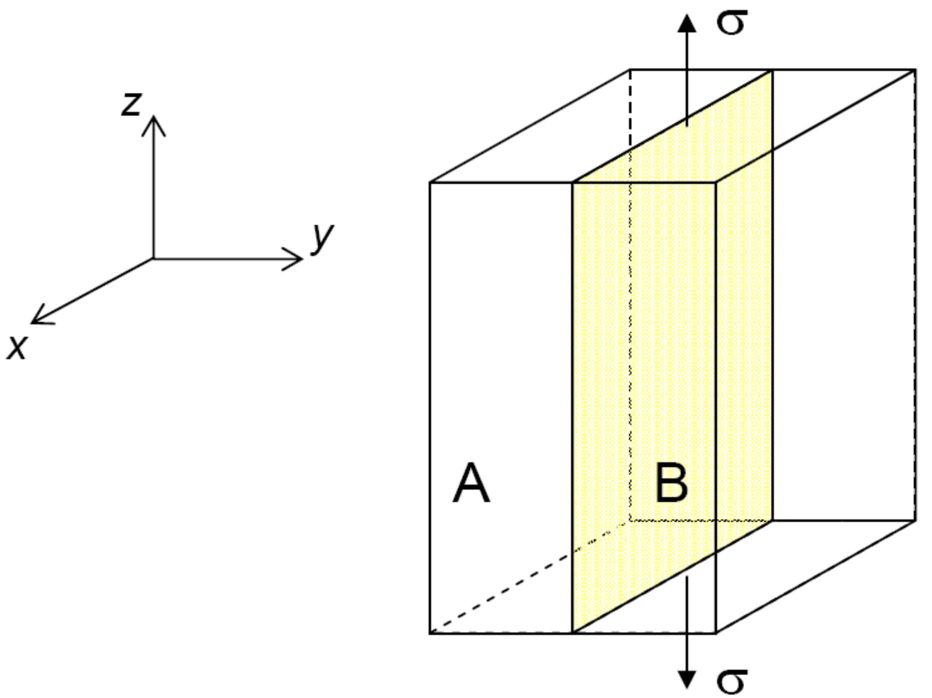


Figure 5.4

Critical resolved shear stress as a function of dislocation density for Cu single crystals and polycrystals. The observed slope of ½ on the logarithmic coordinates verifies that Eq. (5.5) describes the flow strength of work-hardened materials as it relates to dislocation density. □, polycrystalline Cu; ○, single-crystal Cu—one slip system; ◊, single-crystal Cu—two slip systems; △, single-crystal Cu—six slip systems. (*After H. Weidersich, J. Metals*, 16, *425, 1964.*)

Implications for polycrystalline materials

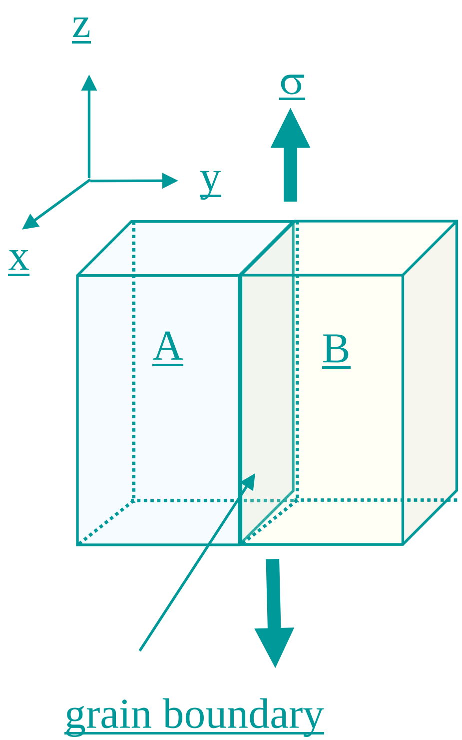
•Plastic deformation within an individual grain is constrained by the neighboring grains.
•Since plastic deformation of a single grain is restrained by its neighboring grain, a polycrystalline material will have an intrinsically greater resistance to plastic flow than would a single crystal.



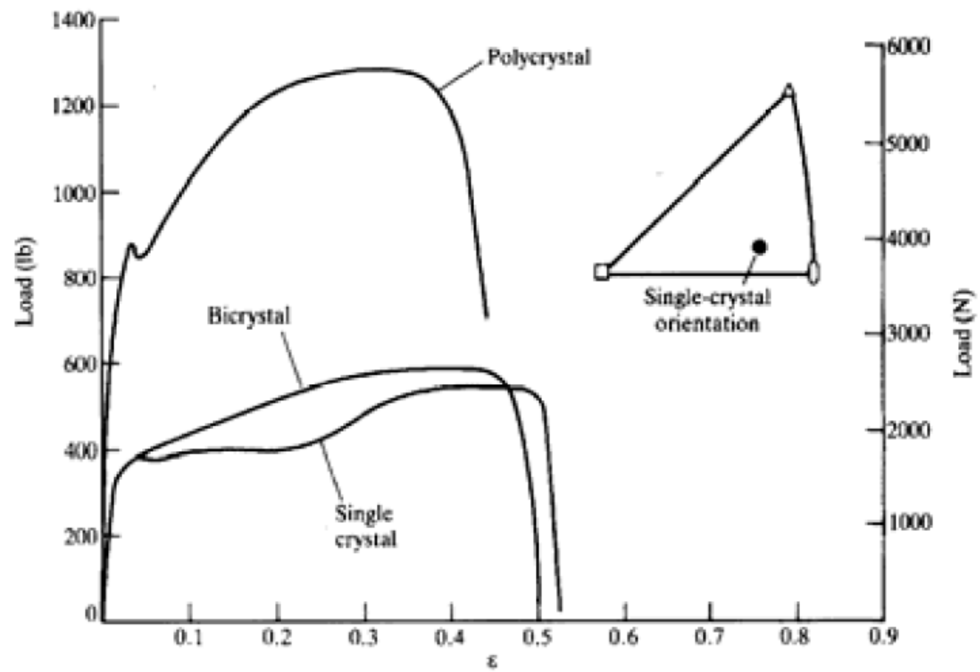
$$\varepsilon_x^A = \varepsilon_x^B$$
$$\varepsilon_z^A = \varepsilon_z^B$$
$$\gamma_{xz}^A = \gamma_{xz}^B$$

Required to maintain continuity of the grain boundary

Implications for polycrystalline materials



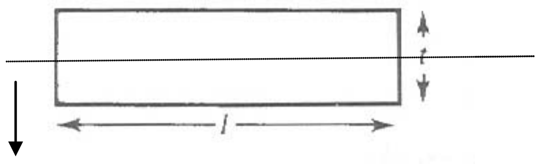
Because *one grain* has a **larger value of cos ϕ cos λ** [smaller Taylor factor (1/m)], the above constraints restrict the deformation of this more favorably oriented grain and result in a *higher Yield Strength* (greater work-hardening response of the bicrystal.

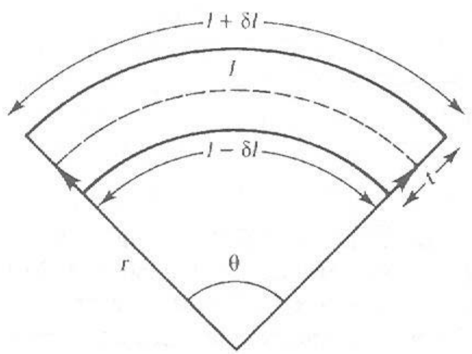


Niobium (bcc)

Geometrically necessary dislocations (↔ statistically stored dislocations)

Plastic bending of a crystal



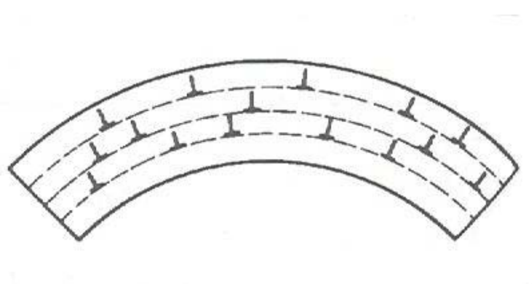


Strain gradient=strain difference/thickness

$$\left(\frac{\delta l}{l}-\frac{-\delta l}{l}\right)/t=\frac{2\delta l}{lt}=\frac{\theta}{l}=r^{-1}$$

$$\text{difference of the atomic planes}$$

$$=\frac{(l+\delta l)}{b}-\frac{(l-\delta l)}{b}$$



$$\text{density of geometrical dislocation}$$

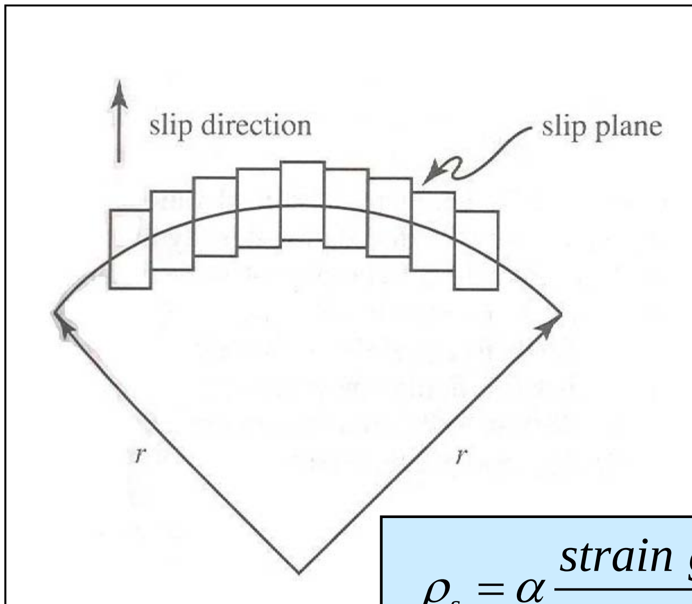
$$\rho_s=\frac{\text{the number of dislocation}}{\text{surface area}}=2\frac{\delta l}{blt}=\frac{\text{strain gradient}}{b}$$

