Lecture 17. Estimating Time Constants of Linear Circuit Networks – Part 2

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Overview

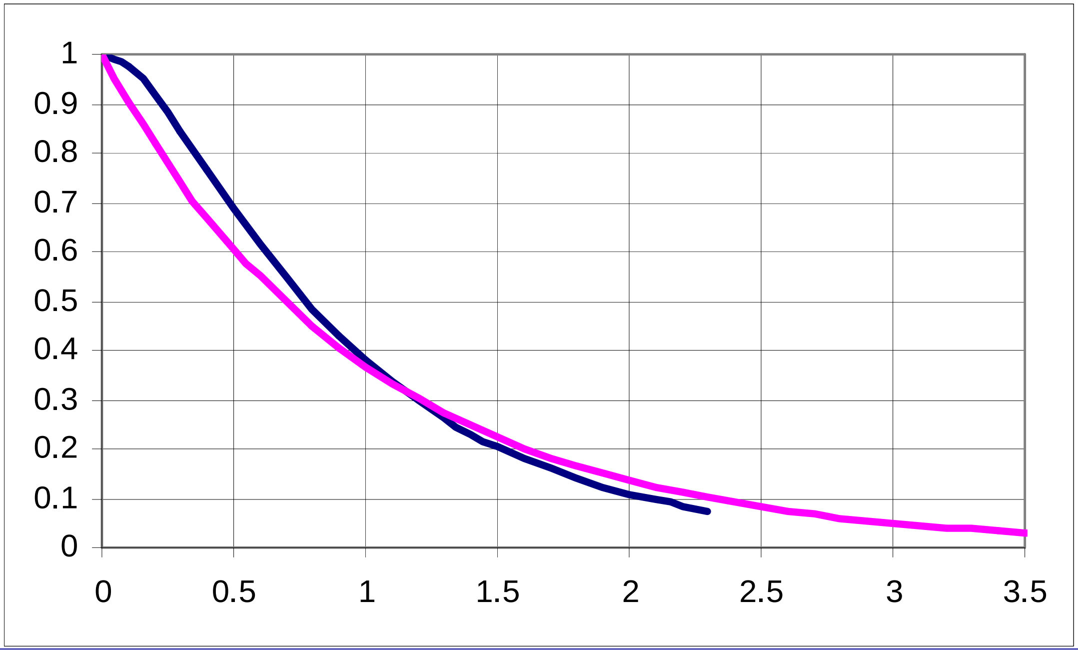
- Readings
  - Y. I. Ismail, et al., “Equivalent Elmore Delay for RLC Trees,” TCAD, 01/2000.

- Introduction
  - We will look at an alternative method to estimate the time constants of linear circuit networks. While the OCT was suitable for estimating the dominant pole position, Elmore delay is more suitable for estimating the delay or rise time. This lecture introduces an extended Elmore delay for linear circuits including inductors.

RC Elmore Delay Review

- Elmore delay estimates 50% delay in the step response by approximating it as that of *a first-order linear system*:

$$H(s) = \frac{1 + a_1 s + a_2 s^2 + \ldots + a_n s^n}{1 + b_1 s + b_2 s^2 + \ldots + b_n s^n} \Rightarrow \frac{1}{1 + b_1 s}$$



Delay of a Single-Pole System

- A single R-C circuit
  - For a step input (e.g. 0 to Vdd)

