Graphs

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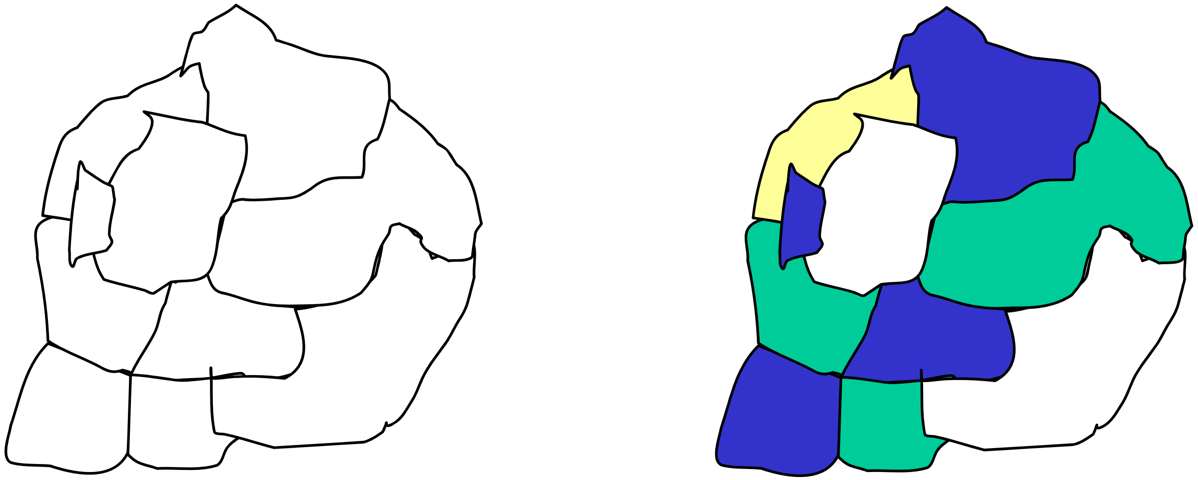
Why Graphs?

- A graph visually represents complex relations among computational objects
  - neighborhood relations among nodes in a computer network
  - conflict relations among sensor nodes
  - dependency relations among concurrent tasks
- In general, graphs formally describe complex problems in visual forms (in all fields including social study and computer science)

Map Coloring Problem

- Imagine a map of a mythical continent that has several countries



- To show the different countries clearly, we want to fill their regions using various colors?
- What is the smallest number of colors we need to color the map?
  - Can this map be colored with fewer than four colors?
    - No (BTW, can you prove?)
  - Is there another map that can be colored with fewer than four colors?
    - Yes
  - Is there a map that require more than four colors?
    - First posed in 1852 by Francis Guthrie and remain unsolved for about a century
    - In the mid-1970s, Appel and Haken proved that every map can be colored using at most four colors

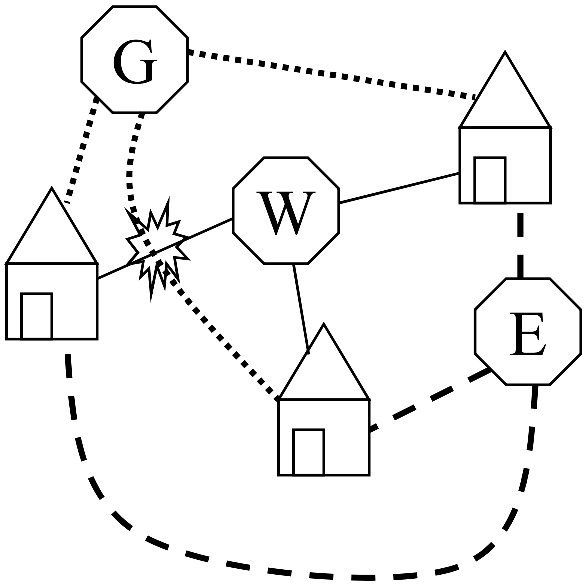
Map Coloring Problem (2)

- Imagine a university in which there are thousands of students and hundreds of courses. As in most universities, at the end of each term there is an examination period. Each course has a 3-hour final exam. On any given day, the university can schedule two final exams.
- It is impossible for a student enrolled in two courses to take both final exams if they were held during the same time slot
- Devise a final examination schedule with the condition that if a student is enrolled in two courses, these courses must get different examination periods. (Want have the smallest possible number of examination slots)
- It is essentially the same as map coloring???

Problem	Map Coloring	Exam Scheduling
Assign	colors	Time slots
to	countries	courses
condition	Common border → different colors	Common students → different slots
objective	Fewest colors	Fewest time slots

Classic Puzzle (Three Utilities Problem)

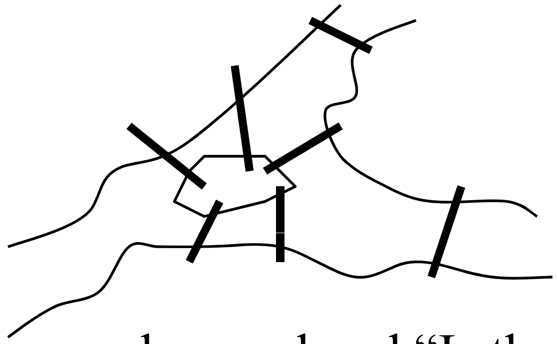
- Imagine a city containing three houses and three utility plants. The three utilities supply gas, water, and electricity. As an urban planar, your job is to run connections from every utility plant to every home. You need three electric wires, three water pipes, and three gas lines. You may place the houses and utility plants anywhere you desire. However, you may not allow two wires/pipes/lines to cross!



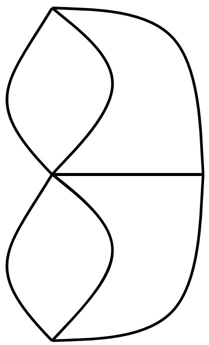
- PCB (Printed Circuit Board) Problem: Can we print the various connecting wires onto the board in such a way that there are no crossing?

