Engineering Mathematics I
- Chapter 1. First-Order ODEs
(1계 상미분 방정식)

민기복

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Introduction

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1. Basic Concepts. Modeling
2. Geometric Meaning of y'=f(x,y). Directional Fields
3. Separable ODEs (변수분리형 상미분방정식). Modeling
4. Exact ODEs (완전 상미분방정식). Integrating Factors.
5. Linear ODEs (선형 상미분방정식). Bernoulli Equation. Population Dynamics
6. Orthogonal Trajectories (직교곡선족). *Optional*
7. Existence and Uniqueness of Solutions (해의 존재성과 유일성)

Introduction
Physical problem vs. mathematical modeling

- Many physical behavior can be expressed as differential equation (containing derivatives of unknown function)

- Three Steps

  1. Deriving them from physical or other problems (modeling)
  2. Solving them by standard methods
  3. Interpreting solutions and their graphs in terms of a given problem

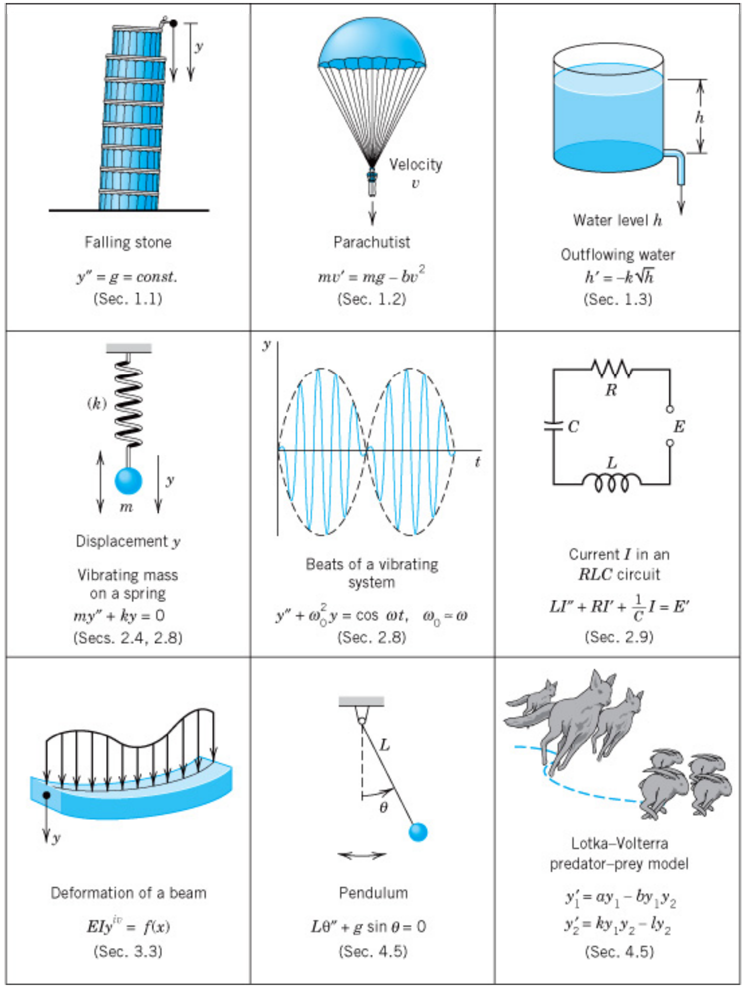

Fig.1 Some applications of differential equations

1.1 Basic Concepts. Modeling
ODE vs PDE

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- Differential Equation: An equation containing derivatives of an unknown function
  - Ordinary Differential Equation (상미분방정식) : contains one or several derivatives of an unknown function of one independent variables (독립변수 1개)
    Ex. $y' = \cos x, \quad y'' + 9y = 0, \quad x^2 y''' y' + 2e^x y'' = (x^2 + 2) y^2$
  - Partial Differential Equation (편미분방정식) : contains partial derivatives of an unknown function of two or more variables (독립변수 2개이상)
    Ex. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

1.1 Basic Concepts. Modeling
Order, Explicit vs. Implicit

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- Order: The highest derivatives of the unknown function
  - Ex. (1) $y' = \cos x \quad \Rightarrow$ 1st Order (1계)
  - (2) $y'' + 9y = 0 \quad \Rightarrow$ 2nd order (2계)
  - (3) $x^2 y''' y' + 2e^x y'' = \left(x^2 + 2\right) y^2 \quad \Rightarrow$ 3rd order (3계)
- First-order ODE: Equations contain only the first derivatives y' and may contain y and any given functions of x
  - Explicit form: $y' = f\left(x, y\right)$
  - Implicit form: $F\left(x, y, y'\right) = 0$

