Lecture 12. Aperture and Noise
Analysis of Clocked Comparators

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Clocked Comparators

- a.k.a. regenerative amplifier, sense-amplifier, flip-flop, latch, etc.

At every clock edge, sample the input (continuous) and decide whether it is 0 or 1 (binary)

- Therefore, it’s inherently nonlinear operation
Comparator Characteristics

- Offset and hysteresis
- Sampling aperture, timing resolution, uncertainty window
- Regeneration gain, voltage sensitivity, metastability
- Random decision errors, input-referred noise

→ Can be analyzed and simulated based on a linear, time-varying (LTV) model of the comparator
Clocked Comparator Operation

- 4 operating phases: reset, sample, regeneration & decision
- Sampling & regeneration phases can be modeled as LTV
An Ideal Comparator Model

\[ V_k = V_i(t_0 + kT), \]

Sampling and decision
- Infinitely-fast tracking of \( V_i(t) \)
- A realistic comparator acts on a filtered version of \( V_i(t) \)
LTV Model for Clocked Comparator

Assumes a noisy, nonlinear filter before the sampling

The filter’s small-signal response is modeled with ISF $\Gamma(\tau)$

ISF for Oscillators

- Impulse sensitivity function (ISF) $\Gamma(\tau)$ is defined as:

$$\Gamma(\tau) = \text{the final shift in the oscillator phase due to a unit impulse arriving at time } \tau$$

**ISF for Oscillators (2)**

- ISF describes the time-varying response of an oscillator
  - Responses to each impulse add up via superposition
  - For arbitrary noise input $n(t)$, the resulting phase shift $\Delta \phi$ is:

  \[ \Delta \phi = \int_{-\infty}^{\infty} \Gamma(\tau) \cdot n(\tau) \, d\tau \]

- ISF led to some key oscillator design idioms:
  - Sharpen the clock edge to lower ISF (i.e. minimize $\Gamma_{\text{RMS}}$)
  - Align noise events within low-ISF period
  - Balance ISF (i.e. $\Gamma_{\text{DC}}=0$) to prevent 1/f-noise up-conversion
ISF for Samplers and Comparators

For sample-and-hold circuits, the sampled voltage $V_s$ can be expressed via a “sampling function” $f(t)$:

$$V_s = \int_{-\infty}^{\infty} f(\tau) \cdot V_i(\tau) \, d\tau$$


For clocked comparators, we simply add the “decision”:

$$D_k = \text{sgn}(V_k) = \text{sgn}\left(\int_{-\infty}^{\infty} \Gamma(\tau) \cdot V_i(\tau) \, d\tau\right)$$

ISF for Clocked Comparators

- ISF shows sampling aperture, i.e. timing resolution
- In frequency domain, it shows sampling gain and BW

\[
\text{ISF } \Gamma(\tau) \quad \text{F.T. } \{ \Gamma(-\tau) \}
\]
Generalized ISF

- In general, ISF is a subset of a so-called time-varying impulse response $h(t, \tau)$ for LTV systems*:

$$ y(t) = \int_{-\infty}^{\infty} h(t, \tau) \cdot x(\tau) \, d\tau $$

- $h(t, \tau)$: the system response at $t$ to a unit impulse arriving at $\tau$
- For LTI systems, $h(t, \tau) = h(t-\tau) \rightarrow$ convolution

- ISF $\mathcal{I}(\tau) = h(t_0, \tau)$

  - $t_0$: the time at which the system response is observed
  - For oscillators, $t_0 = +\infty$
  - For comparators, $t_0$ is before the decision is made (more later)

