Chapter 4

Optimal Linear Filtering

4.1 Discrete-Time Kalman Filter Formulation

The Kalman filter is a recursive, unbiased, minimum error variance estimator.

Given the system and measurement descriptions

$$\underline{x}_k = \Phi_{k-1}\underline{x}_{k-1} + \underline{w}_{k-1},\ k = 1, \cdots \tag{4.1-1}$$

$$\underline{z}_k = H_k\underline{x}_k + \underline{v}_k, k = 1, \cdots \tag{4.1-2}$$

where $\underline{w}_k =$ zero mean, white Gaussian noise with covariance Q_k

$\underline{v}_k =$ zero mean, white Gaussian noise with covariance R_k

$$E\left\{\underline{w}_i \underline{v}_j^T\right\} = 0, E\left\{\underline{x}_0 \underline{w}_i^T\right\} = 0, E\left\{\underline{x}_0 \underline{v}_i^T\right\} = 0, \forall i, j$$

a recursive estimator would have the form

$$\underline{\hat{x}}_k = K_k' \underline{\hat{x}}_k(-) + K_k \underline{z}_k \tag{4.1-3}$$

Define the relations

$$\underline{\hat{x}}_k(+) = \underline{x}_k + \underline{\tilde{x}}_k(+) \tag{4.1-4}$$

$$\underline{\hat{x}}_k(-) = \underline{x}_k + \underline{\tilde{x}}_k(-).$$

Discrete-Time Kalman Filter Formulation (continued)

From Eqs. (4.1-2), (4.1-3), (4.1-4), we obtain

$$
\begin{aligned}
\underline{\tilde{x}}_k(+) &= -\underline{x}_k + \underline{\hat{x}}_k(+) = -\underline{x}_k + K_k'\underline{\hat{x}}_k(-) + K_k\underline{z}_k \\
&= -\underline{x}_k + K_k'\left[\underline{x}_k + \underline{\tilde{x}}_k(-)\right] + K_k\left[H_k\underline{x}_k + \underline{v}_k\right] \quad (4.1\text{-}5) \\
&= \left[K_k' + K_kH_k - I\right]\underline{x}_k + K_k'\underline{\tilde{x}}_k(-) + K_k\underline{v}_k.
\end{aligned}
$$

For the filter to be an unviased estimator,

$$E\left\{\underline{\tilde{x}}_k(-)\right\} = E\left\{\underline{\tilde{x}}_k(+)\right\} = \underline{0}.$$

That is,

$$K_k' = I - K_kH_k. \qquad (4.1\text{-}6)$$

Insert Eq. (4.1-6) to Eq. (4.1-3) to obtain,

$$\underline{\hat{x}}_k(+) = \left(I - K_kH_k\right)\underline{\hat{x}}_k(-) + K_k\underline{z}_k \qquad (4.1\text{-}7)$$

or

$$\underline{\hat{x}}_k(+) = \underline{\hat{x}}_k(-) + K_k\left[\underline{z}_k - H_k\underline{\hat{x}}_k(-)\right] \qquad (4.1\text{-}8)$$

The corresponding estimation error is, from Eqs. (4.1-2), (4.1-4), (4.1-8)

$$
\begin{aligned}
\underline{\tilde{x}}_k(+) = \underline{\hat{x}}_k(+) - \underline{x}_k &= \left(I - K_kH_k\right)\underline{\hat{x}}_k(-) + K_k\left[H_k\underline{x}_k + \underline{v}_k\right] - \underline{x}_k \\
&= \left(I - K_kH_k\right)\underline{\tilde{x}}_k(-) + K_k\underline{v}_k. \qquad (4.1\text{-}9)
\end{aligned}
$$

Discrete-Time Kalman Filter Formulation (continued)

Error Covariance Update

From the definition of the error covariance

$$P_k(+) = E\left[\underline{\tilde{x}}_k(+)\underline{\tilde{x}}_k(+)^T\right] \qquad (4.1\text{-}10)$$

Eq. (5.1-9) gives

$$P_k(+) = E\left\{\left[\left(I - K_k H_k\right)\underline{\tilde{x}}_k(-) + K_k \underline{v}_k\right]\left[\left(I - K_k H_k\right)\underline{\tilde{x}}_k(-) + K_k \underline{v}_k\right]^T\right\}$$

$$= E\left\{\left(I - K_k H_k\right)\underline{\tilde{x}}_k(-)\left[\underline{\tilde{x}}_k(-)^T (I - K_k H_k)^T + \underline{v}_k{}^T K_k{}^T\right] + K_k \underline{v}_k\left[\underline{\tilde{x}}_k(-)^T \left(I - K_k H_k\right)^T + \underline{v}_k{}^T K_k{}^T\right]\right\}$$

$$(4.1\text{-}11)$$

By definition,

$$E\left[\underline{\tilde{x}}_k(-)\underline{\tilde{x}}_k(-)^T\right] = P_k(-) \qquad (4.1\text{-}12)$$

$$E\left[\underline{v}_k \underline{v}_k{}^T\right] = R_k$$

and, with an assumption of measurement errors being uncorrelated,

$$. \qquad E\left[\underline{\tilde{x}}_k(-)\underline{v}_k{}^T\right] = E\left[\underline{v}_k \underline{\tilde{x}}_k(-)^T\right] = 0. \qquad (4.1\text{-}13)$$

Thus,

$$P_k(+) = \left(I - K_k H_k\right) P_k(-)\left(I - K_k H_k\right)^T + K_k R_k K_k{}^T. \qquad (4.1\text{-}14)$$

