Relations

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Relations

- The concept of relations is also commonly used in computer science
  - two of the programs are related if they share some common data and are not related otherwise.
  - two wireless nodes are related if they interfere each other and are not related otherwise
  - In a database, two objects are related if their secondary key values are the same
- What is the mathematical definition of a relation?
- Definition 13.1 (Relation): A relation is a set of ordered pairs
  - The set of ordered pairs is a complete listing of all pairs of objects that "satisfy" the relation
- Examples:
  - GreaterThanRelation = {(2,1), (3,1), (3,2), ...}
  - R = {(1,2), (1,3), (3,0)}

$$(1,2) \in R, \quad 1\, R\, 2 : \text{"}x \text{ is related by the relation } R \text{ to } y\text{"}$$

Relations

- Definition 13.2 (Relation on, between sets) Let R be a relation and let A and B be sets.
  - We say R is a relation on A provided

$$R \subseteq A \times A$$

  - We say R is a relation from A to B provided

$$R \subseteq A \times B$$

Example Relations

- Let $A=\{1,2,3,4\}$ and $B=\{4,5,6,7\}$. Let
  - $R=\{(1,1),(2,2),(3,3),(4,4)\}$
  - $S=\{(1,2),(3,2)\}$
  - $T=\{(1,4),(1,5),(4,7)\}$
  - $U=\{(4,4),(5,2),(6,2),(7,3)\}$, and
  - $V=\{(1,7),(7,1)\}$
- All of these are relations
  - R is a relation on A. Note that it is the equality relation on A.
  - S is a relation on A. Note that the element 4 is never mentioned.
  - T is a relation from A to B. Note that the elements 2, 3 in A and 6 in B are never mentioned.
  - U is a relation from B to A. Note that 1 in A is never mentioned.
  - V is a relation, but it is neither a relation from A to B nor a relation from B to A.

Operations on Relations

- A relation is a set → All the various set operations apply

  $R \cap (A \times A)$: the relation R *restricted* to the set A

  $R \cap (A \times B)$: the relation R *restricted* to a relation from A to B

- Definition 13.4 (Inverse relation) Let *R* be a relation. The inverse of *R*, denoted R^{-1}, is the relation formed by reversing the order of all the ordered pairs in *R*.

$$R^{-1} = \{(x, y) : (y, x) \in R\}$$

- Proposition 13.6: Let *R* be a relation. Then $(R^{-1})^{-1} = R$.
- Proof: ???

Properties of Relations

- Definition 13.7 (Properties of relations) Let *R* be a relation defined on a set *A*.
  - Reflexive: $\forall x \in A, x \, R \, x$
  - Irreflexive: $\forall x \in A, x \not R \, x$
  - Symmetric: $\forall x, y \in A, x \, R \, y \Rightarrow y \, R \, x$
  - Antisymmetric: $\forall x, y \in A, \left(x \, R \, y \wedge y \, R \, x\right) \Rightarrow x = y$
  - Transitive: $\forall x, y, z \in A, \left(x \, R \, y \wedge y \, R \, z\right) \Rightarrow x \, R \, z$
- Example 13.8: "=(equality)" relation on the integers
  - Reflexive, Symmetric, Transitive (also Antisymmetric)
- Example 13.9: "less than or equal to" relation on the integers
  - Reflexive, Transitive, Antisymmetric (not Symmetric)
- Example 13.10: "less than" relation on integers
  - not Reflexive, irreflexive, not symmetric, antisymmetric, transitive
- Example 13.11: "| (divides)" relation on natural numbers
  - Reflexive, not Symmetric, Antisymmetric
  - If "divedes" relation is defined on integers, it is neither symmetric nor antisymmetric.

Equivalence Relations

- Certain relations bear a strong resemblance to the relation *equality*.
- Example: “is-congruent-to” relation on the set of triangles
  - Reflexive
  - Symmetric
  - Transitive
- Definition 14.1 (Equivalence relation) Let R be a relation on a set A. We say R is an “equivalence relation” provided it is “reflexive”, “symmetric”, and “transitive”.
- Example 14.2: “has-the-same-size-as” relation on finite sets
  - not the “equal” relation
  - equivalence relation (share a common property: size): reflexive, symmetric, transitive
  - “like” resemblance to “equal”