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Noise in LTV Systems

- If the input $x(t)$ to an LTV system is a noise process, then the output $y(t)$ is a time-varying noise in general.
  - Expressions become very complex (cyclo-stationary at best)
- We can keep things simple if we are interested in the noise only at one time point (in our case: $t_0 = t_{obs} + kT$)
LTV Output Noise at $t = t_0$

$$\sigma_y^2(t_0) = E[y^2(t_0)] = E[y(t_0) \cdot y(t_0)]$$

$$= E \left[ \left( \int_{-\infty}^{\infty} h(t_0, u) \cdot x(u) du \right) \cdot \left( \int_{-\infty}^{\infty} h(t_0, v) \cdot x(v) dv \right) \right]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[x(u) \cdot x(v)] \cdot h(t_0, u) \cdot h(t_0, v) \, du \, dv$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{xx}(u, v) \cdot h(t_0, u) \cdot h(t_0, v) \, du \, dv$$

- $R_{xx}(u, v)$ is the auto-correlation of the input noise $x(t)$
Response to White and 1/f Noises

- If the input \( x(t) \) is white noise, i.e. \( R_{xx}(u, v) = \sigma_x^2 \cdot \delta(u-v) \):
  \[
  \sigma_y^2(t_0) = \sigma_x^2 \cdot \int_{-\infty}^{\infty} h^2(t_0, u) \, du = \sigma_x^2 \cdot \int_{-\infty}^{\infty} \Gamma^2(\tau) \, d\tau
  \]

- If the input \( x(t) \) is 1/f noise, i.e. \( R_{xx}(u, v) = \sigma_x^2 \):
  \[
  \sigma_y^2(t_0) = \sigma_x^2 \cdot \left[ \int_{-\infty}^{\infty} h(t_0, u) \, du \right]^2
  = \sigma_x^2 \cdot \left[ \int_{-\infty}^{\infty} \Gamma(\tau) \, d\tau \right]^2
  \]

- Agrees with Hajimiri/Lee’s low-noise design idioms:
  - To minimize contribution of white noise, minimize \( \Gamma_{RMS} \)
  - To minimize contribution of 1/f noise, make \( \Gamma_{DC} = 0 \)
Random Decision Error Probability

- If we have multiple noise sources, their contributions add up via RMS sum assuming they are independent:

\[ \sigma_{y,\text{total}}^2(t_o) = \sum_j \sigma_{y,j}^2(t_o) \]

- If the comparator has signal \( V_o \) and noise \( \sigma_{n,o} \) at \( t_{obs} \), the decision error probability \( P(\text{error}) \) can be estimated as:

\[ \sqrt{\text{SNR}} = \frac{V_o(t_{obs})}{\sigma_{n,o}(t_{obs})} \]

\[ P(\text{error}) = Q\left(\sqrt{\text{SNR}}\right) = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\text{SNR}}}^{\infty} \exp\left(-x^2 / 2\right) dx \]
Circuit Analysis Example

- A variant of StrongARM comparator

- When clk is low, the comparator is in reset
  - out+/− are at Vdd
  - X/X' are −Vdd−VTN

- When clk rises (say t=t_c), the comparator goes thru:
  - Sampling phase (t_0~t_1)
  - Regeneration phase (t_1~t_2)
1. Sampling Phase \((t = t_0 \sim t_1)\)

- While out+/- remain high:
  - M1-pair discharges X/X'
  - M2-pair discharges out+/-

- S.S. transfer from \(v_{in}\) to \(v_{out}\):
  \[
  \frac{v_{out}(s)}{v_{in}(s)} = \frac{g_{m1} \cdot g_{m2}}{s C_{out} C_x \left( s + g_{m2} (C_{out} - C_x) / C_{out} C_x \right)}
  \approx \frac{g_{m1} \cdot g_{m2}}{s^2 C_{out} C_x} = \frac{1}{s^2 \tau_{s1} \tau_{s2}}
  \]

- The ISF w.r.t. \(v_{in}\) is:
  \[
  \Gamma(t) \approx \frac{t_1 - t}{\tau_{s1} \tau_{s2}} \cdot G_R
  \]
1. Sampling Phase \((t = 0 \sim t_1)\)

