Lecture 16. Estimating Time Constants in Linear Circuits

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Outlines

- Readings
  - Ali Hajimiri, “Generalized Time- and Transfer-Constant Circuit Analysis,” TCAS-I, 06/2010.
- Overview
  - Deriving circuit equations from small-signal circuit models sound straightforward, but actually carrying it out is a very mechanical process once the circuit becomes big enough. Also, the equations you get in the end are typically very complex, giving few insights about the circuits’ behavior.
  - A generalized method to estimate the coefficients in the transfer function of a linear circuit network has been recently published and it builds on top of the popular open-circuit time constant (OCT) method. It allows one to “incremently” derive the transfer coefficients only when necessary without solving the whole algebraic equations.

Motivation

- Small-signal circuit analysis serves as foundation for analog circuit design
  - The transfer functions (TF) you can derive from the small-signal models describe all the desired characteristics of the intended linear system
- However, deriving TF involves solving large algebraic equations which can be very tedious
  - Solving KCL and KVL equations mechanically
  - You may end up with very complex expressions for high-order circuits (with many poles)
- But, what if all I want is the time constants of the approximate first- or second-order model?
  - Then crunching all these expressions seems an overwork

History

- Open-circuit time constants (OCT)
  - Thornton, Searle, et al. in early 1960s
  - Assume lumped circuits with R's and C's only
  - The coefficient for the first-order term (s) in the denominator is equal to the sum of time constants associated with each capacitor alone when all other capacitors are open-circuited and sources are nulled
  - The coefficient gives an estimate for the dominant pole (BW)
- Zero-value time constants (ZVT)
  - Extends to circuits with inductors
  - Based on the evaluation of the determinant of the Y matrix in the nodal equations

History (2)

- Cochran and Grabel in early 1970s
  - Determine as many denominator coefficients as needed
  - By calculating time constants under different combinations of shorting and opening the energy-storage elements
  - Later cleaned up by Rosenstark in the 1980s
  - Extended to include transcapacitors (by Fox, et al.) and mutual inductors (by Andreani, et al.)
- Davis in late 1970s
  - A method to determine the numerator coefficients as well
  - For lumped RC circuits

Transfer Function of a Linear System

- The canonical expression for the TF of a linear, lumped-element circuit is:

$$H(s) = \frac{a_0 + a_1 s + a_2 s^2 + \ldots + a_m s^m}{1 + b_1 s + b_2 s^2 + \ldots b_n s^n}$$

Or,

$$H(s) = a_0 \cdot \frac{\left(1 - \frac{s}{z_1}\right) \cdot \left(1 - \frac{s}{z_2}\right) \ldots \left(1 - \frac{s}{z_m}\right)}{\left(1 - \frac{s}{p_1}\right) \cdot \left(1 - \frac{s}{p_2}\right) \ldots \left(1 - \frac{s}{p_n}\right)}$$

- So characterizing the TF means either determining the coefficients $\{a_i\}$ and $\{b_i\}$'s or the pole/zero positions

Determining the System Order

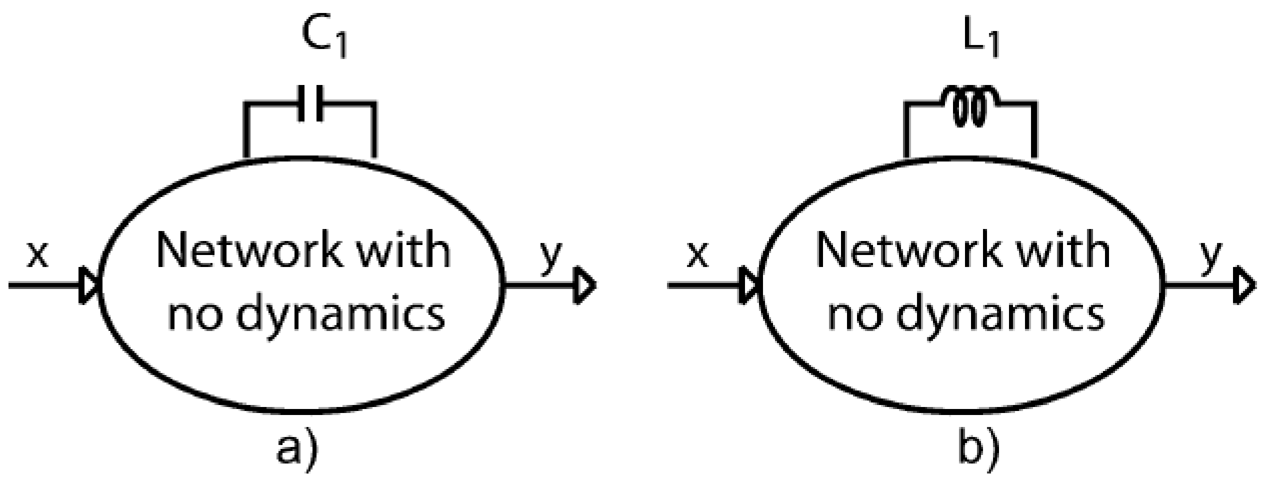
- The order of a linear system is set by the order of the denominator polynomial: “n”
  - Equal to the number of *independent* energy storage elements
  - The maximum number of independent initial condition parameters (e.g. capacitor voltages and inductor currents)
- The order n corresponds to the number of poles
  - Also equal to the number of natural frequencies (eigenmodes)
  - Independent of the choice of input and output ports – the intrinsic characteristic of the circuit
- The zeros are dependent on the choice of input/output

Deriving TF of a First-Order System

- LTI circuit with a single energy-storing element (L or C)
  - The rest of the circuits contain only frequency-independent elements such as resistors and dependent sources
- The circuit can have at most one pole and one zero:

$$H(s) = \frac{a_0 + a_1 s}{1 + b_1 s}$$

  - gain = a_0, pole = $-1/b_1$ and zero = $-a_0/a_1$



Determining the Gain (a_0)

- The DC gain of the circuit (a_0) can be derived from the circuit with C being open and L being shorted

$$a_0 = H^0$$

where H^0 denotes the transfer gain when all reactive elements are zero-valued

Determining the Pole (b_1)

- For a first-order circuit with a capacitor C_1, the only time constant τ_1 is:

$$\tau_1 \equiv R_1^0 C_1 = b_1$$

  where R_1^0 is the resistance seen across the capacitor with all the independent sources and inputs "nulled"

- Nulling a source means:
  - Replacing an independent V-source with a short circuit (V=0)
  - Replacing an independent I-source with an open circuit (I=0)

Determining the Pole (b_1)

- For a first-order circuit with an inductor L_1, the time constant τ_1 is:

$$\tau_1 \equiv \frac{L_1}{R_1^0} = b_1$$

  where R_1^0 is the resistance seen across the inductor with all the independent sources and inputs "nulled"

- Notes on the R_1^0 notation:
  - The superscript (0): all the sources and reactive elements are at their zero values
  - The subscript (1): the index of the energy storing element