Discrete-Time Kalman Filter Formulation (continued)

Optimum Choice of K_k

Set up the following objective function to find an estimator minimizing a norm of the estimation error vector, viz.,

$$J_k = E\left[\underline{\tilde{x}}_k(+)^T \underline{\tilde{x}}_k(+)\right] = trace\left[P_k(+)\right]. \qquad (4.1\text{-}15)$$

To find an optimum K_k, let $\dfrac{\partial J_k}{\partial K_k} = 0$

$$\frac{\partial J_k}{\partial K_k} = \frac{\partial}{\partial K_k} tr\left[\left(I - K_k H_k\right) P_k(-) \left(I - K_k H_k\right)^T + K_k R_k K_k^{\ T}\right]$$

$$= \frac{\partial}{\partial K_k}[trP_k(-) - trP_k(-)H_k^{\ T}K_k^{\ T} - trK_kH_kP_k(-) + trK_kH_kP_k(-)H_k^{\ T}K_k^{\ T} + trK_kR_kK_k^{\ T}]$$

$$= -P_k(-)H_k^{\ T} - P_k(-)^T H_k^{\ T} + 2K_kH_kP_k(-)H_k^{\ T} + 2K_kR_k$$

$$= -2(I - K_kH_k)P_k(-)H_k^{\ T} + 2K_kR_k = 0.$$

Solving for K_k gives

$$K_k = P_k(-)H_k^{\ T}\left[H_kP_k(-)H_k^{\ T} + R_k\right]^{-1}. \qquad (4.1\text{-}16)$$

Discrete-Time Kalman Filter Formulation (continued)

For sufficiency, examine $\dfrac{\partial^2 J_k}{\partial {K_k}^2}$

$$\frac{\partial^2 J_k}{\partial {K_k}^2} = 2H_k P_k(-)^T {H_k}^T + 2R_k > 0.$$

Substitute Eq. (4.1-16) into Eq. (4.1-14) and rearrange

$$\begin{aligned} P_k(+) &= P_k(-) - P_k(-){H_k}^T \left[H_k P_k(-){H_k}^T + R_k\right]^{-1} H_k P_k(-) \\ &= \left[I - K_k H_k\right] P_k(-). \end{aligned} \qquad (4.1\text{-}17)$$

Applied are:

$$tr(A+B) = trA + trB$$

$$tr(AB) = tr(BA)$$

$$\frac{\partial}{\partial A}\left[tr(ABA^T)\right] = 2AB$$

$$\frac{\partial}{\partial A}\left[tr(BA)\right] = B^T$$

$$\frac{\partial}{\partial A}\left[tr(BA^T)\right] = B$$

$${P_k}^T(-) = P_k(-).$$

Discrete-Time Kalman Filter Formulation (continued)

State and Error Covariance Propagation

Repeating the definitions

$$\underline{\tilde{x}}_k = \underline{\hat{x}}_k - \underline{x}_k; \quad P_k = E\left[\underline{\tilde{x}}_k \underline{\tilde{x}}_k^{\,T}\right]$$

Propagating the estimated state

$$\underline{\hat{x}}_k = \Phi_{k-1}\underline{\hat{x}}_{k-1}. \tag{4.1-18}$$

Check if the estimation error is still unbiased

$$\underline{\tilde{x}}_k = \Phi_{k-1}\underline{\tilde{x}}_{k-1} - \underline{w}_{k-1}$$

$$E\left[\underline{\tilde{x}}_k\right] = \Phi_{k-1}E\left[\underline{\tilde{x}}_{k-1}\right] - E\left[\underline{w}_{k-1}\right] = \underline{0} \tag{4.1-19}$$

Now develop the error covariance propagation

$$E\left[\underline{\tilde{x}}_k \underline{\tilde{x}}_k^{\,T}\right] = E\left\{\left(\Phi_{k-1}\underline{\tilde{x}}_{k-1} - \underline{w}_{k-1}\right)\left(\Phi_{k-1}\underline{\tilde{x}}_{k-1} - \underline{w}_{k-1}\right)^T\right\}$$

$$= E\left\{\Phi_{k-1}\underline{\tilde{x}}_{k-1}\underline{\tilde{x}}_{k-1}^{\,T}\Phi_{k-1}^{\,T} - \Phi_{k-1}\underline{\tilde{x}}_{k-1}\underline{w}_{k-1}^{\,T}\Phi_{k-1}^{\,T} - \underline{w}_{k-1}\underline{\tilde{x}}_{k-1}^{\,T}\Phi_{k-1}^{\,T} + \underline{w}_{k-1}\underline{w}_{k-1}^{\,T}\right\} \tag{4.1-20}$$

$$E\left[\underline{\tilde{x}}_{k-1}\underline{w}_{k-1}^{\,T}\right] = E\left\{\Phi_{k-2}\underline{\tilde{x}}_{k-2}\underline{w}_{k-1}^{\,T}\right\} - E\left\{\underline{w}_{k-2}\underline{w}_{k-1}^{\,T}\right\} = \cdots = 0.$$