Classic Puzzle (Seven Bridges Problem)

- In the late 1700s, in the city of Konigsburg located in Russia, there were seven bridges connecting various parts of the city



- The twonspeople wondered “Is there a tour we can take through our city so that we cross every bridge exactly once?
  - No (Proven by Euler)



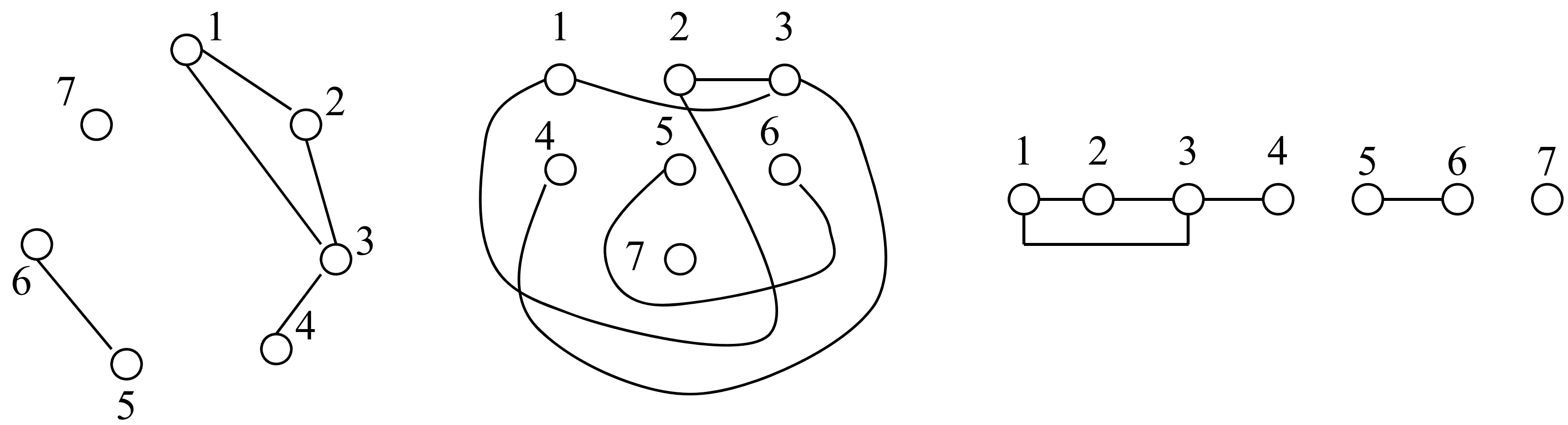
The problem is replaced by the problem of drawing the abstract figure without lifting your pencil from the paper and without redrawing a line

→ The number of points where an odd number of lines meet <= 2

- Urban planning problem (Garbage truck routing): Can we find a route for the garbage truck so that it travels only once down every street?

Graph

- Definition 46.1 (Graph) A graph is a pair $G=(V,E)$ where V is a finite set and E is a set of two-element subsets of V.
- Example 46.2:
  - $G=(\{1,2,3,4,5,6,7\}, \{\{1,2\},\{1,3\},\{2,3\},\{3,4\},\{5,6\}\})$
  - $V = \{1,2,3,4,5,6,7\}$
  - $E=\{\{1,2\},\{1,3\},\{2,3\},\{3,4\},\{5,6\}\}$
- The elements of V are called the vertices (singular: vertex) of the graph, and the elements of E are called the edges of the graph
- More than one drawings for one graph



Adjacency

- Definition 46.3 (Adjacent) Let G=(V,E) be a graph and let $u,v \in V$. We say that u is adjacent to v provided $\{u,v\} \in E$. The notation $u \sim v$ means that u is adjacent to v.
  - If $\{u,v\}$ is an edge of G, we call u and v the endpoints of the edge.
  - We use uv in the place of $\{u,v\}$ provided there is no chance of confusion.
  - Suppose v is a vertex and an endpoint of the edge. Then, $v \in e$. We also say that v is *incident on* (or *incident with*) e.
  - If u and v are adjacent, we also say that u and v are *neighbors*.
- Note that "is-adjacent-to (~)" is a relation defined on the vertex set of a graph G.
  - Reflexive? → No
  - Irreflexive? → Yes, but
    - Self-loop? Multiple edges? (We do not allow them)
  - Symmetric? → Yes
  - Antisymmetric? → Generally No. But, we can make such a graph.
  - Transitive? → Generally No. But, we can make such a graph.

Degree

- The set of all neighbors of a vertex *v* is called the neighborhood of *v* and is denoted *N*(*v*). That is
  - $N(v) = \{u \in V : u \sim v\}$
- Definition 46.4 (Degree) Let $G=(V,E)$ be a graph and let $v \in V$. The degree of *v* is the number of edges with which *v* is incident. The degree of *v* is denoted $d_G(v)$ or, if there is no risk of confusion, simply $d(v)$.
  - $d(v)=|N(v)|$
- Try to add the degrees of vertices of a graph…. find the relation between degrees and number of edges.
- Theorem 46.5 Let G=(V,E). The sum of the degrees of the vertices in G is twice the number of edges; that is

$$\sum_{v \in V} d(v) = 2|E|$$

- Proof ??? --- Hint: Combinatorial proof

Further Notations and Vocabulary

- Maximum and minimum degree of *G*, $\Delta(G)$ and $\delta(G)$
- Regular graph: if all vertices in *G* have the same degree, we call *G* regular. If a graph is regular and all vertices have degree *r*, we also call the graph *r*-regular.
- Vertex and edge set: $V(G)$ and $E(G)$
- Order and size: Let $G=(V,E)$ be a graph. The *order* of *G* is the number of vertices in *G*, that is, $|V|$. The *size* of *G* is the number of edges, that is, $|E|$.
- Complete graphs: Let *G* be a graph. If all pairs of distinct vertices are adjacent in *G*, we call *G complete*. A complete graph on *n* vertices is denoted K_n.
- Edgeless graphs: A graph with no edges is called *edgeless*.
- Empty graph: A graph with no vertices (and hence no edges) is called an *empty graph*.

