Coupled-mode theory for nonlinear optical interactions

Dr Yoonchan Jeong

School of Electrical Engineering, Seoul National University

Tel: +82 (0)2 880 1623, Fax: +82 (0)2 873 9953
Email: yoonchan@snu.ac.kr

Orthogonality of modes

■ **Maxwell's equations**

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0, \quad \nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{J}$$

$$\nabla \cdot \boldsymbol{D} = \rho, \quad \nabla \cdot \boldsymbol{B} = 0$$

■ **Constitutive relations**

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} = \varepsilon_o \boldsymbol{E} + \boldsymbol{P}$$

$$\boldsymbol{B} = \mu \boldsymbol{H} = \mu_o \boldsymbol{H} + \boldsymbol{M}$$

■ **Lorentz reciprocity theorem**

$$\nabla \cdot (\boldsymbol{E}_q \times \boldsymbol{H}_p^* + \boldsymbol{E}_p^* \times \boldsymbol{H}_q) \quad (p, q = 1, 2, 3, \ldots) \Leftarrow \text{ for eigenmodes}$$

$$= \boldsymbol{H}_p^* \cdot (\nabla \times \boldsymbol{E}_q) - \boldsymbol{E}_q \cdot (\nabla \times \boldsymbol{H}_p^*) + \boldsymbol{H}_q \cdot (\nabla \times \boldsymbol{E}_p^*) - \boldsymbol{E}_p^* \cdot (\nabla \times \boldsymbol{H}_q)$$

$$= -\boldsymbol{H}_p^* \cdot \frac{\partial \boldsymbol{B}_q}{\partial t} - \boldsymbol{E}_q \cdot \frac{\partial \boldsymbol{D}_p^*}{\partial t} - \boldsymbol{H}_q \cdot \frac{\partial \boldsymbol{B}_p^*}{\partial t} - \boldsymbol{E}_p^* \cdot \frac{\partial \boldsymbol{D}_q}{\partial t} \Leftarrow \text{ for monochromatic radiation}$$

$$= -i\omega(H_{p,i}^* \mu_{ij} H_{q,j} - E_{q,i} \varepsilon_{ij}^* E_{p,j}^* - H_{q,i} \mu_{ij}^* H_{p,j}^* + E_{p,i}^* \varepsilon_{ij} E_{q,j})$$

$$= -i\omega(H_{p,i}^* \mu_{ij} H_{q,j} - E_{q,j} \varepsilon_{ji}^* E_{p,i}^* - H_{q,j} \mu_{ji}^* H_{p,i}^* + E_{p,i}^* \varepsilon_{ij} E_{q,j})$$

$$= 0$$

Note: Source free and Hermitian tensors of ε and μ

Coupled-mode equations

- **Permittivity perturbation**

$\varepsilon' = \varepsilon + \Delta\varepsilon, \quad \mu' = \mu \quad \Leftarrow$ perturbed medium

$E', H' \quad \Leftarrow$ perturbed fields

- **Coupled-mode equations**

$$\nabla \cdot (E' \times H_p^* + E_p^* \times H') \quad (p = 1, 2, 3, \ldots)$$

$$= H_p^* \cdot (\nabla \times E') - E' \cdot (\nabla \times H_p^*) + H' \cdot (\nabla \times E_p^*) - E_p^* \cdot (\nabla \times H')$$

$$= -H_p^* \cdot \frac{\partial B'}{\partial t} - E' \cdot \frac{\partial D_p^*}{\partial t} - H' \cdot \frac{\partial B_p^*}{\partial t} - E_p^* \cdot \frac{\partial D'}{\partial t}$$

$$= -i\omega(H_p^* \cdot B' - E' \cdot D_p^* - H' \cdot B_p^* + E_p^* \cdot D') \qquad \text{where} \quad D' = \varepsilon'E' = (\varepsilon + \Delta\varepsilon)E' = \varepsilon E' + \Delta P',$$

$$= -i\omega E_p^* \Delta\varepsilon E' = -i\omega E_p^* \Delta P' \qquad\qquad\qquad\qquad B' = \mu H'$$

In result,

$$\nabla \cdot (E' \times H_p^* + E_p^* \times H') = -i\omega E_p^* \Delta\varepsilon E' \quad (p = 1, 2, 3, \ldots)$$

Coupled-mode equations

■ **Derivation of 1st-order differential equations**

Let us take the perturbed fields in the forms

$$\begin{pmatrix} \boldsymbol{E}' \\ \boldsymbol{H}' \end{pmatrix} \equiv \sum_q a_q(z) \begin{pmatrix} \boldsymbol{E}_q \\ \boldsymbol{H}_q \end{pmatrix} = \sum_q a_q(z) \begin{pmatrix} \boldsymbol{e}_q(x,y)\exp(-i\beta_q z) \\ \boldsymbol{h}_q(x,y)\exp(-i\beta_q z) \end{pmatrix}$$

It follows

$$\sum_q a_q(z)\nabla \cdot (\boldsymbol{E}_q \times \boldsymbol{H}_p^* + \boldsymbol{E}_p^* \times \boldsymbol{H}_q) + \sum_q \frac{da_q(z)}{dz}(\boldsymbol{E}_q \times \boldsymbol{H}_p^* + \boldsymbol{E}_p^* \times \boldsymbol{H}_q) \cdot \hat{z}$$

$$= -i\omega \sum_q a_q(z)\boldsymbol{E}_p^* \cdot \Delta\varepsilon(x,y,z)\boldsymbol{E}_q \qquad \text{where}$$

$$\frac{1}{4}\int (\boldsymbol{E}_q \times \boldsymbol{H}_p^* + \boldsymbol{E}_p^* \times \boldsymbol{H}_q) \cdot \hat{z}\, dxdy = \begin{cases} 1, & p=q \\ 0, & p \neq q \end{cases}$$

In result,

$$\frac{da_p(z)}{dz}\exp(-i\beta_p z) = -i\frac{\omega}{4}\sum_q a_q(z)\exp(-i\beta_q z)\int \boldsymbol{e}_p^*(x,y) \cdot \Delta\varepsilon(x,y,z)\boldsymbol{e}_q(x,y)dxdy$$

Mode coupling in periodic media

■ **Periodic permittivity perturbation**

Let us assume the perturbed permittivity in the form

$$\varepsilon'(x, y, z) = \varepsilon_o(x, y) + \Delta\varepsilon(x, y, z)$$

In isotropic medium

$$\Delta\varepsilon(x, y, z) = \varepsilon_o[n'^2(x, y, z) - n^2(x, y)]$$

$$\cong 2\varepsilon_o n(x, y)\Delta n(x, y, z), \quad if \ \Delta n(x, y, z) << n(x, y),$$

$$\Delta n(x, y, z) = \Delta n_{av}(x, y) + \Delta n_{gr}(x, y)\cos(\frac{2\pi}{\Lambda} z + \phi_{gr})$$

$$= \Delta n_{av}(x, y) + \frac{\Delta n_{gr}(x, y)}{2}\left\{\exp[-i(\frac{2\pi}{\Lambda} z + \phi_{gr})] + \exp[i(\frac{2\pi}{\Lambda} z + \phi_{gr})]\right\},$$

Two-mode coupling approximation (TMCA)

■ **Periodic permittivity perturbation**

Let us assume the perturbed permittivity in the form

$$\frac{da_p(z)}{dz} = -i\frac{|\beta_p|}{\beta_p}\sum_q \kappa_{pq}(z)a_q(z)\exp[-i(\beta_q - \beta_p)z], \quad p = 1, 2, 3, \ldots$$

where

$$\kappa_{pq}(z) = \frac{\pi c}{2\lambda}\iint e_p^*(x, y)\cdot\Delta\varepsilon(x, y, z)e_q(x, y)\,dxdy$$