Eq. (4.1-20) can be written

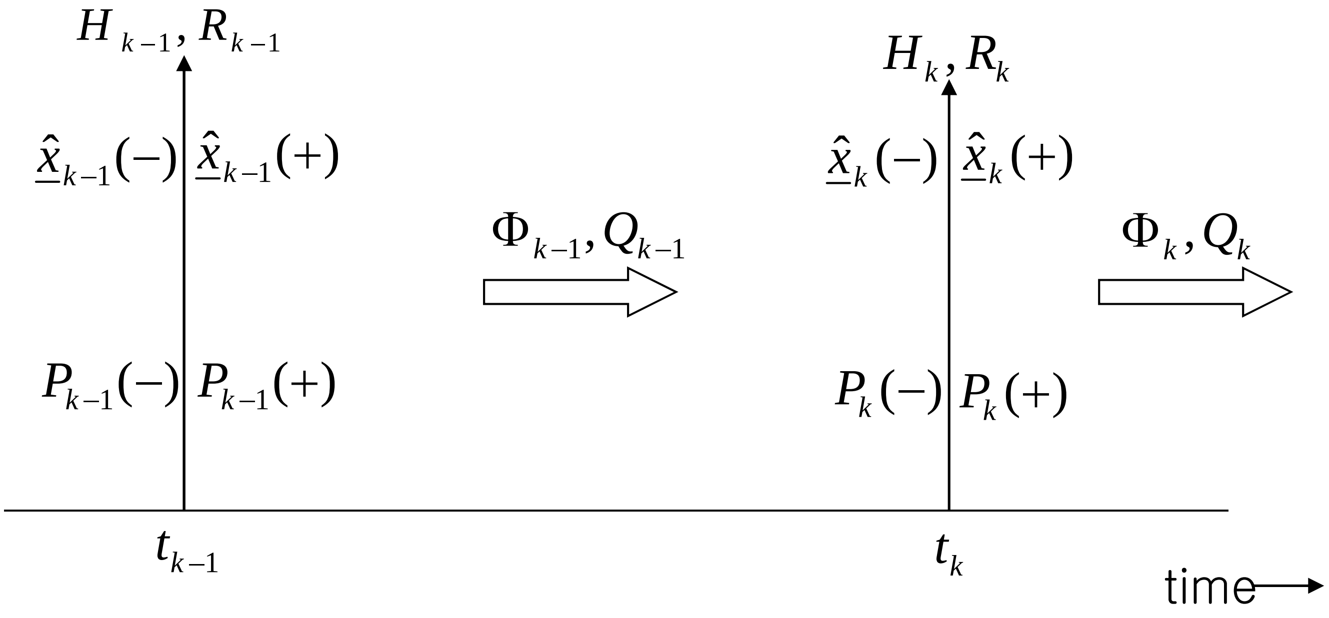
$$P_k = \Phi_{k-1}P_{k-1}\Phi_{k-1}^{\,T} + Q_{k-1} \tag{4.1-21}$$

Express Eqs. (4.1-18) and (4.1-21) using the sign, (+) and (-)

$$\underline{\hat{x}}_k(-) = \Phi_{k-1}\underline{\hat{x}}_{k-1}(+) \tag{4.1-22}$$

$$P_k(-) = \Phi_{k-1}P_{k-1}(+)\Phi_{k-1}^{\,T} + Q_{k-1} \tag{4.1-23}$$

Discrete-Time Kalman Filter Timing Diagram



Summary of Discrete-Time Kalman Filter Equations

System Model Measurement Model	$\underline{x}_k = \Phi_{k-1}\underline{x}_{k-1} + \underline{w}_{k-1}, \underline{w}_k \sim N(\underline{0}, Q_k)$ $\underline{z}_k = H_k \underline{x}_k + \underline{v}_k, \underline{v}_k \sim N(\underline{0}, R_k)$
Initial Conditions Other Assumptions	$E[\underline{x}(0)] = \hat{\underline{x}}_0, E[(\underline{x}(0) - \hat{\underline{x}}_0)(\underline{x}(0) - \hat{\underline{x}}_0)^T] = P_0$ $E[\underline{w}_k \underline{v}_j] = 0$ for all j, k
State Estimate Extrapolation Error Covariance Extrapolation	$\hat{\underline{x}}_k(-) = \Phi_{k-1}\hat{\underline{x}}_{k-1}(+)$ $P_k(-) = \Phi_{k-1}P_{k-1}(+)\Phi_{k-1}{}^T + Q_{k-1}$
State Estimate Update Error Covariance Update Kalman Gain Matrix	$\hat{\underline{x}}_k(+) = \hat{\underline{x}}_k(-) + K_k\left[\underline{z}_k - H_k\hat{\underline{x}}_k(-)\right]$ $P_k(+) = \left(I - K_k H_k\right)P_k(-)$ $K_k = P_k(-)H_k{}^T\left[H_k P_k(-) H_k{}^T + R_k\right]^{-1}$

Example: Ship Navigational Fixes

Example 4.1-1 (Ship Navigational Fixes)

$$d_{k+1} = d_k + s_k$$
$$s_{k+1} = s_k + w_k$$

where,

d_k = easterly position of the ship at hour

s_k = easterly velocity of the ship at hour

w_k = noise from wind and waves.