Subgraphs

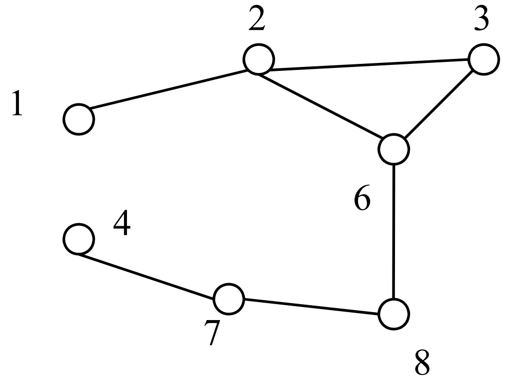
- Definition 47.1 (Subgraph): Let G and H be graphs. We call G a subgraph of H provided V(G)$\subseteq$V(H) and E(G)$\subseteq$E(H).
- Example 47.2: Let G and H be the following graphs:

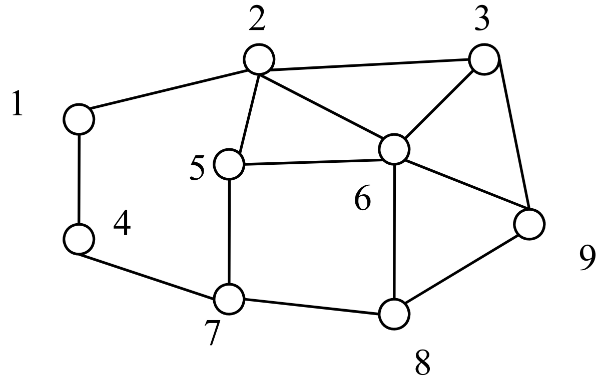
$$V(G) = \{1,2,3,4,6,7,8\}$$

$$V(H) = \{1,2,3,4,5,6,7,8,9\}$$

$$E(G) = \{\{1,2\},\{2,3\},\{2,6\},\{3,6\},\{4,7\},\{6,8\},\{7,8\}\}$$

$$E(H) = \{\{1,2\},\{1,4\},\{2,3\},\{2,5\},\{2,6\},\{3,6\},\{3,9\},\{4,7\},\{5,6\},\{5,7\},\{6,8\},\{6,9\},\{7,8\},\{8,9\}\}$$

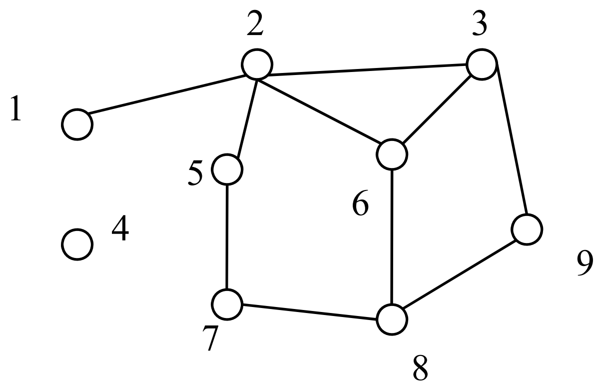


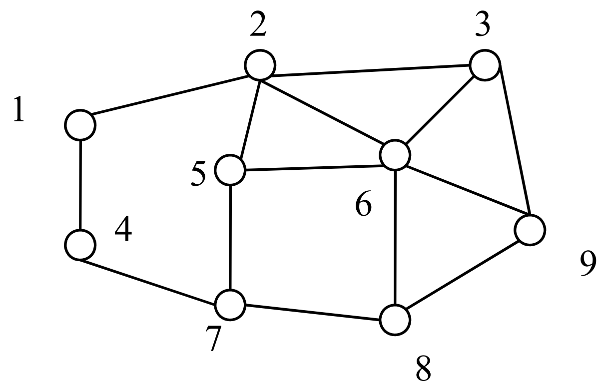


We call H a supergraph of G.

Spanning Subgraph

- "*H-e*" is a new graph with *V(H-e)=V(H)* and *E(H-e)= E(H)-{e}*
- If we form a subgraph of H solely by use of edge deletion, the resulting subgraph is called a *spanning subgraph* of H.
- Definition 47.3 (Spanning subgraph) Let G and H be graphs. We call G a spanning subgraph of H provided G is a subgraph of H, and V(G)=V(H).

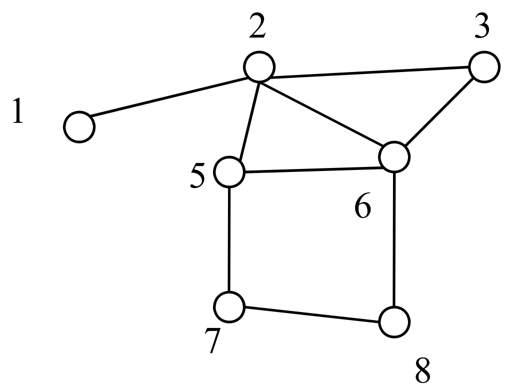


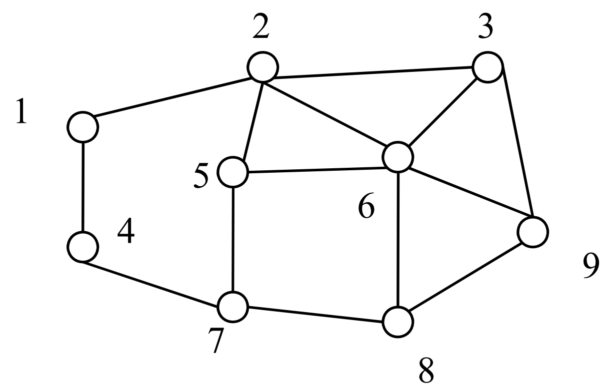


Induced Subgraph

- What about deleting vertices?
- "H-v" is a new graph with V(H-v)=V(H)-{v} and E(H-v)={e$\in$E(H): v$\notin$e}
- If we form a subgraph of H solely by means of vertex deletion, we call the subgraph an *induced subgraph* of H.
- Definition 47.5 (Induced subgraph) Let H be a graph and let A be a subset of vertices of H; that is, A $\subseteq$V(H). The subgraph of H induced on A is the graph H[A] defined by