1.1 Basic Concepts. Modeling
Types of Solution

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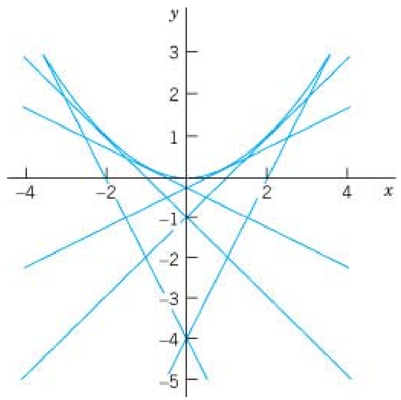
- Solution: functions that make the equation hold true
  - General solution (일반해): a solution that contains an arbitrary constant
  - Particular solution (특수해): a solution in which we choose a specific constants
  - Singular solution (특이해): an additional solution that cannot be obtained from the general solution
    - Ex 16) ODE $y'^2 - xy' + y = 0$, general solution $y = cx - c^2$
      singular solution $y = x^2/4$.



1.1 Basic Concepts. Modeling
Initial Value Problem

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- Initial Value Problem
  - An ordinary differential equation together with specified value of the unknown function at a given point in the domain of the solution

$$y' = f(x, y) \;, \quad y(x_0) = y_0$$

1.1 Basic Concepts. Modeling
Example 4.

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- Solve the initial value problem

$$y' = \frac{dy}{dx} = 3y, \quad y(0) = 5.7$$

- Step 1 Find the general solution.

$$y(x) = ce^{3x}$$

- Step 2 Apply the initial condition.

$$y(0) = ce^{0} = c = 5.7$$

Particular solution : $y(x) = 5.7e^{3x}$

1.1 Basic Concepts. Modeling

Physical phenomena → Mathematical model

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- Typical steps of Modeling
  - Step1 : The transition from the physical situation to its mathematical formulation
  - Step 2: The solution by a mathematical method
  - Step 3: The physical interpretation of differential equations and their applications

1.1 Basic Concepts. Modeling
Example 5.

- Given an amount of a radioactive substance, say 0.5 g(gram), find the amount present at any later time.
  - Physical Information: Experiments show that at each instant a radioactive substance decomposes at a rate proportional to the amount present.
  - Step 1 Setting up a mathematical model(a differential equation) of the physical process.
    By the physical law : $\frac{dy}{dt} \propto y \quad \Rightarrow \quad \frac{dy}{dt} = ky$
    The initial condition : $y(0) = 0.5$
  - Step 2 Mathematical solution.
    General solution : $y(t) = ce^{kt}$
    Particular solution : $y(0) = ce^{0} = c = 0.5 \quad \Rightarrow \quad y(t) = 0.5e^{kt}$
    Always check your result :
  - Step 3 Interpretation of result.
    The limit of y as $t \to \infty$ is zero.

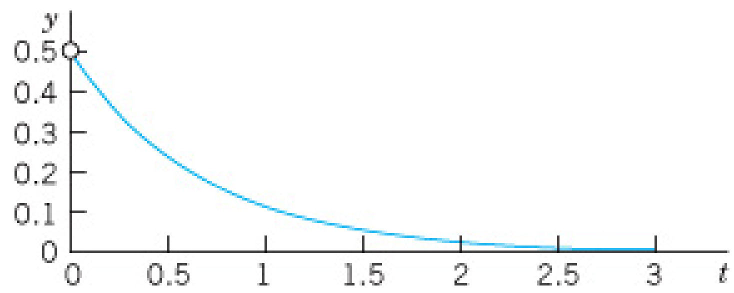

Fig.4. Radioactivity (Exponential decay, $y = 0.5e^{kt}$, with $k = -1.5$ as an example

