

Nonlinear pulse propagation in optical fibres

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Maxwell's equation

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

Constitutive relations for E-field

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P},$$

$$\mathbf{P} = \varepsilon_0 \chi \mathbf{E} = \varepsilon_0 \left(\chi^{(1)} \mathbf{E} + \chi^{(2)} \mathbf{E} \mathbf{E} + \chi^{(3)} \mathbf{E} \mathbf{E} \mathbf{E} + \dots \right)$$

Let $\mathbf{J} = 0$, $\rho = 0$, $\mu = \mu_0$ and $\chi^{(2)} = 0$.

Recall the coupled-mode equations derived from Lorentz reciprocity theorem

$$\nabla \cdot (\mathbf{E}' \times \mathbf{H}_p^* + \mathbf{E}_p^* \times \mathbf{H}') = -i\omega \mathbf{E}_p^* \Delta \varepsilon \mathbf{E}' \quad (p=1, 2, 3, \dots)$$

Dispersion relation

Mode-propagation constant β in a Taylor series:

$$\beta(\omega) = n(\omega) \frac{\omega}{c} = \beta_0 + \beta_1(\omega - \omega_0) + \frac{1}{2} \beta_2(\omega - \omega_0)^2 + \dots,$$

$$\text{where } \beta_m = \left(\frac{d^m \beta}{d\omega^m} \right)_{\omega=\omega_0} \quad (m = 1, 2, 3, \dots)$$

Wave-packet representation

Summation of eigen waves:

$$E(\mathbf{r}, t) = \int \Phi(\omega) \underline{F(x, y, \omega) \exp[i(\omega t - \beta z)]} d\omega$$

Mode-propagation constant β in a Taylor series:

$$\beta(\omega) = n(\omega) \frac{\omega}{c} = \beta_0 + \beta_1(\omega - \omega_0) + \frac{1}{2} \beta_2(\omega - \omega_0)^2 + \dots,$$

$$\text{where } \beta_m = \left(\frac{d^m \beta}{d\omega_m} \right)_{\omega=\omega_0} \quad (m = 1, 2, 3, \dots)$$

In result:

$$\begin{aligned} E(\mathbf{r}, t) &= \int \Phi(\Omega) F(x, y, \Omega) \exp[i(\Omega t - Bz)] d\Omega \cdot \exp[i(\omega_0 t - \beta_0 z)] \\ &\approx \int \Phi(\Omega) \exp(-iBz) \exp(i\Omega t) d\Omega \cdot \underline{F_0(x, y) \exp[i(\omega_0 t - \beta_0 z)]} \\ &= \int \underline{\tilde{A}(z, \Omega) \exp(i\Omega t)} d\Omega \cdot F_0(x, y) \exp[i(\omega_0 t - \beta_0 z)] \\ &= \underline{A(z, t)} \cdot \underline{F_0(x, y) \exp[i(\omega_0 t - \beta_0 z)]}, \\ &\quad \text{Envelop} \qquad \text{Carrier} \end{aligned}$$

$$\text{where } \Omega = \omega - \omega_0, \quad B = \beta - \beta_0 = \beta_1 \Omega + \frac{\beta_2}{2} \Omega^2 + \dots.$$

Envelop evolution

For linear & lossless dispersive media:

$$\begin{aligned}
 A(z,t) &= \int \tilde{A}(z, \Omega) \exp(i\Omega t) d\Omega \\
 &= \int \Phi(\Omega) \exp[i(\Omega t - Bz)] d\Omega \\
 &= \int \Phi(\Omega) \exp[i\Omega t - i(\beta_1 \Omega + \frac{\beta_2}{2} \Omega^2 + \dots)z] d\Omega
 \end{aligned}$$

$$\rightarrow \frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} - \frac{i\beta_2}{2} \frac{\partial^2 A}{\partial t^2} + \dots = 0$$