Equivalence Classes

- An equivalence relation *R* on *A* categorizes the elements into disjoint subsets --- each subset is called an equivalence class
- Definition 14.6 (Equivalence class) Let *R* be an equivalence relation on a set *A* and let *a* be an element of *A*. The equivalence class of *a*, denoted by [*a*], is the set of all elements of *A* related (by *R*) to *a*; that is,

$$[a] = \{x \in Z : x\ R\ a\}$$

- Example 14.8: Let R be the "has-the-same-size-as" relation defined on the set of finite subsets of Z.

$$[\phi] = \{A \subseteq Z : |A| = 0\} = \{\phi\}$$

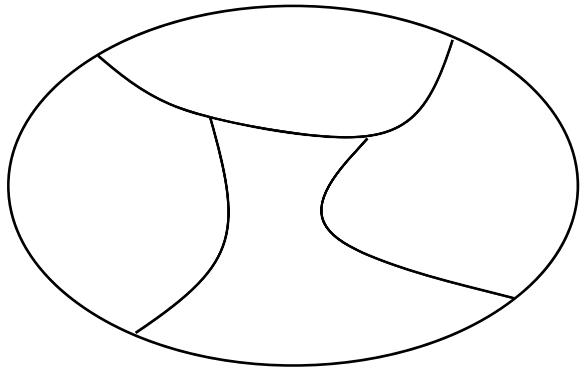
$$[\{2,4,6,8\}] = \{A \subseteq Z : |A| = 4\}$$

Propositions on Equivalence Classes

- Proposition 14.9: Let *R* be an equivalence relation on a set *A* and let $a \in A$. Then $a \in [a]$
- Proposition 14.10: Let *R* be an equivalence relation on a set *A* and let $a, b \in A$. Then *a R b* iff [*a*]=[*b*].
- Proposition 14.11: Let *R* be an equivalence relation on a set *A* and let $a, x, y \in A$. If $x, y \in [a]$, then *x R y*.
- Proposition 14.2: Let *R* be an equivalence relation on *A* and suppose $[a] \cap [b] \neq \phi$. Then $[a] = [b]$.
- Corollary 14.13: Let *R* be an equivalence relation on a set *A*. The equivalence classes of *R* are nonempty, pairwise disjoint subsets of *A* whose union is *A*.

Partitions

- The equivalence classes of *R* “partitions” the set into pairwise disjoint subsets.



- Definition 15.1 (Partition) Let *A* be a set. A partition of (or on) *A* is a set of nonempty, pairwise disjoint sets whose union is *A*.
    - A partition is a set of sets; each member of a partition is a subset of *A*. The members of the partition are called parts.
    - The parts of a partition are nonempty. The empty set is never a part of a partition.
    - The parts of a partition are pairwise disjoint. No two parts of a partition may have an element in common.
    - The union of the parts is the original set.

An Example Partition

- Example 15.2: Let $A = \{1,2,3,4,5,6\}$ and let $P=\{\{1,2\},\{3\},\{4,5,6\}\}$. This is a partition of A into three parts. The Three parts are $\{1,2\}$, $\{3\}$, and $\{4,5,6\}$. These three sets are (1) nonempty, (2) they are pairwise disjoint, and (3) their union is A.
- $\{\{1,2,3,4,5,6\}\}$ is a partition of A into just one part containing all the elements of A
- $\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$ is a partition of A into six parts, each containing just one element.
- Let R be an equivalence relation on a set A. The equivalence classes of R form a partition of the set A.
- An equivalence relation forms a partition and a partition forms an equivalence relation.

Let $\mathcal{P}$ be a partition of a set A. We use $\mathcal{P}$ to form a relation "is-in-the-same-part-as" on A. Formally,

$$a \overset{\mathcal{P}}{\equiv} b \Leftrightarrow \exists P \in \mathcal{P}, a, b \in P$$

Propositions on Partitions and Equivalence Relations

- Proposition 15.3: Let A be a set and let $\mathcal{P}$ be a partition on A. The "is-in-the-same-part-as" relation is an equivalence relation on A.
- Proof???

- Proposition 15.4: Let $\mathcal{P}$ be a partition on a set A. The equivalence classes of the "is-in-the-same-part-as" relation are exactly the parts of $\mathcal{P}$.
- Proof???

Counting Equivalence Classes/Parts

- Example 15.5: In how many ways can the letters in the word WORD be rearranged?
- How about HELLO?
  - Differentiate two Ls as a Large L and small l.
  - Let A be the set of all rearrangements
  - Define a relation R with $a\ R\ b$ provided that a and b give the same rearrangement of HELLO when we shrink the Large L to small l.
  - The number of parts (equivalence classes) are the number of different rearrangements of HELLO.
- How about AARDVARK?
- Theorem 15.6 (Counting equivalence classes) Let R be an equivalence relation on a finite set A. If all the equivalence classes of R have the same size, m, then the number of equivalence classes is $|A|/m$.