Define $x_k \equiv \begin{bmatrix} d_k \\ s_k \end{bmatrix}$, then,

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_k$$

where

$$x_0 \approx N\left(\bar{x}_0, P_0\right) = N\left(\begin{bmatrix} 0 \\ 10 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right); \quad w_k \approx N(0,Q) = N(0,1).$$

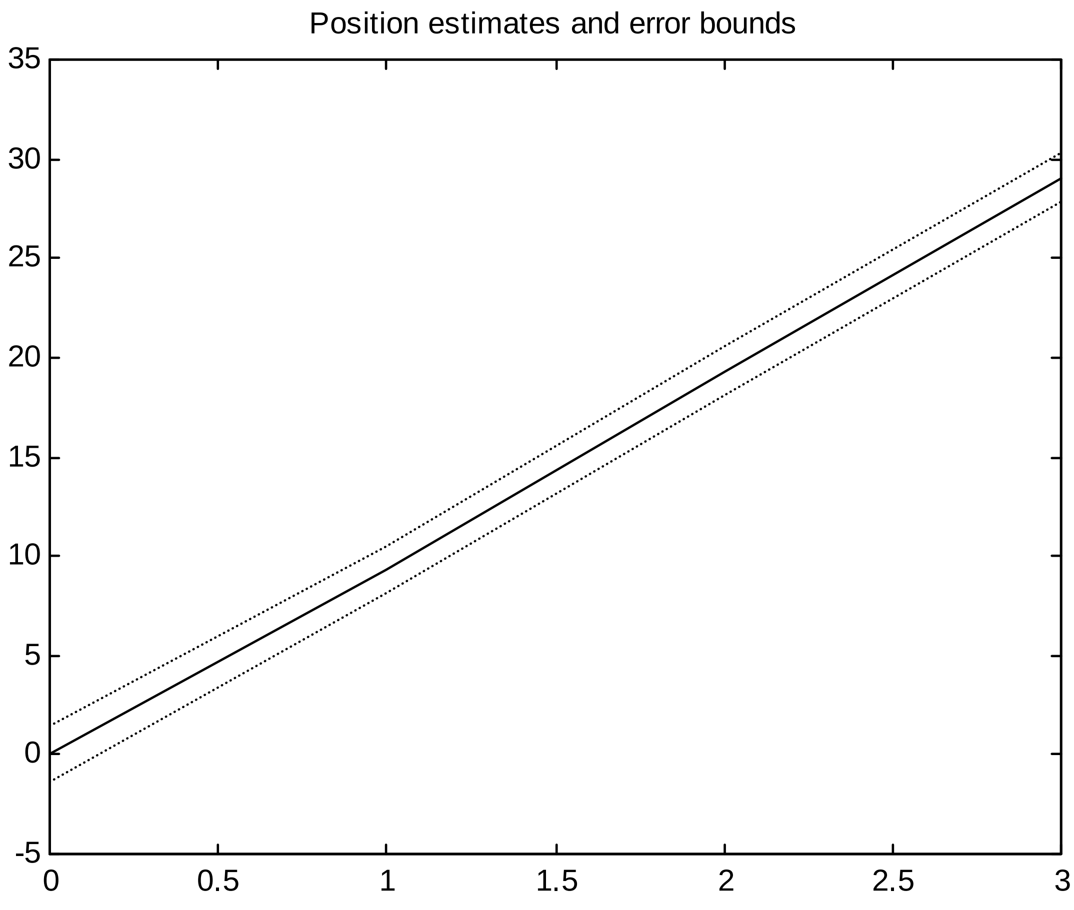
Suppose

$$z_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + v_k, \qquad v_k \approx N(0,2)$$
$$z_1 = 9,\ z_2 = 19.5,\ z_3 = 29.$$

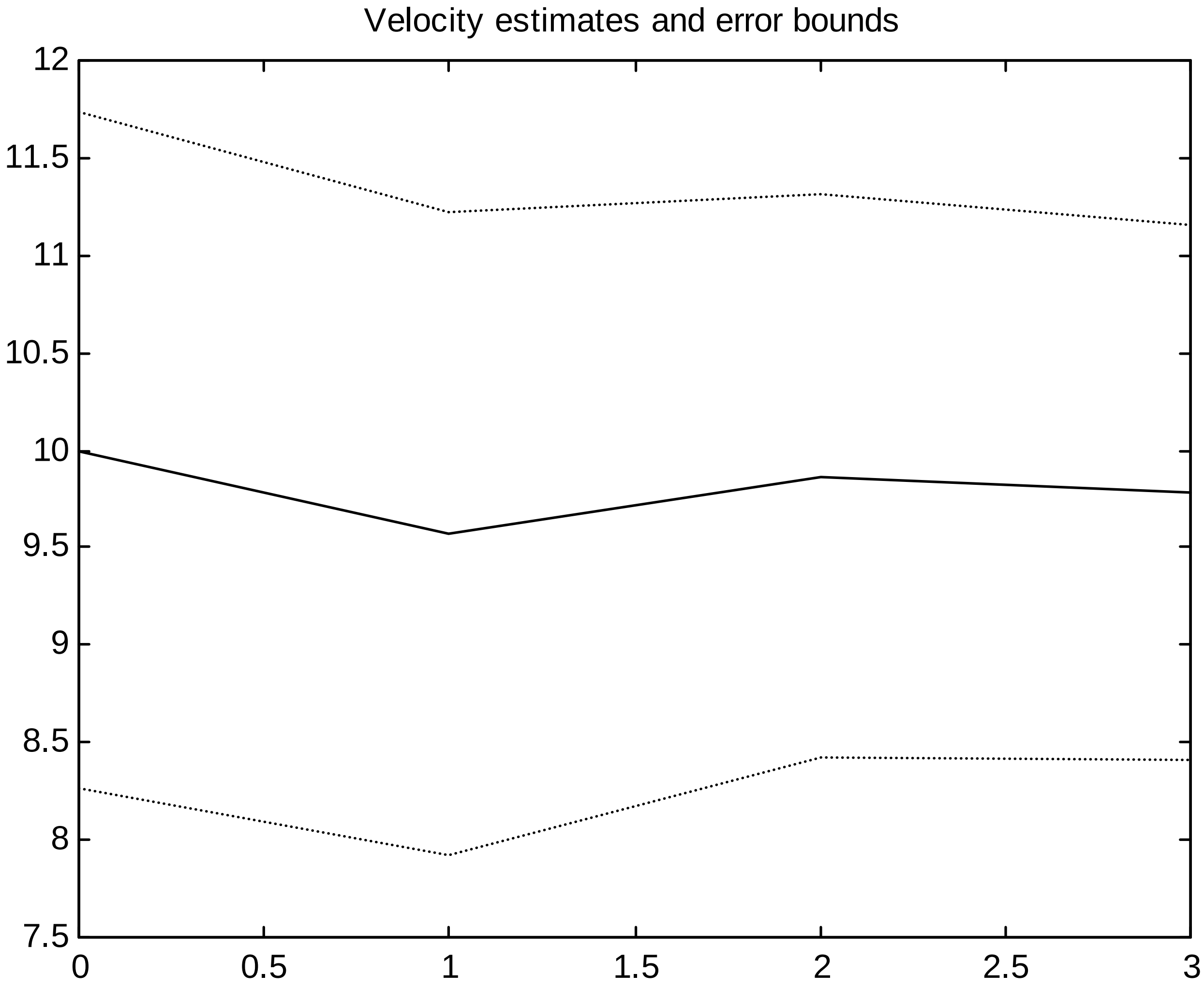
Find $\hat{x}_k$ and P_k for k =1,2,3.