For nonlinear & lossy dispersive media:

$$\begin{aligned}
 A(z,t) &= \int \tilde{A}(z, \Omega) \exp(i\Omega t) d\Omega \\
 &= \int \Phi(\Omega) \exp[i(\Omega t - Bz) - \frac{\alpha}{2} z - i\gamma |A|^2 z] d\Omega \\
 &= \int \Phi(\Omega) \exp[i\Omega t - i(\beta_1 \Omega + \frac{\beta_2}{2} \Omega^2 + \dots)z - \frac{\alpha}{2} z - i\gamma |A|^2 z] d\Omega
 \end{aligned}$$

Loss Nonlinear phase shift due to $\chi^{(3)}$

$$\rightarrow \frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} - \frac{i\beta_2}{2} \frac{\partial^2 A}{\partial t^2} + \frac{\alpha}{2} A + \dots = -i\gamma |A|^2 A$$

Derivation of nonlinear Schrödinger equation (NLSE)

Perturbed
field
↓

$$\tilde{E}'(z, \omega) = \tilde{A}(z, \Omega) \cdot F_0(x, y) \exp(-i\beta_0 z)$$

Slowly
varying
amplitude

Eigen mode

$$\beta(\omega) = n(\omega) \frac{\omega}{c} = \beta_0 + \beta_1(\omega - \omega_0) + \frac{1}{2} \beta_2(\omega - \omega_0)^2 + \dots$$

$$= \beta_0 + \Delta\beta_L, \text{ where } \beta_0 = n(\omega_0) \frac{\omega_0}{c}$$

$$\rightarrow n(\omega) = n(\omega_0) + \Delta n_L$$

$$\Delta n_L \approx \frac{c}{\omega_0} \Delta\beta_L$$

$$\Delta\epsilon_L = \epsilon_0 n^2(\omega) - \epsilon_0 n^2(\omega_0)$$

$$= \epsilon_0 [n(\omega_0) + \Delta n_L]^2 - \epsilon_0 n^2(\omega_0)$$

$$\approx 2\epsilon_0 n(\omega_0) \Delta n_L = \frac{2\epsilon_0 n(\omega_0) c}{\omega_0} \Delta\beta_L$$

If nonlinear perturbation involved:

$$n = n_L + n_2 I$$

$$\Delta \varepsilon_{NL} \approx 2\varepsilon_0 n(\omega_0) n_2 I = 2\varepsilon_0 n(\omega_0) n_2 \cdot \frac{1}{2} \varepsilon_0 c n(\omega_0) |A|^2 |F_0(x, y)|^2$$

$$= \varepsilon_0^2 c n^2(\omega_0) n_2 |A|^2 |F_0(x, y)|^2, \text{ where } n_2 = \frac{3\chi^{(3)}}{4\varepsilon_0 c n^2}$$

$$= \varepsilon_0 \frac{3}{4} \chi^{(3)} |A|^2 |F_0(x, y)|^2$$

Recall the coupled-mode equations!

Without second-order nonlinearity

$$\nabla \cdot (E' \times H_p^* + E_p^* \times H') = -i\omega E_p^* \cdot \Delta \varepsilon E', \quad (p = 1, 2, 3, \dots)$$

$$\Delta \varepsilon_{NL(q)} = \varepsilon_o \frac{3}{4} \chi^{(3)}(x, y, z) \cdot \sum_s \alpha_{(q,s)} |E_s(t, z)|^2 |\hat{e}_s|^2, \quad \alpha_{(q,s)} = \begin{cases} 1 & (q = s), \\ 2 & (q \neq s), \end{cases}$$

$$\frac{da_p(z)}{dz} \exp(-i\beta_p z) = -i \frac{\omega}{4} \sum_q a_q(z) \exp(-i\beta_q z) \int \mathbf{e}_p^*(x, y) \cdot (\Delta \varepsilon_L + \Delta \varepsilon_{NL}) \mathbf{e}_q(x, y) dx dy$$