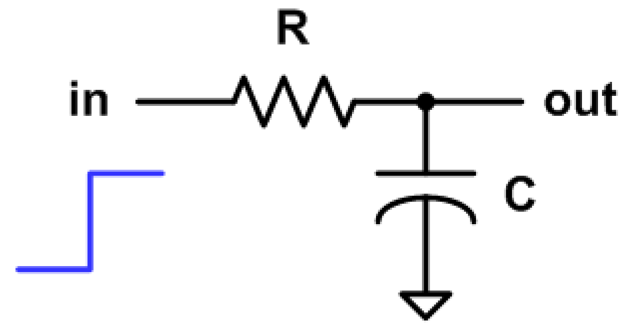
$$H(s) = \frac{1}{1 + sRC}$$

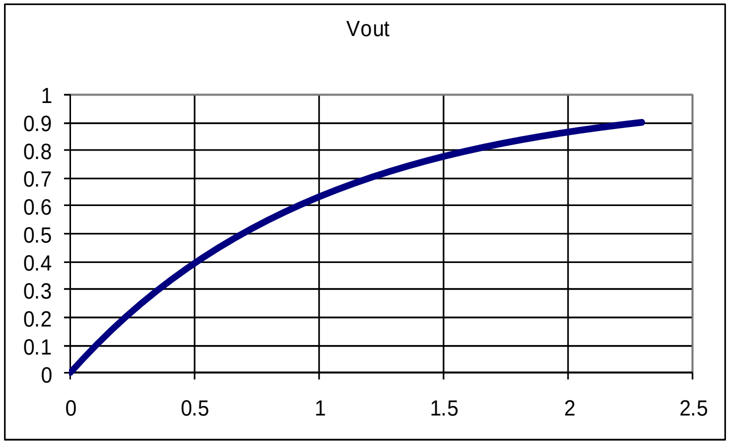
$$Vout = Vdd \cdot [1 - e^{-t/\tau}]$$

where $\tau = RC$

  - Then delay to 50% Vdd is:

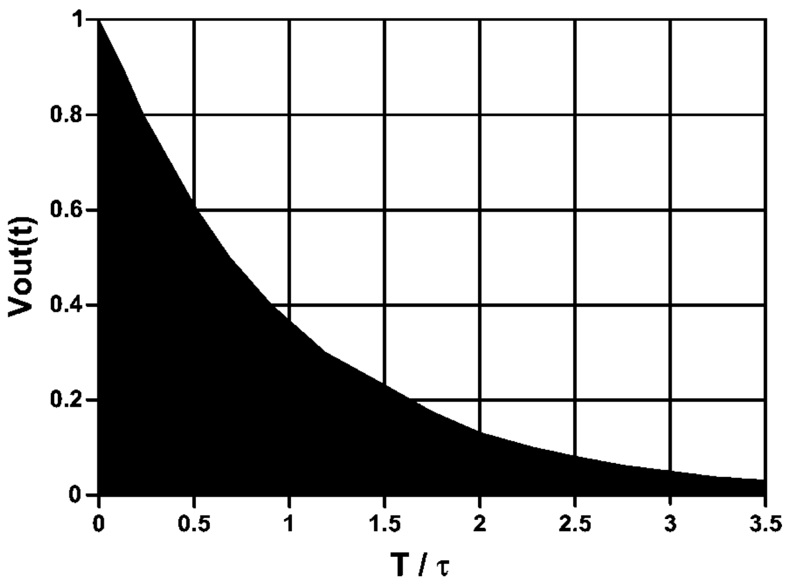
$$ln(2) \cdot RC \approx 0.7 \cdot RC$$

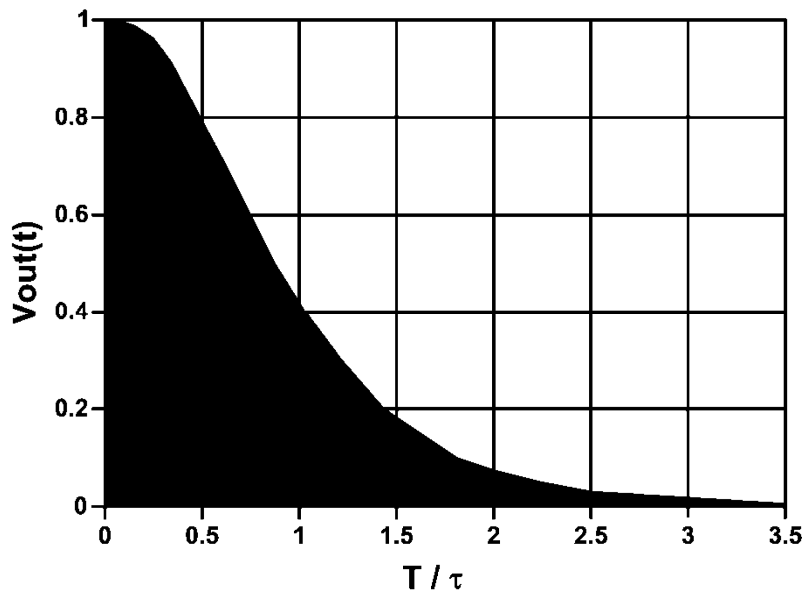




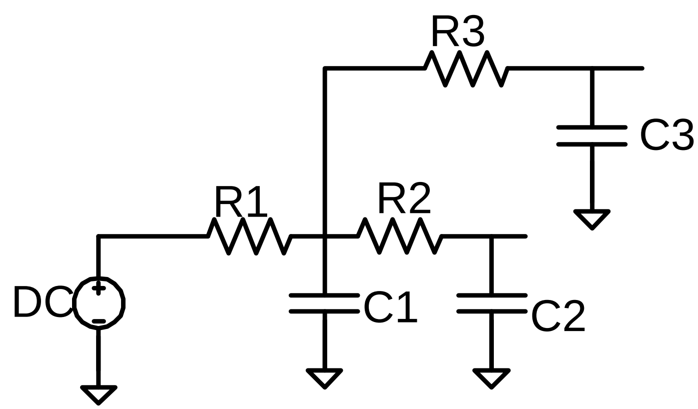
Use “Moment Matching” to Find τ (=b_1)

- Look at the step response waveforms of both circuits
  - Specifically, look at the area below the waveforms (1st moment)
  - Choose the single pole τ such that the areas match
- The area under an exponential is τ: $$\int_0^\infty e^{-t/\tau}\,dt = \tau$$



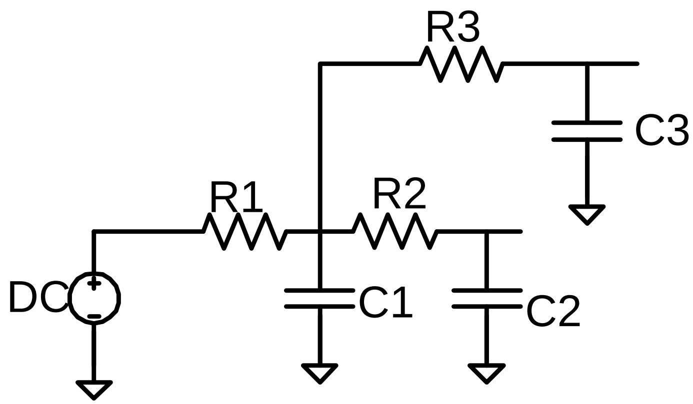


Matching Moments of RC Network



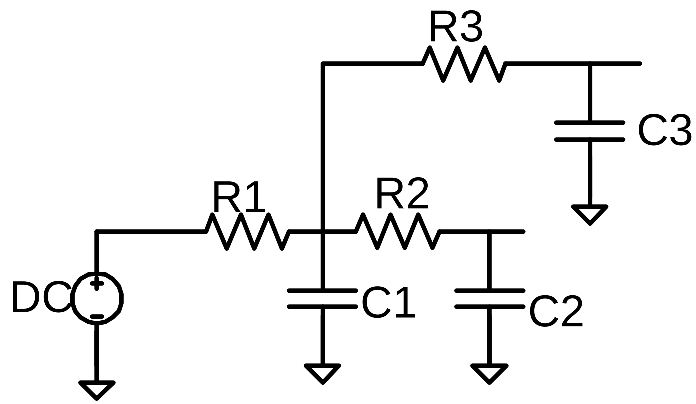
- For a general RC network, we want to find the area under its step response
  - And approximate the area as the optimal time-constant τ for a single-pole model
- Luckily, there is an easy way

Matching Moments of RC Network (2)



- To change its value, a current must flow through each C
  - This current I_k is $C_k \cdot dV_k/dt$, for the capacitor at node 'k'
  - Only way for the charge to leave the system is:
    to flow through the resistors between C_k and Gnd (or Vdd)
  - Voltage is dropped across each resistor that carries I_k
  - But, computing I_k directly is not easy in general

Matching Moments of RC Network (3)

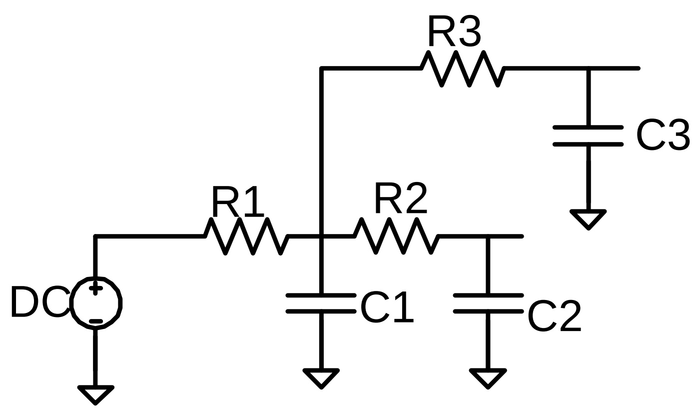


- While I don't know what I_k is
  - I DO know what the integral of this current is → $Q_k = C_k \Delta V_k$!
- And we can compute the area under the response as:

$$V_i = \sum_k R_{ik} I_k$$

$$\int_0^\infty V_i \cdot dt = \sum_k R_{ik} \int_0^\infty I_k \cdot dt = \sum_k R_{ik} C_k \Delta V_i$$

Elmore Time Constant



- The time constant of a single-pole network that estimates the response of the network $\hat{\tau} = \hat{R}_i \hat{C}_i$ is:

$$V_i = \sum_k R_{ik} I_k$$

$$\int_0^\infty V_i \cdot dt = \sum_k R_{ik} C_k \Delta V_i = \hat{R}_i \hat{C}_i \Delta V_i$$

- $R_{ik} \equiv$ the resistance that the resistive path from input to node k and the resistive path from input to node i (=output) have in common
- If all ΔV_i's are equal (e.g. = Vdd), then:

$$\boxed{\sum_k R_{ik} C_k = \hat{R}_i \hat{C}_i}$$

"Elmore time constant"

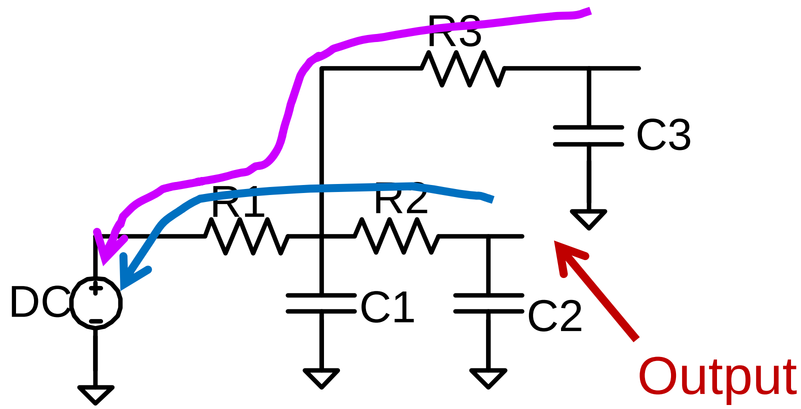
Elmore Time Constant (2)

$$\boxed{\sum_k R_{ik} C_k = \hat{R}_i \hat{C}_i}$$

"Elmore time constant"

- This is the first moment of the RC network
  - And our estimate of the circuit's time constant
  - Hence, the 50% Vdd delay is: $D = ln(2) \cdot \hat{R}_i \hat{C}_i$
- Side note:
  - When the RC circuit is a "tree" (i.e. with no resistor loops)
    - It's easy to find R_{ik} by inspection
    - Find the common resistance of the paths (in, i) and (in, k)
  - Even for other RC networks, R_{ik} always exists
    - R_{ik} is simply the voltage change seen at the output i, by a current injected at node k.

Elmore Delay Example



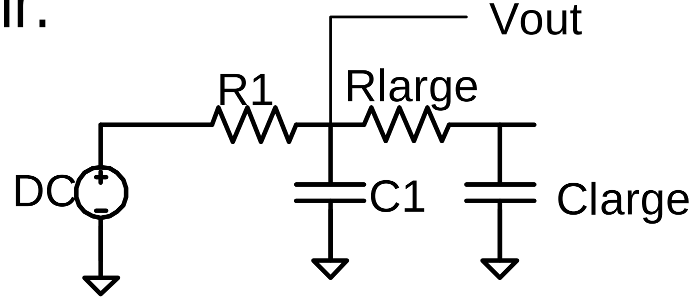
- To determine R_{ik}, just look for the common resistors on the current flow path back to the input
  - The purple line is the current flow from C3 to the input
    - If the output is node '2', $R_{ik}=R_{23} = R1$
  - The blue line is the current flow from C2 to the input
    - If the output is node '2', $R_{ik}=R_{22} = R1+R2$
- Another (more formal) way to determine R_{ik}:
  - Inject a current at node k, and measure the voltage induced at node 'i'; $R_{ik} = V_i$ / injected current