Run Ex4_1_1.m

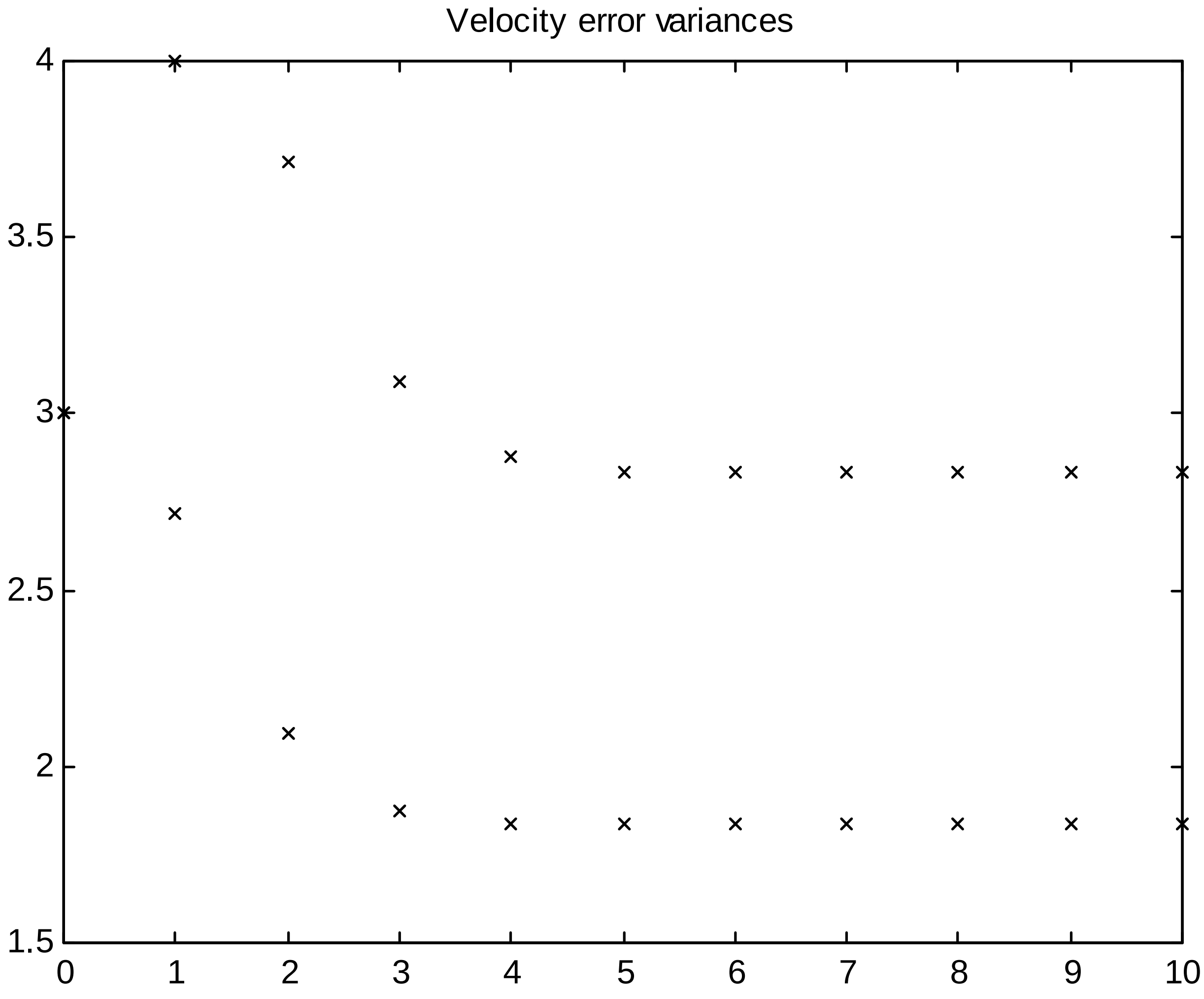
Example: Ship Navigational Fixes (continued)



Example: Ship Navigational Fixes (continued)



Example: Ship Navigational Fixes (continued)



Discrete-Time Kalman Filter Equivalent Form

Equivalent Form

This section presents an equivalent form of the Kalman filter. This form is useful in the derivation of the Kalman smoother. It provides alternative expressions for computing $\underline{\hat{x}}_k(+)$ in Eq. (4.1-8) and $P_k(+)$ in Eq. (4.1-17).