1.2 Geometric meaning of y'=f(x,y). Direction Fields

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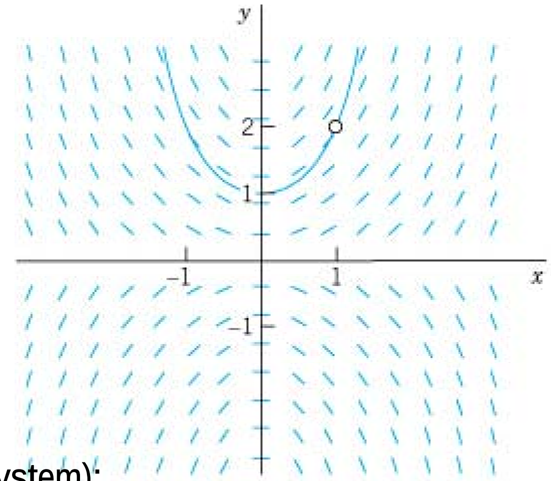
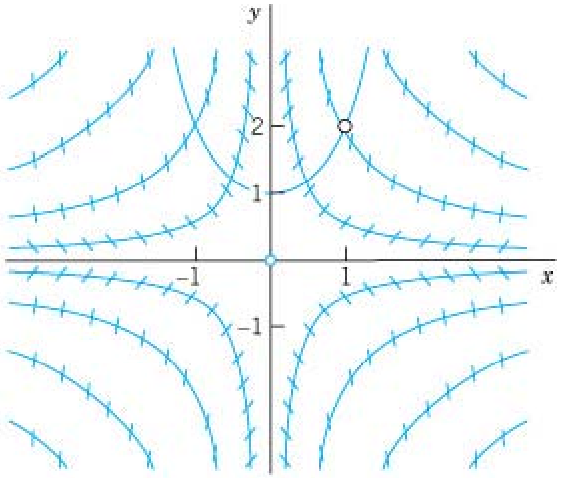
- Direction Field (방향장)
  - The graph with line segments (tangent line to the solution).
- Reason of importance of the direction field
  - You need not solve a ODE
  - The method shows the whole family of solutions and their typical properties.

$$y' = xy$$



(a) By a CAS



(b) By isoclines

Isoclines: curves of equal inclination

Fig. 7 Direction field of *y'* = *xy*

*CAS (Computer Algebra System):
ex) Maple, Mathematica, Matlab

1.3 Separable ODEs. Modeling
Definition

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- Separable Equation

$$g(y)y' = f(x)$$

  - 왼쪽은 y, 오른쪽은 x만으로 구성 가능한 형태

- Method of separation of variables (변수분리법)

$$g(y)y' = f(x) \quad \Rightarrow \quad \int g(y)\,dy = \int f(x)\,dx + c \quad \left(\because \frac{dy}{dx}dx = dy \right)$$

1.3 Separable ODEs. Modeling
Example.1

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Solve $y' = 1 + y^2$

$$\frac{y'}{1+y^2} = 1 \quad \Rightarrow \quad \frac{dy/dx}{1+y^2} = 1 \quad \Rightarrow \quad \frac{dy}{1+y^2} = dx$$ **(변수분리형)**

$$\Rightarrow \quad \int \frac{1}{1+y^2}\,dy = \int dx + c \quad \Rightarrow \quad \arctan y = x + c$$ **(적분)**

$$\Rightarrow \quad y = \tan(x+c)$$ **(정리)**

1.3 Separable ODEs. Modeling
Example 3. Mixing Problem

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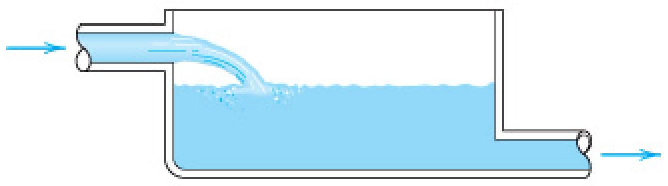
– Initial Condition: 1000 gal of water, 100 lb salt, initially brine runs in 10 gal/min, 5 lb/gal, stirring all the time, brine runs out at 10 gal/min

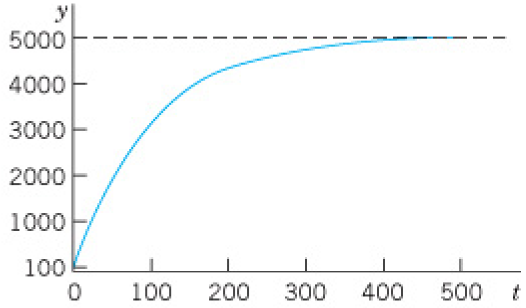
– Amount of salt at t?

– Step1. Setting up a model

– Step2. Solution of the model



Tank



Salt content $y(t)$

1.3 Separable ODEs. Modeling
Extended Method: Reduction to Separable Form

- When equation is not separable???
- Extended Method: Reduction to Separable Form
  - A certain 1st order equation can be made separable by a simple change of variables

$$y' = f\left(\frac{y}{x}\right) \qquad Ex.\ \cos\left(\frac{y}{x}\right)$$

$$\left(y = ux \quad \Rightarrow \quad u = \frac{y}{x} \ \&\ y' = (ux)' = u'x + u\right)$$

$$y' = f\left(\frac{y}{x}\right) \quad \Rightarrow \quad u'x + u = f(u) \quad \Rightarrow \quad \frac{du}{f(u) - u} = \frac{dx}{x}$$

