



5 Determinant (for a square matrix A)
 $= \det A = |A|$

5.1 Properties of Determinant (Strang, page 234)

1. $\det I = 1$.
2. \det changes sign when two rows are exchanged.

$$\text{ex) } \begin{vmatrix} c & d \\ a & b \end{vmatrix} = - \begin{vmatrix} a & b \\ c & d \end{vmatrix} = bc - ad$$

3. \det is a linear ftn of each row separately, when all other rows stay fixed.

$$\text{ex) } \begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

Note Rules 1-3 completely determine the number $\det A$.

Other Properties (These follow from above)

4. If two rows of A are equal, then $\det A = 0$.

$$\text{ex) } \begin{vmatrix} c & d \\ a & b \end{vmatrix} = 0 \quad (\text{use rule 2})$$

5. Subtracting a multiple of one row from another row does not change $\det A$.

$$\text{ex) } \begin{vmatrix} a & b \\ c - la & d - lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Note The usual elimination steps do not change $\det A$.

$$\therefore \underline{\det A} = \underline{\det U}$$

(If rows are exchanged, $\det A = \pm \det U$)

6. A matrix with a row of zeros has $\det A = 0$.

$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0$$

(rule 5 & 4)

7. If A is triangular then,

$$\det A = \text{product of diagonal entries.}$$

(Triangular matrix $\xrightarrow[\text{elimination}]{} \text{diagonal matrix} \rightarrow \text{rule 3 \& 1}$)

8. If A is singular then $\det A = 0$.
If A is invertible then $\det A \neq 0$.

If A is singular then U has a zero row.

$$\Rightarrow \det A = \det U = 0$$

If A is invertible then U has nonzero pivots along its diagonal.

$$\Rightarrow \det A = \pm \det U = \pm (\text{product of the pivots})$$

9. $|AB| = |A||B|$

$$\text{ex) } \begin{vmatrix} a_1 & b_1 \\ c_1 & d_1 \end{vmatrix} \begin{vmatrix} a_2 & b_2 \\ c_2 & d_2 \end{vmatrix} = \begin{vmatrix} a_1 a_2 + b_1 b_2 & a_1 b_2 + b_1 d_2 \\ c a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{vmatrix}$$

$$(\det A)(\det A^{-1}) = \det I = 1$$

$$\therefore |A^{-1}| = \frac{1}{|A|}$$

10. $|A^T| = |A|$

$$\begin{aligned} \text{factorization of } A & : PA = LU \\ \Rightarrow A^T P^T & = U^T L^T \end{aligned}$$

$$\left\{ \begin{array}{l} |P||A| = |L||U| \\ |A^T||P^T| = |U^T||L^T| \end{array} \right\} \quad (1)$$

$$|U| = |U^T| \quad , \quad |L| = |L^T| = 1$$

$$P^T = P^{-1} \quad \Rightarrow \quad |P^T||P| = 1$$

$$(|P^T| = |P| = \text{both } 1 \text{ or } -1)$$

$$\therefore (1) \text{ yields } |A^T| = |A|$$

Note Thanks to Rule 10, every rule for the rows apply also to the columns.
ex) $\cdot \det$ changes sign when two columns are reversed.

- A zero column makes $\det = 0$.
- A column is multiplied by t , so is $\det A$.
- \det is a linear ftn of column separately.

5.2 Determinant by Cofactors

$$\begin{aligned} \begin{vmatrix} a & b \\ c & d \end{vmatrix} &= \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} \\ &= ad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + bc \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \\ &= ad - bc . \end{aligned}$$

$$\begin{aligned} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= \begin{vmatrix} a_{11} & & & \\ & a_{22} & a_{23} & \\ & a_{32} & a_{33} & \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix} \\ &= \begin{vmatrix} a_{11} & & & \\ & a_{22} & a_{23} & \\ & 0 & a_{33} & \end{vmatrix} + \begin{vmatrix} a_{11} & & & \\ & a_{22} & a_{23} & \\ & a_{32} & 0 & \end{vmatrix} + \begin{vmatrix} & a_{12} & & \\ a_{21} & & a_{23} & \\ 0 & & a_{33} & \end{vmatrix} \\ &+ \begin{vmatrix} & a_{12} & & \\ a_{21} & & a_{23} & \\ a_{31} & & 0 & \end{vmatrix} + \begin{vmatrix} & & a_{13} & \\ a_{21} & a_{22} & & \\ 0 & a_{32} & & \end{vmatrix} + \begin{vmatrix} & & a_{13} & \\ a_{21} & a_{22} & & \\ a_{31} & 0 & & \end{vmatrix} \\ &= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} \\ &\quad + a_{31}a_{12}a_{27} + a_{21}a_{32}a_{13} - a_{31}a_{22}a_{13} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(-a_{21}a_{33} + a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &\quad \text{"Cofactors"} \\ &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= \sum_{j=1}^3 a_{1j}C_{1j} \end{aligned}$$

Cofactor $C_{1j} = (-1)^{1+j} \det M_{1j}$ where M_{1j} is a submatrix of A without row 1 and column j .

We can do the same for row i , not just row 1.

For any row i ,

$$\begin{aligned} \det A &= \sum_{j=1}^n a_{ij} C_{ij} \\ \text{where } C_{ij} &= (-1)^{i+j} \det M_{ij} \\ M_{ij} &= \text{submatrix (order } n-1) \\ &\quad \text{without row } i \text{ and column } j \end{aligned}$$

$\det A$ = dot product of any row i with its cofactors

$$= \sum_{j=1}^n a_{ij} C_{ij} \quad (\text{expand down a column})$$

Ex

$$|A| = \begin{vmatrix} -2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{vmatrix} - (-1) \begin{vmatrix} -1 & -1 & \\ & 2 & -1 \\ & -1 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - (-1) \begin{vmatrix} -1 & \\ -1 & 2 \end{vmatrix}$$

$$= 2 \cdot 3 + (-2) = 4$$

$$\begin{vmatrix} -1 & -1 & \\ & 2 & -1 \\ & -1 & 2 \end{vmatrix} = (-1) \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = (-1) \cdot 3 = -3$$

$$\therefore |A| = 2 \cdot 4 + (-3) = 5$$