Some Observation

- The impedance of a capacitor C is:

$$Z_C(s) = \frac{1}{sC}$$

The capacitance “C” and the complex frequency “s” always appear together as a product

- The TF of the single-capacitor circuit can be written as:

$$H(s) = \frac{a_0 + \alpha_1 C_1 s}{1 + \beta_1 C_1 s}$$

and

$$\beta_1 = R_1^0$$

Determining the Zero (a_1)

- As C_1 goes to infinity, the TF converges to:

$$H^1 \equiv H(s)|_{C_1 \to \infty} = \frac{a_0 + \alpha_1 C_1 s}{1 + \beta_1 C_1 s} \longrightarrow \frac{\alpha_1}{\beta_1}$$

where H^1 is the transfer gain with the reactive element at its infinite value (e.g. capacitor C_1 short-circuited)

- Then the coefficient a_1 is:

$$a_1 = \alpha_1 C_1 = R_1^0 C_1 H^1 = \tau_1 H^1$$

Putting It Altogether

- TF of an LTI circuit with one energy-storing element is:

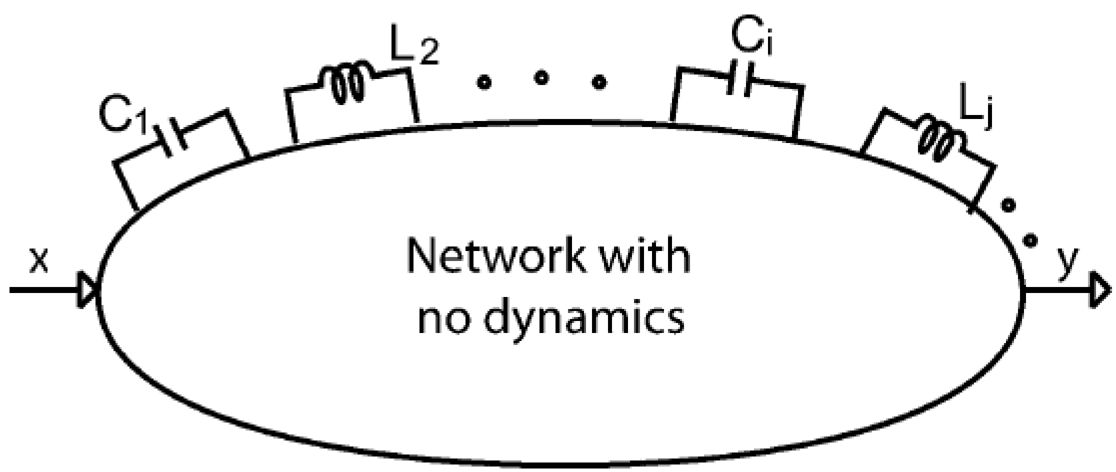
$$\boxed{H(s) = \frac{H^0 + \tau_1 H^1 s}{1 + \tau_1 s}}$$

where

- H^0: the zero-valued transfer gain (C opened, L shorted)
- H^1: the infinite-valued transfer gain (C shorted, L opened)
- τ_1 : the time constant associated with the reactive element and resistance it sees with the independent sources nulled (R_1^0)

Case with N Energy-Storage Elements

- Let's assume the energy-storing elements (L's and C's) are separated from the rest as shown below
  - The rest include only resistors and dependent sources



Some Observations on TF

- The only way for an "*s*" term to occur in the TF of a lumped circuit is as a multiplicative factor to a capacitor or an inductor: *sC* or *sL*
- It implies that for capacitor-only circuits, the coefficients in the following expression should be:

$$H(s) = \frac{a_0 + a_1 s + a_2 s^2 + \ldots + a_m s^m}{1 + b_1 s + b_2 s^2 + \ldots b_n s^n}$$

$$a_1 = \sum_{i=1}^{N} \alpha_1^i C_i, \quad b_1 = \sum_{i=1}^{N} \beta_1^i C_i$$

Linear combination of all capacitors

$$a_2 = \sum_{i}^{1 \le i} \sum_{j}^{< j \le N} \alpha_2^{ij} C_i C_j, \quad b_2 = \sum_{i}^{1 \le i} \sum_{j}^{< j \le N} \beta_2^{ij} C_i C_j$$

Linear combination two-way products of different capacitors

Determining b_1

- If we reduce the circuit to a case where all capacitors but C_i has a value of zero (open-circuited), it should have a TF with the following form:

$$H_i(s) = \frac{a_0 + \alpha_1^i C_i s}{1 + \beta_1^i C_i s}$$

- We previously derived that the time constant of the resulting first-order system is:

$$\tau_i^0 = R_i^0 C_i = \beta_1^i C_i \longrightarrow \therefore R_i^0 = \beta_1^i$$

where R_i^0 is the resistance seen by the capacitor C_i looking into its port with all other reactive elements at their zero values and independent sources nulled

Determining b_1 (2)

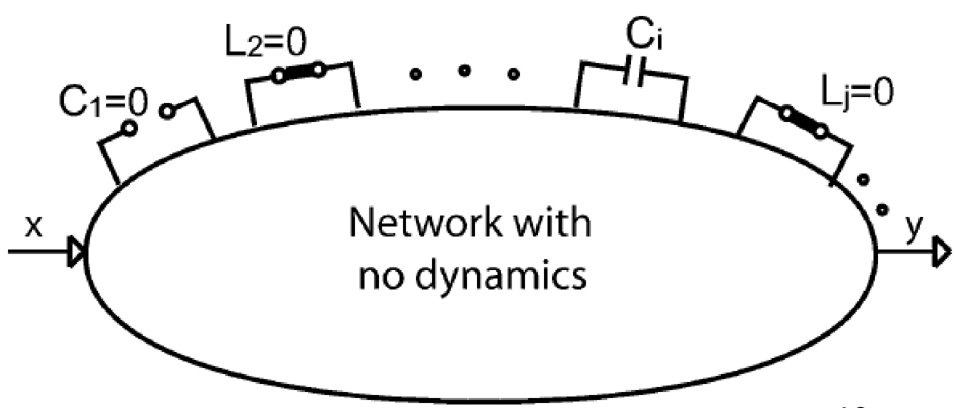
- Since this argument is applicable to any capacitor in the system, the first denominator coefficient b1 is equal to the sum of the *zero-value time constants* (*ZVT*):

$$\boxed{b_1 = \sum_{i=1}^{N} \tau_i^0}$$

where the ZVT's are calculated as:

$$\tau_i^0 = R_i^0 C_i$$

$$\tau_i^0 = \frac{L_i}{R_i^0}$$



Estimating the Dominant Pole from b_1

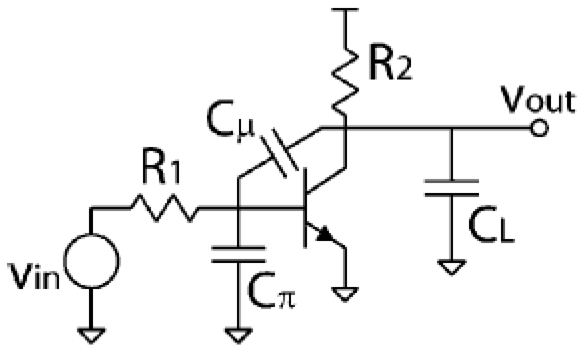
- From $$H(s) = \frac{a_0 + a_1 s + a_2 s^2 + \ldots + a_m s^m}{1 + b_1 s + b_2 s^2 + \ldots b_n s^n}$$
$$= a_0 \cdot \frac{\left(1 - \frac{s}{z_1}\right) \cdot \left(1 - \frac{s}{z_2}\right) \ldots \left(1 - \frac{s}{z_m}\right)}{\left(1 - \frac{s}{p_1}\right) \cdot \left(1 - \frac{s}{p_2}\right) \ldots \left(1 - \frac{s}{p_n}\right)}$$