Elmore Delay Summary

- In summary, the delay estimate is easy to find:
  1. Find every capacitor in the circuit that **changes** its value
  2. Find the voltage that a current injected from this capacitor (k) will produce at the output (i), and define the effective resistance, R_{ik}, equal to this voltage/current
  3. Sum $C_k \cdot R_{ik}$, over all nodes k that change value
     - This sum is an estimate of the time constant
- We can apply this to distributed elements like wires too
  - Look at a small segment of the wire of length dL, and then add up delay contribution from all of these small segment

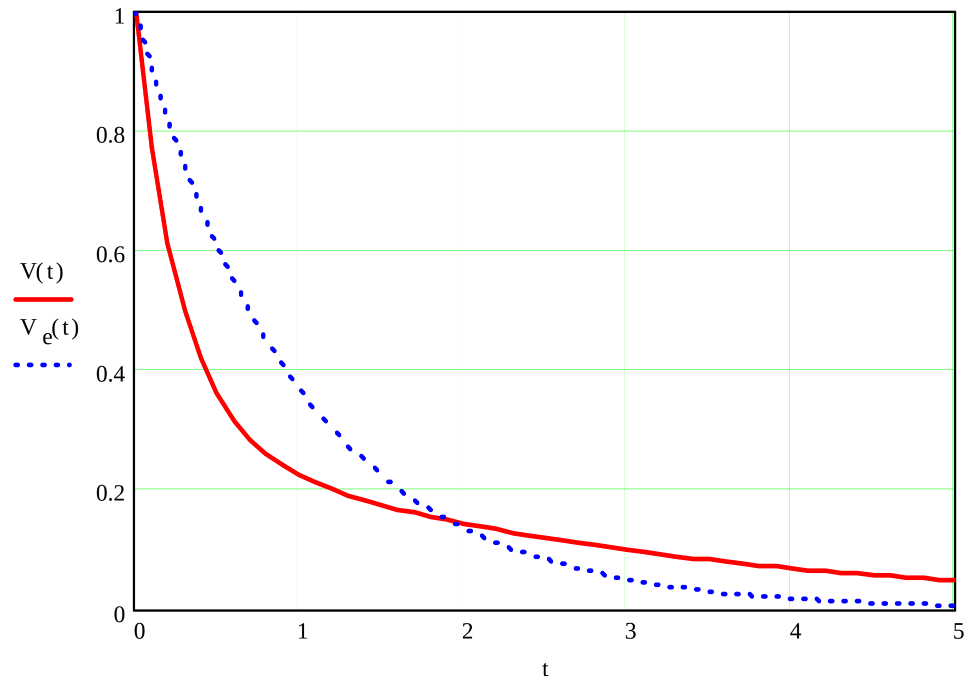
Elmore Delay – Limitation

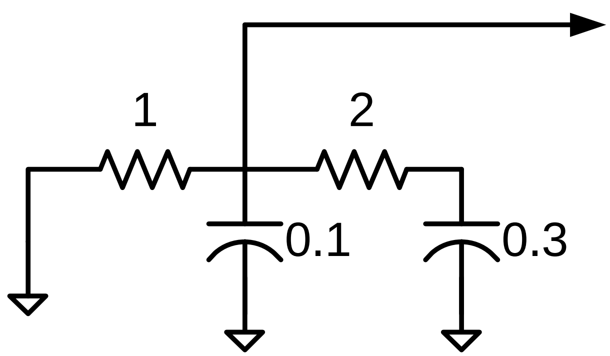
- Works well in most cases
  - Output at the end of a long line can almost always modeled as a single-pole model
  - Works even when two poles are closely spaced
- Poor where there is a pole-zero pair:
  - Output is sum of two exponentials
    - One is slow with low amplitude
    - One is fast with full amplitude
  - In example shown on right
    - Decays with R1C1 to R1/(R1+Rlarge)
    - Then decays with time constant Clarge·(Rlarge +R1)