- **S.S. response to M1 noise:**

\[
\frac{v_{\text{out}}(s)}{i_{n1}(s)} \approx \frac{g_{m2}}{s^2 C_{\text{out}} C_x}
\]

\[
\Gamma_{n1}(t) \approx \frac{t_1 - t}{g_{m1}\tau_s \tau_{s2}} \cdot G_R
\]

- **S.S. response to M2 noise:**

\[
\frac{v_{\text{out}}(s)}{i_{n2}(s)} \approx \frac{1}{sC_{\text{out}}}
\]

\[
\Gamma_{n2}(t) \approx \frac{1}{C_{\text{out}}} \cdot G_R
\]
2. Regeneration Phase \((t = t_1 \sim t_2)\)

- We assume \(X/X' \sim 0V\) and M1-pair is in linear region.
  - The circuit is no longer sensitive to \(v_{in}\) (ISF=0).

- Cross-coupled inverters amplify signals via positive-feedback:

\[
G_R = \exp\left(\frac{t_2 - t_1}{\tau_R}\right)
\]

\[
\tau_R = C_{out}/(g_{m2,r} + g_{m3,r}) \quad -(g_{m2,r} + g_{m3,r}) \cdot v_{out}
\]

- The ISF w.r.t. noise is:

\[
\Gamma_{n,r}(t) \approx \frac{1}{C_{out}} \cdot \exp\left(\frac{t_2 - t}{\tau_R}\right)
\]
Putting It All Together

- The overall gain $G$ is:

$$G = \frac{v_o(t_{obs})}{v_{i,dc}} = \int_{-\infty}^{\infty} \Gamma(t) \, dt$$

$$= \frac{(t_0 - t_1)^2}{2\tau_{s1}\tau_{s2}} \cdot \exp\left(\frac{t_2 - t_1}{\tau_R}\right)$$

- The total input-referred noise is:

$$\sigma_{n,i}^2 = \frac{\sigma_{n,o}^2}{G^2}$$

$$\approx \frac{16kT\gamma}{3C_x} \cdot \frac{\tau_{s1}}{t_1 - t_0} + \frac{16kT\gamma}{C_{out}} \cdot \frac{\tau_{s1}^2 \cdot \tau_{s2}}{(t_1 - t_0)^3}$$

- Most of the noise is contributed by M1 and M2 pairs during the sampling phase.
Design Trade-Offs

- The input-referred noise can be approximated as:

\[
\sigma_{n,i}^2 \approx \frac{16kT\gamma}{3C_x} \cdot \frac{\tau_{s1}}{t_1 - t_0} + \frac{16kT\gamma}{C_{out}} \cdot \frac{\tau_{s1}^2 \cdot \tau_{s2}}{(t_1 - t_0)^3}
\]

where

\[
\frac{\tau_{s1}}{t_1 - t_0} \approx \frac{C_x}{C_{out}} \cdot \frac{\left[\frac{g_{m1}}{I_{d1}} \cdot V_{Tp}\right]}{} , \quad \frac{\tau_{s2}}{t_1 - t_0} \approx \frac{1}{\left[\frac{g_{m2}}{I_{d2}} \cdot V_{Tp}\right]}
\]

- Therefore, noise improves with larger \(g_m/I_d\) ratios and wider sampling aperture \((t_1 - t_0)\)

- However, sampling bandwidth and/or gain may degrade
  - Controlling the tail turn-on rate is a good way to keep high gain
Simulating Aperture & Noise

- RF simulators (e.g. SpectreRF) can simulate small-signal LPTV response and noise efficiently:
  - Simulates linearized responses around a periodic steady-state
  - PAC analysis gives $H(j\omega; t) = \text{Fourier transform of } h(t, \tau)$
  - PNOISE analysis can give the noise PSD at one time point