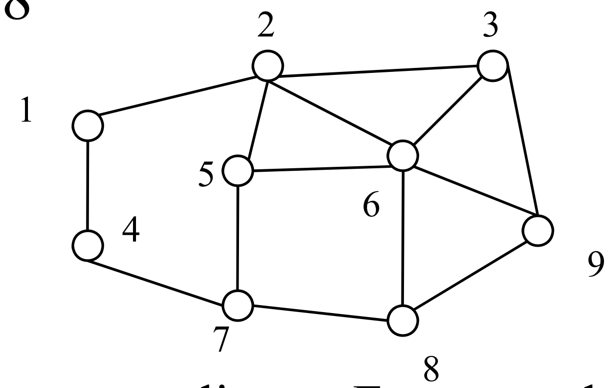
$$V(H[A]) = A$$
$$E(H[A]) = \{xy \in E(H) : x \in A \text{ and } y \in A\}$$





Clique and clique number

- Definition 47.7 (Clique, clique number) Let *G* be a graph. A subset of vertices $S \subseteq V(G)$ is called a clique provided any two distinct vertices in *S* are adjacent. The clique number of *G* is the size of a largest clique; it is denoted $\omega(G)$.
- In other words, a set $S \subseteq V(G)$ is called a clique provided G[S] (subgraph of G induced on S) is a complete graph.
- Example 47.8



This graph has many cliques. For examples,

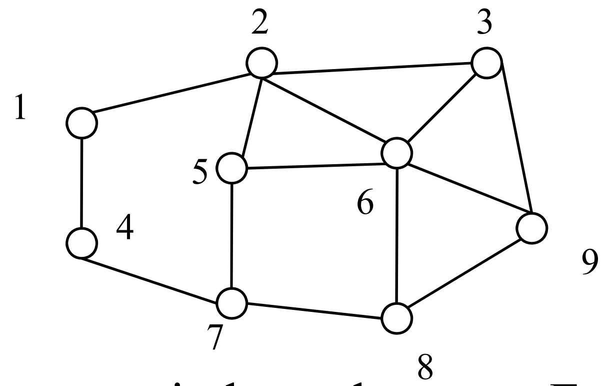
{1,4} {2,5,6} {9} {2,3,6} {6,8,7} {4} { }

The largest size of a clique in H is 3, so $\omega(H)$=3

Note that {1,4} is a maximal (cannot be extended) clique that is not clique of maximum size

Independent set, independence number

- Definition 47.9 (Independent set, independence number) Let G be a graph. A subset of vertices $S \subseteq V(G)$ is called an independent set provided no two vertices in S are adjacent. The independence number of G is the size of a largest independent set; it is denoted $\alpha(G)$.
- In other words, a set $S \subseteq V(G)$ is independent provided G[S] is an edgeless graph.



This graph has many independent sets. For examples,
{1,3,5} {1,7,9} {4} {1,3,5,8} {4,6} {1,3,7} { }
The largest size of an independent set in H is 4, so $\alpha(H)$=4

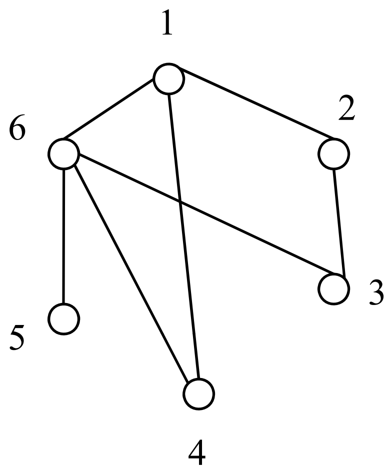
Note that {4,6} is a maximal (cannot be extended) independent set that is not of maximum size

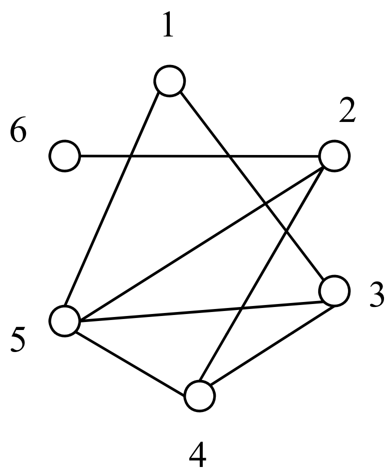
Complements (1)

- The two notions of clique and independent sets are flip sides of the same coin.
- The complement of a graph G is a new graph formed by removing all the edges of G and replacing them by all possible edges that are not in G.
- Definition 47.11 (Complement) Let G be a graph. The complement of G is the graph denoted $\overline{G}$ defined by

$$V\left(\overline{G}\right) = V(G)$$

$$E\left(\overline{G}\right) = \{xy : x, y \in V(G), x \neq y, xy \notin E(G)\}.$$





Complements (2)

- Proposition 47.12: Let G be a graph. A subset of V(G) is a clique of G if and only if it is an independent set of $\overline{G}$. Furthermore,

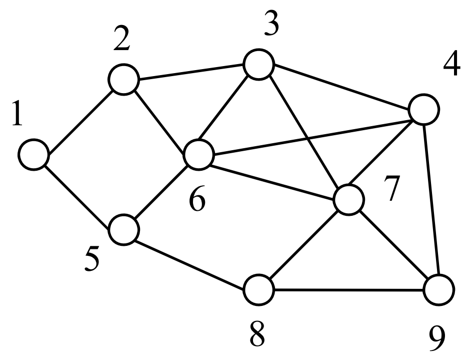
$$\omega(G) = \alpha(\overline{G}) \text{ and } \alpha(G) = \omega(\overline{G}).$$

- Let G be a "very large" graph (i.e., a graph with a great many vertices). A celebrated theorem in graph theory (known as Ramsey's Theorem) implies that either G or its complement, $\overline{G}$, must have a "large" clique.
- Proposition 47.13: Let G be a graph with at least six vertices. Then

$$\omega(G) \geq 3 \text{ or } \omega(\overline{G}) \geq 3.$$

Connection -Walks

- Definition 48.1 (Walk) Let $G=(V,E)$ be a graph. A *walk* in G is a sequence (or list) of vertices, with each vertex adjacent to the next; that is, $W=(v_0, v_1, \ldots, v_l)$ with $v_0 \sim v_1 \sim v_2 \sim \ldots \sim v_l$. The length of this walk is l (Note there are l+1 vertices on this walk).