$$\text{Generally,}$$

$$\rho_s=\alpha\frac{\text{strain gradient}}{b}$$

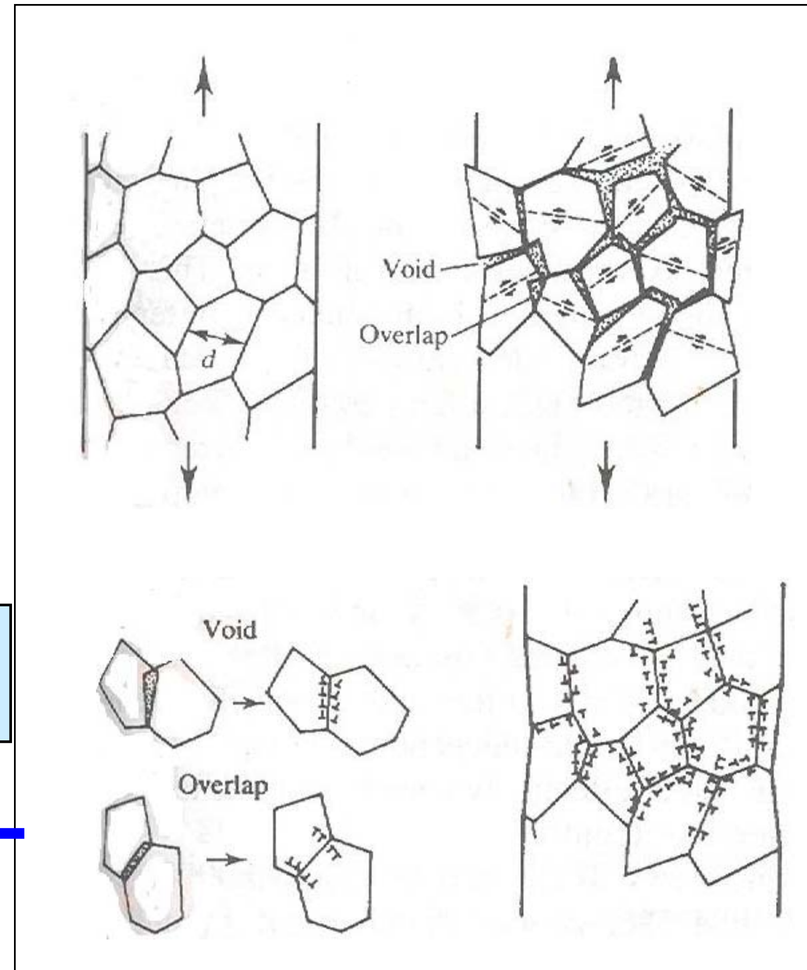
Geometrically necessary dislocations

Single crystal without geometrically necessary dislocation



$$\rho_s = \alpha \frac{strain\ gradient}{b}$$

Polycrystal: grain boundary

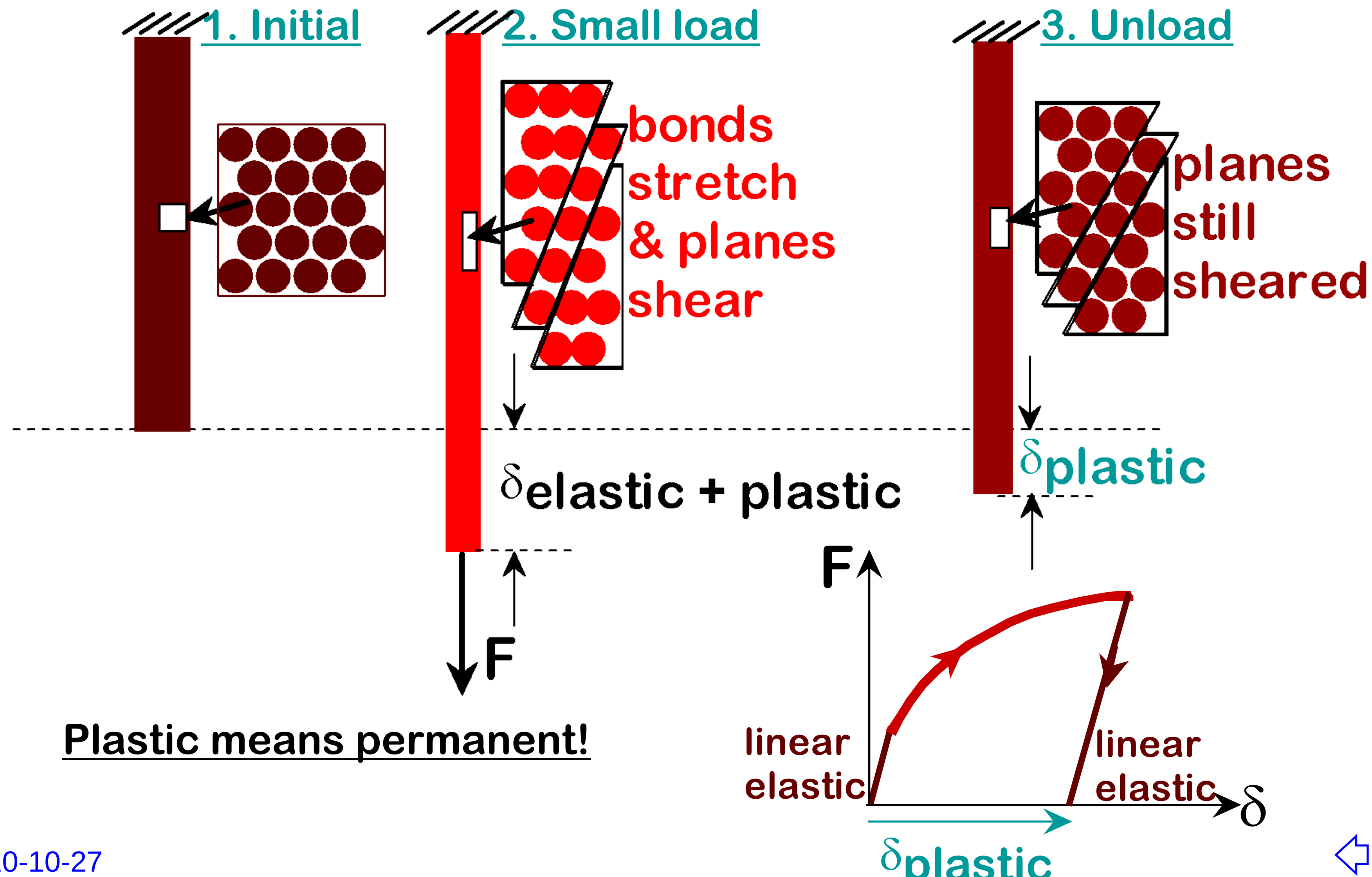


$$\rho_s \cong \frac{\bar{\varepsilon} d / b}{d^2} \Rightarrow \frac{\bar{\varepsilon}}{4bd}$$

d : diameter of grain

$\bar{\varepsilon}$: average strain

Plastic deformation (for metals)



Plastic means permanent!

Plastic Yielding

- **What is yielding?**
  Slip, Glide of Dislocation on Slip System

- **What is yield criterion ?**
  Distinction between elastic region and plastic region.

- **What is yield stress, locus and surface ?**
  Uniaxial stress (1-c) : Yield stress : A value
  Plane Stress (2-c) : Yield Locus : A line
  3 dimensions (6-c) : Yield surface : A surface

"Yield surface divides the stress space into elastic region and plastic region"

Yielding

- Plastic deformation (yielding)
  - Slip process
  - (Maximum) Shear stress

- Yield Criteria
  - Tresca (Maximum-Shear-Stress) Yield Criteria

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} \underline{\mathbf{= k}}$$

  - Von-Mises Yield Criteria

$$J_2' = k^2$$

Yield Criteria

<u>Tresca (Maximum-Shear-Stress) Yield Criteria</u>

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \mathbf{k}$$

- We can determine *k* from a simple tensile test. In uniaxial tension, yielding occurs when $\sigma_1 = \sigma_o$ (yield stress), $\sigma_2 = \sigma_3 = 0$.

$$\mathbf{k = \sigma_0 / 2}$$

$$\boxed{\sigma_1 - \sigma_3 = \sigma_o}$$

Stress state and stress space

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \Rightarrow \boldsymbol{\sigma}_{ij} = \boldsymbol{\sigma}_{ji}$$

(6 component)

Stress space => 6 dimensional space

Three principal stresses

$$\left|\sigma_{ij} - \sigma\delta_{ij}\right| = 0$$

or

$$\begin{vmatrix} \sigma_{xx} - \sigma & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} - \sigma & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} - \sigma \end{vmatrix} = 0$$

$$\Downarrow$$

$$\sigma^3 - J_1\sigma^2 + J_2\sigma - J_3 = 0$$

Stress state and stress space

$$J_1 = \mathrm{tr}(\sigma) = \sigma_{ii} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$J_2 = \frac{1}{2}\delta_{ij}^{\alpha\beta}\sigma_i^{\alpha}\sigma_j^{\beta} = \frac{1}{2}\sigma_{ij}\sigma_{ij} = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \sigma_{xy}^2 - \sigma_{yz}^2 - \sigma_{zx}^2$$

$$J_3 = \frac{1}{3}\delta_{ijk}^{\alpha\beta\gamma}\sigma_i^{\alpha}\sigma_j^{\beta}\sigma_k^{\gamma} = \det(\sigma_{ij})$$

Using the solution of cubic equation

$$Q = \frac{1}{9}(3J_2 - J_1^2) \quad R = \frac{1}{54}(-9J_1J_2 + 27J_3 + 2J_1^3)$$

$$\cos\theta = \frac{R}{\sqrt{-Q^3}}$$

$$\sigma_1 = 2\sqrt{-Q}\cos(\frac{1}{3}\theta) \quad + \frac{1}{3}J_1$$

$$\sigma_2 = 2\sqrt{-Q}\cos(\frac{1}{3}\theta + \frac{2}{3}\pi) + \frac{1}{3}J_1$$

$$\sigma_3 = 2\sqrt{-Q}\cos(\frac{1}{3}\theta + \frac{4}{3}\pi) + \frac{1}{3}J_1$$