- Example 6. Reduction to Separable Form. Solve

$$2xyy' = y^2 - x^2$$

Example 6. Reduction to Separable form

- Example 6. Reduction to Separable Form. Solve

$$2xyy' = y^2 - x^2$$

$$2xyy' = y^2 - x^2 \quad \Rightarrow \quad y' = \frac{1}{2}\left(\frac{y}{x} - \frac{x}{y}\right)$$ **(2*xy*로 나눔)**

$$\Rightarrow \quad y = ux, \quad u = \frac{y}{x}, \quad y' = u'x + u = \frac{1}{2}\left(u - \frac{1}{u}\right)$$

$$u'x = -\frac{1}{2}\left(u + \frac{1}{u}\right) = -\frac{u^2+1}{2u} \quad \Rightarrow \quad \frac{du}{dx}\frac{2u}{u^2+1} = -\frac{1}{x} \quad \Rightarrow \quad \frac{2u}{u^2+1}du = -\frac{1}{x}dx$$

$$\int \frac{2u}{u^2+1}du = -\int \frac{1}{x}dx + c^* \quad \Rightarrow \quad \ln\left|u^2+1\right| = -\ln|x| + c^* = \ln\frac{1}{|x|} + \ln|c| = \ln\left|\frac{c}{x}\right|, \quad c = e^{c^*}$$

$$u^2 + 1 = \frac{c}{x} \quad \Rightarrow \quad \left(\frac{y}{x}\right)^2 + 1 = \frac{c}{x} \quad \Rightarrow \quad x^2 + y^2 = cx$$

$$\left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}$$

1. 4 Exact ODEs. Integrating Factors
Basic Ideas

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For $u(x, y)$, *its differential* (미분) *is*

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$if\ \ u(x, y) = c,\ then\ du = 0$$

$$For\ example,\ if\ \ u = x + x^2y^3 = c$$

$$du = (1 + 2xy^3)dx + (3x^2y^2)dy = 0$$

$$or\ \ y' = \frac{dy}{dx} = -\frac{1 + 2xy^3}{3x^2y^2}$$

1. 4 Exact ODEs. Integrating Factors

Definition

- Exact Differential Equation (완전미분방정식) :

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

  – If the differential form $M(x,y)\,dx + N(x,y)\,dy$ is exact

  – This means that this form is the differential of u(x,y)

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \qquad \Rightarrow \quad du = 0 \quad \Rightarrow \quad u(x,y) = c$$

  – Condition for exactness

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \left(\because \quad \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial N}{\partial x} \right)$$

1. 4 Exact ODEs. Integrating Factors
Solution method

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- Solution method of exact differential equation (완전미분방정식 해법) :

$$M(x,y)\,dx + N(x,y)\,dy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$

Case 1)

$$M(x,y) = \frac{\partial u}{\partial x} \quad \Rightarrow \quad \underline{u(x,y) = \int M(x,y)\,dx + k(y)}$$

$$\Rightarrow \quad \frac{\partial u}{\partial y} = N(x,y) \quad \Rightarrow \quad \frac{dk}{dy} \;\; \& \;\; k(y)$$

Case 2)

$$N(x,y) = \frac{\partial u}{\partial y} \quad \Rightarrow \quad \underline{u(x,y) = \int N(x,y)\,dy + l(x)}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = M(x,y) \quad \Rightarrow \quad \frac{dl}{dx} \;\; \& \;\; l(x)$$

1. 4 Exact ODEs. Integrating Factors
Example 1.

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- Solve $$\cos(x+y)\,dx+\left(3y^2+2y+\cos(x+y)\right)dy=0$$
  - Step 1 Test for exactness. $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

$$M(x,y)=\cos(x+y)\quad\Rightarrow\quad\frac{\partial M}{\partial y}=-\sin(x+y)$$

$$N(x,y)=3y^2+2y+\cos(x+y)\quad\Rightarrow\quad\frac{\partial N}{\partial x}=-\sin(x+y)$$

  - Step 2 Implicit general solution.

$$u(x,y)=\int M(x,y)\,dx+k(y)=\int\cos(x+y)\,dx+k(y)=\sin(x+y)+k(y)$$

$$\Rightarrow\quad\frac{\partial u}{\partial y}=\cos(x+y)+\frac{dk}{dy}=N(x,y)\quad\Rightarrow\quad\frac{dk}{dy}=3y^2+2y\quad\Rightarrow\quad k=y^3+y^2+c^*$$

$$\therefore\quad u(x,y)=\sin(x+y)+y^3+y^2=c$$

  - Step 3 Checking an implicit solution.

1. 4 Exact ODEs. Integrating Factors
Integrating Factors

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- Reduction to Exact Form, Integrating Factors
  - Some equations can be made exact by multiplication by some function (called the Integrating Factor, 적분인자)
- Ex 3. $-ydx + xdy = 0$
  - This equation is not exact. Why?
  - multiplying it by $1/x^2$ ← Integrating factor

$$-\frac{y}{x^2}dx + \frac{1}{x}dy = 0 \left(\because \quad \frac{\partial}{\partial y}\left(-\frac{y}{x^2}\right) = -\frac{1}{x^2} = \frac{\partial}{\partial x}\left(\frac{1}{x}\right) \right)$$

- Issue is then how to find this integrating factor (when it is not simple)?