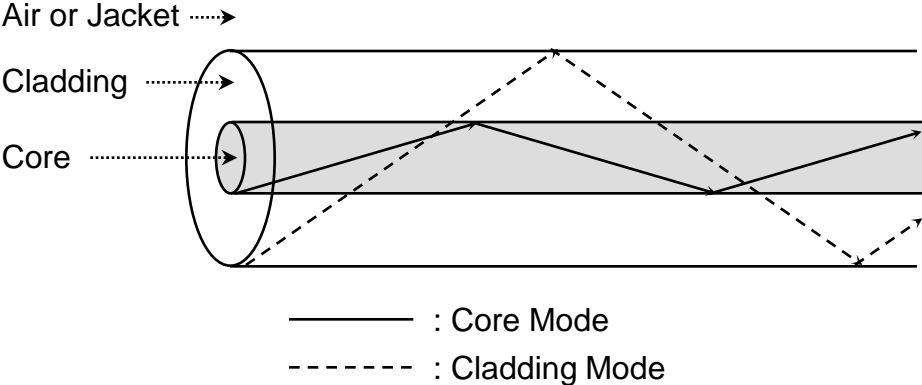■ **Two-mode coupling approximation**

Let us take dominant two modes and their coupling constants
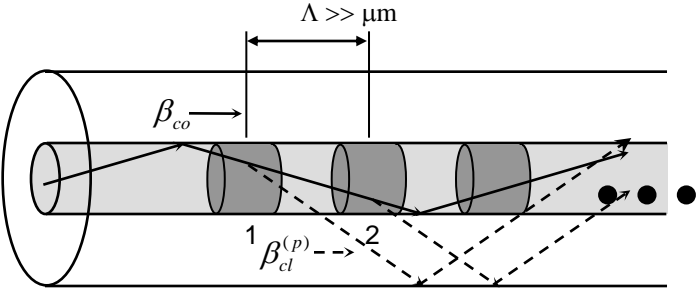on the condition of the longitudinal phase matching

- Codirectional coupling: $\beta_d / \beta_i > 0$
- Contradirectional coupling: $\beta_d / \beta_i < 0$
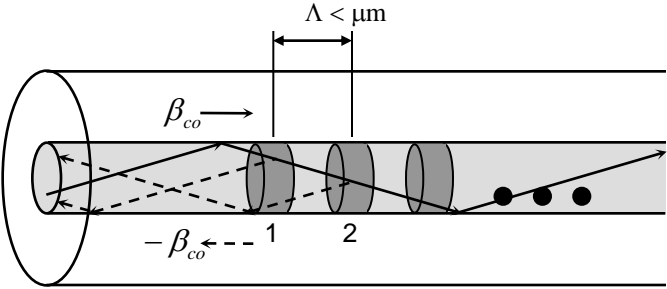
Mode coupling

■ Fiber modes

Air or Jacket ·····→

Cladding ········→

Core ············→

———— : Core Mode

- - - - - : Cladding Mode

Codirectional coupling

$\Lambda \gg \mu m$

$\beta_{co} \longrightarrow$

$1\ \beta_{cl}^{(p)} \dashrightarrow 2$

Contradirectional coupling

$\Lambda < \mu m$

$\beta_{co} \longrightarrow$

$-\beta_{co} \dashleftarrow 1 \quad 2$

Coupling coefficients for TMCA

■ **Coupling coefficients**

Let us take the coupling coefficients in the forms:

for the incident and diffracted modes E_i and E_d

self- and cross-coupling coefficients

$$\kappa_{\sigma,i} = \frac{\pi \varepsilon_o c}{\lambda} \iint e_i^*(x, y) \cdot n(x, y) \Delta n_{av}(x, y) e_i(x, y) \, dxdy$$

$$\kappa_{\sigma,d} = \frac{\pi \varepsilon_o c}{\lambda} \iint e_d^*(x, y) \cdot n(x, y) \Delta n_{av}(x, y) e_d(x, y) \, dxdy$$

$$\kappa_{\xi} = \frac{\pi \varepsilon_o c}{2\lambda} \iint e_i^*(x, y) \cdot n(x, y) \Delta n_{gr}(x, y) e_d(x, y) \, dxdy$$

Codirectional coupling

■ **Coupled-mode equations**

$$\frac{da_i(z)}{dz} = -i\kappa_{\sigma,i}a_i(z) - i\kappa_\xi a_d(z)\exp(i\Delta\beta z - i\phi_{gr})$$

$$\frac{da_d(z)}{dz} = -i\kappa_{\sigma,d}a_d(z) - i\kappa_\xi^* a_i(z)\exp(-i\Delta\beta z + i\phi_{gr})$$

where

$$\Delta\beta = \beta_i - \beta_d - \frac{2\pi}{\Lambda}, \ \ (\beta_i > \beta_d)$$

■ **Boundary conditions**

$$E_i(z)\big|_{z=0} = E_i(0), \quad E_d(z)\big|_{z=0} = E_d(0)$$

Codirectional coupling

■ **Solution**

$$E_i(z) = a_i(z)\exp(-i\beta_i z)$$

$$= \left\{ (\cos\sigma_f z - i\frac{\Delta\beta'}{2\sigma_f}\sin\sigma_f z)E_i(0) - i\frac{\kappa_\xi}{\sigma_f}\sin\sigma_f z\exp(-i\phi_{gr})E_d(0) \right\}\exp[-i(\gamma_f + \pi/\Lambda)z]$$

$$E_d(z) = a_d(z)\exp(-i\beta_d z)$$

$$= \left\{ -i\frac{\kappa_\xi^*}{\sigma_f}\sin\sigma_f z\exp(+i\phi_{gr})E_i(0) + (\cos\sigma_f z + i\frac{\Delta\beta'}{2\sigma_f}\sin\sigma_f z)E_d(0) \right\}\exp[-i(\gamma_f - \pi/\Lambda)z]$$

where

$$\Delta\beta' = \Delta\beta + \kappa_{\sigma,i} - \kappa_{\sigma,d}, \quad \gamma_f = \frac{\beta_i + \beta_d + \kappa_{\sigma,i} + \kappa_{\sigma,d}}{2}, \quad \sigma_f^2 = \kappa_\xi^*\kappa_\xi + \left(\frac{\Delta\beta'}{2}\right)^2$$

Codirectional coupling

■ **Transmissivity**

Boundary conditions: $E_i(z)\big|_{z=0} = E_i(0), \quad E_d(z)\big|_{z=0} = 0$

$$T_i \equiv \left|\frac{E_i(L)}{E_i(0)}\right|^2 = \cos^2 \sigma_f L + \left(\frac{\Delta\beta'}{2\sigma_f}\right)^2 \sin^2 \sigma_f L$$

$$T_d \equiv \left|\frac{E_d(L)}{E_i(0)}\right|^2 = \frac{\kappa_\xi^* \kappa_\xi}{\sigma_f^2} \sin^2 \sigma_f L$$

When $\beta' = 0$

$$T_{i,\min} = \cos^2 |\kappa_\xi| L$$

$$\lambda_{\min} = [\Delta n_{eff} + \delta n_{eff,av}(1-u_\sigma)]\Lambda = \left[1 + \frac{\delta n_{eff,av}(1-u_\sigma)}{\Delta n_{eff}}\right]\lambda_{F,0}$$

Contradirectional coupling

■ **Coupled-mode equations**

$$\frac{da_i(z)}{dz} = -i\kappa_{\sigma,i}a_i(z) - i\kappa_\xi a_d(z)\exp(i\Delta\beta z - i\phi_{gr})$$

$$\frac{da_d(z)}{dz} = i\kappa_{\sigma,d}a_d(z) + i\kappa_\xi^* a_i(z)\exp(-i\Delta\beta z + i\phi_{gr})$$

where

$$\Delta\beta = \beta_i + \beta_d - \frac{2\pi}{\Lambda} \quad (\beta_i > 0, \beta_d > 0)$$