Example – A Pole-Zero Pair

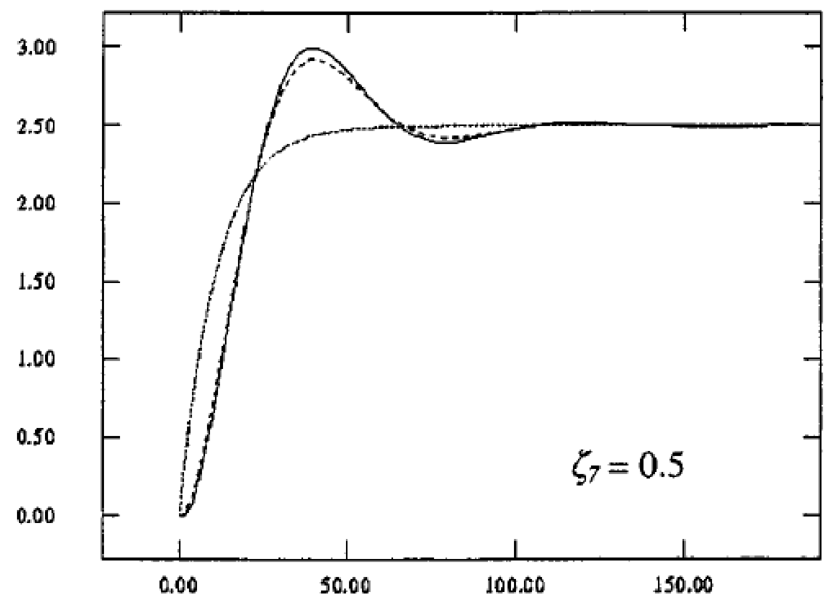
- This plot shows a 'bad' output (circuit below)
  - Tail is 10 times slower
  - With only .3 full swing
- Moments are not the best model for this case



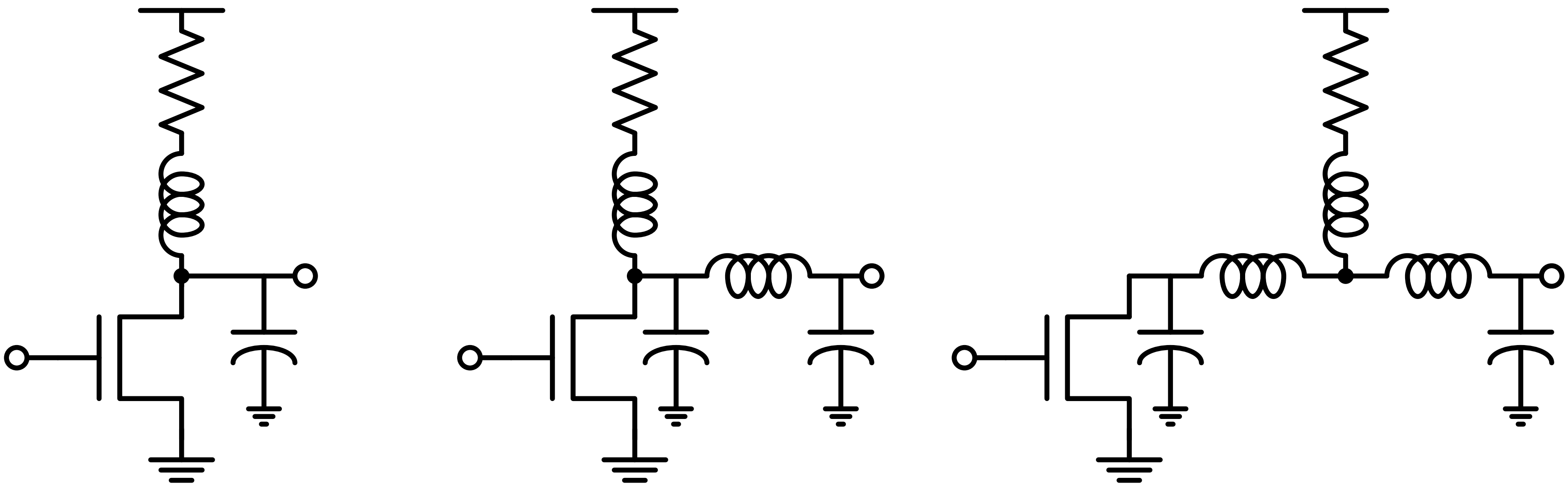


Elmore Delay for RLC Network?