$\Rightarrow$

$$J_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$J_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

$$J_3 = \sigma_1\sigma_2\sigma_3$$

Stress state and stress space

Deviatoric Stress S $\sigma_{ij} = p\delta_{ij} + S_{ij}$ $p = \frac{1}{3}\sigma_{ii}$

$$J'_1 = \mathrm{tr}(\mathbf{S}) = S_1 + S_2 + S_3 = S_{ii}$$

$$J'_2 = \frac{1}{2}(S_1^2 + S_2^2 + S_3^2) = \frac{1}{2}S_{ij}S_{ij}$$

$$J'_3 = S_1 S_2 S_3$$

note, $J'_2 = \frac{1}{3}(J_1^2 - 3J_2)$ => -3Q

$J'_3 = \frac{1}{27}(2J_1^3 - 9J_1J_2 + 27J_3)$ => 2R

$$J'_2 = \frac{1}{3}\left[(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})^2 - 3(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx}) - (\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2)\right]$$

$$= \frac{1}{6}\left[(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2\right] + \sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2$$

$$= \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

Yield Criteria

- We can determine k^2 from a simple tensile test. In uniaxial tension, yielding occurs when $\sigma_1 = \sigma_o$ (yield stress), $\sigma_2 = \sigma_3 = 0$. Thus J_2 becomes:

$$J_2' = \frac{1}{6}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right] = \frac{1}{6}\left[(\sigma_o)^2+(-\sigma_o)^2\right] = \frac{\sigma_o^2}{3}$$

- It represents the condition required to cause yielding. Therefore:

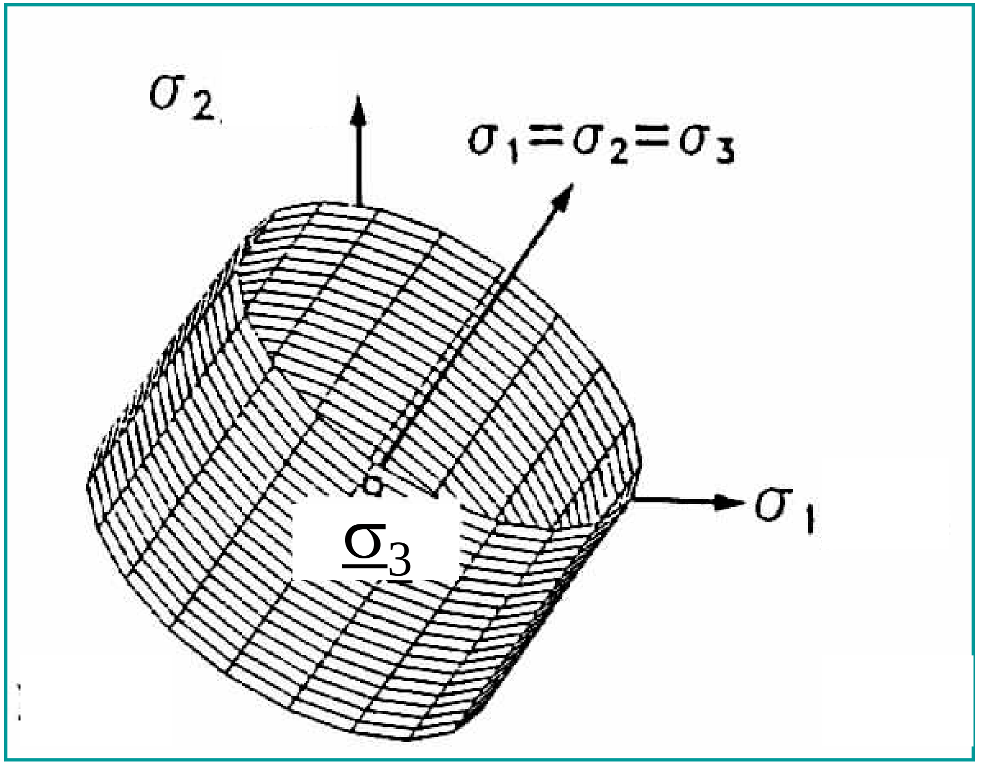
$$k^2 = \frac{1}{6}\left[\sigma_o^2+\sigma_o^2\right] = \frac{\sigma_o^2}{3} = \frac{YS^2}{3}$$

- The von Mises criterion then becomes:

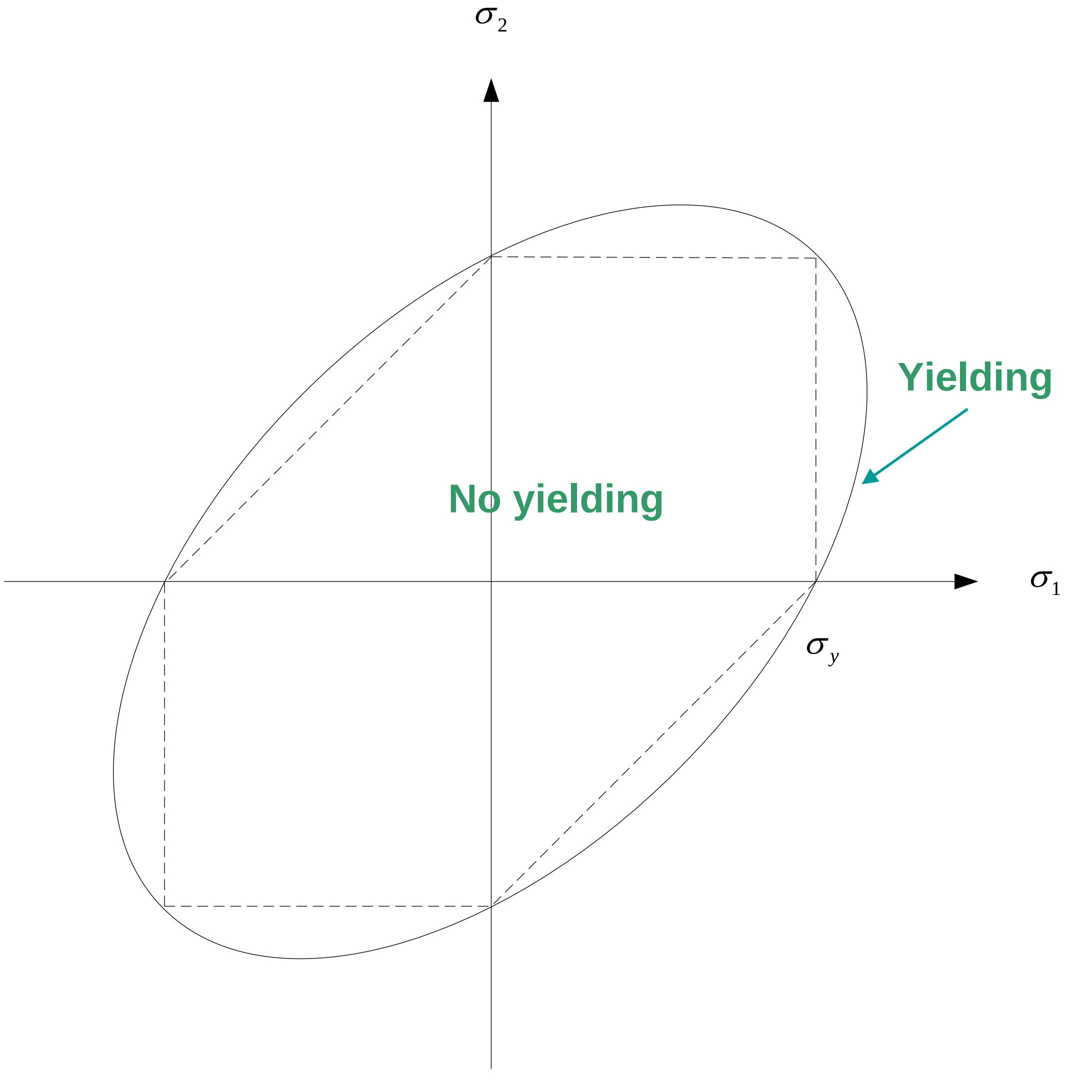
$$\frac{1}{\sqrt{2}}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]^{1/2} = \sigma_{ys} = \sigma_o$$