Revisit Binomial Coefficients

- Theorem 16.12: Let *n* and *k* be integers with $0 \le k \le n$. Then

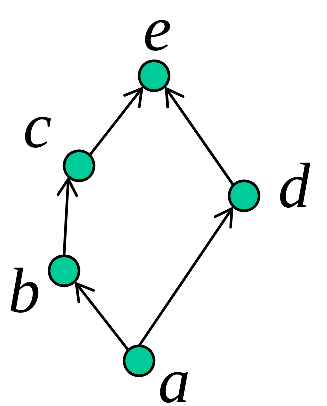
$$\binom{n}{k} = \frac{n!/(n-k)!}{k!}$$

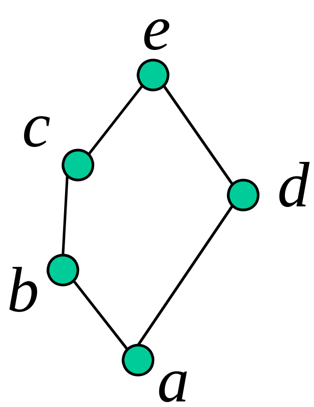
- How the concept of partition helps?
  - The number of *k*-element repetition-free lists: $(n)_k$
  - Partition those lists with the relation "has-the-same-elements-as"
  - Each part (equivalence class) has the same size, $k!$
  - Thus, the number of *k*-element subsets of *n*-element set is $(n)_k / k!$ by Theorem 15.6

Partial Ordering Relations

- A relation is said to be a *partial ordering relation* if it is reflexive, anti-symmetric, and transitive.
  - Example 1: $R = \{(a,b) : a, b \in N, a \mid b\}$
  - Example 2: Let A be a set of foods. Let R be a relation on A such that (a,b) is in R if a is inferior to b in terms of both nutrition value and price.
- Objects in a set are ordered according to the property of R. But, it is also possible that two given objects in the set are not related. → Partial Ordering

{(a,a), (a,b), (a,c), (a,d), (a,e), (b,b), (b,c), (b,e), (c,c), (c,e), (d,d), (d,e), (e,e)}

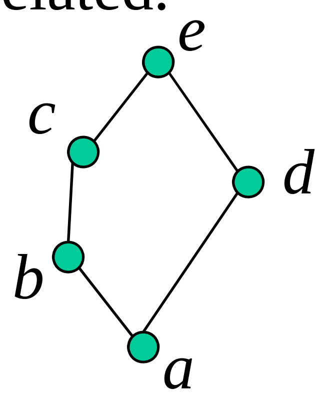




Hasse diagram

Partially Ordered Set (1)

- Set *A*, together with a partial ordering relation *R* on *A*, is called a *partially ordered set* and is denoted by (*A*, *R*).
- Let (*A*, *R*) be a partially ordered set. A subset of *A* is called a *chain* if every two elements in the subset are related.
  - Because of antisymmetry and transitivity, all the elements in a chain form an ordered list.
  - The number of elements in a chain is called the *length* of the chain
- Let (*A*, *R*) be a partially ordered set. A subset of *A* is called an *antichain* if no two distinct elements in the subset are related.



Chains: {a,b,c,e}, {a,b,c}, {a,d,e}, {a}

Antichains: {b,d}, {c,d}, {a}

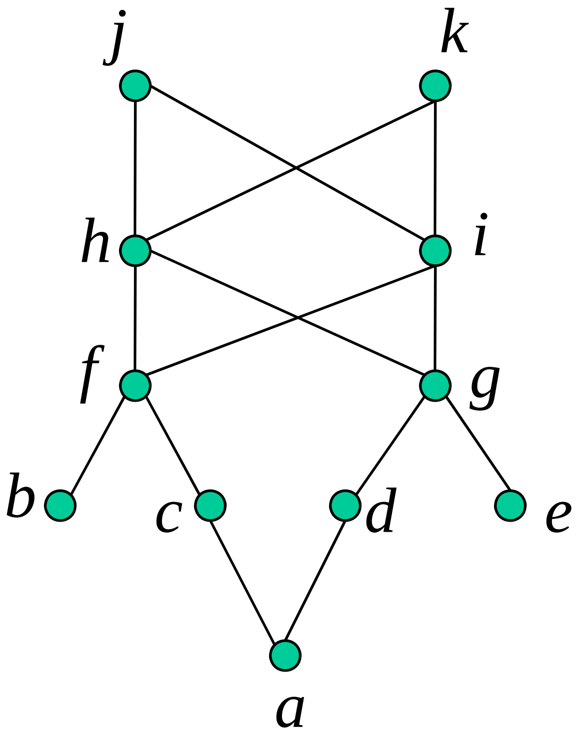
Partially Ordered Set (2)