■ **Boundary conditions**

$$E_i(z)\big|_{z=0} = E_i(0), \quad E_d(z)\big|_{z=L} = E_d(L)$$

Contradirectional coupling

■ **Solution**

$$E_i(z) = a_i(z)\exp(-i\beta_i z)$$

$$= \left\{ \frac{\sigma_b \cosh\sigma_b(L-z) + i(\Delta\beta'/2)\sinh\sigma_b(L-z)}{\sigma_b \cosh\sigma_b L + i(\Delta\beta'/2)\sinh\sigma_b L} E_i(0) \right.$$

$$\left. + \frac{-i\kappa_\xi \sinh\sigma_b z \exp[i(\gamma_b - \pi/\Lambda)L - i\phi_{gr}]}{\sigma_b \cosh\sigma_b L + i(\Delta\beta'/2)\sinh\sigma_b L} E_d(L) \right\} \exp[-i(\gamma_b + \pi/\Lambda)z],$$

$$E_d(z) = a_d(z)\exp(i\beta_d z)$$

$$= \left\{ \frac{-i\kappa_\xi^* \sinh\sigma_b(L-z)\exp(i\phi_{gr})}{\sigma_b \cosh\sigma_b L + i(\Delta\beta'/2)\sinh\sigma_b L} E_i(0) \right.$$

$$\left. + \frac{\sigma_b \cosh\sigma_b z + i(\Delta\beta'/2)\sinh\sigma_b z}{\sigma_b \cosh\sigma_b L + i(\Delta\beta'/2)\sinh\sigma_b L} \exp[i(\gamma_b - \pi/\Lambda)L]E_d(L) \right\} \exp[-i(\gamma_b - \pi/\Lambda)z],$$

where

$$\Delta\beta' = \Delta\beta + \kappa_{\sigma,i} + \kappa_{\sigma,d}, \quad \gamma_b = \frac{\beta_i - \beta_d + \kappa_{\sigma,i} - \kappa_{\sigma,d}}{2}, \quad \sigma_b^2 = \kappa_\xi^* \kappa_\xi - \left(\frac{\Delta\beta'}{2}\right)^2$$

Contradirectional coupling

■ **Reflectivity**

Boundary conditions: $E_i(z)\big|_{z=0} = E_i(0), \quad E_d(z)\big|_{z=L} = 0$

$$R \equiv \left|\frac{E_d(0)}{E_i(0)}\right|^2 = \left|\frac{\kappa_\xi^* \kappa_\xi \sinh^2 \sigma_b L}{\sigma_b^2 \cosh^2 \sigma_b L + (\Delta\beta'/2)^2 \sinh^2 \sigma_b L}\right|$$

When $\beta' = 0$

$$R_{\max} = \tanh^2\left|\kappa_\xi\right|L, \quad \lambda_{\max} = 2(n_{eff} + \delta n_{eff,av})\Lambda = \left(1 + \delta n_{eff,av}/n_{eff}\right)\lambda_{B,0},$$

■ **Bandwidth**

$$R_{edge} = \frac{\kappa_\xi^* \kappa_\xi L^2}{1 + \kappa_\xi^* \kappa_\xi L^2}, \quad \lambda_{edge} = \lambda_{\max} \pm \frac{\delta n_{eff,gr}}{2n_{eff}}\lambda_{B,0}$$

$$\frac{\Delta\lambda_{edge}}{\lambda_{B,0}} = \frac{\delta n_{eff,gr}}{n_{eff}}$$

Numerical and experimental results
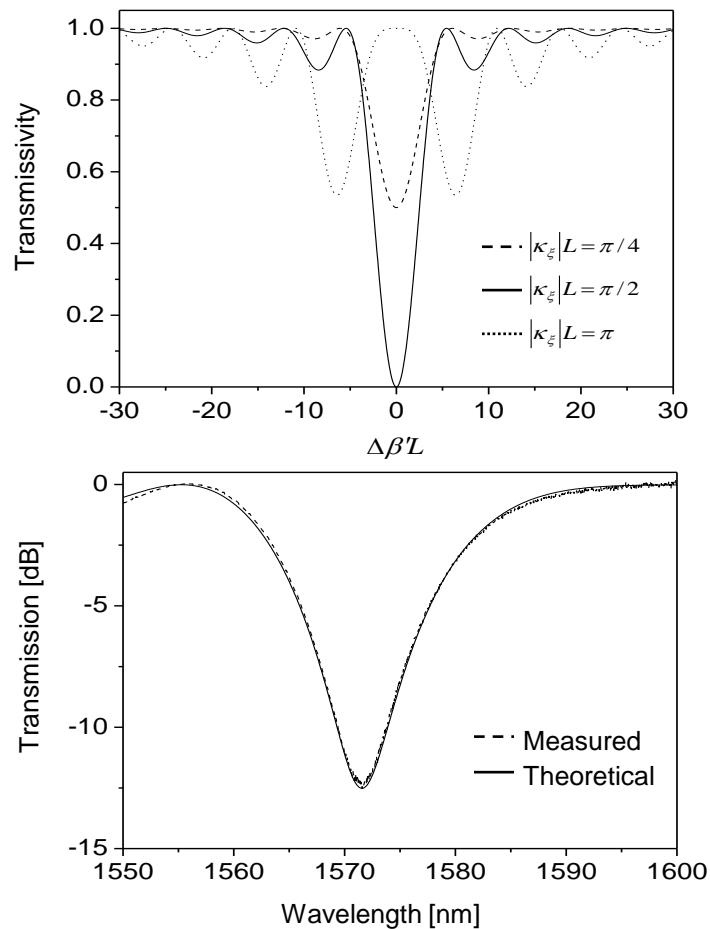
■ **Codirectional Coupling**   ■ **Contradirectional Coupling**
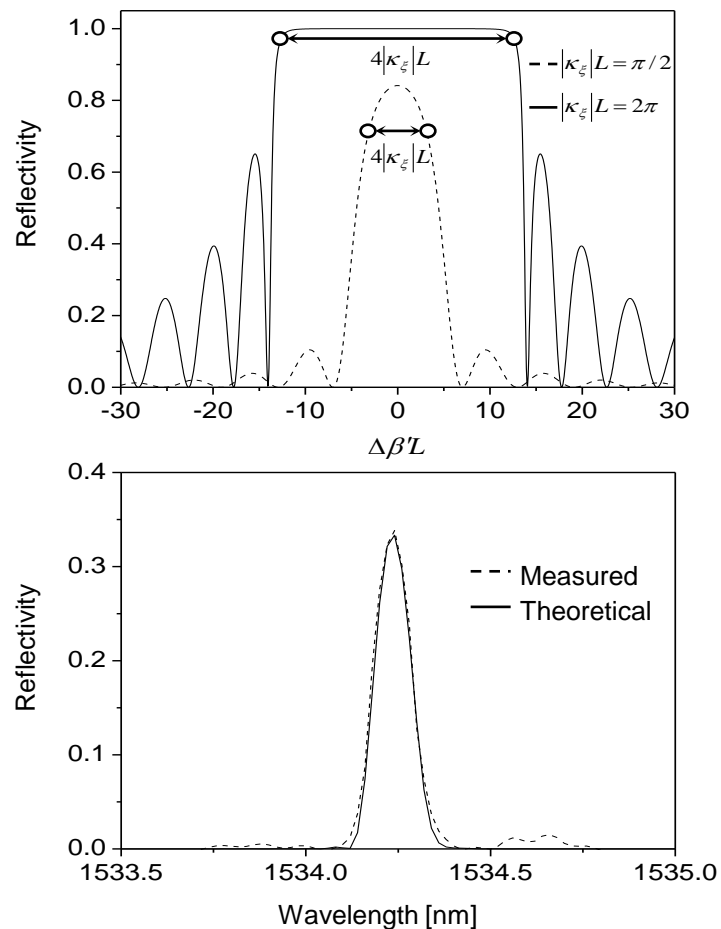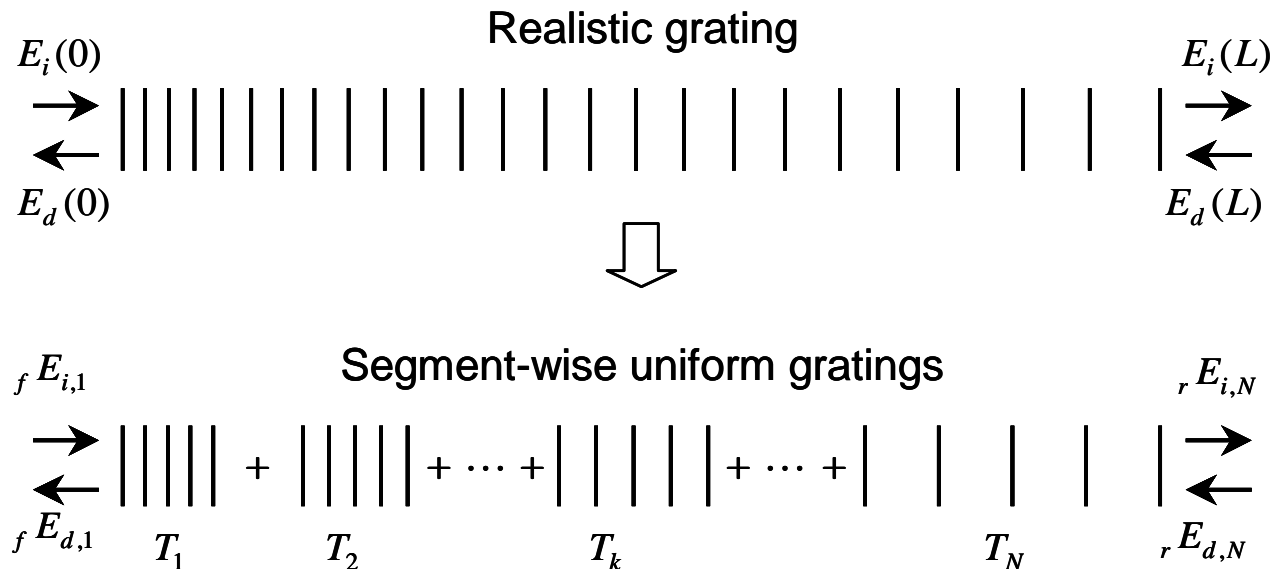
Transfer matrix method (TMM)