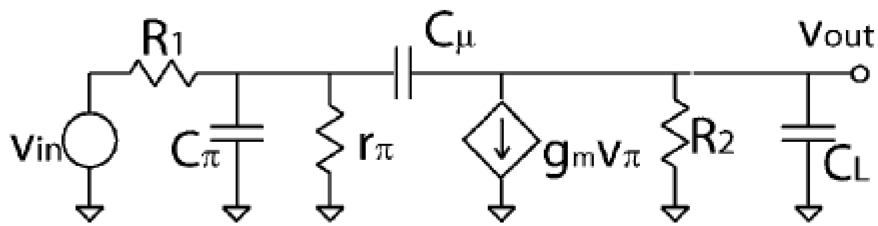
- It follows that: $$b_1 = -\sum_i \frac{1}{p_i}$$

- If $1/p_1$ is much greater than any other $1/p_i$'s, then:
$$b_1 \approx -\frac{1}{p_1}$$

- However, there is no 1:1 correspondence between the ZVT τ_i and the pole frequency p_i

Example: Common-Emitter Stage





- Low-frequency gain is:

$$a_0 = H^0 = -g_m R_2 \cdot \frac{r_\pi}{r_\pi + R_1}$$

- The zero-value time constants are:

$$\tau_\pi^0 = C_\pi R_\pi^0 = C_\pi (R_1 || r_\pi)$$

$$\tau_\mu^0 = C_\mu R_\mu^0 = C_\mu \left(R_1 || r_\pi + R_2 + g_m (R_1 || r_\pi) R_2\right)$$

$$\tau_L^0 = C_L R_L^0 = C_L R_2$$

Example: Common-Emitter Stage

- Therefore,

$$b_1 = \sum_i \tau_i^0 = \tau_\pi^0 + \tau_\mu^0 + \tau_L^0$$

- Analysis results:

$$H^0 = -57$$

$$\tau_\pi^0 \approx 70ps, \quad \tau_\mu^0 \approx 1,200ps \quad \tau_L^0 \approx 400ps$$

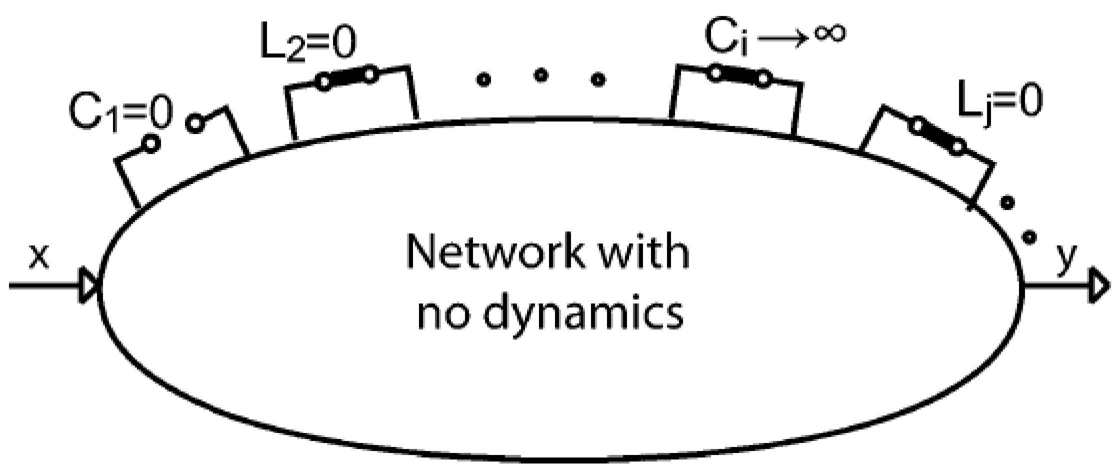
$$\omega_h \approx 1/b_1 \approx 2\pi \cdot 95MHz$$

- SPICE gives -3dB bandwidth of 97MHz

Determining a_1

- When C_i goes to infinity (i.e. short-circuit) while all other reactive elements are at zero value, TF reduces to a constant H^i:

$$H^i \equiv H|_{C_i \to \infty, C_j = 0, i \neq j} = \frac{\alpha_1^i}{\beta_1^i}$$

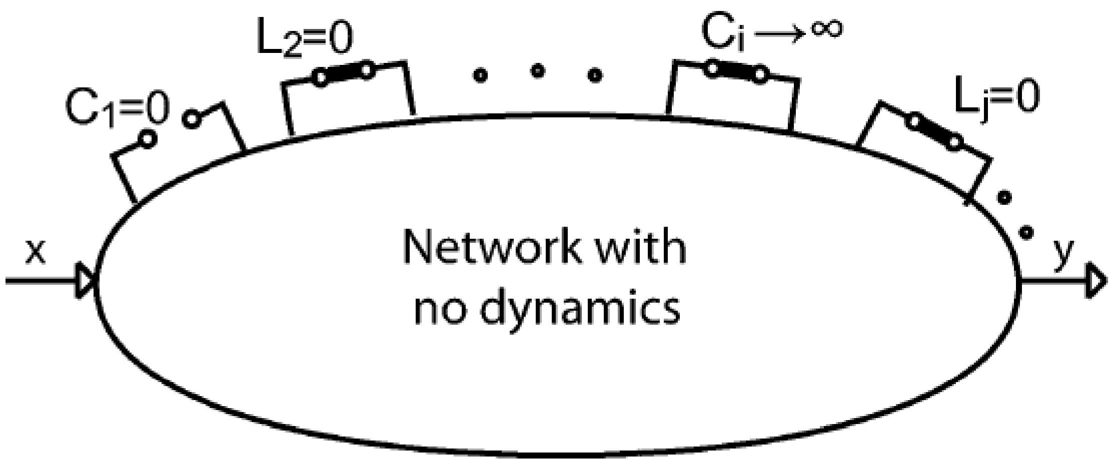


Determining a_1 (2)