- With inductors, there may exist complex poles in the circuits (i.e. resonance)
  - Its step response may exhibit ringing – its non-monotonic response cannot be approximated by a first-order system
- Why worry about L?
  - On-chip interconnect wires do exhibit L effects
  - Due to high-frequency operation and low resistance (thick top metals)



Why Use L? – Inductive Peaking



Shunt Peaking

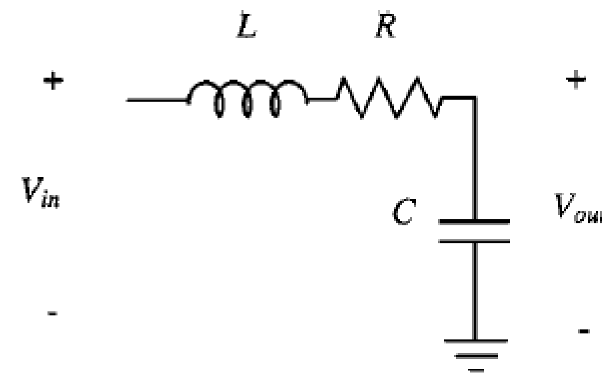
Shunt and Series Peaking

Shunt and Double-Series Peaking

- Inductors delay current flows; each C charges faster

The Simplest RLC Circuit

- Its normalized TF (i.e. gain=1) is:



$$H(s) = \frac{1}{s^2LC + sRC + 1}$$

which can be expressed by the natural frequency (ω_n) and damping factor (ζ):

$$H(s) = \frac{1}{s^2/\omega_n^2 + s2\zeta/\omega_n + 1}$$

where

$$\omega_n = \frac{1}{\sqrt{LC}} \qquad \zeta = \frac{1}{2}\frac{RC}{\sqrt{LC}}$$

The Time-Domain Response

- The step response of the second-order system *H(s)*:

$$s(t) = 1 + \frac{1}{2\sqrt{\zeta^2 - 1}} \left[\frac{e^{\omega_n t(-\zeta + \sqrt{\zeta^2 - 1})}}{-\zeta + \sqrt{\zeta^2 - 1}} - \frac{e^{\omega_n t(-\zeta - \sqrt{\zeta^2 - 1})}}{-\zeta - \sqrt{\zeta^2 - 1}} \right]$$

- And the 50% propagation delay and 10-90% rise time can be estimated as:

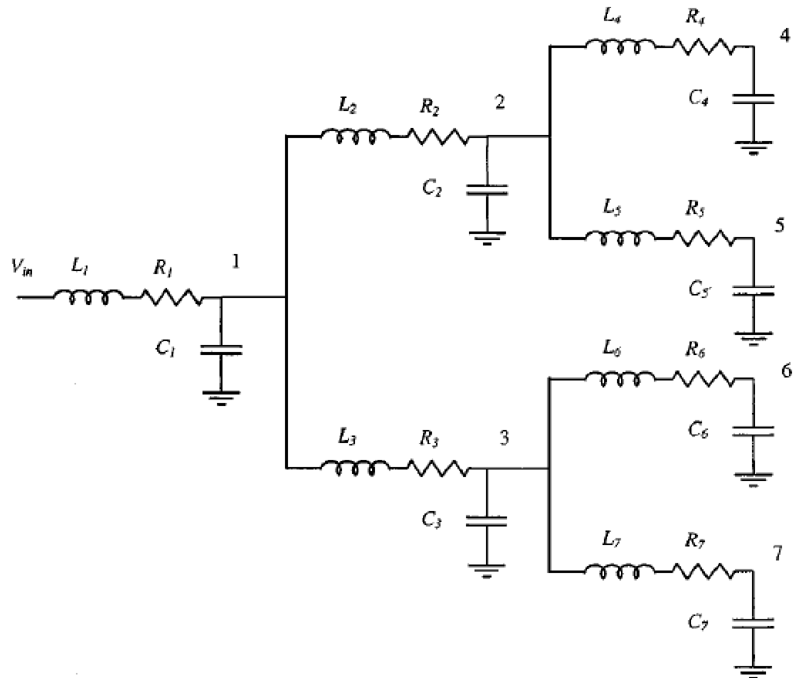
$$t_{pd} \approx (1.047 e^{\zeta/0.85} + 1.39\zeta)/\omega_n$$

$$t_r \approx (6.017 e^{\zeta^{1.35}/0.4} - 5e^{(\zeta^{1.25}/0.64)} + 4.39\zeta)/\omega_n$$

