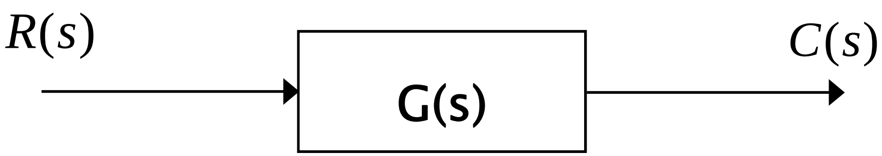
Linear Systems Analysis
in the Time Domain II
- Transient Response -

Second Order Systems



$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$R(s) = \frac{1}{s} \ \ (step\ input), \qquad C(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s} \quad = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{1}{s} - \frac{(s + \zeta\omega_n) + \left(\zeta / \sqrt{1-\zeta^2}\right)\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t - \phi)$$

$$\phi = \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}$$

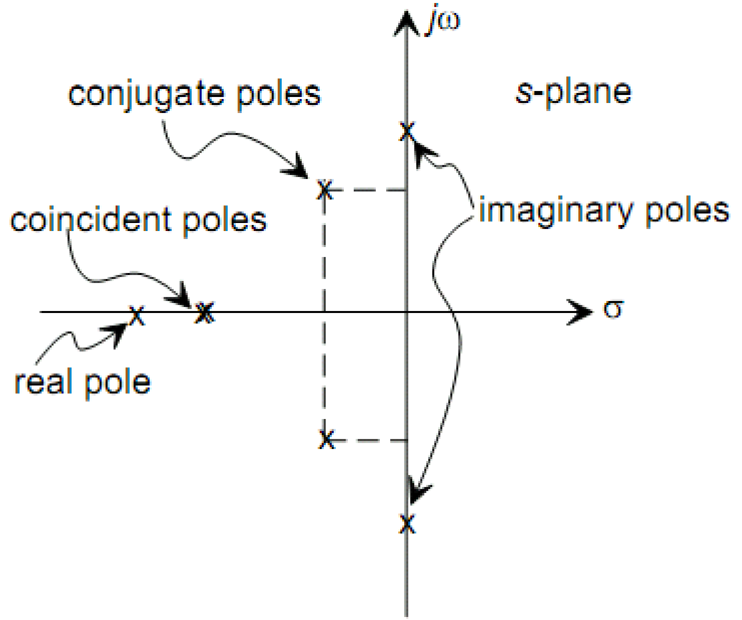
Damping ratio and Pole placement

i) $\zeta > 1$: poles are real and distinct (over damped)