For each Fourier component:

$$\rightarrow \frac{\partial \tilde{A}(z, \Omega)}{\partial z} = -i \frac{\omega_0}{4} \tilde{A} \cdot \int |F_0(x, y)|^2 (\Delta \varepsilon_L + \Delta \varepsilon_{NL}) dx dy$$

$$\text{where } \frac{\varepsilon_o cn(\omega_0)}{2} \int |F_0(x, y)|^2 dx dy = 1.$$

$$\begin{aligned}
\frac{\partial \tilde{A}}{\partial z} &= -i \frac{\omega_0}{4} \tilde{A} \cdot \int |F_0(x, y)|^2 (\Delta \varepsilon_L + \Delta \varepsilon_{NL}) dx dy \\
&= -i \frac{\omega_0}{4} \tilde{A} \cdot (\Delta \varepsilon_L \int |F_0(x, y)|^2 dx dy + \varepsilon_0 \frac{3}{4} \chi^{(3)} |A|^2 \int |F_0(x, y)|^4 dx dy) \\
&= -i \frac{\omega_0}{4} \tilde{A} \cdot \left(\Delta \varepsilon_L \frac{2}{\varepsilon_0 c n} + \varepsilon_0 \frac{3}{4} \chi^{(3)} |A|^2 \frac{4}{\varepsilon_0^2 c^2 n^2} \frac{1}{A_{eff}} \right) \quad \leftarrow A_{eff} = \frac{(\int |F_0(x, y)|^2 dx dy)^2}{\int |F_0(x, y)|^4 dx dy} \\
&= -i \frac{\omega_0}{4} \Delta \varepsilon_L \frac{2}{\varepsilon_0 c n} \tilde{A} - i \frac{\omega_0}{4} \varepsilon_0 \frac{3}{4} \chi^{(3)} \frac{4}{\varepsilon_0^2 c^2 n^2} \frac{1}{A_{eff}} |A|^2 \tilde{A} \\
&= -i \frac{\omega_0}{4} \frac{2 \varepsilon_0 n(\omega_0) c}{\omega_0} \frac{2 c \mu_0}{n} \Delta \beta_L \tilde{A} - i \frac{\omega_0}{4} \varepsilon_0 \frac{3}{4} \chi^{(3)} \frac{4}{\varepsilon_0^2 c^2 n^2} \frac{1}{A_{eff}} |A|^2 \tilde{A} \\
&= -i \Delta \beta_L \tilde{A} - i \frac{3 \omega_0 \chi^{(3)}}{4 \varepsilon_0 c^2 n^2 A_{eff}} |A|^2 \tilde{A} \\
&= -i \Delta \beta_L \tilde{A} - i \frac{n_2 \omega_0}{c A_{eff}} |A|^2 \tilde{A} \quad \leftarrow \gamma = \frac{n_2 \omega_0}{c A_{eff}} \\
&= -i \Delta \beta_L \tilde{A} - i \gamma |A|^2 \tilde{A}
\end{aligned}$$

Transform into the time domain via inverse Fourier transform:

$$\begin{aligned}\frac{\partial \tilde{A}(z, \Omega)}{\partial z} &= -i\Delta\beta_L \tilde{A}(z, \Omega) - i\gamma|A|^2 \tilde{A}(z, \Omega) \\ &= -i(\beta_1\Omega + \frac{\beta_2}{2}\Omega^2 + \dots)\tilde{A}(z, \Omega) - i\gamma|A|^2 \tilde{A}(z, \Omega) \\ \Omega &\rightarrow -i\frac{\partial}{\partial t}\end{aligned}$$

Nonlinear Schrödinger equation:

$$\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} - i\frac{\beta_2}{2} \frac{\partial^2 A}{\partial t^2} + \dots = -i\gamma|A|^2 A \leftarrow \text{for lossless media}$$

or

$$\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} - i\frac{\beta_2}{2} \frac{\partial^2 A}{\partial t^2} + \frac{\alpha}{2} A \dots = -i\gamma|A|^2 A \leftarrow \text{for lossy media}$$