1. 4 Exact ODEs. Integrating Factors
Integrating Factors

- Given a nonexact equation

$$Pdx + Qdy = 0$$

  – Finding Integrating Factors (F)

$$FPdx + FQdy = 0$$

  – The exactness condition

$$\frac{\partial}{\partial y}(FP) = \frac{\partial}{\partial x}(FQ) \quad \Rightarrow \quad \frac{\partial F}{\partial y}P + F\frac{\partial P}{\partial y} = \frac{\partial F}{\partial x}P + F\frac{\partial Q}{\partial x}$$

1. 4 Exact ODEs. Integrating Factors
Integrating Factors

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– Golden Rule: Solve a simpler one. Hence we look for an integrating factor depending *only* on one variable.

$$F = F(x) \quad \Rightarrow \quad \frac{\partial F}{\partial x} = F', \quad \frac{\partial F}{\partial y} = 0$$

$$FP_y = F'Q + FQ_x \quad \Rightarrow \quad \frac{1}{F}\frac{dF}{dx} = R(x) \text{ where } R(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$\Rightarrow \quad \therefore \; F(x) = \exp\left(\int R(x)\,dx\right)$$

$$F^* = F^*(y) \;\Rightarrow\; \frac{1}{F^*}\frac{dF^*}{dx} = R^* \text{ where } R^* = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \;\Rightarrow\; F^*(y) = \exp\left(\int R^*(y)\,dy\right)$$

1. 4 Exact ODEs. Integrating Factors
Integrating Factors

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Integrating Factor $F(x)$

If (12) is such that the right side R of (16), depends only on x, then (12) has an integrating factor $F = F(x)$, which is obtained by integrating (16) and taking exponents on both sides,

(17) $$F(x) = \exp \int R(x)\, dx.$$

$$R(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

Integrating Factor $F^*(y)$

If (12) is such that the right side R^* of (18) depends only on y, then (12) has an integrating factor $F^* = F^*(y)$, which is obtained from (18) in the form

(19) $$F^*(y) = \exp \int R^*(y)\, dy.$$

$$R^* = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

1. 4 Exact ODEs. Integrating Factors
Integrating Factors

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- Integrating factor for Exact Differential Equation???

$$F(x) = \exp\left(\int R(x)\,dx\right) \qquad R(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$F^*(y) = \exp\left(\int R^*(y)\,dy\right) \qquad R^* = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

1. 4 Exact ODEs. Integrating Factors
Example 5 - Integrating Factors

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- EX. 5 Find Integrating Factor and solve the following initial value problem

$$\left(e^{x+y} + ye^{y}\right)dx + \left(xe^{y} - 1\right)dy = 0, \quad y(0) = -1$$

  – Step 1. Nonexactness

  – Step 2. Integrating factor. General Solution

  – Step 3. Particular Solution

1. 4 Exact ODEs. Integrating Factors

Example 5 - Integrating Factors

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$$\left(e^{x+y}+ye^{y}\right)dx+\left(xe^{y}-1\right)dy=0,\quad y(0)=-1$$

- Step 1. Nonexactness

$$P(x,y)=e^{x+y}+ye^{y}\ \Rightarrow\ \frac{\partial P}{\partial y}=e^{x+y}+e^{y}+ye^{y}$$

$$Q(x,y)=xe^{y}-1\ \Rightarrow\ \frac{\partial Q}{\partial x}=e^{y}$$

$$\frac{\partial P}{\partial y}\neq\frac{\partial Q}{\partial x}$$

- Step 2.

$$R=\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=\frac{1}{xe^{y}-1}\left(e^{x+y}+e^{y}+ye^{y}-e^{y}\right)=\frac{1}{xe^{y}-1}\left(e^{x+y}+ye^{y}\right)\ \Rightarrow\ \text{fails}$$

$$R^{*}=\frac{1}{P}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)=\frac{1}{e^{x+y}+ye^{y}}\left(e^{y}-e^{x+y}-e^{y}-ye^{y}\right)=-1\ \Rightarrow\ F^{*}(y)=e^{-y}$$

$$\therefore\quad \left(e^{x}+y\right)dx+\left(x-e^{-y}\right)dy=0$$

$$u=\int\left(e^{x}+y\right)dx=e^{x}+xy+k(y)\ \Rightarrow\ \frac{\partial u}{\partial y}=x+k'(y)=x-e^{-y}\ \Rightarrow\ k'(y)=-e^{-y},\ k(y)=e^{-y}$$

$$u(x,y)=e^{x}+xy+e^{-y}=c$$

- Step 3. Particular Solution

$$y(0)=-1\ \Rightarrow\ u(0,-1)=e^{0}+0+e=3.72\quad\therefore\ u(x,y)=e^{x}+xy+e^{-y}=3.72$$