- The remaining question is how to choose $t_{obs}$?
  - We’d like to choose it to mark the end of the regeneration
  - Since $\Gamma(\tau)$ in our LTV model captures sampling + regeneration

Comparator Periodic Steady-State (PSS)

- PSS response of the comparator for a small DC input
  - Near the clock’s rising edge; return to reset not shown
Comparator Sampling Aperture (PAC)

\[ \Gamma(\tau) = h(t_{obs}, \tau) \approx \frac{\partial V_o(t_{obs})}{\partial V_i(\tau)} \]

Time (ps)

Diff Output (V)

ISF (V / V.s)

Reset → Sampling → Regeneration → Decision / Compression
Comparator Noise (PNOISE)

- Magenta line plots the rms output noise $\sigma(t)$ vs. time, obtained by integrating the noise PSD at each time point.
- This is not “transient noise analysis”— it’s a time sample of cyclo-stationary noise (much more efficient).
Deciding on $t_{obs}$

- How to choose $t_{obs}$ that marks the end of regeneration

- Most of the noise is contributed during the sampling phase
  - Noise that enters during the sampling phase sees the full gain
  - Noise that enters later during the regeneration phase sees an exponentially decreasing gain with time

- For the purpose of estimating decision errors, selection of $t_{obs}$ is not critical as long as it’s within regeneration phase
  - The SNR and decision error probability stay ~constant
  - I choose $t_{obs}$ when the comparator has the max. small-signal gain (i.e. before the nonlinearity starts suppressing the gain)
Measurement Results

Receiver A (90nm)  Receiver B (65nm)

- Both receivers are based on StrongARM comparators
- Differential $C_{\text{in}} \sim 2\,\text{pF} \Rightarrow$ thermal noise from the input termination resistors $< 100\,\mu\text{Vrms}$
- Excess noise factor $\gamma$ not spec’d by foundries $\Rightarrow$ simulated at multiple values
Simulation of the decision error (BER) = \( Q(V_o(t_{obs})/\sigma_o(t_{obs})) \) versus the DC input level (excess noise factor \( \gamma = 2 \))
Measurement of the decision errors (BER) based on the density of the wrong outputs (0’s) versus the DC input level
Fit both sets of points to the Gaussian BER model

Compare the estimated $\sigma$’s (input-referred rms noise)
## Simulation vs. Measurement

<table>
<thead>
<tr>
<th>Receiver</th>
<th>Simulated (mV, rms)</th>
<th>Measured (mV, rms)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\gamma=1$</td>
<td>$\gamma=2$</td>
</tr>
<tr>
<td>(A) 90nm Direct Sampling</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Front-End</td>
<td>0.59</td>
<td>0.79</td>
</tr>
<tr>
<td>(B) 65nm w/ Linear</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Front-End</td>
<td>0.62</td>
<td>0.73</td>
</tr>
</tbody>
</table>

(Pos. / Neg. / Avg.) refers to measurement results for positive $V_{IN}$, negative $V_{IN}$, and their average.
In receiver B, the noise contributed by the linear front-end is filtered by the finite aperture of the comparator.
Conclusions

- The linear time-varying (LTV) system model is a good tool for understanding the key characteristics of clocked comparators:
  - Sampling aperture and bandwidth
  - Regeneration gain and metastability
  - Random decision errors and input-referred noise

- The impulse sensitivity function (ISF) has a central role in it:
  - As it did for oscillators
  - Guides design trade-offs between noise, bandwidth, gain, etc.

- The LTV framework is demonstrated on the analysis, simulation, and measurement of clocked comparators
Back-Up Slides
Extracting ISF from $h(t, \tau)$

- Choose $t_{\text{obs}}$ as the maximum small-signal gain point
- ISF: $\Gamma(\tau) = h(t_{\text{obs}}, \tau)$
Effects of the Bridging Device

- Improves hold time and metastability
Effects of Input and Output Loading

- Increased Hold Time
- Increased Setup Time