Nonlinear pulse propagation

NLSE:

$$\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} - i \frac{\beta_2}{2} \frac{\partial^2 A}{\partial t^2} + \frac{\alpha}{2} A = -i\gamma |A|^2 A$$

$$\begin{aligned} \text{Let } A(z, t) &= \int \Phi(z, \Omega) \exp\left\{ \left[-i\left(\beta_1 \Omega + \frac{\beta_2}{2} \Omega^2\right) - \frac{\alpha}{2} \right] z \right\} \exp(i\Omega t) d\Omega \\ &= \int T(z, \Omega) \cdot \Phi(z, \Omega) \exp(i\Omega t) d\Omega, \end{aligned}$$

$$\text{where } \Phi(0, \Omega) = \frac{1}{2\pi} \int A(0, t) \exp(-i\Omega t) dt$$

$$\text{Let } \gamma |A|^2 A = \int T(z, \Omega) G(z, \Omega) \exp(i\Omega t) d\Omega,$$

$$\text{where } G(z, \Omega) = \frac{1}{2\pi} \int T^{-1}(z, \Omega) \gamma |A|^2 A \exp(-i\Omega t) dt$$

$$\text{In result: } \frac{\partial}{\partial z} \Phi(z, \Omega) = -iG(z, \Omega),$$

which can be solved by the fast Fourier transform method in a predictor-corrector scheme.

Predictor-corrector scheme

For a first-order differential equation:

$$\frac{dy}{dz} = f(z, y),$$

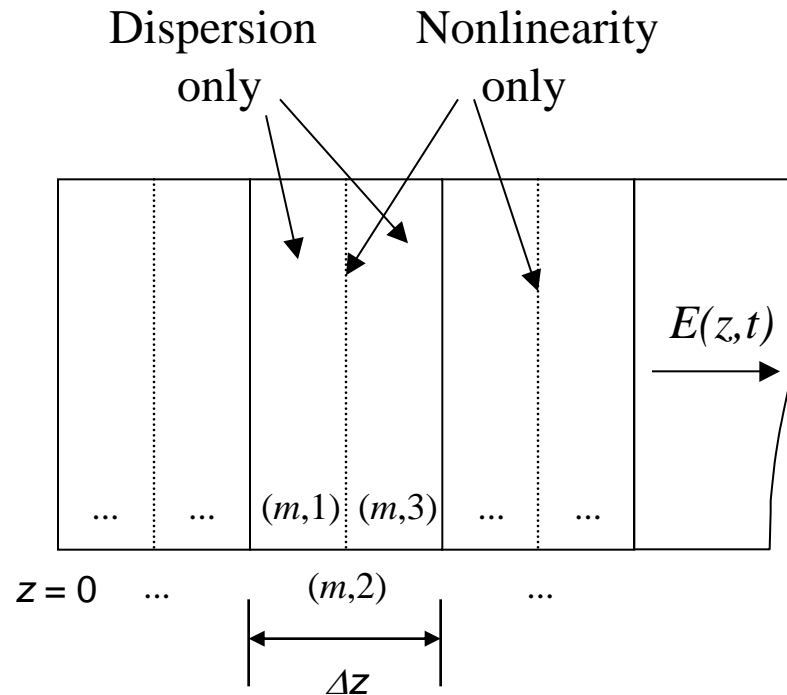
$$y_p(z + \Delta z) = y(z) + f[z, y(z)]\Delta z,$$

$$y(z + \Delta z) = y(z) + \frac{1}{2}\{f[z, y(z)] + f[z, y_p(z + \Delta z)]\}\Delta z.$$

The value predicted by an initial rough estimation is corrected by iterations.

Split-step Fourier method

Fibre length is divided into a large number of segments of with Δz . Within a segment, the effect of nonlinearity is included at the midplane.



$$\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} - i \frac{\beta_2}{2} \frac{\partial^2 A}{\partial t^2} + \frac{\alpha}{2} A = -i\gamma |A|^2 A$$

$$\left. \begin{array}{l} \frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} - i \frac{\beta_2}{2} \frac{\partial^2 A}{\partial t^2} + \frac{\alpha}{2} A = 0 \\ \frac{\partial A}{\partial z} = -i\gamma |A|^2 A \end{array} \right\}$$