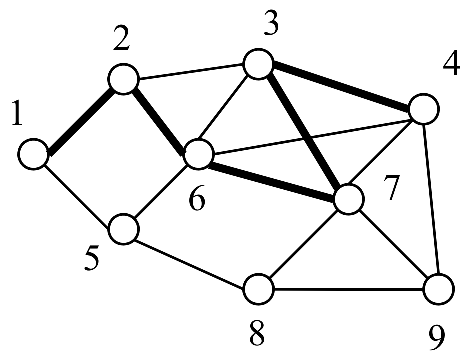
- 1~2~3~4: (1,4)-walk, (u,v)-walk in general
- 1~2~3~6~2~1~5: revisits are permitted
- 5~1~2~6~3~2~1: reversal of the previous walk W^{-1}
- 9: Walk of length zero
- 1~5~1~5~1: a closed walk, begins and ends at the same vertex
- (1,1,2,3,4) is not a walk
- (1,6,7,9) is not a walk

- Definition 48.2 (Concatenation) Let G be a graph. Consider two walks $W_1=v_0 \sim v_1 \sim \ldots \sim v_l$ and $W_2=w_0 \sim w_1 \sim \ldots \sim w_k$ with $v_l=w_0$. Their concatenation, denoted W_1+W_2, is the walk

$$v_0 \sim v_1 \sim \cdots \sim (v_l = w_0) \sim w_1 \sim \cdots \sim w_k$$

Connection - Paths (1)

- Definition 48.3 (Path) A path in a graph is a walk in which no vertex is repeated.
- 1~2~6~7~3~4 is a path. It is called a (1,4)-path. (u,v)-path in general.



- Proposition 48.4 Let P be a path in a graph G. Then P does not traverse any edge of G more than once.
- Proof:

$$P = \cdots \sim u \sim v \sim \cdots \sim u \sim v \sim \cdots \quad \text{or}$$

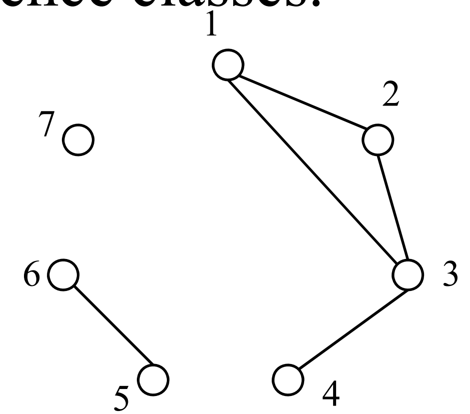
$$P = \cdots \sim u \sim v \sim \cdots \sim v \sim u \sim \cdots$$

Connection - Paths (2)

- We often think a path as a graph or as a subgraph of a given graph.
- Definition 48.5 (Path graph) A path is a graph with vertex set $V=\{v_1, v_2, \ldots, v_n\}$ and edge set $E=\{v_i v_{i+1}: 1 \leq i < n\}$. A path on n vertices is denoted P_n.
- Definition 48.6 (Connected to) Let G be a graph and let $u, v \in V(G)$. We say that u is connected to v provided there is a (u, v)-path in G (i.e., a path whose first vertex is u and whose last vertex is v).
- "is-connected-to" relation vs. "is-adjacent-to" relation
  - the former is reflexive but the latter is irreflexive.
  - both are symmetric.
  - the former is transitive but the latter is not. (proof???)
    - If there exists a (x,y)-path P and a (y,z)-path Q, P+Q is a (x,z)-path. Right?
    - Wrong!
    - P+Q is not necessary a path. It can be a walk!
    - Prove that if there is an (x,z)-walk in G, then there is an (x,y)-path in G.

Connection - Paths (3)

- Theorem 48.8 Let *G* be a graph. The is-connected-to relation is an equivalence relation on *V(G)*.
- By "is-connected-to" relation, a graph is partitioned into equivalence classes.

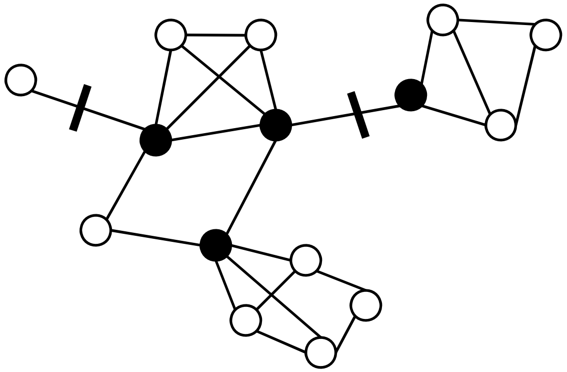


{1,2,3,4}, {5,6}, and {7}

- The equivalence classes of is-connected-to decompose a graph into what we call *components*.
- Definition 48.9 (Component) A *component* of *G* is a subgraph of G induced on an equivalence class of the is-connected-to relation on *V(G)*
- Definition 48.10 (Connected) A graph is called connected provided each pair of vertices in the graph are connected by a path; that is, for all $x, y \in V(G)$, there is an (x,y)-path.

Disconnection

- Definition 48.11 (Cut vertex, cut edge) Let *G* be a graph. A vertex $v \in V(G)$ is called a cut vertex of *G* provided *G-v* has more components than *G*. Similarly, an edge $e \in E(G)$ is called a cut edge of *G* provided *G-e* has more components than *G*.



- If *G* is a connected graph, a cut vertex *v* is a vertex such that *G-v* is disconnected. Likewise *e* is a cut edge if *G-e* is disconnected.
- Theorem 48.12 Let *G* be a connected graph and suppose $e \in E(G)$ is a cut edge of *G*. Then *G-e* has exactly two components.
- What about G-v if v is a cut vertex?
  - G-v can have many components.