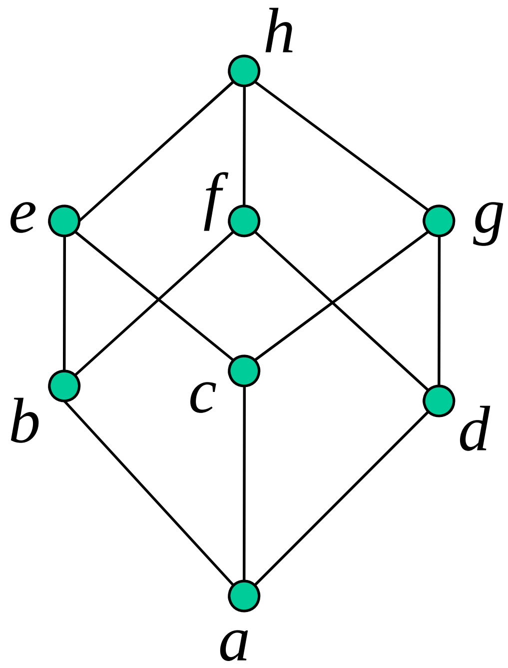
- A partially ordered set (*A*, *R*) is called a totally ordered set if A is a chain.
  - In this case, the relation *R* is called a total ordering relation.
- An element a in (*A*,*R*) is called a *maximal element* if for no *b* in *A*, $a \neq b, a \leq b$.
- An element a in (*A*,*R*) is called a *minimal element* if for no *b* in *A*, $a \neq b, b \leq a$.
- An element *a* is said to *cover* another element *b* if $b \leq a$ and for no other element *c*, $b \leq c \leq a$.
- An element *c* is said to be an *upper bound* of *a* and *b* if

$$a \leq c \text{ and } b \leq c.$$

- An element *c* is said to be a *least upper bound* of *a* and *b* if *c* is an upper bound of *a* and *b* and if there is no other upper bound *d* of *a* and *b* such that $d \leq c$.
- *Lower bound*, *greatest lower bound*

Partially Ordered Set (3)





A partially ordered set is said to be a *lattice* if every two elements in the set have a unique least upper bound and a unique greatest lower bound.