■ **Aperiodic Media**

Realistic grating

$E_i(0)$ $E_i(L)$

$E_d(0)$ $E_d(L)$

Segment-wise uniform gratings

$_f E_{i,1}$ $_r E_{i,N}$

$_f E_{d,1}$ T_1 T_2 T_k T_N $_r E_{d,N}$

■ **Formulation**

$$\begin{pmatrix} _r E_{i,N} \\ _r E_{d,N} \end{pmatrix} = \boldsymbol{T}_N \cdots \boldsymbol{T}_{k+1} \cdot \boldsymbol{T}_k \cdots \boldsymbol{T}_1 \cdot \begin{pmatrix} _f E_{i,1} \\ _f E_{d,1} \end{pmatrix}$$

Transfer matrix method

■ Codirectional coupling

$$T_{k,(1,1)} = (\cos \sigma_{f,k} L_k - i \frac{\Delta \beta'_k}{2\sigma_{f,k}} \sin \sigma_{f,k} L_k) \exp[-i(\gamma_{f,k} + \pi / \Lambda_k) L_k]$$

$$T_{k,(1,2)} = -i \frac{\kappa_{\xi,k}}{\sigma_{f,k}} \sin \sigma_{f,k} L_k \exp[-i(\gamma_{f,k} + \pi / \Lambda_k) L_k - i\phi_{gr,k}]$$
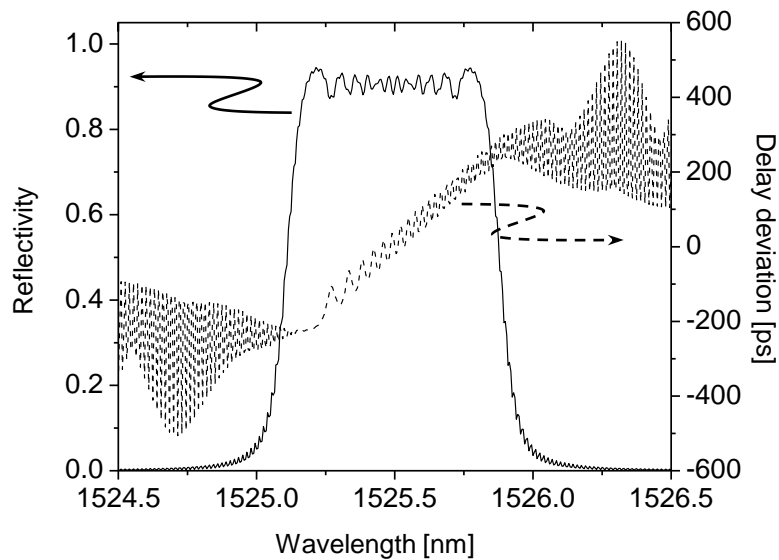
$$T_{k,(2,1)} = -i \frac{\kappa^*_{\xi,k}}{\sigma_{f,k}} \sin \sigma_{f,k} L_k \exp[-i(\gamma_{f,k} - \pi / \Lambda_k) L_k + i\phi_{gr,k}]$$

$$T_{k,(2,2)} = (\cos \sigma_{f,k} L_k + i \frac{\Delta \beta'_k}{2\sigma_{f,k}} \sin \sigma_{f,k} L_k) \exp[-i(\gamma_{f,k} - \pi / \Lambda_k) L_k]$$

■ Contradirectional coupling

$$T_{k,(1,1)} = (\cosh \sigma_{b,k} L_k - i \frac{\Delta \beta'_k}{2\sigma_{b,k}} \sinh \sigma_{b,k} L_k) \exp[-i(\gamma_{b,k} + \pi / \Lambda_k) L_k]$$
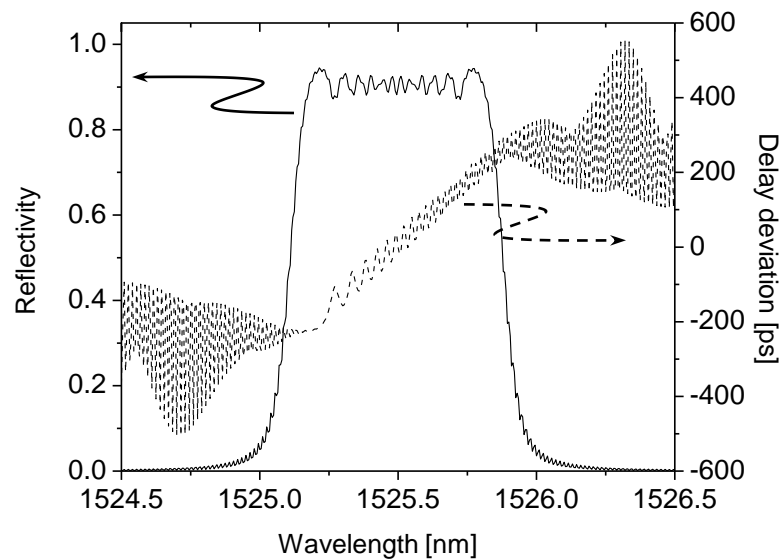
$$T_{k,(1,2)} = -i \frac{\kappa_{\xi,k}}{\sigma_{b,k}} \sinh \sigma_{b,k} L_k \exp[-i(\gamma_{b,k} + \pi / \Lambda_k) L_k - i\phi_{gr,k}]$$

$$T_{k,(2,1)} = +i \frac{\kappa^*_{\xi,k}}{\sigma_{b,k}} \sinh \sigma_{b,k} L_k \exp[-i(\gamma_{b,k} - \pi / \Lambda_k) L_k + i\phi_{gr,k}]$$

$$T_{k,(2,2)} = (\cosh \sigma_{b,k} L_k + i \frac{\Delta \beta'_k}{2\sigma_{b,k}} \sinh \sigma_{b,k} L_k) \exp[-i(\gamma_{b,k} - \pi / \Lambda_k) L_k]$$