Trees

- Threes are connected graphs that have no cycles.
- Definition 49.1 (Cycle) A cycle is a walk of length at least three in which the first and last vertex are the same, but no other vertices are repeated. The term cycle also refers to a (sub)graph consisting of the vertices and edges of such a walk. In other words, a cycle is a graph of the form G=(V,E) where

$$V = \{v_1, v_1, \cdots v_n\} \quad \text{and}$$

$$E = \{v_1 v_2, v_2 v_3, \cdots, v_{n-1} v_n, v_n v_1\}.$$

- A cycle (graph) on n vertices is denoted C_n.
- Definition 49.2 (Forest) Let G be a graph. If G contains no cycles, then we call G acyclic. Alternatively, we call G a forest.
- Definition 49.3 (Tree) A *tree* is a connected acyclic graph

Properties of Trees

- Theorem 49.4 Let T be a tree. For any two vertices a and b in $V(T)$, there is a unique (a,b)-path. Conversely, if G is a graph with the property that for any two vertices, u, v, there is exactly one (u,v)-path, then G must be a tree.
- Theorem 49.5 Let G be a connected graph. Then G is a tree if and only if every edge of G is a cut edge.

Leaves

- Definition 49.6 (Leaf) A leaf of a graph is a vertex of degree 1.
- Does every three have leaves?
  - No. empty graph and K_1 do not have leaves.
  - Other than these, every tree has a leaf.
- Theorem 49.7 Every tree with at least two vertices has a leaf
- Proposition 49.8 Let T be a tree and let v be a leaf of T. Then T-v is a tree.
- This proposition forms the basis of a proof technique for trees.

Proof Template 25

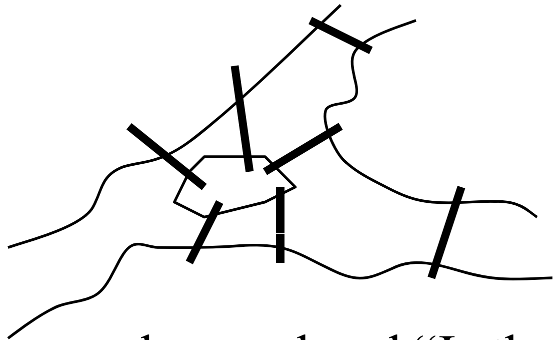
- To prove: some theorem about trees
- Proof: We prove the result by induction on the number of vertices in T
  - Base case: Claim the theorem is true for all trees on n=1 vertices
  - Induction hypothesis: Suppose the theorem is true for all trees on n=k vertices
  - Induction: Let T be a tree on n=k+1 vertices. Let v be a leaf of T. Let T'=T-v. Note that T' is a tree with k vertices, so by induction T' satisfies the theorem. Now we use the fact that the theorem is true for T' to somehow prove that the conclusion of the theorem holds for T. The the result is proved by induction.
- Theorem 49.9 Let T be a tree with $n \geq 1$ vertices. Then T has $n-1$ edges.
- Proof ????
  - It is true when n=1
  - Suppose that it is true when n=k
  - Let T be a tree on n=k+1 vertices. Let v be a leaf of T and let T'=T-v. T' has k-1 edges
  - degree of v =1. Hence, T has (k-1)+1 edges.

Spanning Trees

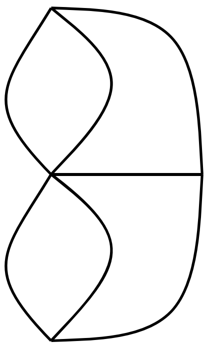
- Trees are, in a sense, minimally connected graphs. By definition, they are connected, but deletion of any edge disconnects a tree.
- Definition 49.10 (Spanning tree) Let G be a graph. A spanning tree of G is a spanning subgraph of G that is a tree.
- Theorem 49.11 A graph has a spanning tree if and only if it is connected.
- Theorem 49.12 Let G be a connected graph on $n \geq 1$ vertices. Then G is a tree if and only if G has exactly $n\text{-}1$ edges.

Revisit: Seven Bridges Problem

- In the late 1700s, in the city of Konigsburg located in Russia, there were seven bridges connecting various parts of the city



- The townspeople wondered "Is there a tour we can take through our city so that we cross every bridge exactly once?
  - No (Proven by Euler)



The problem is replaced by the problem of drawing the abstract figure without lifting your pencil from the paper and without redrawing a line

→ The number of points where an odd number of lines meet <= 2

- Urban planning problem (Garbage truck routing): Can we find a route for the garbage truck so that it travels only once down every street?

Eulerian Graphs (1)

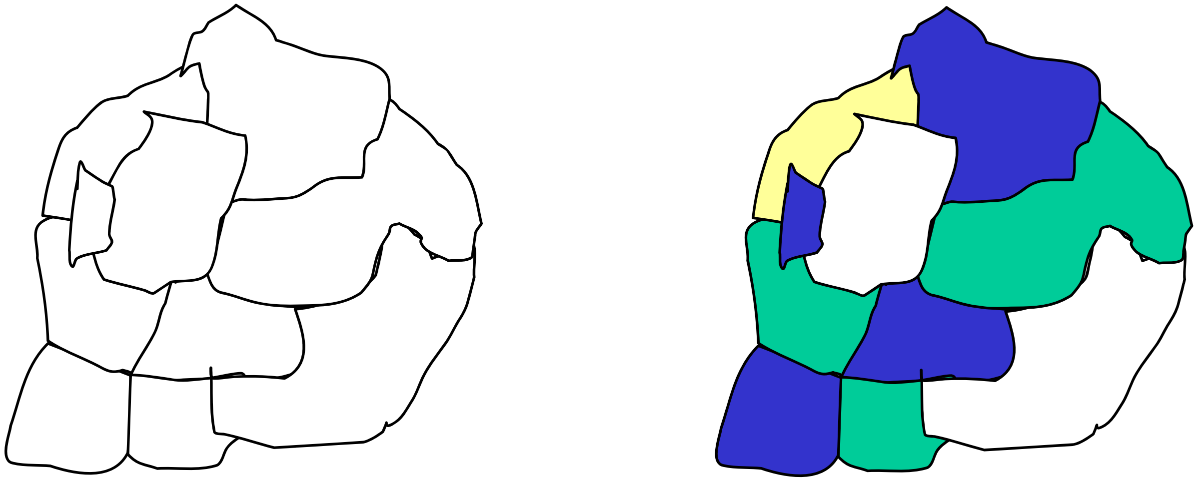
- Definition 50.1 (Eulerian trail, tour) Let G be a graph. A walk in G that traverses every edge exactly once is called an Eulerian trail. If, in addition, the trail begins and ends at the same vertex, we call the walk an Eulerian tour. Finally, if G has an Eulerian tour, we call G Eulerian.
- Necessary condition for G be Eulerian: If G is Eulerian, then G has at most one nontrivial component (component with more than one vertex)
- Theorem 50.2 Let G be a connected graph all of whose vertices have even degrees. For every vertex $v \in V(G)$, there is an Eulerian tour that begins and ends at v.
- Theorem 50.3 Let G be a connected graph with exactly two vertices of odd degree: a and b. Then G has an Eulerian trail that begins at a and ends at b.