Lemma 4.1 (Matrix Inversion Lemma) Let A_{11} be a nonsingular $p \times p$ matrix, A_{12} and A_{21} be $p \times q$ and $q \times p$ matrices, respectively, and A_{22} be a nonsingular $q \times q$ matrix. Then,

$$\left[A_{11}^{-1} + A_{12}A_{22}A_{21}\right]^{-1} = A_{11} - A_{11}A_{12}\left[A_{21}A_{11}A_{12} + A_{22}^{-1}\right]^{-1} A_{21}A_{11}. \qquad (4.1\text{-}24)$$

First we find a new expression for $\underline{\hat{x}}_k(+)$. In Eq. (4.1-7) we multiply by $P_k(-)$ and $P_k^{-1}(-)$ to obtain

$$\begin{aligned} \underline{\hat{x}}_k(+) &= \left[I - K_k H_k\right] P_k(-) P_k^{-1}(-) \underline{\hat{x}}_k(-) + K_k \underline{z}_k \\ &= P_k(+) P_k^{-1}(-) \underline{\hat{x}}_k(-) + K_k \underline{z}_k \\ &= P_k(+)\left[P_k^{-1}(-) \underline{\hat{x}}_k(-) + P_k^{-1}(+) K_k \underline{z}_k\right] \qquad (4.1\text{-}25) \end{aligned}$$

Discrete-Time Kalman Filter Equivalent Form (continued)

For $P_k(+)$, apply the matrix inversion lemma to Eq. (4.1-17)

$$\begin{aligned} P_k^{-1}(+) &= P_k^{-1}(-) + P_k^{-1}(-)P_k(-)H^T[-HP_k(-)P_k^{-1}(-)P_k(-)H^T \\ &\quad + HP_k(-)H^T + R_k]^{-1}HP_k(-)P_k^{-1}(-) \\ &= P_k^{-1}(-) + H^T R_k^{-1} H \end{aligned} \qquad (4.1\text{-}26)$$

Next we multiply by K_k and substitute for K_k to obtain

$$\begin{aligned} P_k^{-1}(+)K_k &= [P_k^{-1}(-) + H^T R_k^{-1} H]P_k(-)H^T[HP_k(-)H^T + R_k]^{-1} \\ &= H^T\left[I + R_k^{-1}HP_k(-)H^T\right]\left[HP_k(-)H^T + R_k\right]^{-1} \\ &= H^T R_k^{-1}\left[R_k + HP_k(-)H^T\right]\left[HP_k(-)H^T + R_k\right]^{-1} \\ &= H^T R_k^{-1}. \end{aligned}$$

Therefore, Eq. (4.1-25) can be written as

$$\underline{\hat{\mathrm{x}}}_{\mathrm{k}}(+) = P_k(+)\left[P_k^{-1}(-)\underline{\hat{\mathrm{x}}}_{\mathrm{k}}(-) + H^T R_k^{-1}\underline{\mathrm{z}}_{\mathrm{k}}\right]. \qquad (4.1\text{-}27)$$

Discrete-Time Kalman Filter Equivalent Form (continued)

Next we find an equivalent form for $P_k(+)$. Rewrite Eq. (4.1-17)

$$P_k(+) = P_k(-) - P_k(-)H_k^{\ T}\left[H_k P_k(-)H_k^{\ T} + R_k\right]^{-1} H_k P_k(-).$$

The right-hand side of this equation fits the form of the right-hand side of Eq. (4.1-24). Then we conclude

$$P_k(+) = \left[P_k^{-1}(-) + H_k^{\ T} R_k^{\ -1} H_k\right]^{-1}. \qquad (4.1\text{-}28)$$

Eqs. (4.1-22), (4.1-23), (4.1-27), and (4.1-28) represent an alternative form of the Kalman filter.

4.2 Discretization of Continuous System

Consider the following continuous system

$$\dot{x}(t) = Ax(t) + Bu(t) + Gw(t) \tag{4.2-1}$$

$$z(t) = Hx(t) + v(t). \tag{4.2-2}$$

Let $x(0) \sim (\overline{x}_0, P_0)$, $w(t) \sim (0, Q)$, $v(t) \sim (0, R)$, where $\{w(t)\}$ and $\{v(t)\}$ are white and uncorrelated with each other and with $x(0)$.

The solution to Eq. (4.2-1) is

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau + \int_{t_0}^{t} e^{A(t-\tau)}Gw(\tau)d\tau \tag{4.2-3}$$

Let $t_0 = kT, t = (k+1)T$ and define $x_k \equiv x(kT)$. Then,

$$x_{k+1} = e^{AT}x_k + \int_{kT}^{(k+1)T} e^{A[(k+1)T-\tau]}Bu(\tau)d\tau + \int_{kT}^{(k+1)T} e^{A[(k+1)T-\tau]}Gw(\tau)d\tau. \tag{4.2-4}$$

Discretization of Continuous System (continued)

Let

$$u_k = u(kT)$$

$$w_k = \int_{kT}^{(k+1)T} e^{A[(k+1)T-\tau]} G w(\tau) d\tau$$

Eq. (4.2-4) becomes,

$$x_{k+1} = e^{AT} x_k + \int_{kT}^{(k+1)T} e^{A[(k+1)T-\tau]} B d\tau \cdot u_k + w_k$$