Numerical results by TMM
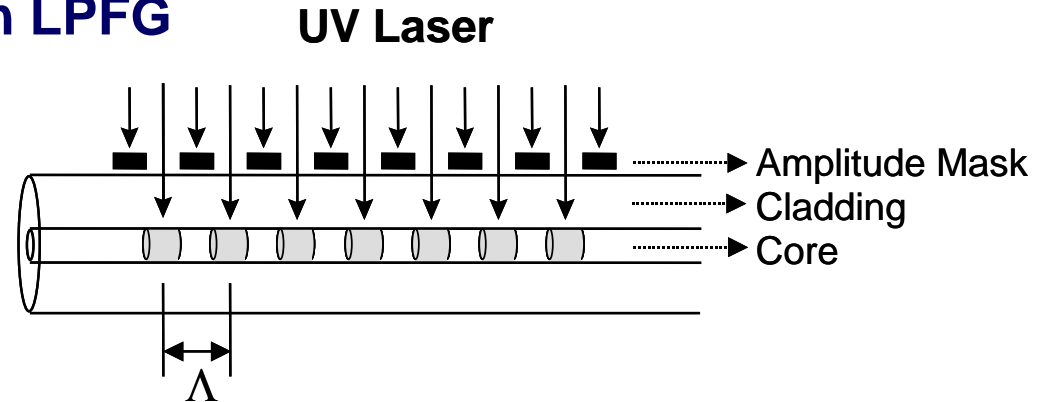
■ **Chirped gratings**
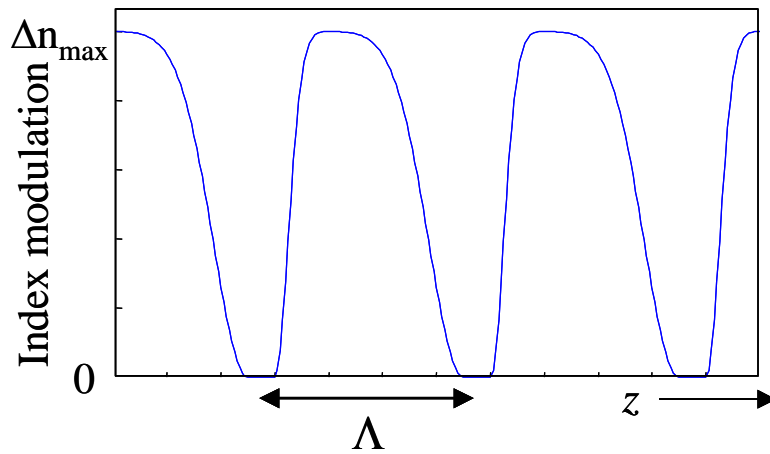
Period modulation

DC-Index modulation

Discretization method

- **Index profile in LPFG**

UV Laser

Amplitude Mask
Cladding
Core

Λ

Real index profile

Δn_{max}

Index modulation

0

Λ

z

Discretized index profile

Δn_{max}

Index modulation

0

Λ

z
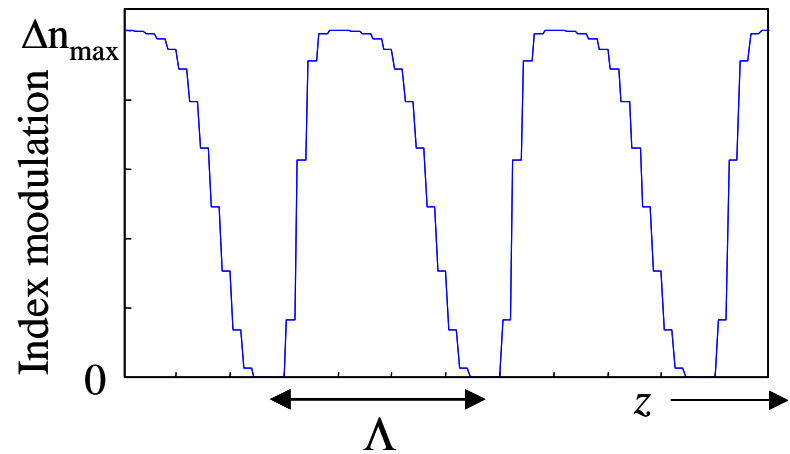
Discretization method

■ **Coupled-mode theory by discretization method**

$$\nabla \cdot (E' \times H_p^* + E_p^* \times H') = -i\omega E_p^* \cdot \Delta\varepsilon_L(m)E', \quad (p = 1,2,3,...)$$

where

$$\boldsymbol{E}_p = \hat{e}_p(r,\phi)\exp(-i\beta_p z) : eigen\ mode,$$

$$\boldsymbol{E}'(m) = \sum_q a_q(z;m)\hat{e}_q(r,\phi)\exp(-i\beta_q z) : perturbed\ mode.$$

In result,

$$\frac{d\boldsymbol{F}_L(z;m)}{dz} = -i\big(\boldsymbol{B} + \boldsymbol{K}(m)\big) \cdot \boldsymbol{F}_L(z;m) \equiv \boldsymbol{D}(m) \cdot \boldsymbol{F}_L(z;m),$$

where

$$F_{L,(p)}(z;m) = a_p(z;m)\exp(-i\beta_p z),$$

$$B_{(p,q)} = \begin{cases} \beta_p & (p = q), \\ 0 & (p \neq q), \end{cases} \qquad K_{(p,q)} = \frac{\omega}{4}\int \hat{e}_p^* \cdot \varepsilon(m)\hat{e}_q\ dS.$$

Discretization method

■ Solution

$$F_L(z_f; m) = V(m) \cdot \exp[U(m) \cdot l(m)] \cdot V(m)^{-1} \cdot F_L(z_i; m) \equiv T(m) \cdot F_L(z_i; m),$$

where 'z_i' and 'z_f' denote the left-end and the right-end positions of the *m*-th section whose length is $l(m)$, respectively, $U(m)$ and $V(m)$ denote the diagonal eigenvalue matrix of $D(m)$ and the corresponding eigenvector matrix, respectively.
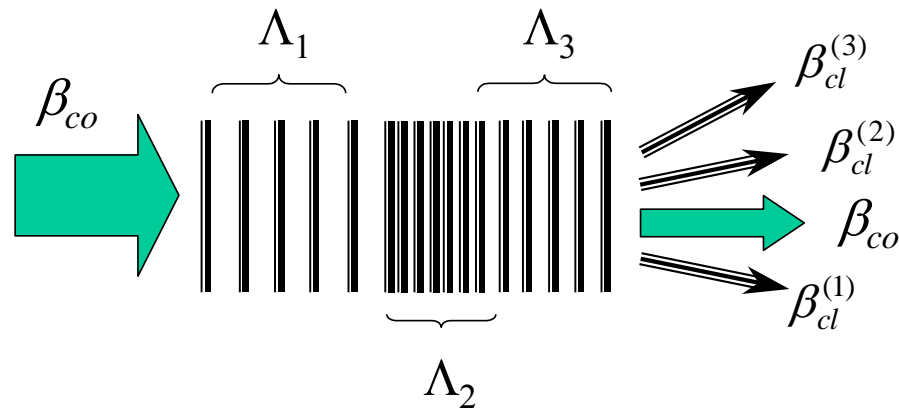
$$F_L^{right} = \cdots T(m+1) \cdot T(m) \cdot T(m-1) \cdots T(1) \cdot F_L^{left}.$$

Therefore, the overall spectral response of the fiber grating can be obtained by cascading each transfer relation of discretized section with initial incident conditions.