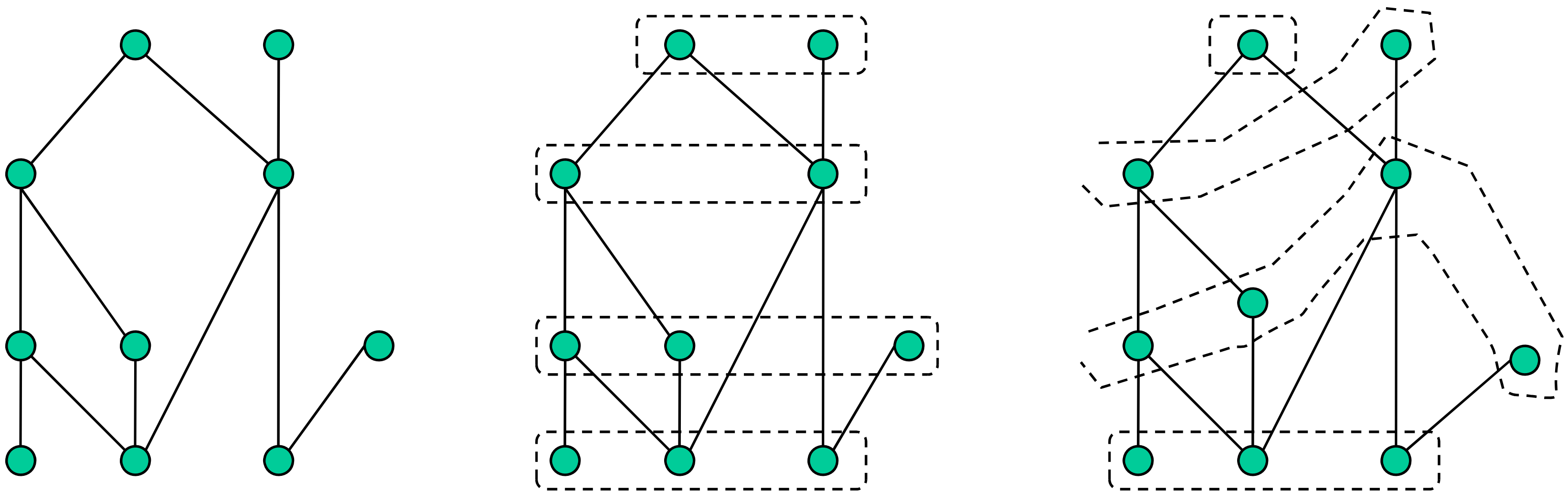
Chains and Antichains

- Example: Let $A=\{a_1, a_2, ..., a_r\}$ be the set of all courses required for graduation. Let R be a reflexive relation on A such that (a_i , a_j) is in R if and only if course a_i is a prerequisite of course a_j. Then, R is a partial ordering relation.
  - What is the minimum number of semesters for graduation?
    - the length of the longest chain in the partially ordered set (A, R).
  - What is the maximum number of courses that a student can take in a semester?
    - the size of the largest antichain in the partially ordered set (A,R).

Chains and Antichains

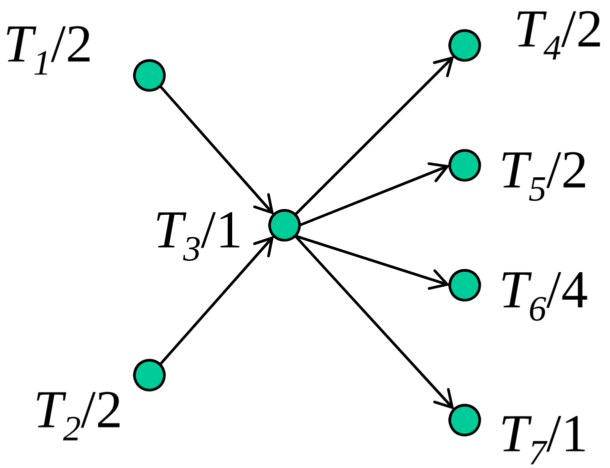
- Theorem: Let (A, R) be a partially ordered set. Suppose the length of the longest chains in A is n. Then the elements in A can be partitioned into n disjoint antichains.
- Proof: by induction
  - for n=1, true
  - Suppose it holds for n-1. Let A be a partially ordered set with the length of its longest chain being n. Let M denote the set of maximal elements in A. Clearly, M is a nonempty antichain. Consider now the partially ordered set $(A-M, R)$. The length of its longest chain is at most n-1. On the other hand, if the length of the longest chains in A-M is less than n-1, M must contain two or more elements that are members of the same chain, which is not possible. Consequently, the length of the longest chain in A-M is n-1. According to the induction hypothesis, A-M can be partitioned into n-1 disjoint antichains. Thus, A can be partitioned into n disjoint antichains.
- Corollary: Let (A,R) be a partially ordered set consisting of mn+1 elements. Either there is an antichain consisting of m+1 elements or there is a chain of length n+1 in A.
  - Proof: Suppose the length of the longest chains in A is n. According to the above theorem, A can be partitioned into n disjoint anticahins. If each of these antichains consists of m or fewer elements, the total number of elements in A is at most mn. Contradiction!