Change variables twice, $\lambda = \tau - kT$ and then $\tau = T - \lambda$. For $\lambda = \tau - kT$,

$kT \le \tau \le (k+1)T \quad \Rightarrow \quad 0 \le \lambda \le T$ and $d\tau = d\lambda$. Therefore, the above equation may be written

$$x_{k+1} = e^{AT} x_k + \int_0^T e^{A(T-\lambda)} B d\tau \cdot u_k + w_k.$$

For $\tau = T - \lambda$, $0 \le \lambda \le T \quad \Rightarrow \quad T \ge \tau \ge 0$ and $d\lambda = -d\tau$. The above equation can now be written

$$x_{k+1} = e^{AT} x_k + \int_0^T e^{A\tau} B d\tau \cdot u_k + w_k \equiv A^s x_k + B^s u_k + w_k \qquad (4.2\text{-}5)$$

with

$$A^s = e^{AT} = I + AT + \frac{A^2T^2}{2!} + \cdots$$

$$B^s = \int_0^T e^{A\tau} B d\tau = \int_0^T \left(I + A\tau + \frac{A^2\tau^2}{2!} + \cdots \right) B d\tau = BT + \frac{ABT^2}{2} + \frac{A^2BT^3}{3!} + \cdots \qquad (4.2\text{-}6)$$

<u>Discretization of Continuous System</u> (continued)

Find the covariance Q^s

$$Q^s = E\left\{w_k w_k^{\ T}\right\} = \iint_{kT} e^{A[(k+1)T-\tau]} GE\left\{w(\tau)w(\sigma)^T\right\} G^T e^{A^T[(k+1)T-\sigma]} d\tau d\sigma$$

$$= \int_{kT}^{(k+1)T} e^{A[(k+1)T-\tau]} GQG^T e^{A^T[(k+1)T-\tau]} d\tau = \int_0^T e^{A\tau} GQG^T e^{A^T\tau} d\tau$$

$$= \int_0^T \left[I + A\tau + \frac{(A\tau)^2}{2!} + \cdots\right] GQG^T \left[I + A\tau + \frac{(A\tau)^2}{2!} + \cdots\right]^T d\tau$$

$$= GQG^T T + \frac{\left(AGQG^T + GQG^T A^T\right)T^2}{2!} + \cdots \qquad (4.2\text{-}7)$$

Discretizing the measurement equation is easy since it has no dynamics:

$$z_k = Hx_k + v_k$$

From the following relations,

$$E\left\{v_k v_k^{\ T}\right\} = R^s \delta(k); \quad E\left\{v(t)v(\tau)^T\right\} = R\delta(t-\tau); \quad \delta(t) = \lim_{T\to 0}\left(1/T\right)\Pi\left(t/T\right)$$

$$\Pi(t) = \begin{cases} 1, & -\dfrac{1}{2} \le t \le \dfrac{1}{2} \\ 0, & otherwise \end{cases}.$$

In the limit, $R\delta(t) = \lim_{T\to 0}\left(R^s T\right)\left(1/T\right)\Pi\left(t/T\right)$ or

$$R^s = \frac{R}{T}. \qquad (4.2\text{-}8)$$

<u>Example: $\alpha - \beta$ tracker</u>

Consider the following range tracker,

$$\ddot{r}(t) = w_r$$

$$z = r + v \qquad (4.2\text{-}9)$$

where v is the tracking error and $v \sim (0, \sigma_r^{\ 2})$. And w_r is the disturbance accelerations of target. Suppose the disturbance accelerations are independent and uniformly distributed between $\pm a$. Then their variances are,

$$E\left\{w_r^{\ 2}\right\} = \int w_r^{\ 2} f_{w_r} dw_r = \int_{-a}^{a} w_r^{\ 2} \cdot \frac{1}{2a} dw_r = \frac{a^2}{3}.$$

Let $x_1 = r(t)$, $x_2 = \dot{r}(t)$. Eq. (4.2-9) can be rewritten,

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + w$$

$$z = \begin{bmatrix} 1 & 0 \end{bmatrix} x + v \qquad (4.2\text{-}10)$$

where $x(0) \sim (\overline{x}_0, P_0)$, $w \sim (0, Q)$, $Q = \begin{bmatrix} 0 & 0 \\ 0 & a^2/3 \end{bmatrix}$.

Example: $\alpha - \beta$ tracker (continued)

Suppose measurements are made at intervals of T units. Then the discretized system becomes,

$$A^s = e^{AT} = I + AT = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \quad (A^2 = 0)$$

$$Q^s = QT + \frac{\left(AQ + QA^T\right)T^2}{2} + \frac{AQA^T}{3}T^3 = \frac{a^2}{3}\begin{bmatrix} T^3/3 & T^2/2 \\ T^2/2 & T \end{bmatrix}$$

$$R^s = R/T = {\sigma_r}^2/T \tag{4.2-11}$$

The discretized model of the $\alpha - \beta$ tracker is,

$$x_{k+1} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x_k + w_k$$

$$z_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + v_k \tag{4.2-12}$$

where $w_k \sim (0, Q^s)$ and $v_k = (0, {\sigma_r}^2/T)$.