- Since $\beta_1^i = R_i^0$, $\alpha_1^i = R_i^0 H^i$ and $\alpha_1^i C_i = R_i^0 C_i H^i = \tau_i^0$

$$\boxed{a_1 = \sum_{i=1}^{N} \tau_i^0 H^i}$$

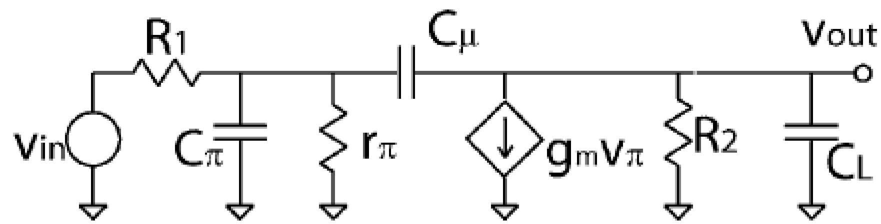
which can be evaluated using low-frequency calc. only



Example: Common-Emitter Stage

- Determining a_1:

$$\boxed{a_1 = \sum_{i=1}^{N} \tau_i^0 H^i}$$



$$H^\pi = 0$$

$$H^L = 0$$

$$H^\mu = \frac{r_\pi || 1/g_m || R_2}{R_1 + r_\pi || 1/g_m || R_2} = \frac{r_\pi}{r_\pi + R_1} \cdot \frac{R_2}{R_\mu^0}$$

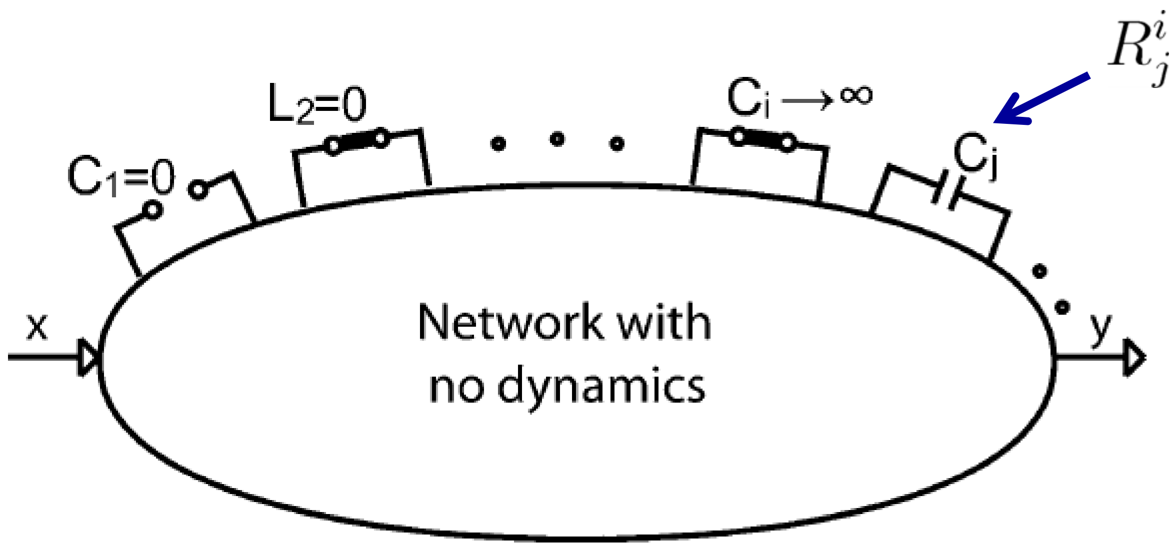
- Combining all the results:

$$H(s) = H^0 \cdot \frac{1 + \frac{H^\mu}{H^0} \tau_\mu^0 s}{1 + b_1 s + b_2 s^2} = H^0 \cdot \frac{1 - \frac{C_\mu}{g_m} s}{1 + b_1 s + b_2 s^2}$$

Determining the High-Order Terms

- Let's set C_i to infinity while all reactive elements but C_j are at their zero values
- Define the time constant of this reduced first-order system as:

$$\tau_j^i = R_j^i C_j$$



Determining the High-Order Terms (2)

- It can be found that (see the paper):

$$b_2 = \sum_{i}^{1 \le j} \sum_{j}^{< j \le N} \tau_i^0 \tau_j^i$$

$$a_2 = \sum_{i}^{1 \le j} \sum_{j}^{< j \le N} \tau_i^0 \tau_j^i H^{ij}$$

where H^{ij} is the input-to-output transfer gain when both the i-th and j-th reactive elements are at their infinite value

Determining the High-Order Terms (3)

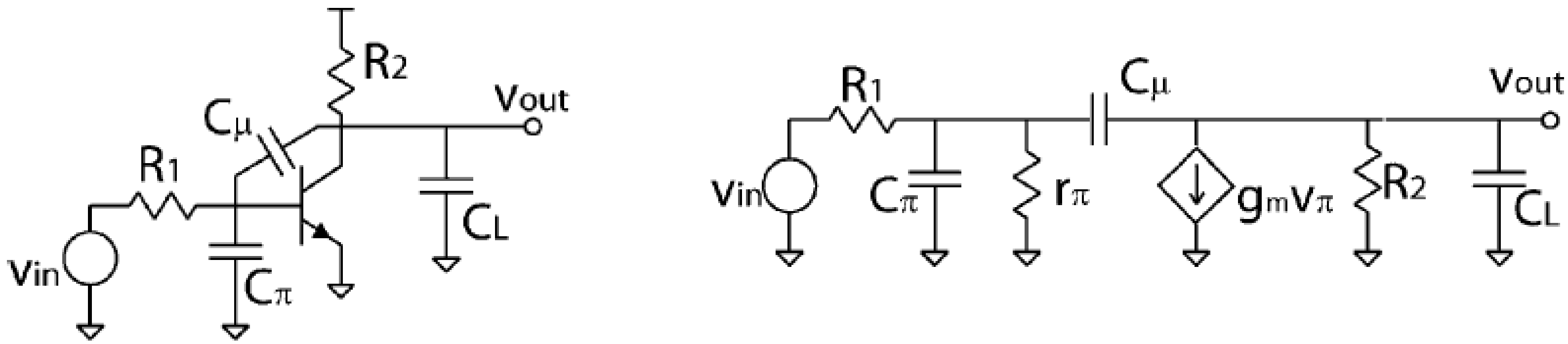
- And in general

$$b_n = \sum_{i}^{1 \le j <} \sum_{j}^{j<k} \sum_{k\ldots}^{\ldots \le N} \ldots \tau_i^0 \tau_j^i \tau_k^{ij} \ldots$$

$$a_n = \sum_{i}^{1 \le j <} \sum_{j}^{j<k} \sum_{k\ldots}^{\ldots \le N} \ldots \tau_i^0 \tau_j^i \tau_k^{ij} \ldots H^{ijk\ldots}$$

where τ_k^{ij} corresponds to the time constant due to k-th reactive element when the indexed elements (i, j, …) are infinite valued while the rest are zero valued and $H^{ijk\ldots}$ is the gain when all the indexed elements (i, j, k, …) are at their infinite value and all others are zero valued