Nonlinear phase shift

Recall NLSE:

$$\begin{aligned} A(z, t) &= \int \tilde{A}(z, \Omega) \exp(i\Omega t) d\Omega \\ &= \int \Phi(\Omega) \exp\left[i(\Omega t - Bz) - \frac{\alpha}{2} z - i\gamma |A|^2 z\right] d\Omega \\ &= \int \Phi(\Omega) \exp\left[i\Omega t - i(\beta_1 \Omega + \frac{\beta_2}{2} \Omega^2 + \dots)z - \frac{\alpha}{2} z - i\gamma |A|^2 z\right] d\Omega \\ &\rightarrow \frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} - \frac{i\beta_2}{2} \frac{\partial^2 A}{\partial t^2} + \frac{\alpha}{2} A + \dots = -i\gamma |A|^2 A \end{aligned}$$

Loss (blue arrow pointing to $-\frac{\alpha}{2} z$)

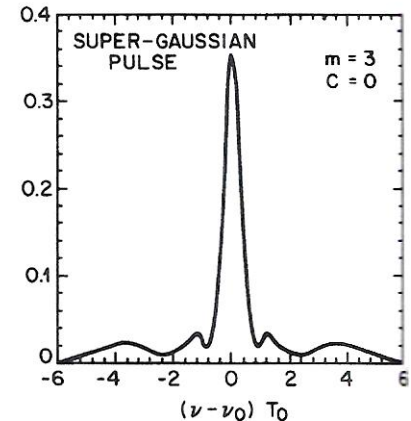
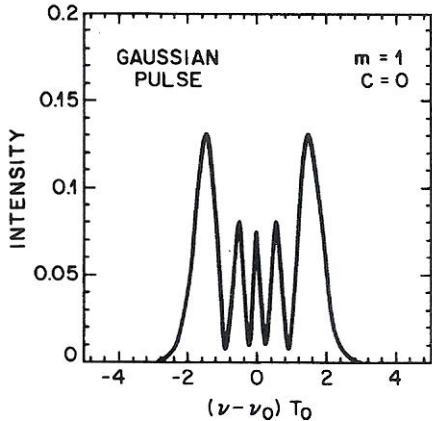
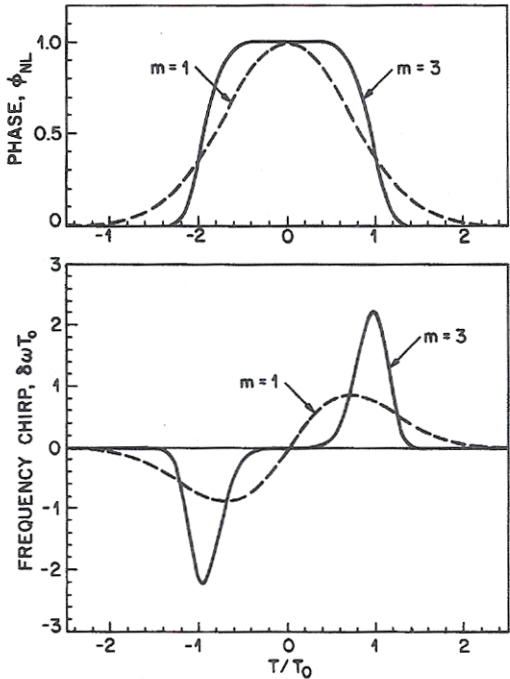
Nonlinear phase shift due to $\chi^{(3)}$ (red arrow pointing to $-i\gamma |A|^2 z$)

Nonlinear phase shift:

$$\begin{aligned} \phi_{NL}(z, t) &= -\gamma |A(z, t)|^2 z \\ \delta\omega(z, t) &= \frac{\partial}{\partial t} \left(-\gamma |A(z, t)|^2 z \right) \end{aligned}$$

Self-phase modulation (SPM)

For Gaussian and super-Gaussian pulses:



$$\phi_{NL,max} = 4.5\pi$$

Frequency chirp incurred

Source: Nonlinear Fiber Optics, G. P. Agrawal