4 Rank & Solutions of Linear Systems

4.1 Rank

Recall G-J elimination. Elimination matrices are multiplied to put A into its reduced row echelon form.

$$i.e. \qquad \mathrm{E}\,[\mathrm{A} \quad \mathrm{I}\,] \quad = \quad [\mathrm{R} \quad \mathrm{E}]$$

If A is invertible,

$$\mathrm{E}\,[\mathrm{A} \quad \mathrm{I}\,]' \quad = \quad [\,\underset{\mathrm{R}}{\overset{/\!/}{\mathrm{I}}} \quad \underset{\mathrm{E}}{\overset{/\!/}{\mathrm{A}^{-1}}}\,]$$

Example .

$$\mathrm{A} = \begin{bmatrix} 1 & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix} \quad \longrightarrow \quad \mathrm{R} = \begin{bmatrix} \underline{1} & 3 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

only one pivot

$\uparrow$

Ax = 0 is <u>just one eqn.</u>, not three.

(1 independent row
1 independent column

The "true" size of A is given by its rank.

Definition. rank(A) = the number of pivots.

Example . above example : rank(A) = 1

$$\text{Let} \quad \mathrm{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \text{then } \mathrm{A} = \begin{bmatrix} | & | & | \\ \mathrm{u} & 3\mathrm{u} & 10\mathrm{u} \\ | & | & | \end{bmatrix} = \mathrm{u}\,[\,1 \quad 3 \quad 10\,]$$

<u>C(A) is 1-dim.</u>

Remark If A is $m \times n$, then $r \leq m$, and $r \leq n$.

rank(A)

· A has full row rank if every row has a pivot.

$(r = m$, No zero rows in R) $[\qquad]$

· A has full column rank if every column has a pivot.

$(r = n$, No free variables) $\begin{bmatrix} \\ \\ \end{bmatrix}$

$$\underset{\llcorner\, m\times n}{\underline{\mathrm{A}}\mathrm{x}} = 0 \quad \longrightarrow \quad \mathrm{Rx} = 0$$

$$i.e \quad \begin{cases} r & \text{pivot columns.} \\ n-r & \text{free variables.} \end{cases}$$

$$\begin{cases} r & \text{indept eqns.} \\ n-r & \text{special solns (indpt).} \end{cases}$$

Example .

$$\mathrm{A} = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{bmatrix} \longrightarrow \mathrm{R} = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rank(A) $= 2$
Ax $= 0$ (or Rx $= 0$) has two independent eqns.

Solution of

$$\mathrm{Rx} = 0 = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \quad \Leftarrow \left(\begin{array}{l} x_1, x_3 \ : \ \text{pivot variables} \\ x_2, x_4, x_5 \ : \ \text{free variables} \end{array} \right.$$

$$\begin{cases} (1) & \text{set } x_2 = 1,\ x_4 = x_5 = 0 \quad \text{then,} \quad x_1 = -3 \\ & \quad \Rightarrow \mathrm{s}_1 = (-3,\ 1,\ 0,\ 0,\ 0) \\ (2) & \text{set } x_4 = 1,\ x_2 = x_5 = 0 \quad \text{then,} \quad x_3 = -4,\ x_1 = -2 \\ & \quad \Rightarrow \mathrm{s}_2 = (-2,\ 0, -4,\ 1,\ 0) \\ (3) & \text{set } x_5 = 1,\ x_2 = x_4 = 0 \quad \text{then,} \quad x_3 = 3,\ x_1 = 1 \\ & \quad \Rightarrow \mathrm{s}_3 = (\ 1,\ 0,\ 3,\ 0,\ 1) \end{cases}$$

$\rightarrow$ Three independent solns. $\quad (n = 5,\ r = 2)$
N(A) is spanned by s_1,s_2,s_3

(Strang, page144)

4.2 Ax = b ≠0

· Suppose $m \times n$ matrix A has rank r.
Then the $n-r$ special solns solve $\underline{\mathrm{Ax}_h = 0}$.
And suppose we found a soln for$\underline{\mathrm{Ax}_p = \mathrm{b}}$.
Then

$$\mathrm{A}(\mathrm{x}_h + \mathrm{x}_p) = \mathrm{b}$$

$$i.e. \qquad \mathrm{x}_h + \mathrm{x}_p \text{ is a soln fot } \mathrm{Ax} = \mathrm{b}$$

· If A is a square, invertaible matrix, the only vector in $N(\mathrm{A})$ is $\mathrm{x}_h = 0$. And $\mathrm{Ax}_p = \mathrm{b}$ has only one soln $\mathrm{x}_p = \mathrm{A}^{-1}\mathrm{b}$.