Chains and Antichains

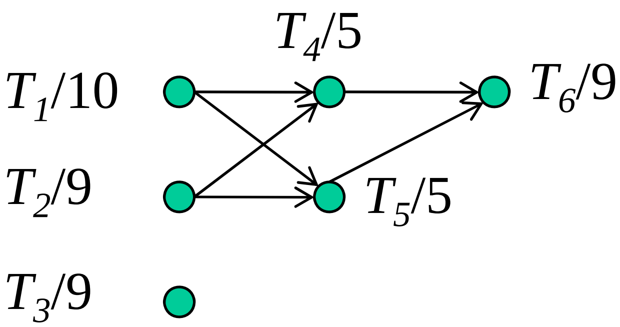


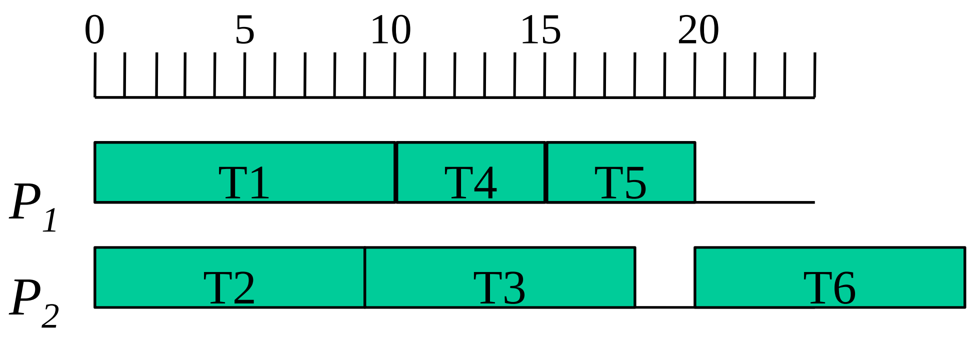
Job-Scheduling Problem

- Scheduling the execution of a set of tasks on n identical processors.
- The set of tasks may have a partial ordering relation R, "$T_i \; R \; T_j$" if and only if the execution of task T_j cannot begin until the execution of task T_i has been completed. (Precedence relation)

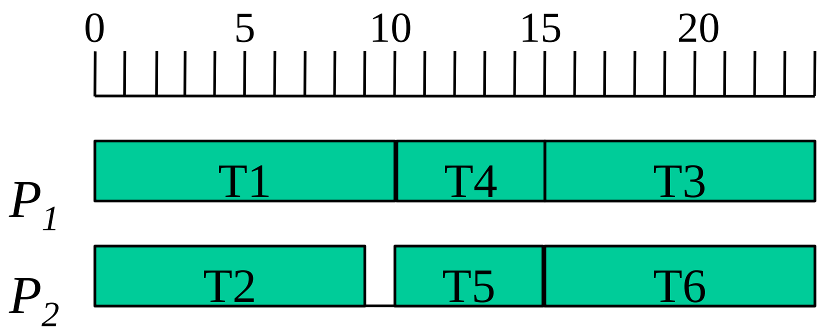


Best Schedule?





work conserving schedule



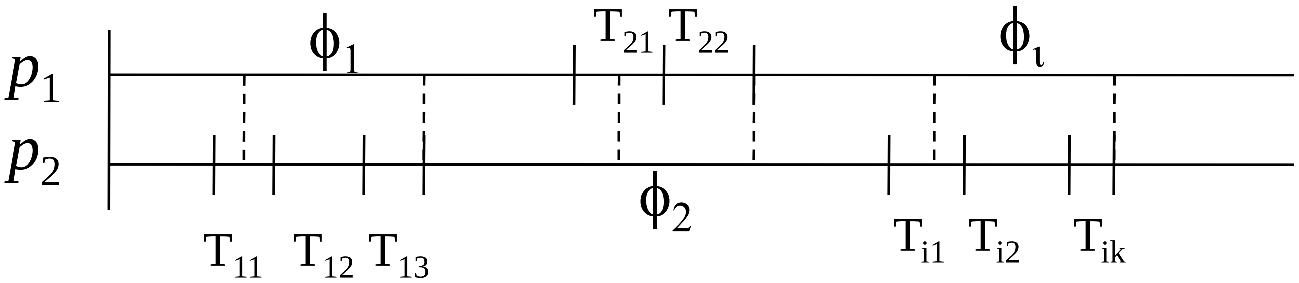
optimal schedule

Work Conserving Schedule

- Theorem: For a given set of tasks, let w denote the total elapsed time of a work conserving schedule and let w_0 denote the minimum possible total elapsed time. Then

$$\frac{w}{w_0} \leq 2 - \frac{1}{n}, \quad \text{where } n \text{ is the number of processors .}$$

- Proof: ???



  – There is a chain ζ such that $\sum_{T_a \in \zeta} \mu(T_a) \geq \sum_{\phi_i \in \Phi} \mu(\phi_i)$

$$w = \frac{1}{2}\left[\sum_{T_j \in T} \mu(T_j) + \sum_{\phi_i \in \Phi} \mu(\phi_i)\right] \leq \frac{1}{2}\left[\sum_{T_j \in T} \mu(T_j) + \sum_{T_a \in \zeta} \mu(T_a)\right]$$

$$w_0 \geq \frac{1}{2} \sum_{T_j \in T} \mu(T_j) \text{ and } w_0 \geq \sum_{T_a \in \zeta} \mu(T_a)$$

$$w \leq w_0 + \tfrac{1}{2} w_0$$

Homework

- 13.1, 13.2, 13.9, 13.13
- 14.1, 14.5, 14.7, 14.13
- 15.2, 15.10