Last Lecture

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- Exact differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \qquad \leftarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$M(x,y) = \frac{\partial u}{\partial x} \quad \Rightarrow \quad u(x,y) = \int M(x,y)\,dx + k(y) \quad \Rightarrow \quad \frac{\partial u}{\partial y} = N(x,y) \quad \Rightarrow \quad \frac{dk}{dy} \ \ \& \ \ k(y)$$

  - Non exact differential equation (finding integrating factors)

$$Pdx + Qdy = 0 \qquad FPdx + FQdy = 0$$

$$F(x) = \exp\left(\int R(x)\,dx\right), \ \ where \ \ R(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

1.5 Linear ODEs
Introduction

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- 미분방정식 (ODEs)
  - 선형 미분방정식 (Linear ODEs)
    - 제차 선형 미분방정식 (Homogeneous Linear ODEs)
    - 비제차 선형미분방정식 (Nonhomogeneous Linear ODEs)
  - 비선형 미분방정식 (Nonlinear ODEs)

Linear ODE (선형미분방정식): 종속변수와 그 도함수가 모두 1차인 미분방정식. 각 계수는 독립변수에만 의존

1.5 Linear ODEs
Definition and Standard Form

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– Linear ODEs are models of various phenomena

– Linear ODEs: linear in both the unknown function and its derivatives. p(x) and r(x) can be any given functions of x.

– ex)

$$y' + p(x)y = r(x)$$ → Standard Form (start with y')

$$y'\cos x + y\sin x = x$$

- Homogeneous Linear ODEs (r(x) is zero for all x) $r(x) \equiv 0$
  ex) $y' + p(x)y = 0$

- Nonhomogeneous Linear ODEs
  ex) $y' + p(x)y = r(x) \neq 0$

– Nonlinear ODEs = Not linear ODEs.
  Ex) $y' + p(x)y = r(x)y^2$

1.5 Linear ODEs
Solution method

- Homogeneous Linear ODE.(Apply the method of separating variables)

$$y' + p(x)y = 0$$

- Nonhomogeneous Linear ODE.(Find integrating factor and solve)

$$y' + p(x)y = r(x)$$

1.5 Linear ODEs
Solution method

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– Homogeneous Linear ODE.(Apply the method of separating variables)

$$y'+p(x)y=0 \quad \Rightarrow \quad \frac{dy}{y}=-p(x)dx \quad \Rightarrow \quad \ln|y|=-\int p(x)dx+c^*$$

$$y=ce^{-\int p(x)dx}$$

When c=0 → trivial solution (자명해)

– Nonhomogeneous Linear ODE.(Find integrating factor and solve)

$$y'+p(x)y=r(x) \quad \Rightarrow \quad (py-r)dx+dy=0$$

Not exact! $\left(\because \frac{\partial}{\partial y}(py-r)=p\neq 0=\frac{\partial}{\partial x}(1)\right)$

$$R=\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=p \quad \Rightarrow \quad \frac{1}{F}\frac{dF}{dx}=p \quad \Rightarrow \quad \therefore \; F=e^{\int pdx}$$

$$e^{\int pdx}(py-r)dx+e^{\int pdx}dy=0$$

$$u=ye^{\int pdx}+l(x) \quad \Rightarrow \quad \frac{\partial u}{\partial x}=pye^{\int pdx}+l'(x)=e^{\int pdx}(py-r) \quad \Rightarrow \quad l'(x)=-re^{\int pdx}, \; l(x)=-\int re^{\int pdx}dx+c$$

$$\Rightarrow \quad u=ye^{\int pdx}-\int re^{\int pdx}dx=c \quad \Rightarrow \quad ye^{\int pdx}=\int re^{\int pdx}dx+c$$

$$y=e^{-\int pdx}\left[\int re^{\int pdx}dx+c\right]$$

1.5 Linear ODEs.
Example 1. Linear ODE

- EX.1 solve the linear ODE $y' - y = e^{2x}$

$$p = -1, \quad r = e^{2x}, \quad h = \int p\,dx = -x \qquad \Rightarrow$$

$$\therefore \quad y = e^{-h}\left[\int e^{h} r\,dx + c\right] = e^{x}\left[\int e^{-x} e^{2x}\,dx + c\right] = e^{x}\left[e^{x} + c\right] = e^{2x} + ce^{x}$$