Yield Surface

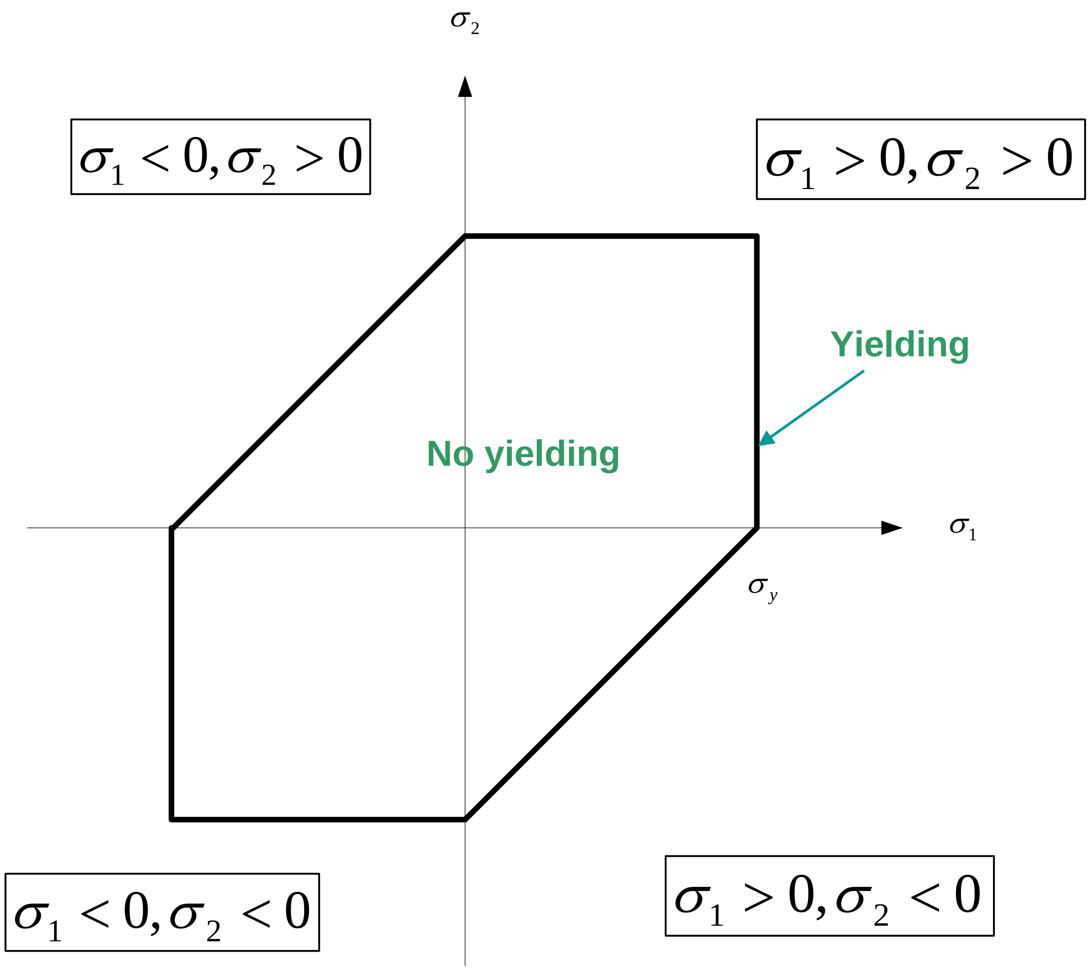
$$\frac{1}{\sqrt{2}}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]^{1/2}=\sigma_{ys}=\sigma_o$$



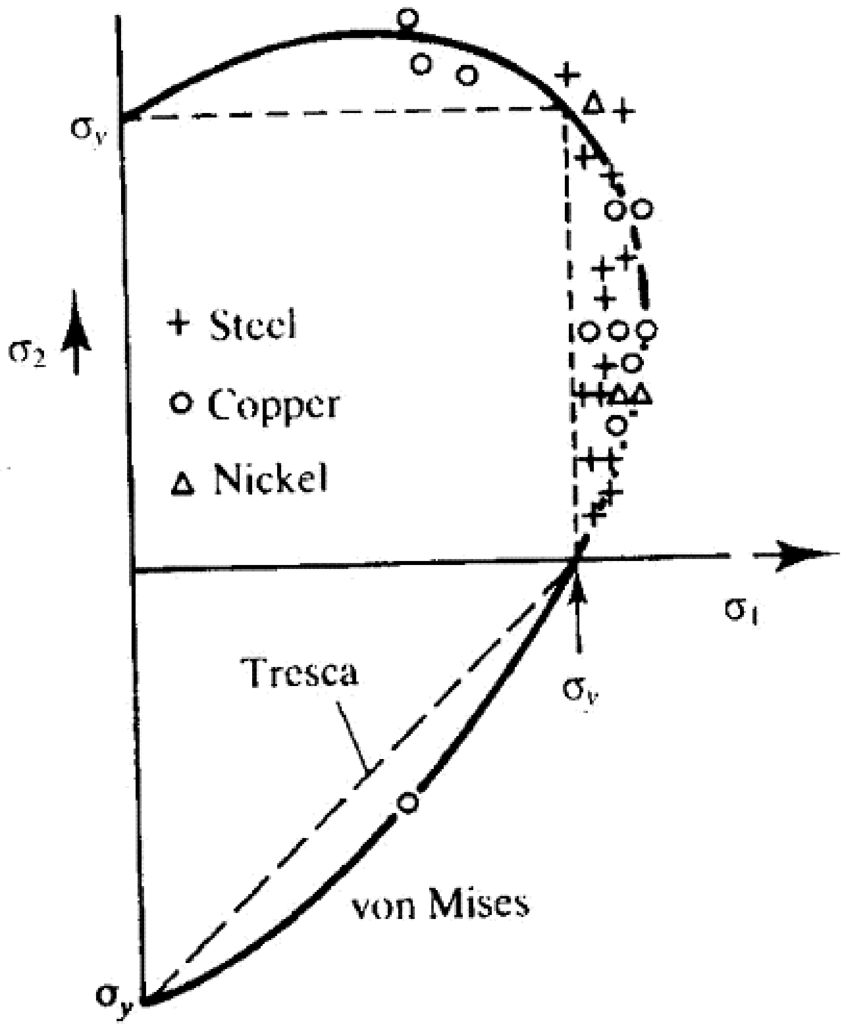
von-Mises Yield Locus



Tresca Yield Locus



Experimental comparison of Yield Locus



Yield surface

$$F(\sigma_{ij}) = 0$$

Elastic Deformation : F < 0
Plastic Deformation : F = 0

Using the representation of stress in principal stress space,

$$F(\sigma_1, \sigma_2, \sigma_3) = F(J_1, J_2, J_3) = 0$$

For metallic material, hydrostatic stress do not effect on yielding.

$$F(J'_2, J'_3) = 0$$

Yield surface

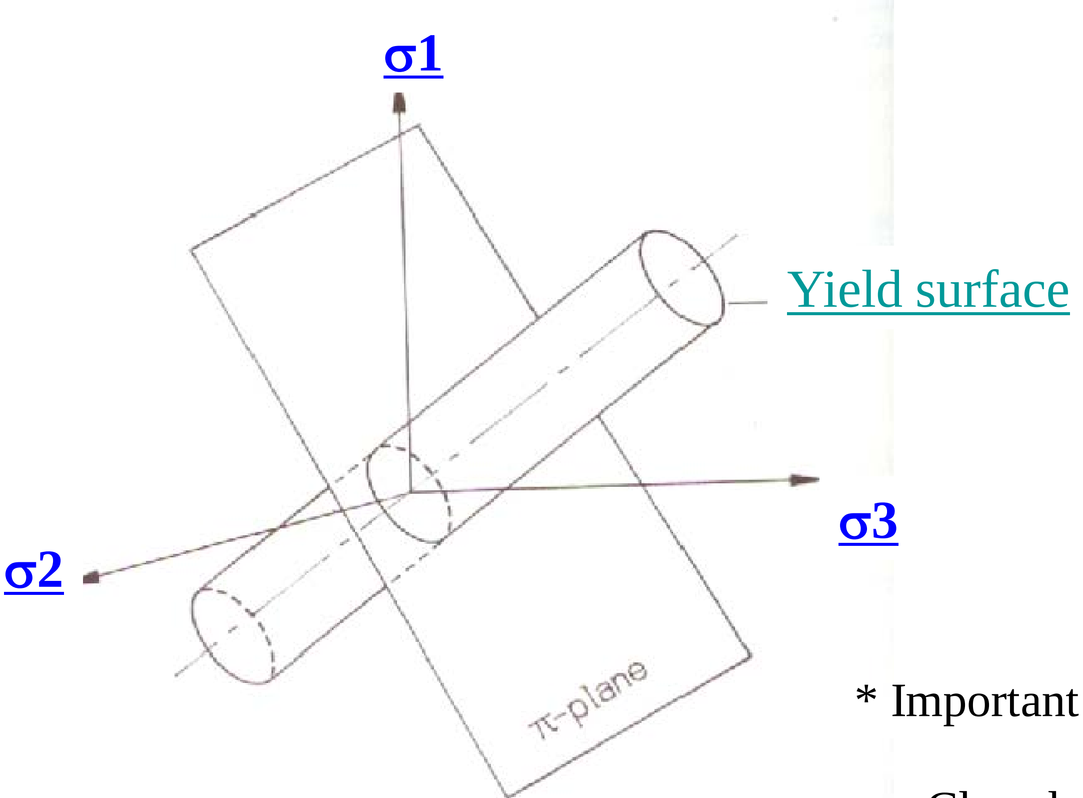


Fig. 4.3 Yield surface for isotropic materials

Decompose the yield function into

Stress dependent part : $f(\sigma_{ij})$

Stress independent part : C

$$F(\sigma_{ij}) = f(\sigma_{ij}) - C = 0$$

* Important Note on Yield surface

Closed surface in Six dimensional space

$F(\sigma_{ij}) = 0$: No physical meaning : $F(\sigma_{ij}) > 0$

Yield Criteria for Isotropic Metals (Maxwell-Huber-von Mises Criterion)

Maxwell (1856) : Initial idea
Huber : 1st published
von Mises : published
Hencky : interpret the criterion

called von Mises criterion or called 'J_2' theory'

$$J_2' - k^2 = J_2' - \frac{1}{3}\sigma_y^2 = \frac{1}{2}S_{ij}\ S_{ij} - \frac{1}{3}\sigma_y^2 = 0$$

k: some critical value

$$J_2' = \frac{1}{6}\left[(\sigma_{xx}-\sigma_{yy})^2 + (\sigma_{yy}-\sigma_{zz})^2 + (\sigma_{zz}-\sigma_{xx})^2\right] + \sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2$$

$$J_2' = \frac{1}{6}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]$$

Yield Criteria for Isotropic Metals (Maxwell-Huber-von Mises Criterion)