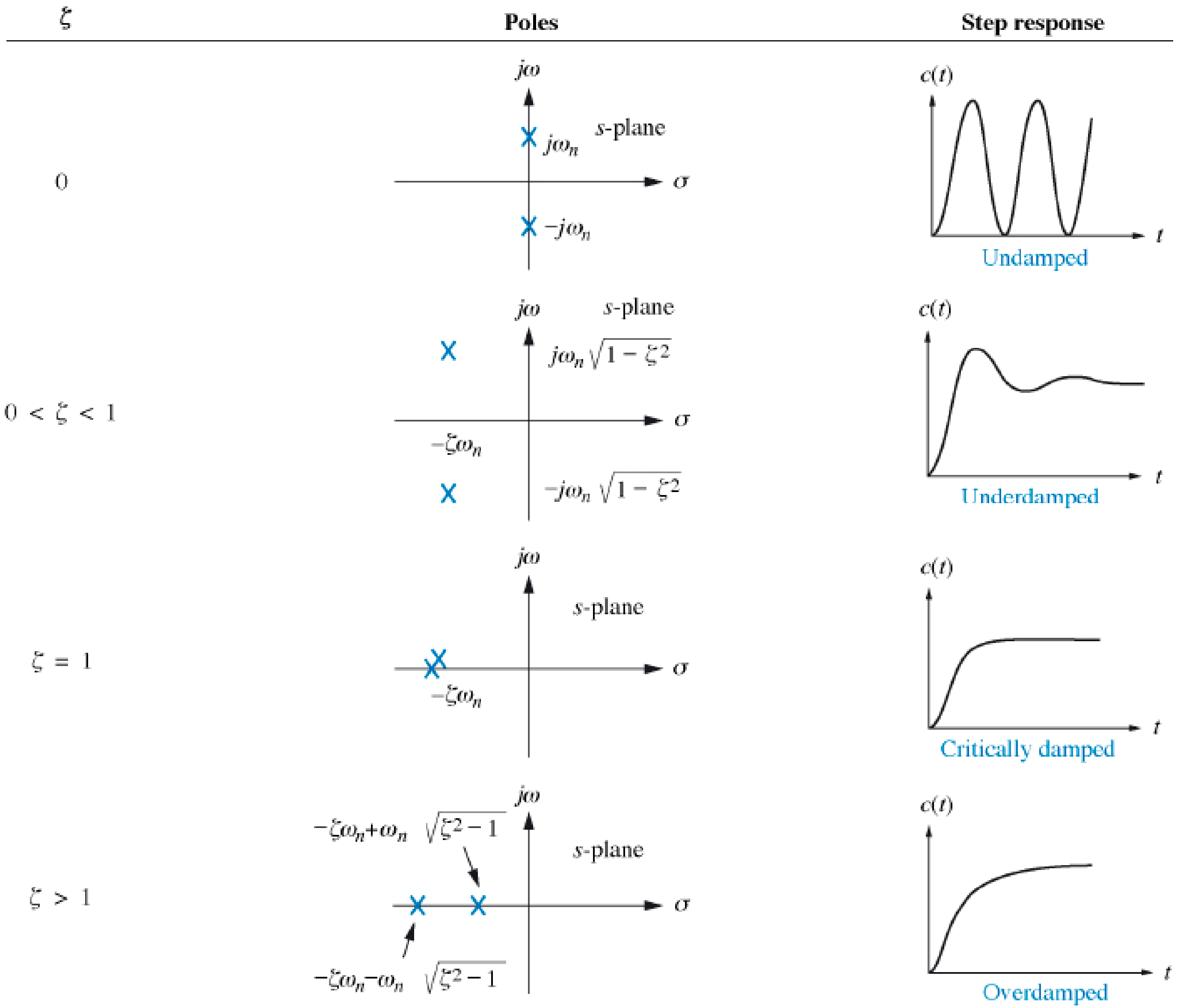
ii) $\zeta = 1$: poles are real and coincident (critically damped)

iii) $0 < \zeta < 1$: pole are complex conjugates (under damped)

iv) $\zeta = 0$: The pole are purely imaginary (undamped)



Step Response of Second-Order Systems



Step Response of Second-Order Systems

1. Over damped Case $\quad p_1, p_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

$$y_{step}(t) = 1 - C_1 e^{-p_1 t} - C_2 e^{-p_2 t}$$

2. Critically damped Case $\quad p_1, p_2 = -\zeta\omega_n$

$$y_{step}(t) = 1 - C_1 e^{-pt} - C_2 t e^{-pt}$$

3. Under damped Case $\quad p_1, p_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$

$$y_{step}(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t - \phi) \qquad \phi = \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}$$

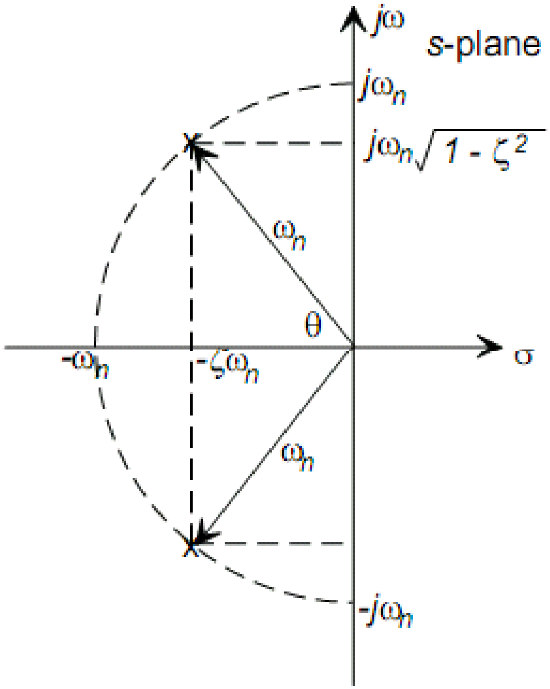
4. Undamped Case $\quad p_1, p_2 = \pm j\omega_n$