Example 2. Linear ODE

- Example 2)

$$y' + y\tan x = \sin 2x \qquad\qquad y(0) = 1$$

1.5 Linear ODEs.
Bernoulli Equation

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- Bernoulli Equation (nonlinear → linear)

$$y' + p(x)y = g(x)y^a \qquad (a \neq 0 \ \& \ a \neq 1)$$

$$u(x) = [y(x)]^{1-a}$$

$$\Rightarrow \quad u' = (1-a)y^{-a}y' = (1-a)y^{-a}(gy^a - py) = (1-a)(g - py^{1-a}) = (1-a)(g - pu)$$

$$\Rightarrow \quad u' + (1-a)pu = (1-a)g$$

1.5 Linear ODEs.
Bernoulli Equation (Logistic Equation)

- Ex. 4 Logistic Equation $y' = Ay - By^2$

1. 6 Orthogonal Trajectories (직교궤적)

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- Orthogonal (직교의) ~ perpendicular
- Orthogonal Trajectory:
  - A family of curves in the plane that intersect a given family of curves at right angles.
- Find the orthogonal trajectories by using ODEs.
  - Step 1 Find an ODE $y' = f(x, y)$ for which the give family is a general solution.
  - Step 2 Write down the ODE $\tilde{y}' = -\dfrac{1}{f(x, \tilde{y})}$ of the orthogonal trajectories.
  - Step 3 Solve it.

1. 6 Orthogonal Trajectories.
Example

- Example. A one-parameter family of quadratic parabolas is given by

$$y = cx^2$$

  - Step 1 find an ODE

$$\frac{y}{x^2} = c \quad \Rightarrow \quad \frac{y'x^2 - 2xy}{x^4} = 0 \quad \Rightarrow \quad y' = \frac{2y}{x}$$

  - Step 2 Write down the ODE of the orthogonal trajectories

$$\tilde{y}' = -\frac{x}{2\tilde{y}}$$

  - Step 3 solve above ODE

$$2\tilde{y}\tilde{y}' + x = 0 \quad \Rightarrow \quad \tilde{y}^2 + \frac{1}{2}x^2 = c^*$$

1. 6 Orthogonal Trajectories.
Example – Problem set 1.6 – 16.

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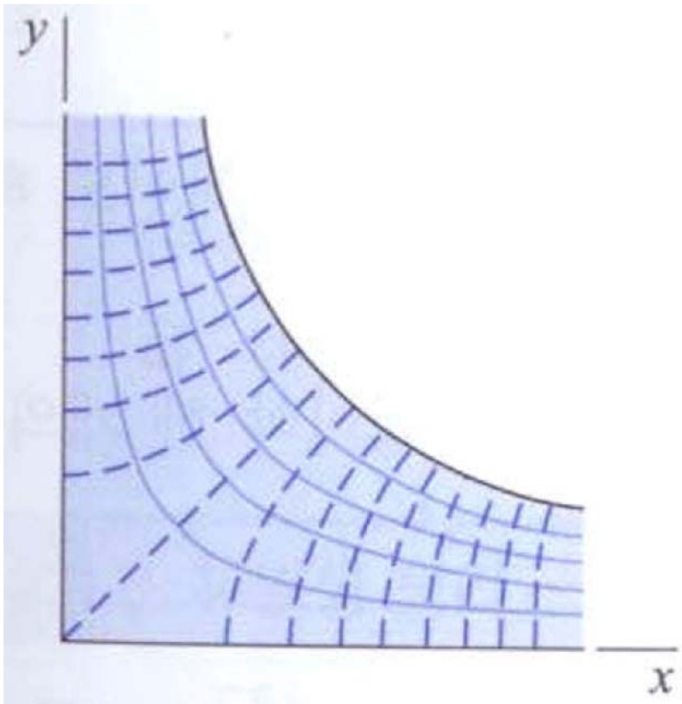
- Streamlines and equipotential lines



Flow in a channel

1. 7 Existence and Uniqueness of Solutions

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- An initial value problem may have no solution, precisely one solution, or more than one solution

$|y'|+|y|=0, \quad y(0)=1 \quad \Rightarrow$ No solution

$y'=2x, \quad y(0)=1 \quad \Rightarrow$ Precisely one solution $\quad \Rightarrow \quad y=x^2+1$

$xy'=y-1, \quad y(0)=1 \quad \Rightarrow$ Infinitely many solutions $\quad \Rightarrow \quad y=1+cx$

- Problem of Existence
  - Under what conditions does an initial value problem have at least one solution (hence one or several solutions)?
- Problem of Uniqueness
  - Under what conditions does that problem have at most one solution (hence excluding the case that is has more than one solution)?

