3 Vector Space, Linear Independence

3.1 Vector Space

Definition. Vector Space : a nonempty set V of vectors such that with any two vectors all their linear combinations $C_1a + c_2b \in V(c_1, c_2 =$ any real numbers), and all these vectors obey the following rules.

(a) $a + b = b + a$.

(b) $a + b) + c = a + (b + c) = a + b + c$.

(c) There is a unique zero vector such that $a + 0 = a$ for all a.

(d) For each a, there is a unique vector-a such that $a + (-a) = 0$.

(e) $1a = a$.

(f) $c_1(c_2a) = (c_1c_2)a = c_1c_2a$.

(g) $c_1(a + b) = c_1a + c_2b$.

(h) $(c_1 + c_2)a = c_1a + c_2a$.

Example 1.

$\mathbb{R}^n$, The vector space that consists only of a zero vector, the vector space of all real n by n matrices, ...

Definition. Subspace of a vector space is a set of vectors (including 0) that satisfies the following. If v and w are vectors in the subspace and c_1 is any scalar, then $v + w$, cv is in the subspace.

Example 2. three-dimensional space $\mathbb{R}^3$.

- A plane through (0,0,0) is a subspace of the full vector space $\mathbb{R}^3$.
- A line through (0,0,0)
- The single vector (0,0,0)
- The whole space $\mathbb{R}^3$

Definition. The column space of a matrix A : $C(A)$

= all linear combinations of the columns

= span of the columns

Note

- The combinations are all possible vectors Ax.
- To solve $Ax = b$, b needs to be a combination of the columns.

$$\therefore Ax = b \text{ is solvable iff } b \in C(A)$$

- If A is m-by-n, columns belong to $\mathbb{R}^m$.

$$\therefore C(A) \text{ is a subspace of } \mathbb{R}^m.$$

Example 3.

$$A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix} \qquad Ax = x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

$C(A) =$ plane in $\mathbb{R}^3$.

- $Ax = b$ is solvable when b is on that plane.
- $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is in $C(A)$ $\quad \therefore Ax = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ solvable.

Example 4. In $\mathbb{R}^2$,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

- $C(I) = \mathbb{R}^2$ $\quad$ ($Ax = b$ always solvable)
- $C(A) =$ all linear combinations of $c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

  $= c_0 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad (c_0 \in \mathbb{R})$

  $\therefore C(A)$ is only a line.

  ($Ax = b$ solvable only when b is on the line.)
- $C(B) = \mathbb{R}^2$ $\quad$ (every b is attainable)

  But $Bx = b = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ has multiple solutions.

  (2 eqns, 3 unknowns)

Definition. The row space of a matrix A : $R(A)$

= all linear combinations of the vectors

= span of the rows

- If A is m-by-n, rows belong to $\mathbb{R}^n$.

  $\therefore$ The row space is a subspace of $\mathbb{R}^n$.

Note

- The row space of $A = C(A^T)$

Definition. The null space of a matrix $A : N(A)$
$= \text{all solutions to } Ax = 0$

- If A is m-by-n, the solution vectors x are in $\mathbb{R}^n$.
  $\therefore N(A)$ is a subspace of $\mathbb{R}^n$.
- Consider $Ax = b$.
  If $b \neq 0$, then the solutions do not form a subspace.
  ($\because x = 0$ is only a solution if $b = 0$.)

Example 5.

$$A = [\,1 \quad 2 \quad 3\,]$$

- $N(A)$ is the plane through the origin

$$x + 2y + 3z = 0$$

- $s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $s_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ lie on the plane

$$x + 2y + 3z = 0$$

All vectors on the plane are combinations of s_1 and s_2.

Example 6.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \Longrightarrow U = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} : \text{ all columns have pivots.}$$
$$N(A) = \{(0,0)\}$$

Example 7.

$$B = \begin{bmatrix} A \\ 2A \end{bmatrix}$$

Extra rows impose more conditions on the vectors x in the nullspace, i.e., $N(B) = \{(0,0)\}$

Example 8.

$$\mathrm{C} = [\ \mathrm{A} \quad \mathrm{B}\ ] = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}$$

Now the solution vector x has 4 components.

$$\Longrightarrow U = \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}}_{\text{pivot cols}} \hspace{-4pt} \underbrace{\begin{bmatrix} 2 & 4 \\ 0 & 4 \end{bmatrix}}_{\text{free cols}}$$

$$\Longrightarrow R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \text{(reduced row echelon form)}$$

Special solutions to $Rx = 0$:

$$s_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad s_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

(The free variables $x_3,\ x_4$ can be given any values whatsoever. Then the pivot variables $x_1,\ x_2$ can be found by back substitution.)

All solutions are linear combination of s_1 and s_2.

Note

- If A has more columns than rows, then $Ax = 0$ has more unknowns than equations, and it has nonzero solutions. (There must be free columns without pivots.)

$$m \underbrace{\begin{bmatrix} \quad & \quad & \quad \\ & & \end{bmatrix}}_{n} \begin{bmatrix} \ \\ \ \\ \ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- $N(A) = N(U) = N(R)$

3.2 Linear Independence

Definition. Vectors $v_1, \cdots, v_n$ are the linearly independent.
if. $x_1 v_1 + \cdots + x_n v_n = 0$ only happens when all x's are zero.
(If a combination is 0 when the x's are not all zero, the vectors are dependent.)

- The columns of A are linear independent when the only solution to $Ax = 0$ is $x = 0$.
- The columns of $A \in \mathbb{R}^{m \times n}$ are linear independent when the rank is $r = n$.
  - n pivots and no free variables.
  - only $\vec{x} = 0$ is in the nullspace.
- Any set of n vectors in $\mathbb{R}^n$ must be linearly dependent if n > m.

Definition. A basis for a vector space is a set of linearly independent vectors that span the space.

- $v_1, \cdots, v_n$ are a basis for $\mathbb{R}^n$ exactly when they are the columns of an n > n invertible matrix.

Q. Given m vectors in $\mathbb{R}^n$, how do you find a basis for the space they span ?

$$m \begin{bmatrix} - & v_1 & - \\ - & v_2 & - \\ - & \cdots & - \\ - & v_n & - \end{bmatrix}$$

$$\Downarrow$$

eliminate to find the nonzero rows.
or, put them in columns $\Longrightarrow$ find pivot columns.