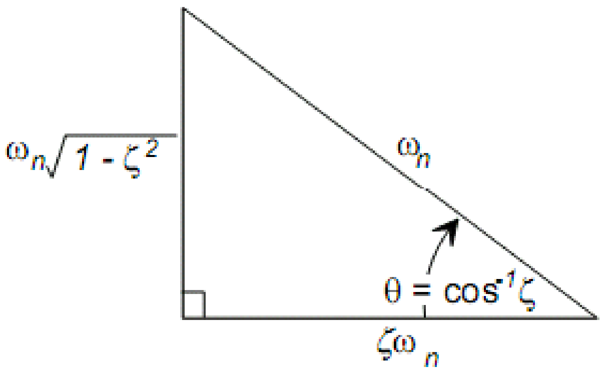
$$y_{step}(t) = 1 - \cos(\omega_n t)$$

Under-damped Second-Order System

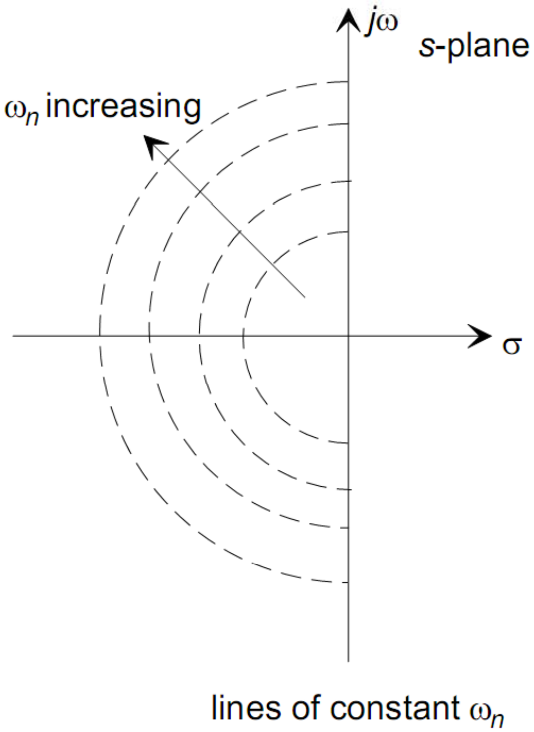
$$p_1, p_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

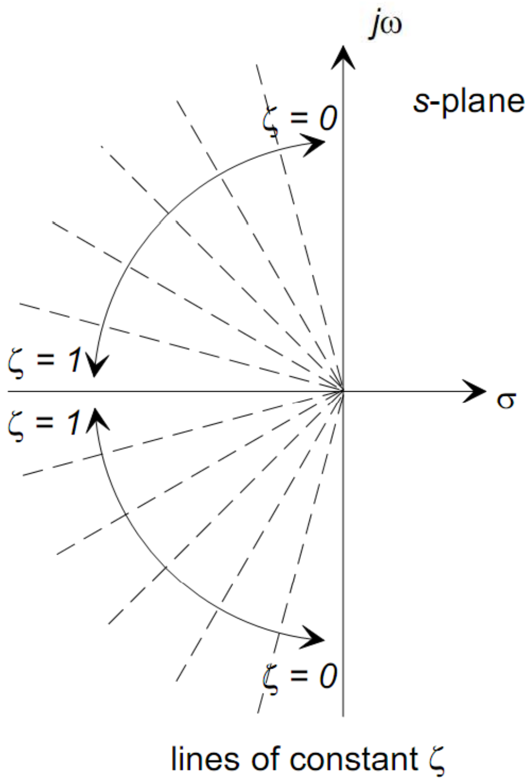
Damped Natural Frequency: $\omega_d = \omega_n\sqrt{1-\zeta^2}$