Multi-mode coupling

■ **Easy handling with multi-mode coupling**

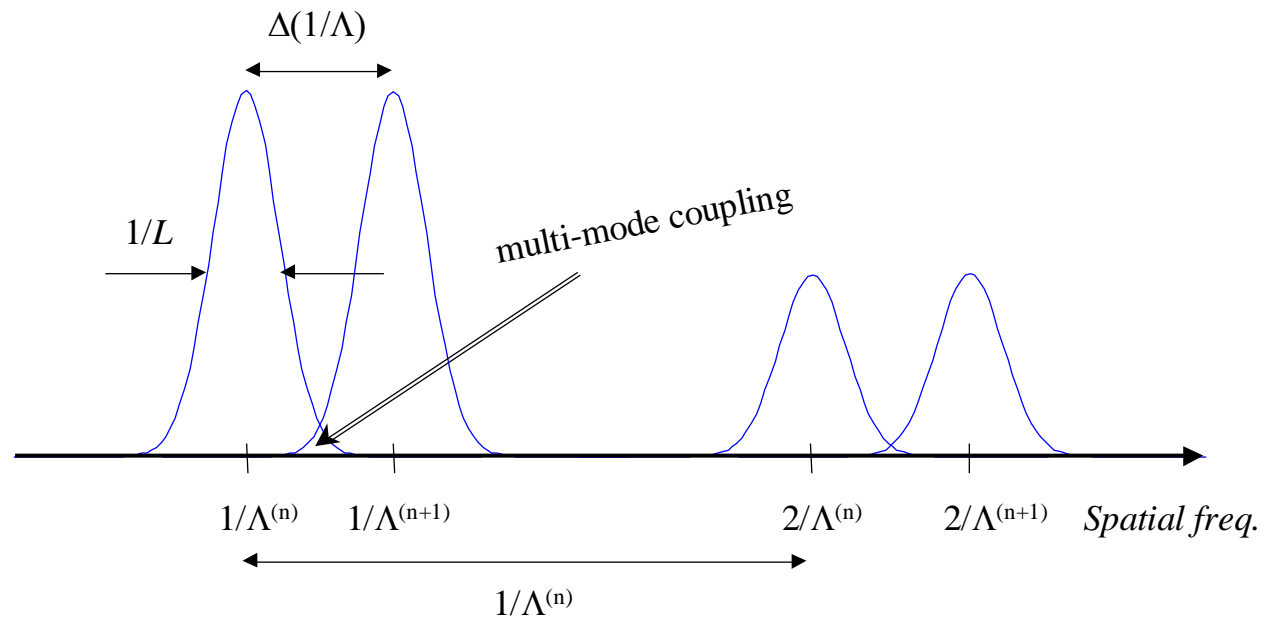Non-uniform long-period grating involving three resonant couplings

cf. **Transfer matrix method**

⇒ Mode selection rule (two-mode coupling)

⇒ Cumbersome with multi-mode couplings

Multi-mode coupling

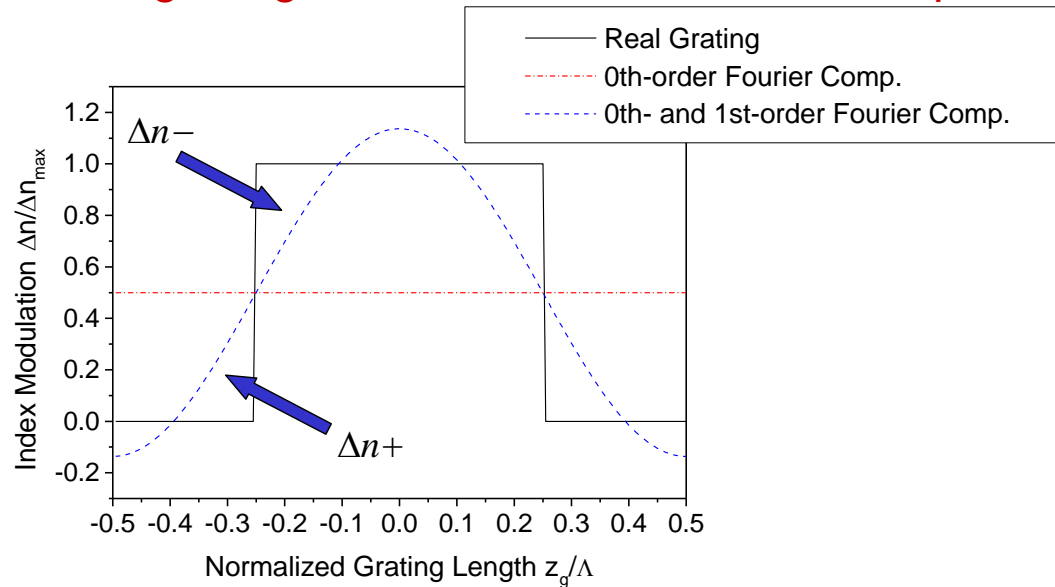Fourier transform of the index modulation having multiple resonance periods

Required multi-mode treatment $\Rightarrow$ *Discretization method*

Nonuniformity consideration

- **Easy handling with non-sinusoidal index modulation**

 A non-sinusoidal grating and its Fourier harmonic components

  Legend:
  - Real Grating
  - 0th-order Fourier Comp.
  - 0th- and 1st-order Fourier Comp.

  Axes: Index Modulation $\Delta n / \Delta n_{max}$ vs. Normalized Grating Length z_g/Λ

  $\Delta n-$

  $\Delta n+$

  cf. **Transfer matrix method**
  $\Rightarrow$ Mode selection rule (two-mode coupling)
  $\Rightarrow$ Single Fourier harmonic