1) circle on π-plane
2) radius $r = \sqrt{2}k$ on π
3) No J_1 effect

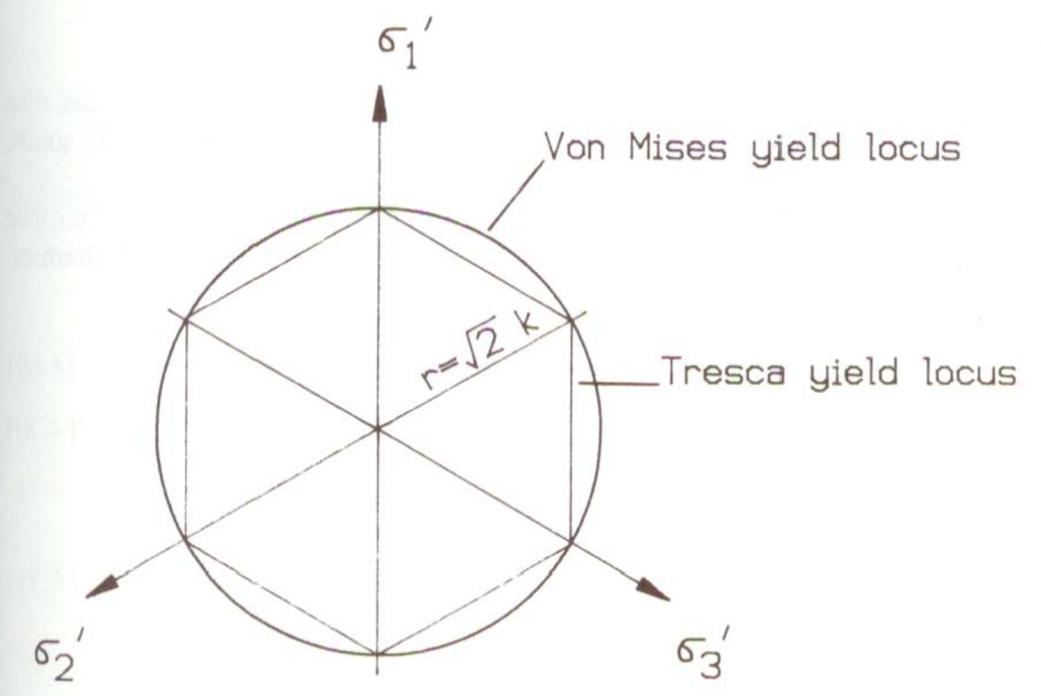


Fig. 4.5 Von Mises and Tresca yield locus on the π-plane

for uniaxial tension $\sigma_1 = \sigma_y, \sigma_2 = \sigma_3 = 0 \quad \therefore \frac{1}{3}\sigma_y^2 = k^2 \quad \therefore k = \frac{\sigma_y}{\sqrt{3}}$

for pure shear, $\sigma_{xy} = \tau_y$, others zero. $\quad \tau_y^2 = k^2 \quad k = \tau_y \quad \tau_y = \frac{\sigma_y}{\sqrt{3}}$

Effective stress

Under von-Mises Yield Criterion

$$\bar{\sigma} = \frac{1}{\sqrt{2}}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2}$$

$$\bar{\sigma} = \frac{1}{\sqrt{2}}[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2$$
$$+ 6(\sigma_{12} + \sigma_{23} + \sigma_{31})]^{1/2}$$

$$\bar{\sigma} = \sqrt{\frac{3}{2}\sigma_{ij}'\sigma_{ij}'}$$

Yield Criteria for Isotropic Metals
(Tresca condition)

$$\tau_{max} = k$$ 1864 : Tresca

in principal stress space

$$\max\left\{ \frac{1}{2}\left|(\sigma_1 - \sigma_2)\right|, \frac{1}{2}\left|(\sigma_2 - \sigma_3)\right|, \frac{1}{2}\left|(\sigma_3 - \sigma_1)\right| \right\} = k$$

$$\frac{1}{2}(\sigma_{max} - \sigma_{min}) = k$$

Yield Criteria for Isotropic Metals (Tresca condition)

Other form of Tresca condition

$$F(\sigma_{ij}) = \left\{(\sigma_1 - \sigma_2)^2 - 4k^2\right\} \cdot \left\{(\sigma_2 - \sigma_3)^2 - 4k^2\right\} \cdot \left\{(\sigma_3 - \sigma_1)^2 - 4k^2\right\}$$

$$F(J_2, J_3) = 4J_2^3 - 27J_3^2 - 36k^2J_2^2 + 96k^4J_2 - 64k^6$$

$$= 4(J_2 - k^2)(J_2 - 4k^2)^2 - 27J_3^2$$

for uniaxial tension $\sigma_1 = \sigma_y, \quad \sigma_2 = \sigma_3 = 0 \qquad \sigma_y = 2k$

for pure shear $\tau = \tau_y, \qquad \tau_y = k \qquad \therefore \tau_y = \frac{1}{2}\sigma_y$

Yield criteria for anisotropic materials

General yield function for anisotropic material

$$F(\sigma, L) = 0$$

σ : Cauchy Stress

L : Material Property Tensor generally called **Anisotropic coefficient**

* **Plastic strain ratio R**

During tensile testing of sheet, plastic strain ratio R is defined as

$$R = \frac{d\varepsilon_w}{d\varepsilon_t}$$

$d\varepsilon_w$: Strain increment along width direction

$d\varepsilon_t$: Strain increment along thickness direction

Yield criteria for anisotropic materials

In isotropic material , $F = J_2' - k^2 = 0$,

$R = 1$ in all direction

In anisotropic material,

$R = R(\theta)$ θ : Angle from rolling direction

The **best anisotropic yield function** is one of the most important goal for the material engineers and scientists.

The **best anisotropic yield function** is the yield function which can describe **exact yielding** behavior with **limited number of coefficient.**

Yield criteria for anisotropic materials (Continuum Based Function)

* Hill(1948)

$$2f(\sigma_{ij}) = F(\sigma_y - \sigma_x)^2 + G(\sigma_z - \sigma_x)^2 + H(\sigma_x - \sigma_y)^2 + 2L\tau_{yz}^2 + 2M\tau_{zx}^2 + 2N\tau_{xy}^2$$

* Hosford(1979)

$$F|\sigma_y - \sigma_x|^m + G|\sigma_z - \sigma_x|^m + H|\sigma_x - \sigma_y|^m = \sigma_o^m$$

* Hill(1979)

$$F|\sigma_y - \sigma_x|^m + G|\sigma_z - \sigma_x|^m + H|\sigma_x - \sigma_y|^m$$
$$+ L|2\sigma_1 - \sigma_2 - \sigma_3|^m + M|2\sigma_1 - \sigma_2 - \sigma_3|^m + N|2\sigma_1 - \sigma_2 - \sigma_3|^m = \sigma_o^m$$

* Hill(1990)

$$|\sigma_x + \sigma_y|^m + (\sigma^m / \tau^m)\left|(\sigma_x - \sigma_y)^2 + 4\sigma_{xy}^2\right|^{m/2}$$
$$+ \left|\sigma_x^2 + \sigma_y^2 + 2\sigma_{xy}^2\right|^{(m/2)-1}\left\{-2a(\sigma_x^2 + \sigma_y^2) + b(\sigma_x - \sigma_y)^2\right\} = (2\sigma_b)^m$$

Yield criteria for anisotropic materials (Continuum Based Function)

* Gotoh

$$C_1\sigma_x^4 + C_2\sigma_x^3\sigma_y + C_3\sigma_x^2\sigma_y^2 + C_4\sigma_x\sigma_y^3 + C_5\sigma_y^4 + \sigma_{xy}^2(C_6\sigma_x^2 + C_7\sigma_x\sigma_y + C_8\sigma_y^2) + C_9\sigma_{xy}^4 = \sigma_o^4$$

* Bassani

$$\left|\sigma_1 + \sigma_2\right|^m + \frac{n}{m}(1+R)\sigma_o^{n-m}\left|\sigma_1 - \sigma_2\right|^m = \left\{1 + \frac{n}{m}(1+2R)\right\}\sigma_o^m$$

* CMTP

$$F = \alpha\{\left|S_{11} - S_{22}\right|^n + \left|S_{22} - S_{33}\right|^n + \left|S_{33} - S_{11}\right|^n\} + 2\beta\{\left|S_{12}\right|^m + \left|S_{23}\right|^m + \left|S_{31}\right|^m\},$$

* Barlat(1991)

$$\Phi = \left|S_1 - S_2\right|^m + \left|S_3 - S_1\right|^m + \left|S_1 - S_2\right|^m = 2\overline{\sigma}^m$$

$S_i = L_{ij}\sigma_j$ L : Symmetry Operation tensor

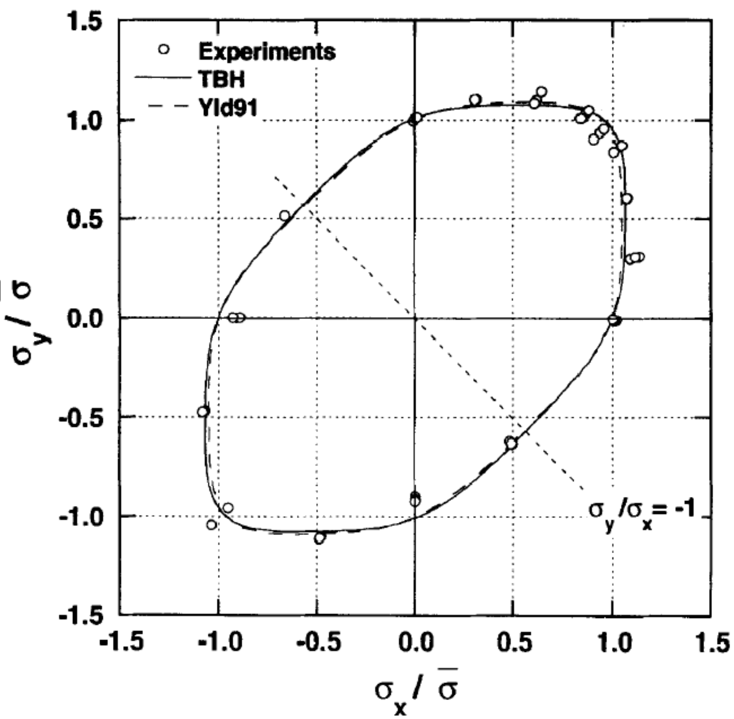


Fig. 5. Experimental, polycrystal and phenomenological (Yld91) yield loci for material BLA (low cold reduction followed by SHT, 2.5% Mg, 150 μm grain size). Coefficients for Yld91: $a = 8$, $c_1 = 1.017$, $c_2 = 1.023$, $c_3 = 0.976$; c_4, c_5 and c_6 irrelevant in this case.