Example: Common-Emitter Stage



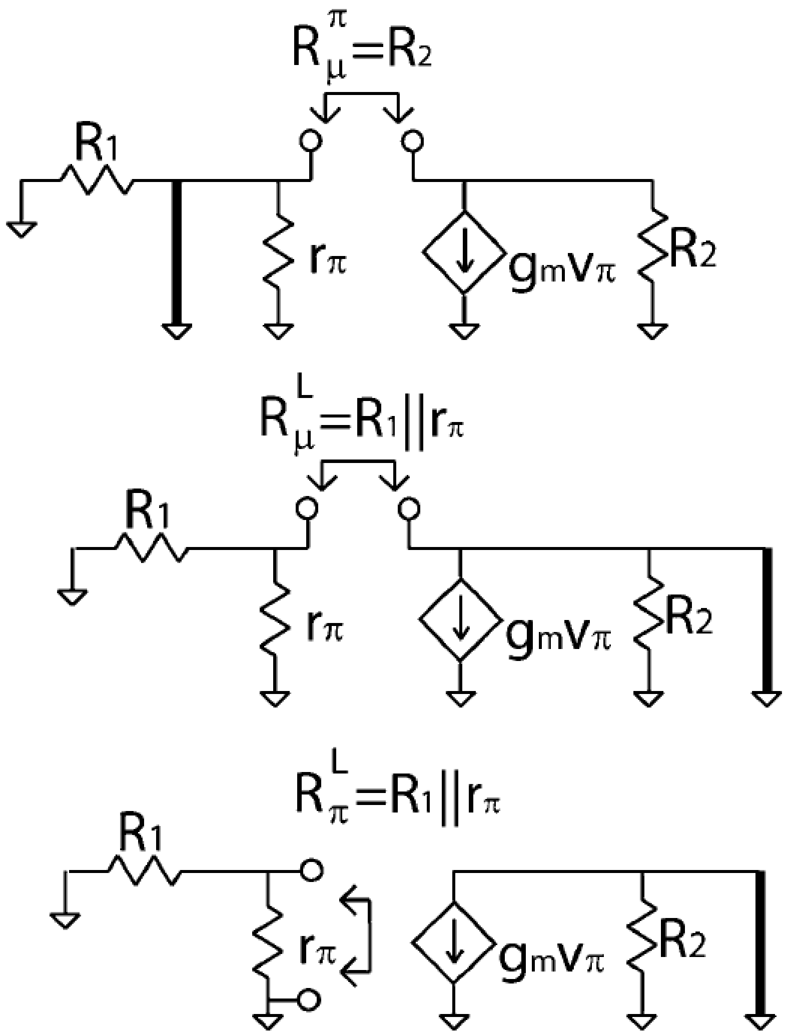
- Determining b_2:

$$b_2 = \sum_{i}^{1\leq i} \sum_{j}^{<j\leq 3} \tau_i^0 \tau_j^i$$

- Choose τ_j^i 's that are easiest to compute:

$$\tau_\mu^\pi, \tau_\mu^L, \tau_\pi^L, \tau_\pi^\mu, \tau_L^\mu, \tau_L^\pi$$

Example: Common-Emitter Stage



$$\tau_\mu^\pi = C_\mu R_2$$

$$\tau_\pi^L = C_\pi (r_\pi || R_1)$$

$$\tau_\mu^L = C_\mu (r_\pi || R_1)$$

Example: Common-Emitter Stage

- Then:

$$
\begin{aligned}
b_2 &= \sum_{i}^{1\leq i} \sum_{j}^{<j\leq 3} \tau_i^0 \tau_j^i \\
&= \tau_L^0 \tau_\pi^L + \tau_\pi^0 \tau_\mu^\pi + \tau_L^0 \tau_\mu^L \\
&= (r_\pi || R_1) R_2 \cdot (C_\pi C_\mu + C_\pi C_L + C_\mu C_L)
\end{aligned}
$$

Insights: Number of Zeros

$$H(s) = \frac{a_0 + a_1 s + a_2 s^2 + \ldots + a_m s^m}{1 + b_1 s + b_2 s^2 + \ldots b_n s^n}$$

- The number of zeros is determined by the order of the numerator polynomial, which is in turn determined by the highest order non-zero transfer gain $H^{ijk\ldots}$

$$a_n = \sum_{i}^{1 \le j <} \sum_{j}^{j<k} \sum_{k\ldots}^{\ldots \le N} \ldots \tau_i^0 \tau_j^i \tau_k^{ij} \ldots H^{ijk\ldots}$$

- The number of zeros is equal to the maximum number of energy-storing elements that can be simultaneously infinite-valued while producing a nonzero transfer gain

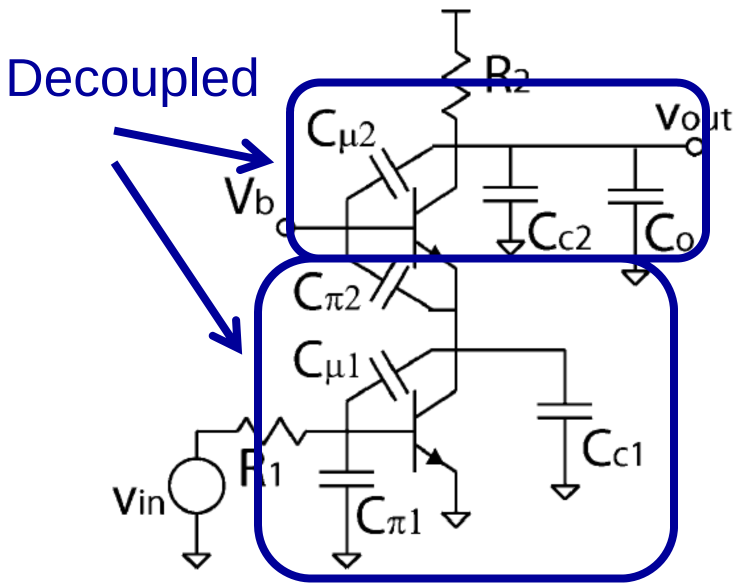
Insights: Decoupled Time Constants

- In general, there is no 1:1 correspondence between the ZVT time constants and the pole frequencies
- However, an exception is when a time-constant is *decoupled*, that is, it does not change for any shorting or opening of other energy-storing elements:

$$\tau_N^0 = \tau_N^i = \tau_N^{ij} = \ldots = \tau_N^{ij\ldots m}$$

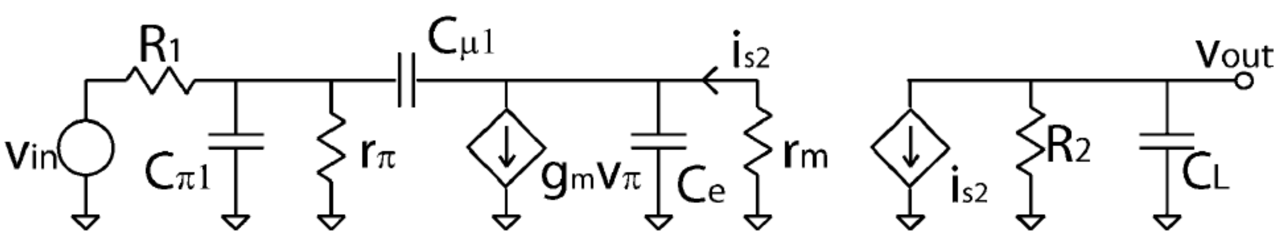
- This concept can be generalized to a group of time constants as well, in which case, we can factor the TF into a product of low-order TFs