Numerical examples

Comparison in case of a non-sinusoidal grating
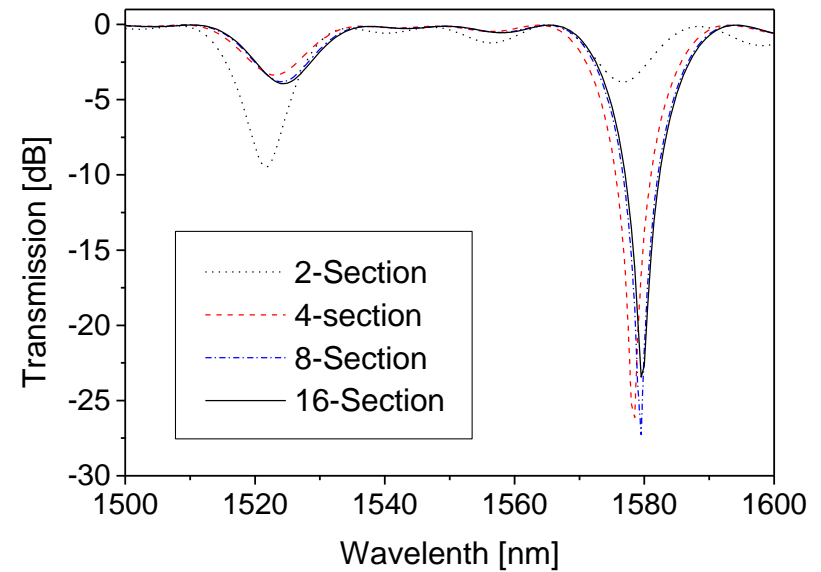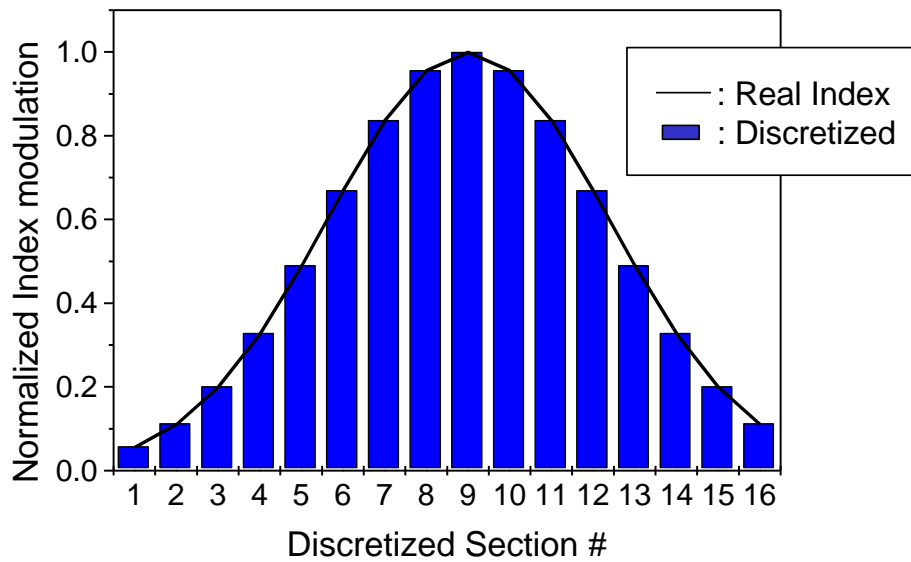

Narrow bandwidth
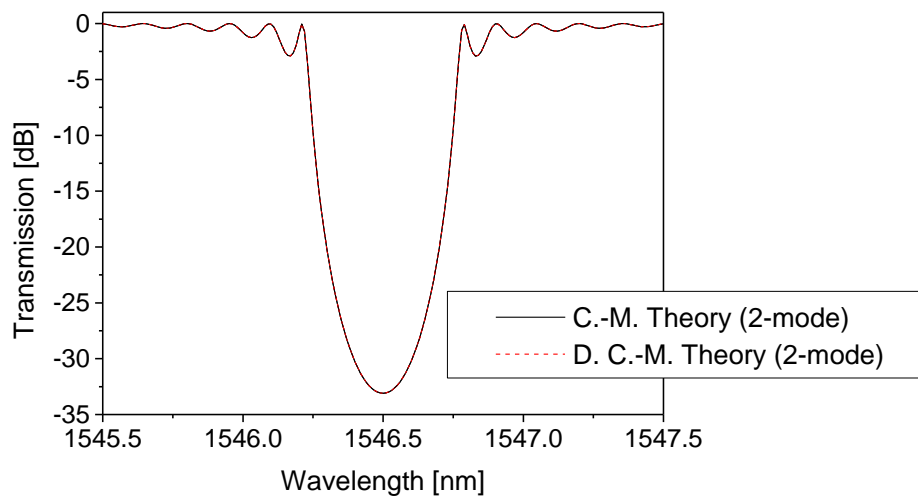
Wide bandwidth
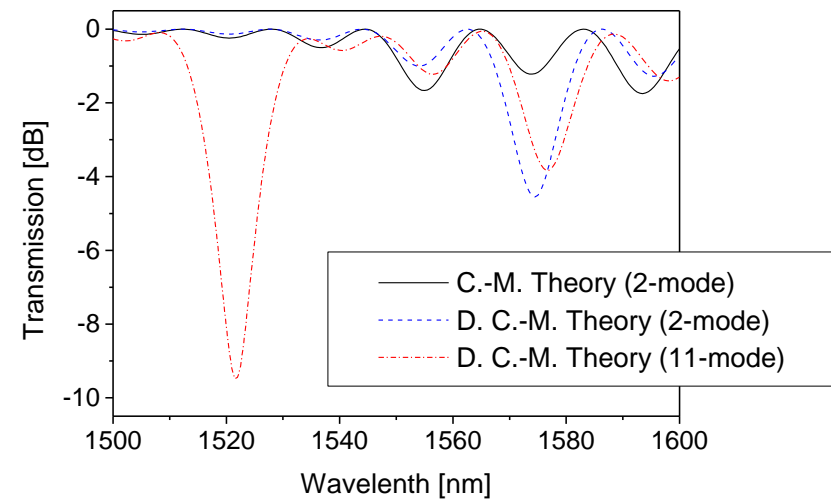
Mode selection rule ⇒ One Fourier harmonic representation
⇐ **Valid only for narrow-bandwidth gratings**

For wide-bandwidth gratings ⇒ Insufficient with one Fourier harmonic.
e.g. multi-mode coupling in LPFGs

Numerical examples

Results for a Gaussian-shape LPFG based on discretizing sections

Numerical examples

Comparison among the coupled-mode theory, transfer matrix method, and discretization method

	Couple-mode theory	Transfer matrix method	Discretization method
Non-uniform grating	No	Yes	Yes
Calculation time for non-uniform grating	-	Relatively fast	Relatively slow
Multi-mode coupling	No	No	Yes
Mode selection rule	Required	Required	Not required
Efficient area	Uniform grating	Non-uniform gratings in two-mode coupling regime	Non-uniform grating in multi-mode coupling regimes

Nonlinear perturbation

- **Coupled-mode theory with nonlinear perturbation terms**

  With second-order nonlinearity

  $$\nabla \cdot (\boldsymbol{E}'_\omega \times \boldsymbol{H}^*_{\omega,p} + \boldsymbol{E}^*_{\omega,p} \times \boldsymbol{H}'_\omega) = -i\omega \boldsymbol{E}^*_{\omega,p} \cdot \Delta \boldsymbol{P}'_\omega, \quad (p = 1, 2, \ldots),$$

  $$\Delta \boldsymbol{P}'_{\omega_1} = \Delta \varepsilon_{ij}(\omega_1) E'_{\omega_1,j} + 2 d_{ijk}(-\omega_1, \omega_2, -\omega_1) E'_{\omega_2,j} E'^*_{\omega_1,k}$$
  $$+ \left\{ 3\chi_{ijkl}(-\omega_1, \omega_1, -\omega_1, \omega_1) E'_{\omega_1,j} E'^*_{\omega_1,k} + 6\chi_{ijkl}(-\omega_1, \omega_2, -\omega_2, \omega_1) E'_{\omega_2,j} E'^*_{\omega_2,k} \right\} E'_{\omega_1,l},$$

  $$\Delta \boldsymbol{P}'_{\omega_2} = \Delta \varepsilon_{ij}(\omega_2) E'_{\omega_2,j} + d_{ijk}(-\omega_2, \omega_1, \omega_1) E'_{\omega_1,k} E'_{\omega_1,k}$$
  $$+ \left\{ 3\chi_{ijkl}(-\omega_2, \omega_2, -\omega_2, \omega_2) E'_{\omega_2,j} E'^*_{\omega_2,k} + 6\chi_{ijkl}(-\omega_2, \omega_1, -\omega_1, \omega_2) E'_{\omega_1,j} E'^*_{\omega_1,k} \right\} E'_{\omega_2,l},$$