Influence of ω_n and ζ on the pole locations

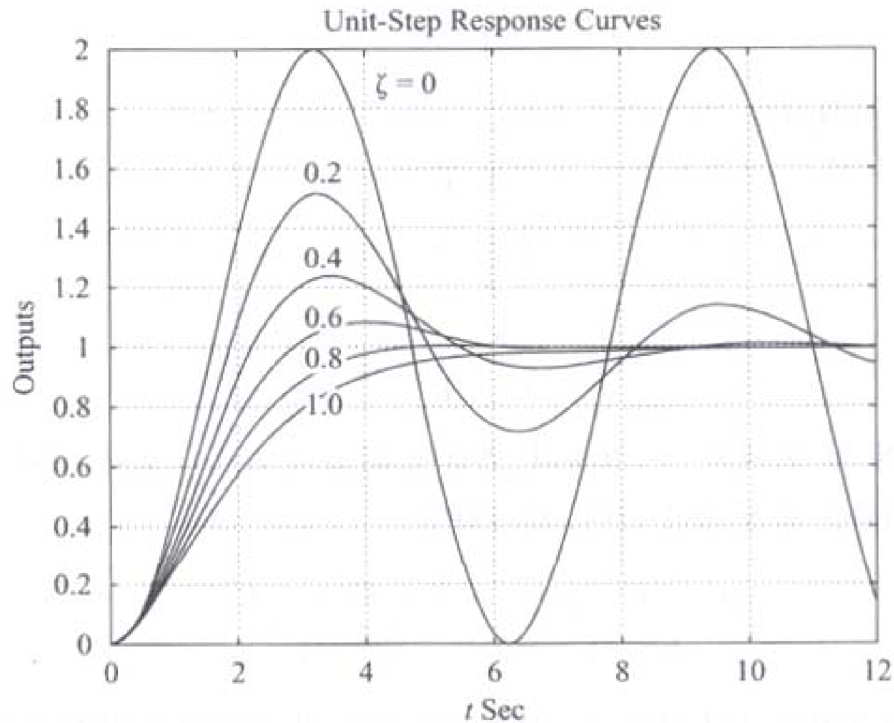




Influence of ζ

$$y_{step}(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_d t = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \eta)$$

$$\eta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$



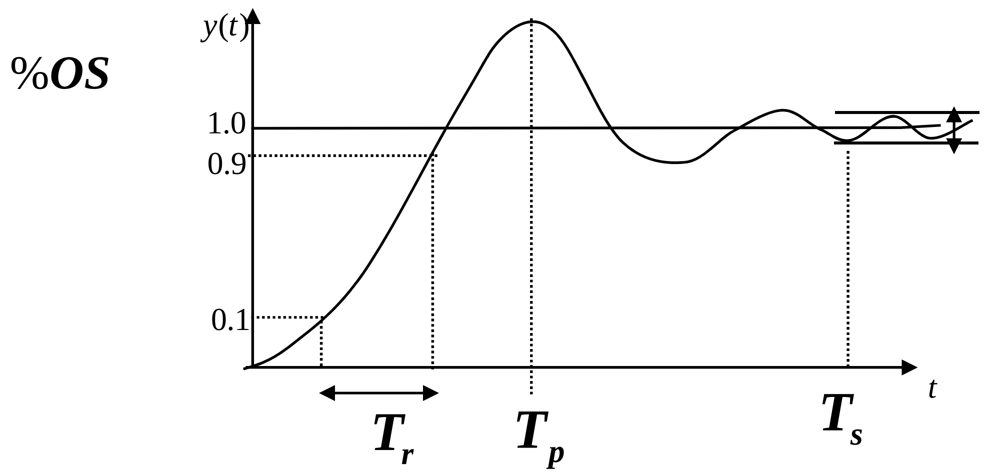
Step Response Based Second -Order System Specifications

1) Peak Time: T_p The time required to reach the first or maximum peak

2) Settling Time: T_s The time required for the transeints' damped oscillations to reach and stay within $\pm 2\%$ of the steady-state value.

3) Rise time : T_r The time required to go from 0.1 to 0.9 of the final value

4) Percent Overshoot: % OS The amount that the waveform overshoots the steady-state at the peak time, expressed as a percentage of the steady-state value



Step Response Based Second -Order System Specifications

1) Peak Time T_p

$$sY(s) = \frac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}} = \frac{\omega_n^{\ 2}}{(s+\zeta\omega_n)^2 + \omega_n^{\ 2}(1-\zeta^2)} = \frac{\dfrac{\omega_n}{\sqrt{1-\zeta^2}}\,\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n^{\ 2}(1-\zeta^2)}$$

$$\dot{y}(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}\,t\right)$$

Set $\dot{y}(t) = 0$, $\omega_d t = \pi, 2\pi, \cdots$

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Step Response Based Second -Order System Specifications