- Note: I think these expressions have typos – fix them

Extended Elmore Delay

- Let's approximate a general RLC network as a second-order system, since it can express non-monotonic responses (when $\zeta < 0.7$)
- How do we determine ω_n and ζ?
  - In the spirit of Elmore delay, match the first and second moments (m_1 and m_2)



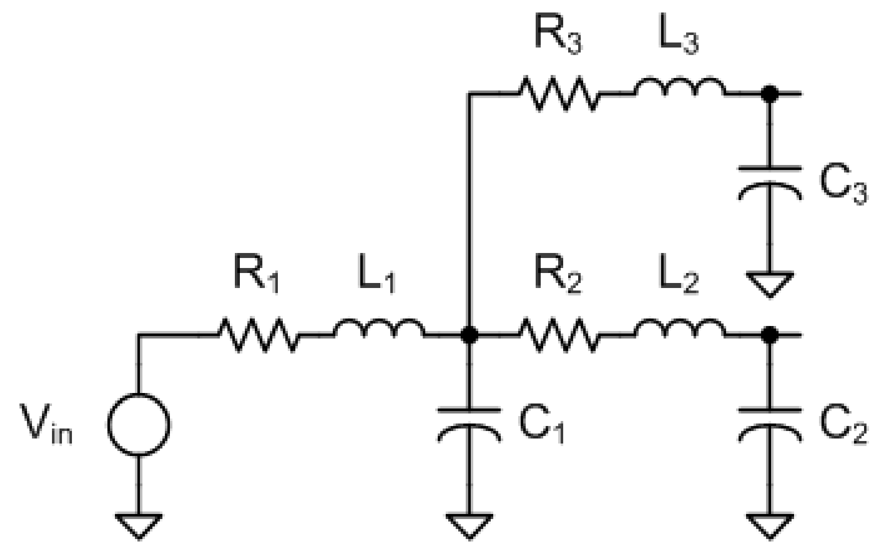
$$\omega_n = \frac{1}{\sqrt{m_1^2 - m_2}}$$

$$\zeta = \frac{1}{2}\frac{m_1}{\sqrt{m_1^2 - m_2}}$$

RLC Elmore Delay – First Moment

- Now with inductors,

$$V_i(t) = \sum_k R_{ik} I_k + L_{ik} \frac{dI_k}{dt}$$



- Since the first moment m_1 is the area under $V_i(t)$:

$$m_1 \equiv \int_0^{\infty} V_i(t)dt = \sum_k \left[R_{ik} \int_0^{\infty} I_k \, dt + L_{ik} \int_0^{\infty} \frac{dI_k}{dt} \, dt \right]$$

$$= \sum_k R_{ik} C_k \cdot \Delta V \, (\because I_k(\infty) = 0)$$

$$\equiv \hat{R}_i \hat{C}_i \cdot \Delta V$$

The same expression as before!

RLC Elmore Delay – Second Moment

- Skipping some intermediate steps…

$$m_1^2 - m_2 \equiv \left[\int_0^\infty V_i(t)dt\right]^2 - \int_0^\infty V_i^2(t)dt$$

???

$$= \left[\sum_k R_{ik}C_k\right]^2 \Delta V^2 - \left[\sum_k \sum_j C_k R_{ik} C_j R_{kj} - \sum_k C_k L_{ik}\right] \Delta V^2$$

$$\approx \sum_k L_{ik}C_k \cdot \Delta V^2$$

$$\equiv \hat{L}_i \hat{C}_i \cdot \Delta V^2$$

Summary

- You can approximate an arbitrary RLC network as a second-order system with:

$$\hat{\omega_n} = \frac{1}{\sqrt{\hat{L}_i\hat{C}_i}} = \frac{1}{\sqrt{\sum_k L_{ik}C_k}} \qquad \hat{\zeta} = \frac{1}{2}\frac{\hat{R}_i\hat{C}_i}{\sqrt{\hat{L}_i\hat{C}_i}} = \frac{1}{2}\frac{\sum_k R_{ik}C_k}{\sqrt{\sum_k L_{ik}C_k}}$$

- Important observation:
  - For a given ζ (for the desired settling response), there is a determined ratio between RC and sqrt(LC)
  - It implies that the delay is still a linear function of RC
  - Therefore, the Logical Effort framework will still apply!!

$$t_{pd} \approx (1.047e^{\zeta/0.85} + 1.39\zeta)/\omega_n$$