Example . Ax = b

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix}$$

Augmented matrix

$$[\,\mathrm{A} \quad |\, \mathrm{b}\,] = \left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 1 & 3 & 1 & 6 & 7 \end{array}\right]$$

Elimination $\downharpoonleft$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = [\,\mathrm{R} \quad |\, \mathrm{d}\,]$$

$\mathrm{Rx}_h = 0$: free variables x_2, x_4

(1) set $x_2 = 1,\ x_4 = 0$ then, $x_1 = -3,\ x_3 = 0$
$\Rightarrow \ \mathrm{s}_1 = (-3,\ 1,\ 0,\ 0)$

(2) set $x_4 = 1,\ x_2 = 0$ then, $x_1 = -2,\ x_3 = -4$
$\Rightarrow \ \mathrm{s}_2 = (-2,\ 0, -4,\ 1)$

Example .

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{bmatrix} A & b \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{array}\right]$$

$\Downarrow$ elimination

$$\left[\begin{array}{cc|c} 1 & 0 & b_1 - b_2 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 + 2b_2 \end{array}\right]$$

$$\Downarrow \text{elimination}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2b_1 - b_2 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 + b_1 + b_2 \end{array}\right] = \left[\begin{array}{c|c} R & d \end{array}\right]$$

$\Longrightarrow$ Row3, For $Ax = b$ to be solvable, $b_1 + b_2 + b_3 = 0$
(otherwise, x_p does not exist)

$\Longrightarrow$ Row1,2, The only particular solution

$$x_p = \begin{bmatrix} 2b_1 - b_2 \\ b_2 - b_1 \end{bmatrix}$$

complete solution :

$$X = x_p + x_n = \begin{bmatrix} 2b_1 - b_2 \\ b_2 - b_1 \end{bmatrix} + \begin{bmatrix} o \\ o \end{bmatrix}$$

Note that every column has a pivot $\Longrightarrow$ $r = n$
<u>full column rank</u>

$$A = m\underbrace{\begin{bmatrix} \quad & \quad \\ & \\ & \end{bmatrix}}_{n} \quad \Longrightarrow \text{elimination} \Longrightarrow \qquad R = \begin{bmatrix} I(n \times n) \\ 0(m-n) \end{bmatrix}$$

(tall and thin)

- $x_n = 0$ is the only nullspace solution. (no free variables, no special solutions)
- columns are linear independent !

- Every matrix A with full column rank (r = n) has all these properties.

  1. All columns are pivot columns.
  2. No free variables or special solutions.
  3. N(A) contains only the zero vector.
  4. If Ax=b has a solution (it might not) then it has only one solution.

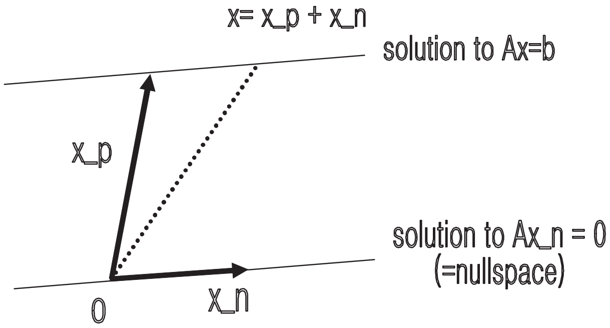
Example . $\left(\begin{array}{c} x + y + z = 3 \\ x + 2y - z = 4 \end{array} \right.$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & -1 & 4 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -2 & 1 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & 1 \end{array}\right] = \left[\begin{array}{c|c} R & d \end{array}\right]$$

$Rx_h = 0$: free variable x_3
set $x_3 = 1$, then $x_1 = -3$, $x_2 = 2$
$\therefore$ s = (-3,2,1)

x_p has free variable $x_3 = 0$. x_p comes directly from d.
$\therefore x_p$ = (2,1,0)

complete solution $\quad x = x_p + x_n = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$



$\Longrightarrow$ Any points on this line could be chosen as x_p. (We chose one with x_3=0)

- Every matrix A with full row rank (r =m) has all these properties.

    1. All rows have pivots, and R has no zero rows.
    2. $Ax = b$ has a solution for every right side b.
    3. $C(A) = \mathbb{R}^m$
    4. $n - r = n - m$ special solutions in $N(A)$