Eulerian Graphs (2)

- Proof by Induction using the following two lemmas
  - Lemma 50.4 Let G be a graph all of whose vertices have even degrees. Then no edge of G is a cut edge.
  - Lemma 50.5 Let G be a connected graph with exactly two vertices of odd degree. Let a be a vertex of odd degree and suppose $d(a) \neq 1$. Then at least one of the edges incident with a is not a cut edge.

Revisit: Map Coloring Problem

- Imagine a map of a mythical continent that has several countries



- To show the different countries clearly, we want to fill their regions using various colors?
- What is the smallest number of colors we need to color the map?
  - Can this map be colored with fewer than four colors?
    - No (BTW, can you prove?)
  - Is there another map that can be colored with fewer than four colors?
    - Yes
  - Is there a map that require more than four colors?
    - First posed in 1852 by Francis Guthrie and remain unsolved for about a century
    - In the mid-1970s, Appel and Haken proved that every map can be colored using at most four colors

Coloring Problem

- Let G be a graph. To each vertex of G, we wish to assign a color. The restriction is that adjacent vertices must receive different colors. The objective is to use as few colors as possible.
- Definition 51.1 (Graph coloring) Let G be a graph and let k be a positive integer. A k-coloring of G is a function

$$f: V(G) \rightarrow \{1, 2, ..., k\}.$$

We call this coloring proper provided

$$\forall xy \in E(G), f(x) \neq f(y).$$

If a graph has a proper k-coloring, we call it *k-colorable*.

- In the above, $\{1,2, ..., k\}$ is called a palette.
- The coloring function f is not onto.
- k refers to the size of the palette.
- if k-colorable, it is also $(k+1)$-colorable.
- The goal in graph coloring is to use as few colors as possible.

Chromatic number

- Definition 51.2 (Chromatic number) Let G be a graph. The smallest positive integer k for which G is k-colorable is called the chromatic number of G. The chromatic number of G is denoted $\chi(G)$.
- Example 51.3 Consider the complete graph K_n. We can properly color K_n with n colors by giving every vertex a different color.
  – Can we do better?
  – No
  – $\chi(K_n)=n$
- Proposition 51.4 Let G be a subgraph of H. Then $\chi(G) \leq \chi(H)$
- Proposition 51.5Let G be a graph with maximum degree Δ. Then $\chi(G) \leq \Delta + 1$.
- Proposition 51.6 For a circle C_n,

$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$$

One-colorable graph

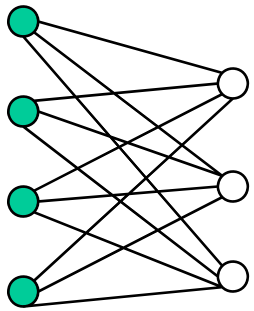
- Which graphs are one-colorable?
- Proposition 51.7: A graph G is on-colorable if and only if it is edgeless

Two-colorable graph (1)

- Definition 51.8 (Bipartite graphs) A graph G is called bipartite provided it is 2-colorable.
- Let G=(V,E) be a bipartite graph. V is partitioned into X and Y where X is the set of vertices assigned with one color and Y with the other color.
- The partition of V into the sets of X and Y is called a bipartition of the bipartite graph.
- Which graphs are bipartite?
  – even cycles are bipartite
  – odd cycles are not
- Proposition 51.9: Trees are bipartite
  – Basis case: Clearly a tree with only one vertex is bipartite.
  – Induction hypothesis: Every tree with n vertices is bipartite. Let T be a tree with n+1 vertices. Let v be a leaf of T and let T' = T-v. T' is bipartite. In T, v has a single neighbor w. Whatever color w has, we can give v the other color. Thus, T is also two-colorable.

Two-colorable graph (2)

- Definition 51.10 (Complete bipartite graphs) Let n, m be positive integers. The complete bipartite graph $K_{n,m}$ is a graph whose vertices can be partitioned $V=X \cup Y$ such that
  - $|X| = n$,
  - $|Y| = m$,
  - for all $x \in X$ and for all $y \in Y$, xy is an edge, and
  - no edge has both its endpoints in X or both its endpoints in Y



How to know bipartite or not?

- How can we convince that a graph is bipartite?
  - show that it can be colored with two colors.
- How can we convince that a graph is NOT bipartite?

- Theorem 51.11 A graph is bipartite if and only if it does not contain an odd cycle

Procedure of two coloring

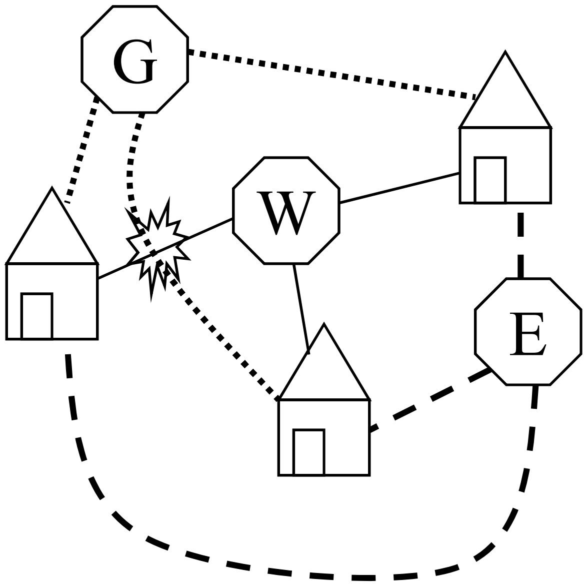
- We begin with a graph, all of whose vertices are uncolored.
- We arbitrarily color one vertex white.
- Then we color all its neighbors black.
- Now we color all neighbors of black vertices white, and then all neighbors of white vertices black.
  - At some point in this procedure, we may color two adjacent vertices the same color. If we do, we tries to find an odd cycle, proving the graph is not bipartite.
  - At some point, we may find that this procedure finds no new vertices to color, but yet, there remain uncolored vertices. In this case, the graph is not connected. and we restart this procedure in another component.
- This procedure is simple and efficient. We know that once we color a vertex, say black, all its neighbors must be white. <u>There is no choice</u> in this matter because there are only two colors.