2) % OS M_o

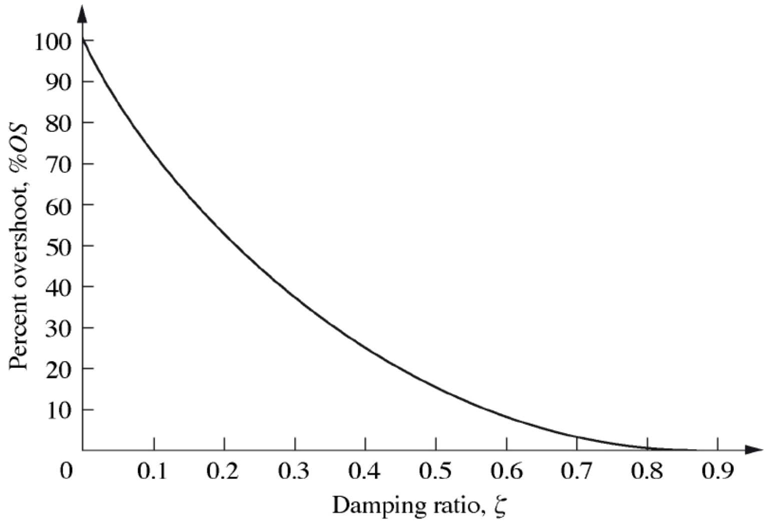
$$\%\,\mathrm{OS} = \frac{y(T_p) - y_{steady-state}}{y_{steady-state}} \times 100$$

$$M_p = y(T_p) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n \cdot \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}} \sin\left(\omega_n\sqrt{1-\zeta^2} \cdot \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} + \phi \right)$$

$$= 1 + \exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}} \right)$$

percent overshoot

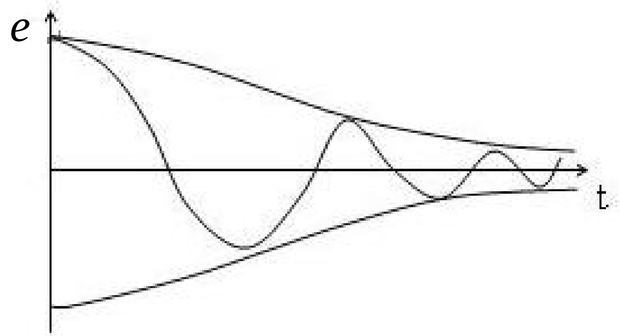
$$M_o = \frac{M_p - y_s}{y_s} \times 100 = \exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}} \right) \times 100$$



Step Response Based Second -Order System Specifications

3) Settling time :

$$e = y - r = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi)$$



$$\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} = 0.2$$

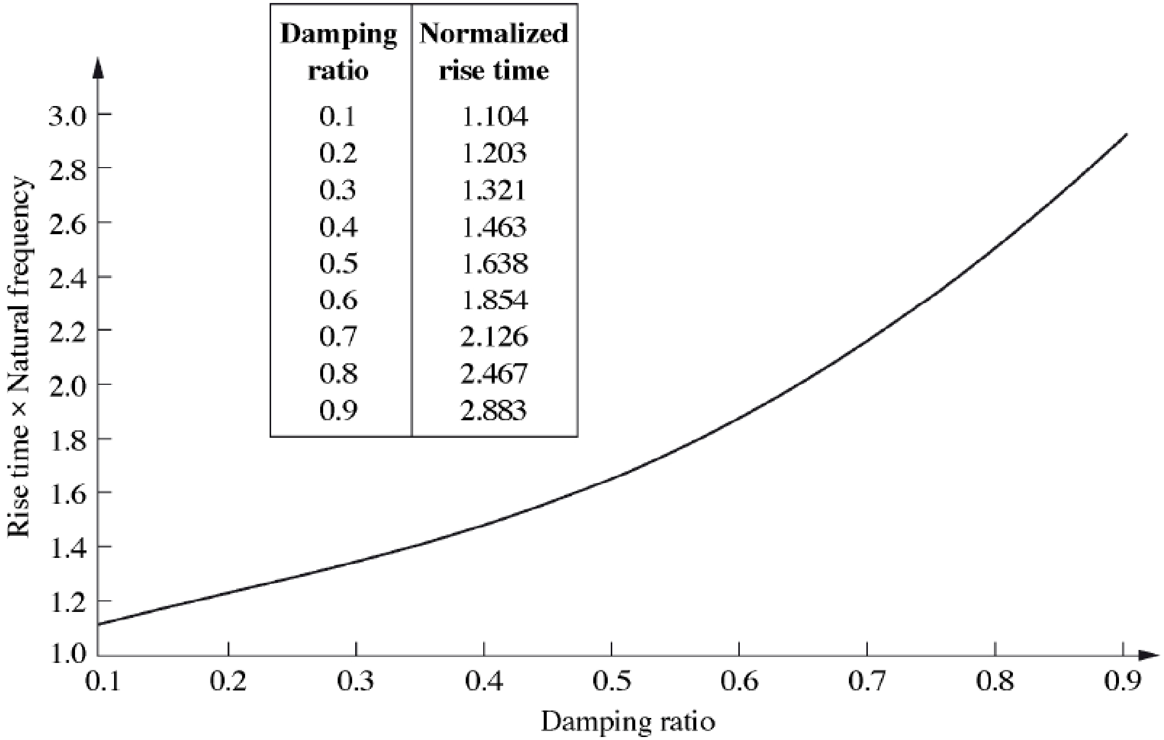
$$T_s = \frac{-\ln(0.2\sqrt{1-\zeta^2})}{\zeta\omega_n}$$