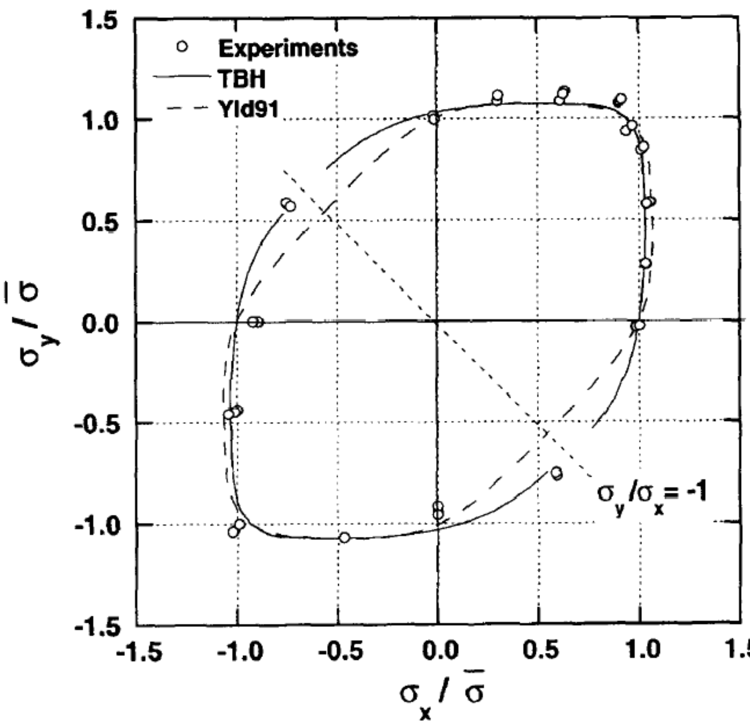


Fig. 6. Experimental, polycrystal and phenomenological (Yld91) yield loci for material BHE (high cold reduction followed by SHT, 2.5% Mg, 150 μm grain size). Coefficients for Yld91: $a = 8$, $c_1 = 1.005$, $c_2 = 1.036$, $c_3 = 0.963$; c_4, c_5 and c_6 irrelevant in this case.

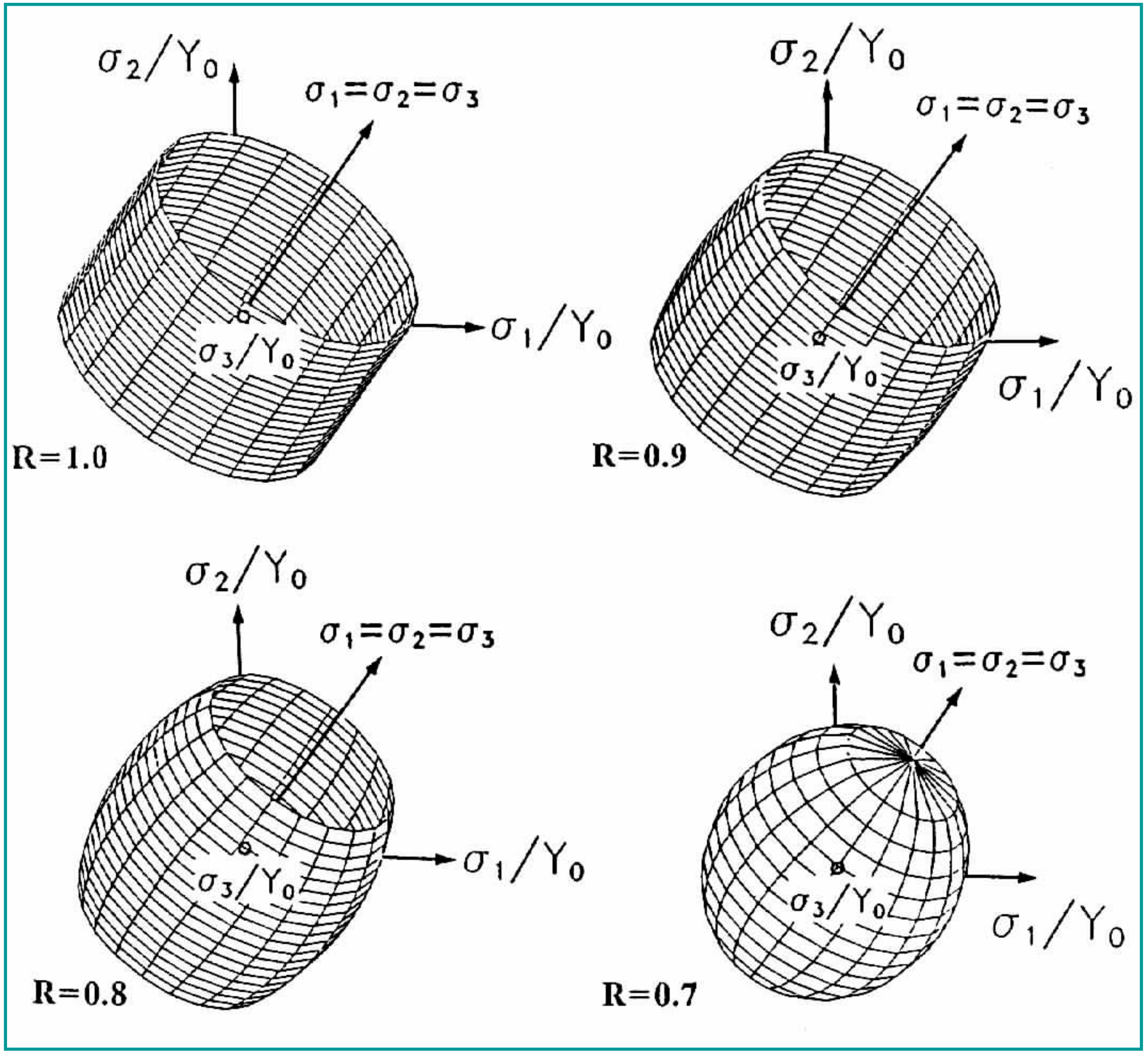
Yield criterion of Powder

$$\boxed{AJ_2' + BJ_1^2 = \eta Y_0^2}$$

$$\nu = -\frac{\dot{\varepsilon}_{11}^{p}}{\dot{\varepsilon}_{33}^{p}} = \frac{A-2}{2} = \frac{1-3B}{2} \quad \Rightarrow \quad 2(1+\nu)J_2' + \frac{(1-2\nu)}{3}J_1^2 = \eta Y_0^2$$

Authors	η	ν
Green [10]	$\frac{\delta}{1+\alpha}$	$\frac{1-2\alpha}{2(1+\alpha)}$
	where $\alpha = \frac{1}{4}\left[\frac{3\left\{1-(1-R)^{1/3}\right\}}{\left\{3-2(1-R)^{1/4}\right\}\ln(1-R)}\right]^2$	and $\delta = \left[\frac{3\left\{1-(1-R)^{1/3}\right\}}{3-2(1-R)^{1/4}}\right]^2$
Shima and Oyane [11]	$\frac{R^5}{1+(2.49/3)^2(1-R)^{1.028}}$	$\frac{1-2(2.49/3)^2(1-R)^{1.028}}{2\left\{1+(2.49/3)^2(1-R)^{1.028}\right\}}$
Gurson [12]	$\frac{4R^2}{5-R}$	$\frac{1+R}{5-R}$
Doraivelu *et al.* [13]	$2R^2-1$	$0.5R^2$
Lee and Kim [14]	$\left(\frac{R-R_C}{1-R_C}\right)^2$	$0.5R^2$
Park and Han	$[(R-R_T)/(1-R_T)]^m$	$0.5R^2$

Yield criterion of powder



Isotropic hardening and kinematic hardening

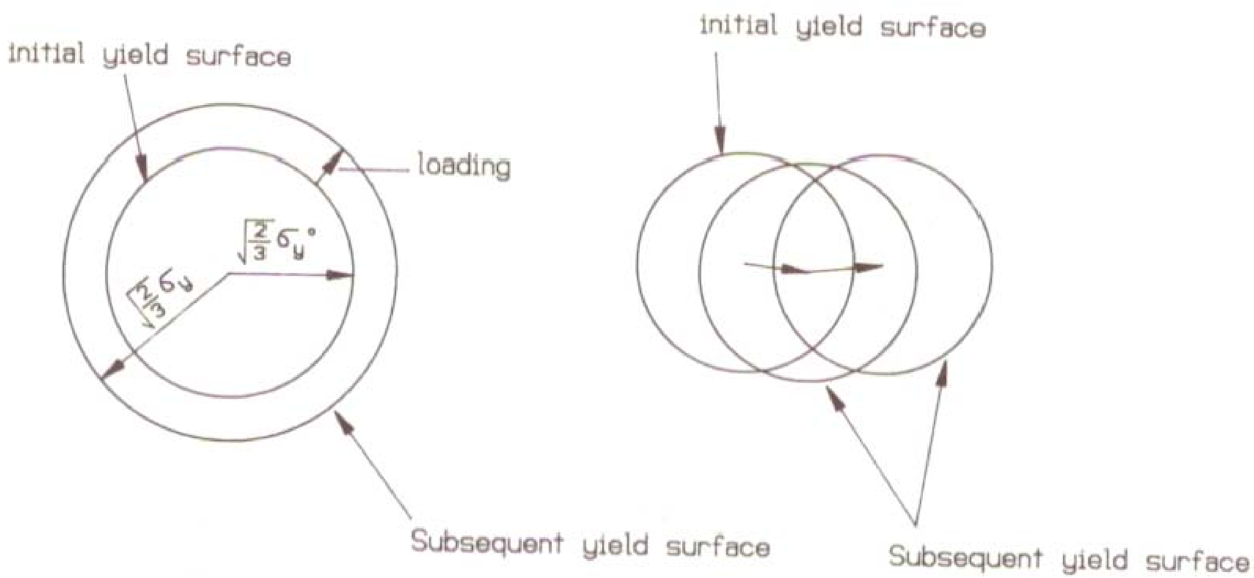
Yield surface movement in stress space

- Expansion

- Translation

- Distortion

Isotropic hardening	**Kinematic hardening**
i) yield locus center is fixed.	**i) yield locus center is translated.**
ii) Expansion of size	**ii) same size**