Example: CE Stage with Cascode



- With $C_L = C_{\mu 2} + C_{c2} + C_o$ decoupled from the rest, the TF can be computed with less efforts:

$$H(s) = H^0 \cdot \frac{1 + \frac{a_1'}{a_0} s}{1 + b_1' s + b_2' s^2} \cdot \frac{1}{1 + \tau_L s}$$



Estimating Bandwidth with ZVT's

- In case with no dominant zeros, the TF can be approximated as:

$$H(s) \approx \frac{a_0}{1 + b_1 s + b_2 s^2 + \ldots + b_n s^n} \approx \frac{a_0}{1 + b_1 s}$$

- Implying that the bandwidth ω_h can be estimated as:

$$\omega_h \approx \frac{1}{b_1} = \frac{1}{\sum_{i=1}^{N} \tau_i^0}$$

  - This estimate is conservative and underestimate the true bandwidth
  - Also, it assumes the dominant pole is real (gross underestimation is resulted when it is complex)

Estimating BW for System with Zeros

- For systems with zeros, we approximate them as:

$$H(s) \approx a_0 \cdot \frac{1 + \frac{a_1}{a_0} s}{1 + b_1 s}$$

where

$$\frac{a_1}{a_0} = \sum_{i=1}^{N} \tau_i^0 \frac{H^i}{H^0}$$

- And one can approximate the bandwidth ω_h as:

$$\omega_h \approx \frac{1}{b_1 - \frac{a_1}{a_0}} = \frac{1}{\sum_{i=1}^{N} \tau_i^0 \left(1 - \left|\frac{H^i}{H^0}\right|\right)}$$

Effect of Zeros in a First-Order System

- From

$$H(s) = a_0 \cdot \frac{1 - s/z}{1 - s/p} = \frac{H^0 + \tau_1 H^1 s}{1 + \tau_1 s}$$

we can get:

$$z = \frac{H^0}{H^1} \cdot p$$

- Therefore the ratio H^0/H^1 tells which one is at the lower frequency

Effect of Zeros in a First-Order System

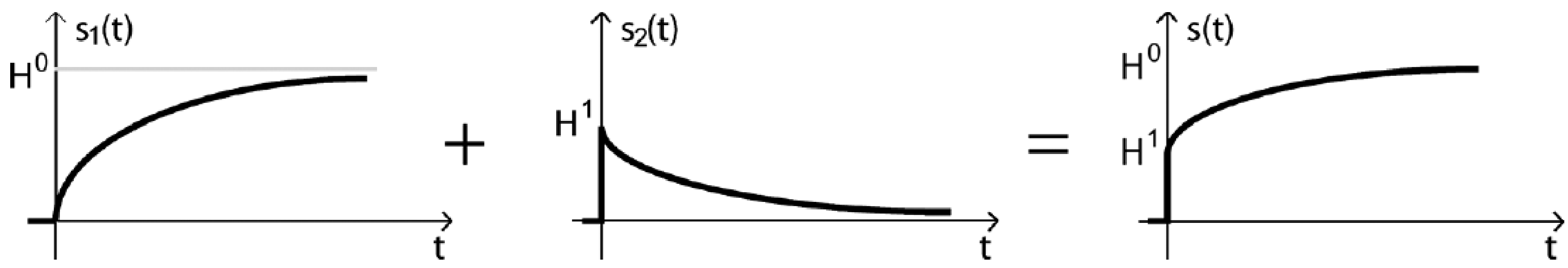
- The TF can be decomposed into two parallel first-order systems:

$$H(s) = \frac{H^0}{1+\tau s} + \frac{H^1}{1+\frac{1}{\tau s}}$$

which has a step response of:

$$s(t) = H^0(1-e^{-t/\tau})u(t) + H^1 e^{-t/\tau}u(t)$$

- The ratio H^0/H^1 determines the overall shape

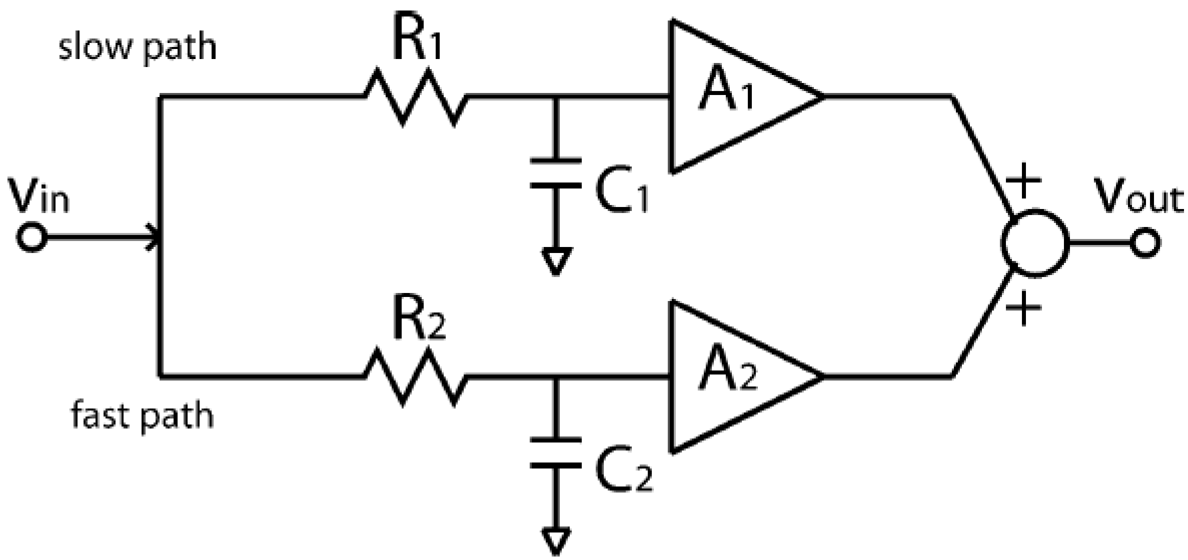


Effect of Zeros in a Second-Order System

- For a second-order system with two *real* poles:

$$H(s) = H^0 \cdot \frac{1 - s/z}{(1 - s/p_1)(1 - s/p_2)} = H^0 \cdot \frac{1 + \tau_z s}{(1 + \tau_1 s)(1 + \tau_2 s)}$$