1. 7 Existence and Uniqueness of Solutions

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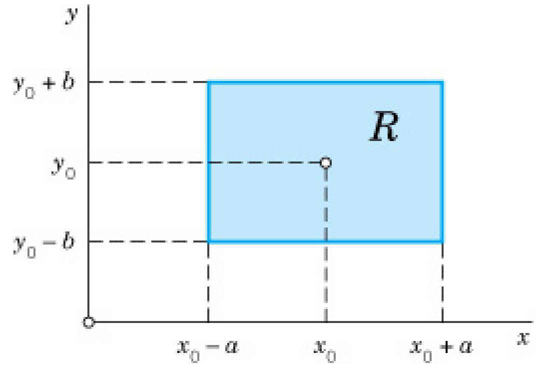
● Theorem 1 Existence Theorem



Let the right side *f* (*x*,*y*) of the ODE in the initial value problem

(1) $$y' = f(x,y), \quad y(x_0) = y_0$$

be continuous at all points (*x*,*y*) in some rectangle

$$R \; : \; |x - x_0| < a, \quad |y - y_0| < b$$

and bounded in *R* ; that is, there is a number *K* such that

(2) $$|f(x,y)| \le K$$ for all (*x*,*y*) in *R*.

Then the initial value problem (1) has at least one solution *y*(*x*). This solution exists at least for all *x* in the subinterval $|x - x_0| < \alpha$ of the interval $|x - x_0| < a$; here, α is the smaller of the two numbers *a* and *b*/*K*.

1. 7 Existence and Uniqueness of Solutions

● Theorem 2 Uniqueness Theorem

Let *f* and its partial derivative $f_y = \partial f / \partial y$ be continuous for all (*x*,*y*) in the rectangle *R* and bounded, say,

(3) (a) $|f(x,y)| \le K$ (b) $|f_y(x,y)| \le M$ for all (*x*,*y*) in *R*.

Then the initial value problem (1) has at most one solution *y*(*x*). Thus, by the Existence Theorem, the problem has precisely one solution. This solution exists at least for all *x* in that subinterval

. $|x - x_0| < \alpha$

1. First-Order ODEs
Summary (1)

- Differential Equation?
  - ODE vs. PDE,
  - 1st order, 2nd order, …
  - linear vs. nonlinear
- Physical behavior → mathematical model and solution
- Solution method
  - Separation of variables & Reduction to separable form
  - Exact differential Equation & Integrating Factors
  - Linear ODEs & Bernoulli Eq (nonlinear)

1. First-Order ODEs
Summary (2)

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- Separation of variables & Reduction to separable form
  - Separation of variables

$$g(y)y' = f(x)$$

$$g(y)y' = f(x) \qquad \Rightarrow \qquad \int g(y)\,dy = \int f(x)\,dx + c$$

  - Extended Method: Reduction to separable form

$$y' = f\left(\frac{y}{x}\right)$$

$$y' = f\left(\frac{y}{x}\right) \;\Rightarrow\; u'x + u = f(u) \;\Rightarrow\; \frac{du}{f(u) - u} = \frac{dx}{x}$$

1. First-Order ODEs
Summary (3)

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- Exact differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \leftarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$M(x,y) = \frac{\partial u}{\partial x} \quad \Rightarrow \quad u(x,y) = \int M(x,y)\,dx + k(y) \quad \Rightarrow \quad \frac{\partial u}{\partial y} = N(x,y) \quad \Rightarrow \quad \frac{dk}{dy} \;\;\&\;\; k(y)$$

  - Non exact differential equation (finding integrating factors)

$$Pdx + Qdy = 0 \qquad FPdx + FQdy = 0$$

$$F(x) = \exp\left(\int R(x)\,dx\right), \; where \; R(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

1. First-Order ODEs
Summary (4)

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- Linear ODEs $y' + p(x)y = r(x)$
  - Homogeneous ODEs

$$y' + p(x)y = 0 \quad \Rightarrow \quad y = ce^{-\int p(x)dx}$$

  - Nonhomgeneous ODEs

$$y' + p(x)y = r(x) \quad \Rightarrow \quad (py - r)dx + dy = 0$$

$$y = e^{-h}\left[\int e^h r dx + c\right], \ where = h = \int pdx$$

  - Bernoulli Equation (reduction to linear ODEs)

$$y' + p(x)y = g(x)y^a \quad (a \neq 0 \ \& \ a \neq 1)$$

$$u' + (1-a)pu = (1-a)g \leftarrow u(x) = [y(x)]^{1-a}$$

Acknowledgement and References

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- All the figures and problems sets are from the following book unless otherwise stated.
  - Erwin Kreyszig, 2006, Advanced Engineering Mathematics, 9th Ed., Wiley