4.3 Continuous-Time Kalman Filter Formulation

Given the system and Measurement description

$$\dot{x} = Fx + Gw \tag{4.3-1}$$

$$z = Hx + v \tag{4.3-2}$$

the following equivalences are valid in the limit as $t_k - t_{k-1} = \Delta t \to 0$

$$\Phi_k \to I + F\Delta t; \quad Q_k \to GQG^T \Delta t; \quad R_k \to \frac{R}{\Delta t}\ . \tag{4.3-3}$$

Apply these relations to Eq. (4.1-21)

$$\begin{aligned} P_{k+1}(-) &= \Phi_k P_k(+) \Phi_k{}^T + Q_k = [I + F\Delta t] P_k(+) [I + F\Delta t]^T + GQG^T \Delta t \\ &= P_k(+) + [FP_k(+) + P_k(+)F^T + GQG^T]\Delta t + O(\Delta t^2) \end{aligned} \tag{4.3-4}$$

Also apply $P_k(+) = [I - K_k H_k] P_k(-)$ (Eq. (5.1-17)) to the above equation then,

$$\begin{aligned} \frac{P_{k+1}(-) - P_k(-)}{\Delta t} &= FP_k(-) + P_k(-)F^T + GQG^T - \frac{1}{\Delta t} K_k H_k P_k(-) - FK_k H_k P_k(-) \\ &\quad - K_k H_k P_k(-) F^T + O(\Delta t). \end{aligned} \tag{4.3-5}$$

Continuous-Time Kalman Filter Formulation (continued)

Rearrange Eq. (4.1-16) to investigate $\dfrac{1}{\Delta t}K_k$

$$\begin{aligned}\frac{1}{\Delta t}K_k &= \frac{1}{\Delta t}P_k(-)H_k{}^T[H_kP_k(-)H_k{}^T + R_k]^{-1}\\ &= P_k(-)H_k{}^T[H_kP_k(-)H_k{}^T\Delta t + R_k\Delta t]^{-1}\\ &= P_k(-)H_k{}^T[H_kP_k(-)H_k{}^T\Delta t + R]^{-1}.\end{aligned}$$

At the limit, the above equation becomes

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}K_k = PH^TR^{-1}. \qquad (4.3\text{-}6)$$

Furthermore,

$$\lim_{\Delta t\to 0}K_k = \lim_{\Delta t\to 0}\left(PH^TR^{-1}\Delta t\right) = 0. \qquad (4.3\text{-}7)$$

We obtain the Riccati equation by applying Eqs. (4.3-6) and (4.3-7) to Eq. (4.3-5)

$$\dot{P} = FP + PF^T + GQG^T - PH^TR^{-1}HP. \qquad (4.3\text{-}8)$$

Continuous-Time Kalman Filter Formulation (continued)

Now we rewrite Eq. (4.1-8) applying the relation, $\widehat{x}_k(-) = \Phi_{k-1}\hat{x}_{k-1}(+)$,

$$\widehat{x}_k(+) = \widehat{x}_k(-) + K_k\left[z_k - H_k\hat{x}_k(-)\right]$$

$$= \Phi_{k-1}\widehat{x}_{k-1}(+) + K_k[z_k - H_k\Phi_{k-1}\hat{x}_{k-1}(+)].$$

Applying Eq. (4.3-3) to the above equation gives

$$\widehat{x}_k(+) \approx (I + F\Delta t)\widehat{x}_{k-1}(+) + K_k\left[z_k - H_k(I + F\Delta t)\hat{x}_{k-1}(+)\right]$$

$$= \widehat{x}_{k-1}(+) + F\widehat{x}_{k-1}(+)\Delta t + K_k[z_k - H_k\widehat{x}_{k-1}(+)] - K_kH_kF\hat{x}_{k-1}(+)\Delta t. \quad (4.3\text{-}9)$$

Rearrange Eq. (4.3-9) and take the limit

$$\lim_{\Delta t\to 0}\frac{\widehat{x}_k(+) - \hat{x}_{k-1}(+)}{\Delta t} = \lim_{\Delta t\to 0}\left[F\widehat{x}_{k-1}(+) + \frac{K_k}{\Delta t}[z_k - H_k\widehat{x}_{k-1}(+)] - K_kH_kF\hat{x}_{k-1}(+)\right].$$

Applying Eqs. (4.3-6) and (4.3-7) to the above equation gives

$$\widehat{x} = F\widehat{x} + PH^TR^{-1}[z - H\hat{x}]$$

$$= F\widehat{x} + K[z - H\hat{x}]. \quad (4.3\text{-}10)$$

Eqs. (4.3-8) and (4.3-10) form the continuous-time Kalman filter.

Continuous-Time Kalman Filter Summary

System Model Measurement Model	$\dot{x}(t) = F(t)x(t) + G(t)w(t), \quad w(t) \sim N(0, Q(t))$ $z(t) = H(t)x(t) + v(t), \quad v(t) \sim N(0, R(t))$
Initial Conditions Other Assumptions	$E\{x(0)\} = \hat{x}_0,\ E\{[x(0) - \hat{x}_0][x(0) - \hat{x}_0]^T\} = P_0$ $R^{-1}(t)$ exists, $E\{w(t)v^T(\tau)\} = 0$
State Estimate Error Covariance Propagation Kalman Gain Matrix	$\dot{\hat{x}}(t) = F(t)\hat{x}(t) + K(t)[z(t) - H(t)\hat{x}(t)]$ $\hat{x}(0) = \hat{x}_0$ $\dot{P}(t) = F(t)P(t) + P(t)F^T(t) + G(t)Q(t)G^T(t) - K(t)R(t)K^T(t)$ $K(t) = P(t)H^T(t)R^{-1}(t)$