Isotropic hardening and kinematic hardening

Typical Example of Yield Function with Hardening

Isotropic : $J_2' = k \cdot \varepsilon_p^n$

Kinematic : $f(\mathbf{S} - \boldsymbol{\alpha}) = k$ **α : back stress**

To consider Bauschinger effect

General yield surface with hardening

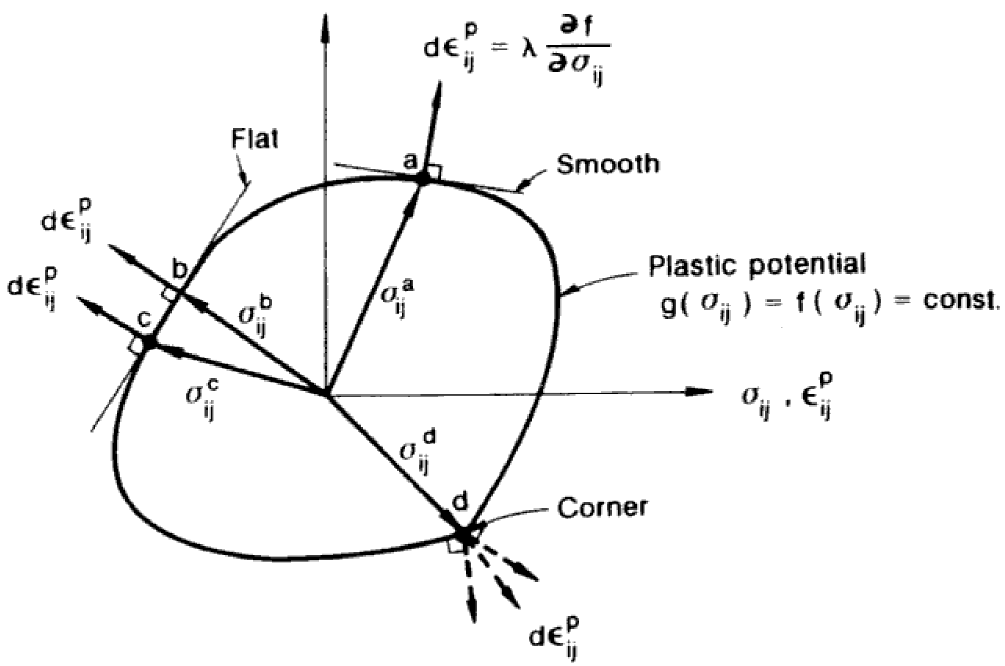
$$F(L, \sigma, \alpha_i) = 0$$

Plastic Potential Theory and Plastic work

1928, von Mises proposed a Potential function $Q = Q(\sigma_{ij})$ which satisfies

$$\dot{\varepsilon}_{ij}^{p} = \dot{\lambda}\frac{\partial Q}{\partial \sigma_{ij}}$$ **Plastic Potential Theory**

Q ; Plastic Potential Function (Scalar Function)



Plastic Potential Theory and Plastic work

1928, von Mises proposed a Potential function $Q = Q(\sigma_{ij})$ which satisfies

$$\dot{\varepsilon}_{ij}^{p} = \dot{\lambda}\frac{\partial Q}{\partial \sigma_{ij}} \qquad \textbf{Plastic Potential Theory}$$

Q ; Plastic Potential Function (Scalar Function)

1. Geometric representation

$Q(\sigma_{ij}) = C$ 에 수직한 방향으로 $\dot{\varepsilon}_{ij}^{p}$ 가 정해짐

(use definition of gradient) thus called as “Normality Rule”

2. Isotropy : $Q = Q(\sigma_{ij}) = Q(J_1, J_2, J_3)$ Anisotropy : $Q = Q(\sigma_{ij}, \beta_k)$ k=1,2,...,n

3. Incompressibility of plastic flow : $Q = Q(J_2', J_3')$

Porous material (J_1 sensitive) : $Q = Q(J_1, J_2, J_3)$

Plastic Potential Theory and Plastic work

*** Flow Rule :** $\dot{\varepsilon}_{ij}^{p} = \dot{\lambda}\dfrac{\partial Q}{\partial \sigma_{ij}}$

Associated Flow Rule

$$\dot{\varepsilon}_{ij}^{p} = \dot{\lambda}\frac{\partial F}{\partial \sigma_{ij}}$$

$$Q \equiv F$$

Nonassociated Flow Rule

$$\dot{\varepsilon}_{ij}^{p} = \dot{\lambda}\frac{\partial Q}{\partial \sigma_{ij}}$$

$$Q \neq F$$

Typically deformation of metal - associated flow rule

Concrete, soil - nonassociated

*** Prantl-Reuss Equation (or Levy-Mises Equation)**

Associated flow rule based on von Mises yield function

$$F = \frac{1}{2}S_{ij}S_{ij} - k \quad \rightarrow \quad \frac{\partial F}{\partial \sigma_{ij}} = S_{ij}$$

$$\dot{\varepsilon}_{ij}^{p} = \dot{\lambda}\frac{\partial F}{\partial \sigma_{ij}} = \dot{\lambda}S_{ij} \quad \rightarrow \quad \text{Prantl-Reuss equation}$$

Plastic constitutive equation (flow rule)

Levy and von Mises suggested this relationship under von-Mises yield criterion. dλ : a positive constant.

$$\frac{d\varepsilon^p{}_{11}}{\sigma_{11}'} = \frac{d\varepsilon^p{}_{22}}{\sigma_{22}'} = \frac{d\varepsilon^p{}_{33}}{\sigma_{33}'} = \frac{d\varepsilon^p{}_{12}}{\sigma_{12}} = \frac{d\varepsilon^p{}_{23}}{\sigma_{23}} = \frac{d\varepsilon^p{}_{31}}{\sigma_{31}} = d\lambda$$

$$d\varepsilon^p{}_{11} = \frac{2}{3} d\lambda[\sigma_{11} - \frac{1}{2}(\sigma_{22} + \sigma_{33})]$$

$$d\varepsilon^p{}_{22} = \frac{2}{3} d\lambda[\sigma_{22} - \frac{1}{2}(\sigma_{33} + \sigma_{11})]$$

$$d\varepsilon^p{}_{33} = \frac{2}{3} d\lambda[\sigma_{33} - \frac{1}{2}(\sigma_{11} + \sigma_{22})]$$

$$d\varepsilon^p{}_{12} = d\lambda \sigma_{12}$$

$$d\varepsilon^p{}_{23} = d\lambda \sigma_{23}$$

$$d\varepsilon^p{}_{31} = d\lambda \sigma_{31}$$

$$d\lambda = \frac{d\bar{\varepsilon}^p}{\bar{\sigma}}$$

$$dW = \bar{\sigma} d\bar{\varepsilon}^p = \sigma_{ij} d\varepsilon^p{}_{ij}$$

$$d\bar{\varepsilon}^p = \sqrt{\frac{2}{3} d\varepsilon^p{}_{ij}' d\varepsilon^p{}_{ij}'}$$

General Stress-Strain Relations

Note that plastic flow relation must be incremental form

$$d\boldsymbol{\sigma} = \mathbf{C}^{ep} : d\boldsymbol{\varepsilon}$$

$\mathbf{C}^{ep}$; the elastic -plastic stiffness tensor (4th order)

$d\boldsymbol{\sigma}$; stress increment

$d\boldsymbol{\varepsilon}$; strain increment

$F(\boldsymbol{\sigma},\boldsymbol{\alpha}) = 0$; yield condition

plastic deformation cause $\quad \begin{aligned} \boldsymbol{\sigma} &\to \boldsymbol{\sigma} + d\boldsymbol{\sigma} \\ \boldsymbol{\alpha} &\to \boldsymbol{\alpha} + d\boldsymbol{\alpha} \end{aligned}$

$\boldsymbol{\sigma} + d\boldsymbol{\sigma}$, $\boldsymbol{\alpha} + d\boldsymbol{\alpha}$ must be on the subsequent yield surface $\rightarrow$ $F(\boldsymbol{\sigma} + d\boldsymbol{\sigma}, \boldsymbol{\alpha} + d\boldsymbol{\alpha}) = 0$

$$F(\sigma + d\sigma, \alpha + d\alpha) = F(\sigma,\alpha) + \frac{\partial F}{\partial \sigma} : d\sigma + \frac{\partial F}{\partial \alpha} : d\alpha = 0$$

$$dF = \frac{\partial F}{\partial \boldsymbol{\alpha}} : d\boldsymbol{\sigma} + \frac{\partial F}{\partial \boldsymbol{\alpha}} : d\boldsymbol{\alpha} = 0$$

General Stress-Strain Relations (Isotropic Hardening)

$$F(\boldsymbol{\sigma},\boldsymbol{\alpha}) = f(\boldsymbol{\sigma}) - k = 0$$

$$\alpha = \varepsilon_e^p \quad \text{or} \quad \alpha = W^p \qquad k = k(\alpha)$$

$$d\varepsilon^e = \mathbf{C}^{e^{-1}} : d\sigma, \qquad d\varepsilon^p = \dot{\lambda}\frac{\partial F}{\partial \sigma}$$

$$dF = \frac{\partial F}{\partial \boldsymbol{\sigma}} : d\boldsymbol{\sigma} + \frac{\partial F}{\partial \alpha} d\alpha = 0 \qquad = \frac{\partial F}{\partial \boldsymbol{\sigma}} : d\boldsymbol{\sigma} + \left(-\frac{dk}{d\varepsilon_e^p}\right) d\varepsilon_e^p = 0$$

$$= \frac{\partial f}{\partial \boldsymbol{\sigma}} : d\boldsymbol{\sigma} + \left(-\frac{dk}{d\varepsilon_e^p}\right)\frac{2}{\sqrt{6}}\dot{\lambda}\left(\frac{\partial f}{\partial \boldsymbol{\sigma}} : \frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^{1/2}$$

$$\therefore \dot{\lambda} = \frac{\dfrac{\partial f}{\partial \boldsymbol{\sigma}} : d\boldsymbol{\sigma}}{\dfrac{2}{\sqrt{6}}\dfrac{dk}{d\varepsilon_e^p}\left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}} : \dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^{1/2}}$$