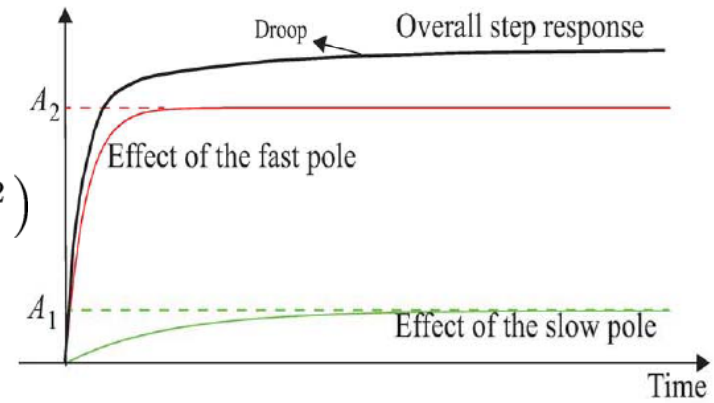
- The system can be decomposed into:

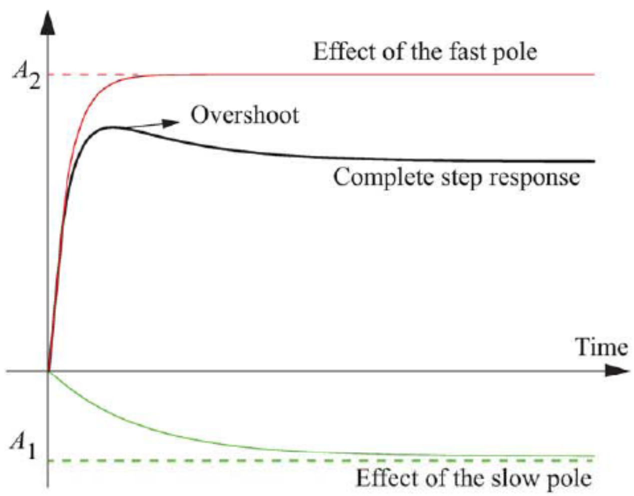


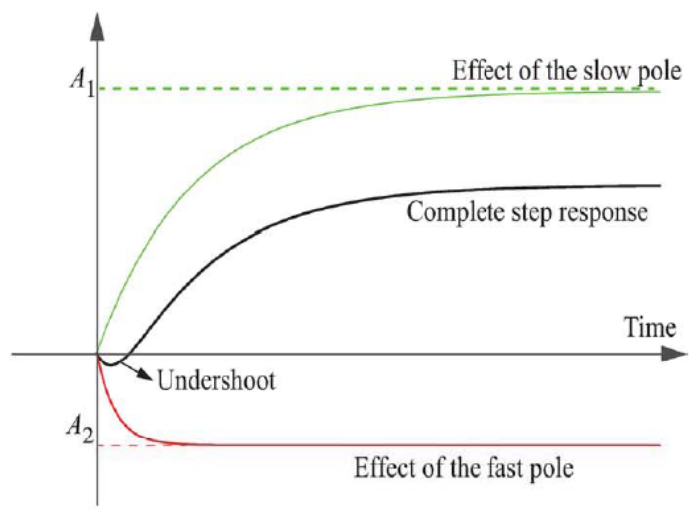
Effect of Zeros in a Second-Order System

- Again, the ratio A_1/A_2 determines the overall shape of s(t)

$$s(t) = A_1(1 - e^{-t/\tau_1}) + A_2(1 - e^{-t/\tau_2})$$







A General Second-Order System

- The TF of a general second-order system can be expressed by its natural frequency ω_n and the quality factor Q:

$$H(s) = \frac{N(s)}{1 + \frac{s}{Q\omega_n} + \frac{s^2}{\omega_n^2}}$$

where

$$Q = \frac{1}{2\zeta} = \frac{\sqrt{b_2}}{b_1} = \underline{\frac{\sqrt{\tau_1^0 \tau_2^1}}{\tau_1^0 + \tau_2^0}}$$

$$\omega_n = \frac{1}{\sqrt{b_2}} = \underline{\frac{1}{\sqrt{\tau_1^0 \tau_2^1}} = \frac{1}{\sqrt{\tau_2^0 \tau_1^2}}}$$

For a second-order system

Infinite Value Time Constant (IVT)

- For a high-pass system whose TF can be expressed as:

$$H(s) \approx \frac{a_n s^n}{1 + b_1 s + b_2 s^2 + ... + b_n s^n}$$
$$= \frac{a_{mid}}{1 + \frac{b_{n-1}}{b_n s} + ... + \frac{1}{b_n s^n}}$$

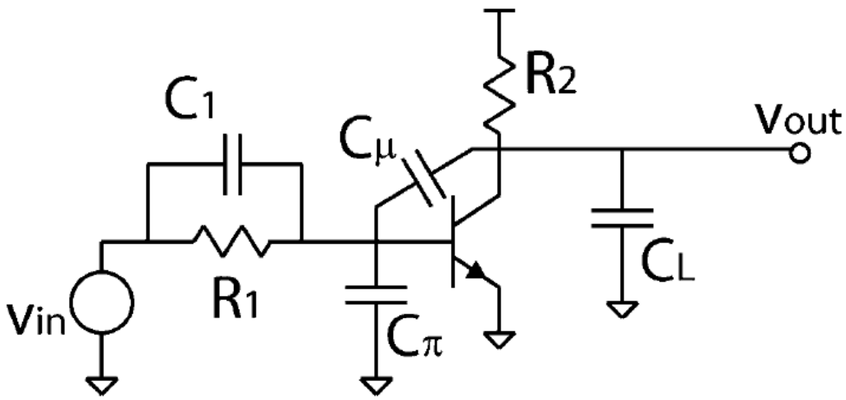
- Then the low-side –3dB frequency ω_l can be estimated:

$$\omega_l \approx \frac{b_{n-1}}{b_n} = \sum_{i=1}^{N} \frac{1}{\tau_i^\infty}$$

$$\tau_i^\infty = C_i R_i^\infty \quad or \quad L_i / R_i^\infty$$

where R_i^∞ is the resistance looking into i-th port when all the reactive elements are at their *infinite* values

Example: CE Stage with Input Zero



$$\tau_1^0 = C_1 R_1^0 = C_1(R_1||r_\pi)$$
$$\tau_\pi^0 = C_\pi R_\pi^0 = C_\pi(R_1||r_\pi)$$
$$\tau_\mu^0 = C_\mu R_\mu^0$$
$$= C_\mu\left(R_1||r_\pi + R_2 + g_m(R_1||r_\pi)R_2\right)$$
$$\tau_L^0 = C_L R_L^0 = C_L R_2$$

- With C_1 = 4.3pF,

$$\tau_1^0 \approx 3.07ns$$
$$\omega_h \approx 2\pi \cdot 34MHz$$

- But SPICE says BW = 482MHz!
  - Due to the zero that cancels the first pole

Example: CE Stage with Input Zero

- Using the modified ZVT's:

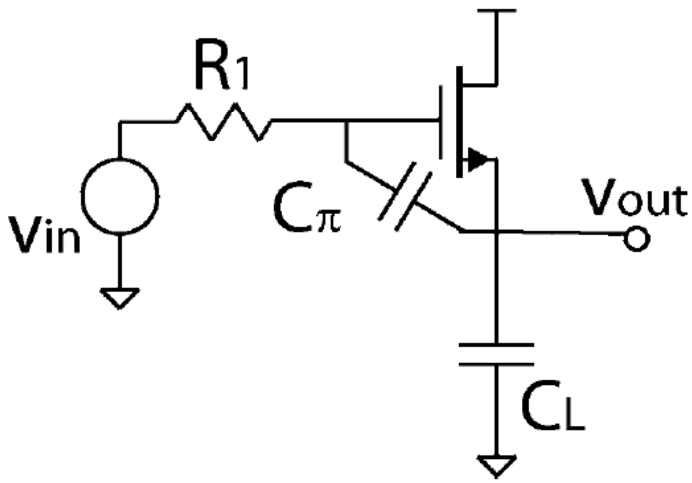
$$H^{\pi} = 0 \qquad H^{\mu} = \frac{r_{\pi}||1/g_m||R_2}{R_1 + r_{\pi}||1/g_m||R_2}$$

$$H^{L} = 0 \qquad H^{1} = -g_m R_2$$

$$\omega_h \approx \frac{1}{\sum_{i=1}^{N} \tau_i^0 \left(1 - |\frac{H^i}{H^0}|\right)} \approx 2\pi \cdot 362MHz$$

- Closer to the simulated 482MHz
  - The more accurate results can be obtained by using the second-order system analysis

Example: Source Follower



$$\tau_\pi^0 = C_\pi / g_m$$

$$\tau_L^0 = C_L / g_m$$

$$\tau_\pi^L = R_1 C_\pi$$

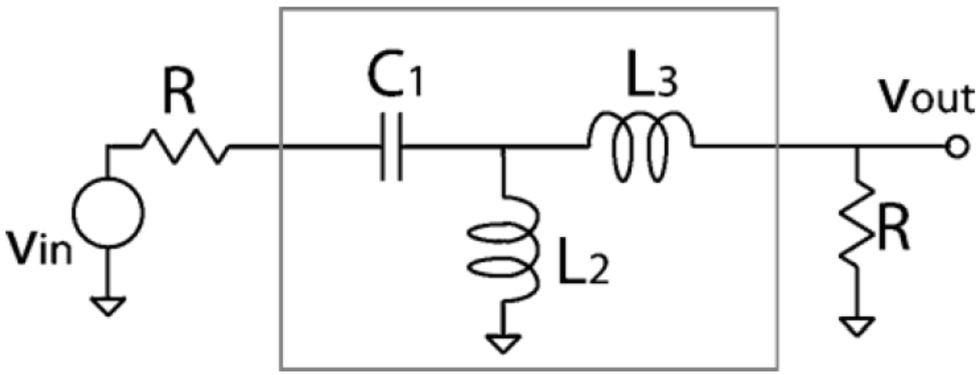
- Then we get: $b_1 = \tau_\pi^0 + \tau_L^0 = 5ps$

$$b_2 = \tau_L^0 \tau_\pi^L = 250(ps)^2$$

$$Q = 3.16, \quad \omega_n = 2\pi \cdot 10GHz, \quad \omega_h \approx 2\pi \cdot 15.9GHz$$

- SPICE gives peaking at 9.8GHz and ω_h of 15.5GHz

Example: Reactive Bandpass Filter



- Time constants are:

$$\tau_1^0 = RC_1 \qquad \tau_2^0 = L_2/R \quad \tau_3^0 = L_3/R$$
$$\tau_2^1 = 2L_2/R \quad \tau_3^1 = L_2/R \quad \tau_3^2 = 0$$
$$\tau_3^{12} = L_3/2R.$$

- And all transfer gains are zero except:

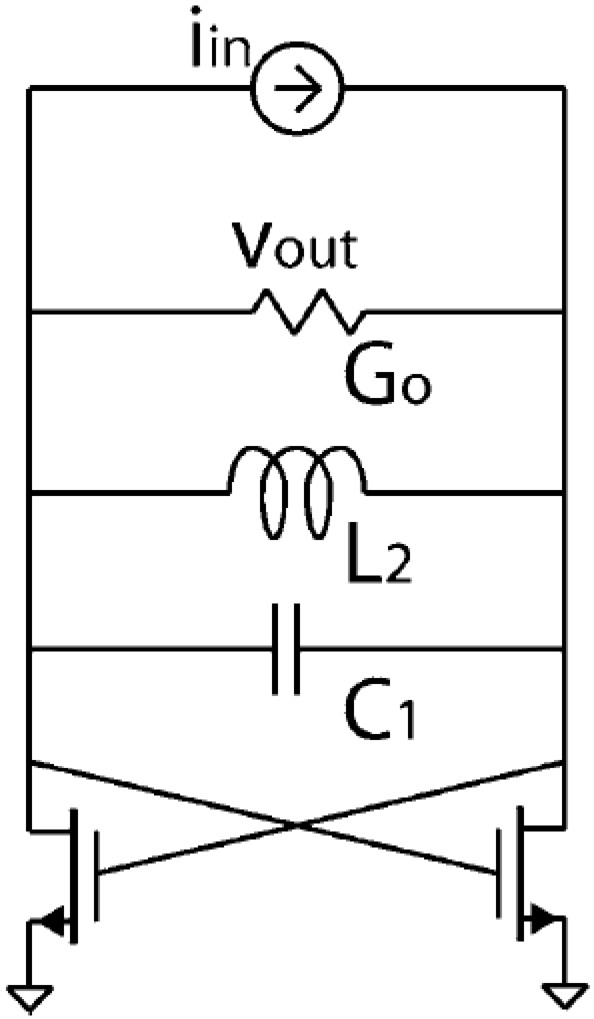
$$H^{12} = 1/2$$

Example: Reactive Bandpass Filter

- It immediately results in the following TF:

$$H(s)=\frac{L_2C_1s^2}{1+\left(RC_1+\frac{L_2+L_3}{R}\right)s+(2L_2C_1+L_3C_1)s^2+\frac{L_2L_3C_1}{R}s^3}$$

Example: LC Oscillator



- Time constants are:

$$\tau_C^0 = 0$$
$$\tau_L^0 = L(-g_m/2 + G_o) \equiv -L \cdot G_{eff}$$
$$\tau_C^L = -C/G_{eff}$$

- The only non-zero transfer gain:

$$H^L = 1/G_{eff}$$

- And the TF is:

$$H(s) \equiv \frac{v_{out}}{i_{in}} = \frac{Ls}{1 - G_{eff}Ls + LCs^2}$$