$$2\% \ \ case, \qquad T_s \cong 4 \cdot \frac{1}{\zeta\omega_n}$$

$$5\% \ \ case, \qquad T_s \cong 3 \cdot \frac{1}{\zeta\omega_n}$$

Step Response Based Second -Order System Specifications

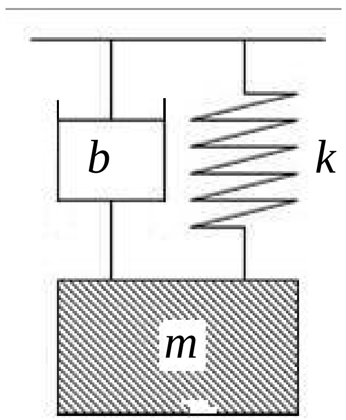
4) Rise time :



Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

Experimental Determination of Damping Ratio

$$m\ddot{x} + b\dot{x} + kx = 0, \qquad \dot{x}(0) = 0$$



$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n{}^2 x = 0$$

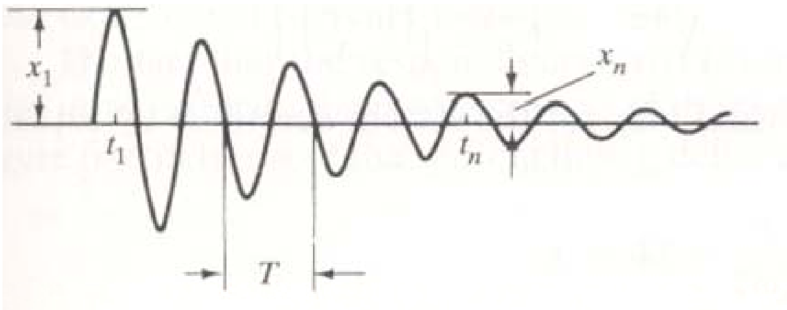
$$\zeta = \frac{1}{2\omega_n}\frac{b}{m} = \frac{b}{2\sqrt{mk}}$$

$$\left[s^2X(s) - sx(0) - \dot{x}(0)\right] + 2\zeta\omega_n\left[sX(s) - x(0)\right] + \omega_n{}^2X(s) = 0$$

$$X(s) = \frac{\left(s + 2\zeta\omega_n\right)x(0)}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}$$

$$x(t) = e^{-\zeta\omega_n t}\left\{\frac{\zeta}{\sqrt{1-\zeta^2}}x(0)\sin\omega_d t + x(0)\cos\omega_d t\right\} = \frac{x(0)}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos\left(\omega_d t - \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$

$$\frac{x_1}{x_n} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n(t_1+(n-1)T)}} = e^{(n-1)\zeta\omega_n T}$$



Experimental Determination of Damping Ratio

Logarithmic decrement

$$\ln\frac{x_1}{x_2} = \zeta\omega_n T = \zeta\omega_n \cdot \frac{2\pi}{\omega_d} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{1}{n-1}\left(\ln\frac{x_1}{x_n}\right)$$

$$\ln\frac{x_1}{x_n} = (n-1)\zeta\omega_n T$$

$$\Rightarrow \ \zeta = \frac{\dfrac{1}{n-1}\left(\ln\dfrac{x_1}{x_n}\right)}{\sqrt{4\pi^2 + \left\{\dfrac{1}{n-1}\left(\ln\dfrac{x_1}{x_n}\right)\right\}^2}}$$

Estimate of Response Time

$$x(t) = \frac{x(0)}{\sqrt{1-\zeta^2}} e^{-\zeta w_n t} \cos\left(\omega_d t - \tan^{-1} \frac{\zeta}{\sqrt{1-\zeta^2}} \right)$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2}, \qquad t = \frac{1}{\zeta \omega_n}, \qquad \omega_d t = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\tan^{-1} \frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{\pi}{2} - \eta$$