  Without second-order nonlinearity

  $$\nabla \cdot (E' \times H^*_p + E^*_p \times H') = -i\omega E^*_p \cdot (\Delta \varepsilon_L + \Delta \varepsilon_{NL}) E', \quad (p = 1, 2, 3, \ldots)$$

  $$\Delta \varepsilon_{NL(q)} = \varepsilon_o \frac{3}{4} \chi^{(3)}(r, \phi, z) \cdot \sum_s \alpha_{(q,s)} \left| E_s(t,z) \right|^2 \left| \hat{e}_s \right|^2, \quad \alpha_{(q,s)} = \begin{cases} 1 & (q = s), \\ 2 & (q \neq s), \end{cases}$$

Nonlinear pulse propagation

■ **Wave packet representation**

$$\frac{d\boldsymbol{E}(t,z',m)}{dz'} = \frac{d}{dz'}\int \boldsymbol{F}(\Omega,z';m)\exp(i\Omega t)d\Omega$$

$$= \int \boldsymbol{D}(\Omega,m)\cdot\boldsymbol{F}(\Omega,z';m)\exp(i\Omega t)d\Omega - i\boldsymbol{X}(t,z';m)\cdot\boldsymbol{E}(t,z',m),$$

where

$$X_{(p,q)}(t,z';m) = \frac{\omega_c}{4}\int \hat{e}_p^* \cdot \Delta\varepsilon_{NL(q)}(t,z';m)\ \hat{e}_q\ dS.$$

Nonlinear pulse propagation

■ **Solution by Fourier transform method**

$$F(\Omega, z'; m) = T(\Omega, z'; m) \cdot \Phi(\Omega, z'; m),$$

where

$$\Phi(\Omega, z' = 0; m) = \frac{1}{2\pi} \int E(t, z' = 0; m) \exp(-i\Omega t) dt.$$

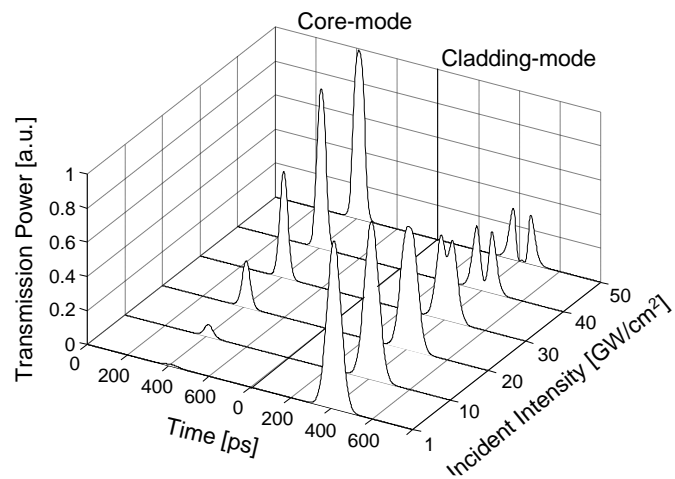$$G(\Omega, z'; m) = \frac{1}{2\pi} \int T^{-1}(\Omega, z'; m) \cdot X(t, z'; m) \cdot E(t, z'; m) \exp(-i\Omega t) dt.$$
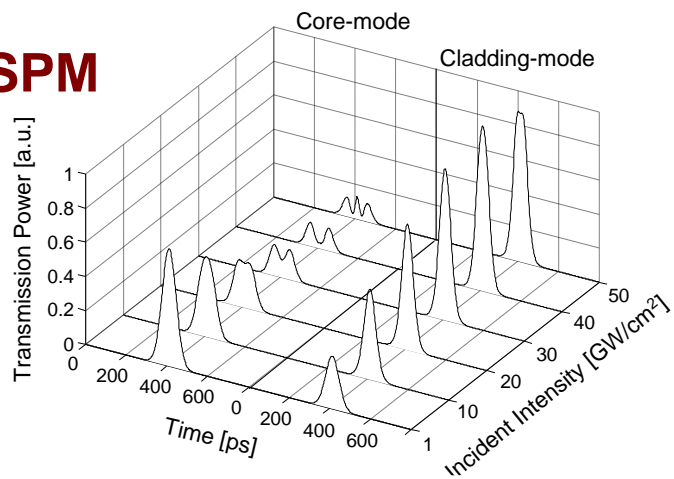
In result,

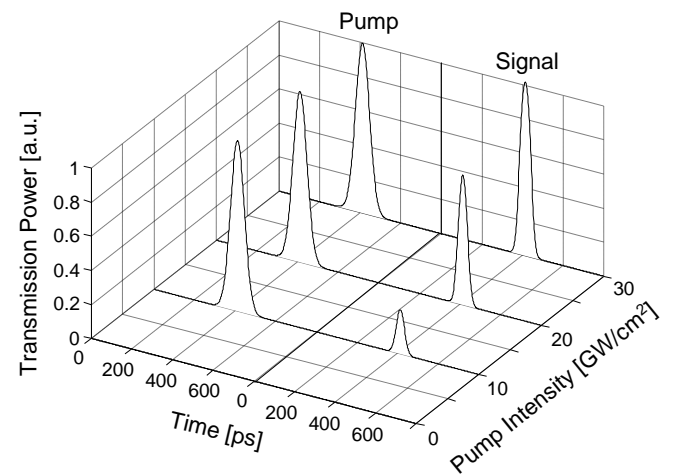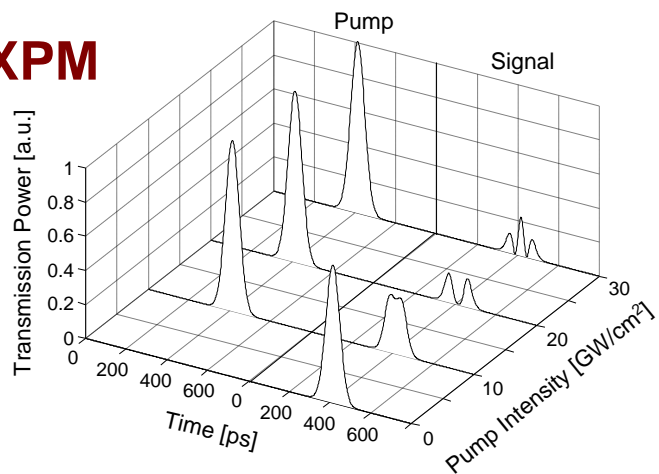$$\frac{d\Phi(\Omega, z'; m)}{dz'} = -iG(\Omega, z'; m),$$

which can be solved by fast Fourier transform method with a predictor-corrector scheme.

Numerical examples

■ SPM

■ XPM

Volume holographic gratings

■ **Born and paraxial approximation** $\Rightarrow$ **Single diffraction**

■ **Coupled-mode theory for volume holographic gratings**
- Applicable to arbitrary cases of multiplexed holograms
- Continuous spectrum $\Rightarrow$ discrete spectrum (FFT)

$$\nabla \cdot (E' \times H_p^* + E_p^* \times H') = -i\omega E_p^* \cdot \Delta\varepsilon E', \quad (p = 1, 2, \ldots)$$

$$\boldsymbol{E}'(z) = \sum_q a_q(z)\boldsymbol{e}_q(x, y)\exp(-i\beta_q z)$$

$$\Rightarrow \quad \boldsymbol{E}'(z) = \int a(k_x', k_y', k_z'; z)\exp[-i(k_x'x + k_y'y + k_z'z)]dk_x'dk_y'$$

$$\frac{da(k_x, k_y, k_z'; z)}{dz} = -i\frac{\omega^2\mu}{2k_z}\int a(k_x', k_y', k_z'; z)G(k_x', k_y', k_x, k_y; z)\exp[-i(k_z' - k_z)z]dk_x'dk_y'$$

Numerical examples

Uniform Phase

Random Phase