General Stress-Strain Relations (Isotropic Hardening)

$$l = \frac{\partial f}{\partial \boldsymbol{\sigma}} : d\boldsymbol{\sigma}\,, \qquad \mathbf{n} = \frac{\dfrac{\partial f}{\partial \boldsymbol{\sigma}}}{\left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}} : \dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^{1/2}}$$ (unit normal to the yield surface)

Thus $d\boldsymbol{\varepsilon}^p = \dot{\lambda}\dfrac{\partial F}{\partial \boldsymbol{\sigma}}$ $\rightarrow$

$$d\boldsymbol{\varepsilon}^p = \frac{\dfrac{\partial f}{\partial \sigma} : d\sigma}{\dfrac{2}{\sqrt{6}}\dfrac{dk}{d\varepsilon_e^p}\left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}} : \dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^{1/2}}\frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{\sqrt{6}\,l}{2\dfrac{dk}{d\varepsilon_e^p}}\mathbf{n} = \frac{\sqrt{6}}{2\dfrac{dk}{d\varepsilon_e^p}}(\mathbf{n} : d\sigma)\frac{\partial f}{\partial \sigma}$$

using $d\varepsilon_e^p = \dfrac{2}{\sqrt{6}}(d\boldsymbol{\varepsilon}^p : d\boldsymbol{\varepsilon}^p)^{1/2}$ $\rightarrow$

$$d\varepsilon_e^p = \frac{\dfrac{\partial f}{\partial \boldsymbol{\sigma}} : d\sigma}{\left(\dfrac{dk}{d\varepsilon_e^p}\right)} = \frac{1}{\left(\dfrac{dk}{d\varepsilon_e^p}\right)} l$$

General Stress-Strain Relations (Isotropic Hardening)

using additive decomposition of strain

$$d\boldsymbol{\varepsilon} = d\boldsymbol{\varepsilon}^{e} + d\boldsymbol{\varepsilon}^{p} = \mathbf{C}^{e^{-1}} : d\boldsymbol{\sigma} + \frac{\sqrt{6}}{2(\frac{dk}{d\varepsilon_{e}^{p}})}\frac{\partial f}{\partial \boldsymbol{\sigma}}\mathbf{n} : d\boldsymbol{\sigma} = \left[\mathbf{C}^{e^{-1}} + \frac{\sqrt{6}}{2(\frac{dk}{d\varepsilon_{e}^{p}})}\frac{\partial f}{\partial \boldsymbol{\sigma}}\mathbf{n}\right] : d\boldsymbol{\sigma}$$

$$d\boldsymbol{\sigma} = \left[\mathbf{C}^{e^{-1}} + \frac{\sqrt{6}}{2(\frac{dk}{d\varepsilon_{e}^{p}})}\frac{\partial f}{\partial \boldsymbol{\sigma}}\mathbf{n}\right]^{-1} : d\boldsymbol{\varepsilon} \quad \rightarrow \quad \mathbf{C}^{ep} = \left[\mathbf{C}^{e^{-1}} + \frac{\sqrt{6}}{2(\frac{dk}{d\varepsilon_{e}^{p}})}\frac{\partial f}{\partial \boldsymbol{\sigma}}\mathbf{n}\right]^{-1}$$

General Stress-Strain Relations (Isotropic Work Hardening)

$$dF = \frac{\partial F}{\partial \boldsymbol{\sigma}} : d\boldsymbol{\sigma} + \frac{\partial F}{\partial W^p} dW^p = 0$$

$$F(\boldsymbol{\sigma}, W^p) = f(\boldsymbol{\sigma}) - k(W^p) = 0$$

Using $\dfrac{\partial F}{\partial W^p} = -\dfrac{dk}{dW^p}$, $\quad dW^p = \boldsymbol{\sigma} : d\boldsymbol{\varepsilon}^p = \mathbf{S} : d\boldsymbol{\varepsilon}^p = \mathbf{S} : \dot{\lambda} \dfrac{\partial f}{\partial \boldsymbol{\sigma}}$

$$dF = 1 + \left(-\frac{dk}{dW^p}\right) \dot{\lambda} \frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{S} = 0 \quad \rightarrow \quad \therefore \dot{\lambda} = \frac{1}{\left(\frac{dk}{dW^p}\right) \frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{S}}$$

$$d\boldsymbol{\varepsilon}^p = \frac{\frac{\partial f}{\partial \boldsymbol{\sigma}}}{\left(\frac{dk}{dW^p}\right) \frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{S}} 1 \qquad \text{(associated flow rule)}$$

$$d\boldsymbol{\varepsilon} = \left[\mathbf{C}^{e^{-1}} + \frac{\frac{\partial f}{\partial \boldsymbol{\sigma}} \frac{\partial f}{\partial \boldsymbol{\sigma}}}{\left(\frac{dk}{dW^p}\right) \frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{S}} \right] : d\boldsymbol{\sigma} \quad \rightarrow \quad \mathbf{C}^{ep} = \left[\mathbf{C}^{e^{-1}} + \frac{\frac{\partial f}{\partial \sigma} \frac{\partial f}{\partial \sigma}}{\frac{dk}{dW^p} \frac{\partial f}{\partial \sigma} : \mathbf{S}} \right]^{-1}$$

General Stress-Strain Relations (Isotropic Work Hardening)

using von Mises $\quad J_2' - \frac{1}{3}\sigma_y^2 = 0$

* Strain hardening

$$J_2' - \frac{1}{3}\sigma_y^2(\varepsilon_e^p) = 0 \quad \rightarrow \quad \frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{\partial J_2'}{\partial \boldsymbol{\sigma}} = \mathbf{S}, \qquad \frac{dk}{d\varepsilon_e^p} = \frac{1}{3}\frac{d(\sigma_y^2)}{d\varepsilon_e^p} = \frac{2}{3}\sigma_y \frac{d\sigma}{d\varepsilon_e^p}$$

$$\rightarrow \quad \mathbf{C}^{ep} = \left[\mathbf{C}^{e^{-1}} + \frac{9\mathbf{SS}}{4\sigma_y^2 \dfrac{d\sigma_y}{d\varepsilon_e^p}} \right]^{-1}$$

* Work hardening

$$J_2' - \frac{1}{3}\sigma_y^2(W^p) = 0 \quad \rightarrow \quad \frac{\partial f}{\partial \boldsymbol{\sigma}} = \mathbf{S}, \qquad \frac{dk}{dW^p} = \frac{2}{3}\sigma_y \frac{d\sigma_y}{dW^p}$$

$$\rightarrow \quad \mathbf{C}^{ep} = \left[\mathbf{C}^{e^{-1}} + \frac{9\mathbf{SS}}{4\sigma_y^3 \dfrac{d\sigma_y}{dW^p}} \right]^{-1}$$

General Stress-Strain Relations (Kinematic Hardening)

using constitutive eqn.($d\boldsymbol{\alpha} = c\,d\boldsymbol{\varepsilon}^p$) and associated flow rule($d\boldsymbol{\varepsilon}^p = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}}$)

For the von Mises criterion,

$$f(\boldsymbol{\sigma} - \boldsymbol{\alpha}) - k_0 = \frac{1}{2}(\mathbf{s} - \boldsymbol{\alpha}) : (\mathbf{s} - \boldsymbol{\alpha}) - \frac{1}{3}\boldsymbol{\sigma}_y^{\circ\,2} = 0$$

$$\mathbf{n} = \frac{(\mathbf{s} - \boldsymbol{\alpha})}{\left[(\mathbf{s} - \boldsymbol{\alpha}) : (\mathbf{s} - \boldsymbol{\alpha})\right]^{1/2}}$$

Thus $\mathbf{C}^{ep}$ is

$$\mathbf{C}_{ep} = \left[\mathbf{C}^{e^{-1}} + \frac{1}{c}\frac{(\mathbf{s} - \boldsymbol{\alpha})}{(\mathbf{s} - \boldsymbol{\alpha}) : (\mathbf{s} - \boldsymbol{\alpha})}\right]^{-1}$$

General Stress-Strain Relations (Combined Isotropic and Kinematic)

$$f(\boldsymbol{\sigma}-\boldsymbol{\alpha})+k(W_p)=0$$

$$\frac{\partial f}{\partial \boldsymbol{\sigma}}:d\boldsymbol{\sigma}+\frac{\partial f}{\partial \boldsymbol{\alpha}}:d\boldsymbol{\alpha}-\frac{dk}{dW_p}dW^p=0$$

$$\dot{\lambda}=\frac{\frac{\partial f}{\partial \boldsymbol{\sigma}}:d\boldsymbol{\sigma}}{d\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}:\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)+\frac{dk}{dW^p}\frac{\partial f}{\partial \boldsymbol{\sigma}}:\mathbf{s}}=\frac{1}{k_p}\frac{1}{\frac{\partial f}{\partial \boldsymbol{\sigma}}:\frac{\partial f}{\partial \boldsymbol{\sigma}}}$$

$$k_p=c+\frac{\frac{dk}{dW^p}+\frac{\partial f}{\partial \boldsymbol{\sigma}}:\mathbf{s}}{\frac{\partial f}{\partial \boldsymbol{\sigma}}:\frac{\partial f}{\partial \boldsymbol{\sigma}}}$$

$$d\boldsymbol{\varepsilon}^p=\dot{\boldsymbol{\lambda}}\frac{\partial f}{\partial \boldsymbol{\sigma}}=\frac{1}{k_p}(\mathbf{n}:d\boldsymbol{\sigma})\mathbf{n}$$

$$\mathbf{C}^{ep}=\left[\mathbf{C}^{e^{-1}}+\frac{1}{k_p}\mathbf{nn}\right]^{-1}$$