What about three-coloring?

- We have more than one choice at each step. This complicates the problem.
- There are no known efficient procedures to determine whether or not a graph is tree-colorable (or k-colorable for any fixed value of $k > 2$): NP-complete
- There is no known efficient procedure for calculating $\chi(G)$: NP-complete
- There are, however, heuristic and approximate methods that often give good results.

Revisit: Three Utilities Problem

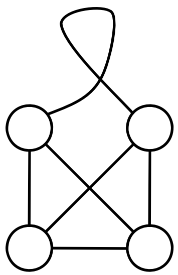
- Imagine a city containing three houses and three utility plants. The three utilities supply gas, water, and electricity. As an urban planar, your job is to run connections from every utility plant to every home. You need three electric wires, three water pipes, and three gas lines. You may place the houses and utility plants anywhere you desire. However, you may not allow two wires/pipes/lines to cross!

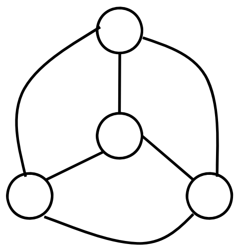


- PCB (Printed Circuit Board) Problem: Can we print the various connecting wires onto the board in such a way that there are no crossing?

Planar Graphs

- Definition (Simple curve): A simple curve is a curve that joins two distinct points in the plane and does not cross itself.
- Definition (Closed) If a curve returns to its starting point, we call the curve closed.
- Definition (Simple closed curve) If the first/last point of the curve is the only point on the curve that is repeated, then we call the curve a simple closed curve.
- A graph has many different drawings by drawing edges with different curves.

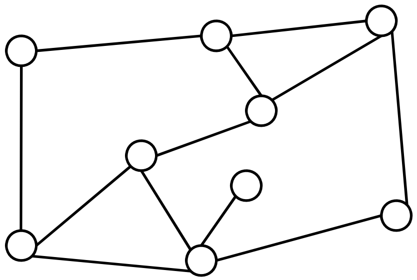




- Definition 52.2 (Planar graph) A planar graph is a graph that has a crossing-free drawing in the plane.

How do we know planar or not?

- Euler's formula gives a relation among numbers of faces, vertices, and edges of a crossing-free drawing of a planar graph.



n=9 vertices
m=12 edges
f=5 faces

- Theorem 52.3 (Euler's formula) Let *G* be a connected planar graph with *n* vertices and *m* edges. Choose a crossing-free drawing for *G*, and let *f* be the number of faces in the drawing. Then

$$n - m + f = 2$$

- Proof by induction
    - Basis case: G with n vertices and m=n-1 edges. It is a tree
    - Induction hypothesis: For G with n vertices and m edges, it holds
        - Show that it also holds for G' with n vertices and (m+1) edges
            - From G', delete non cut-edge e
            - G'-e is a planar graph. It has f-1 faces. Thus, n - m + (f-1) = 2
            - Thus, n - (m+1) +f = 2

More properties of planar graphs

- Degree of a face is the number of edges that are on the boundary of the face (an edge is counted twice if both sides of it are on the boundary of the same face)

- Proposition 52.4: Let G be a planar graph. The sum of the degrees of the faces in a crossing-free drawing of G in the plane is equals $2|E(G)|$

$$n - m + f = 2$$

- Corollary 52.5: Let G be a planar graph with at least two edges. Then

$$|E(G)| \leq 3\,|V(G)| - 6$$

Furthermore, if G does not contain K_3 as a subgraph, then

$$|E(G)| \leq 2\,|V(G)| - 4$$

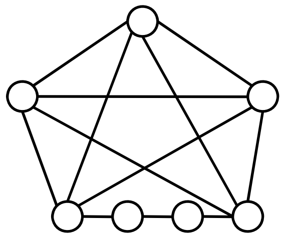
- Proof: every face has degree at least 3, so the sum of the face degrees is at least 3f. Therefore, $2|E(G)| \geq 3f$. Putting this into Euler's formula, we have

$$2 - |V(G)| + |E(G)| = f \leq 2/3\,|E(G)|$$

$$|E(G)| \leq 3|V(G)| - 6$$

Nonplanar graphs

- We can use Corollary 52.5 to prove that certain graphs are nonplanar
- Proposition 52.7: The graph K_5 is nonplanar.
- Proposition 52.8: The graph $K_{3,3}$ is nonplanar.
- A subdivision of G is formed from G by replacing edges with paths.



- If a graph is planar, so are its subdivisions. If a graph is nonplanar, then all of its subdivisions are also nonplanar.
- Theorem 52.9 (Kuratowski) A graph is planar if an only if it does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph.
- How to say a graph is planar?
  - show a crossing-free drawing
- How to say a graph is nonplanar?
  - find a subdivision of K_5 or $K_{3,3}$

Coloring Planar Graphs

- The problem of coloring a map is equivalent to the problem of coloring a graph.
- A map has a special property.
  - It is a planar graph
- Is every planar graph four-colorable?
- Theorem 52.10 (Four color) If G is a planar graph, then $\chi(G) \leq 4$.
- 4 is the best possible number, that is 4 cannot be replaced by a smaller value in the above theorem.

Homework

- 46.1, 46.9, 46.10, 46.16
- 47.1, 47.3, 47.8
- 48.1, 48.6, 48.11
- 49.1, 49.4, 49.9
- 50.1, 50.3, 50.4
- 51.1, 51.2, 51.6, 51.13
- 52.2, 52.3, 52.6