Frequency Domain Analysis II
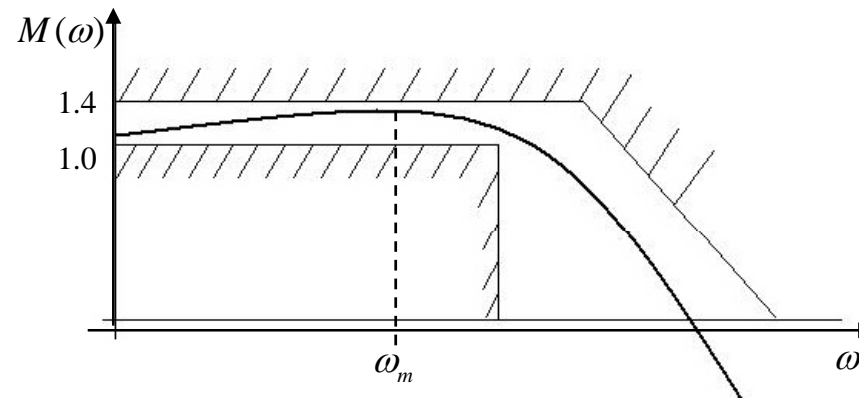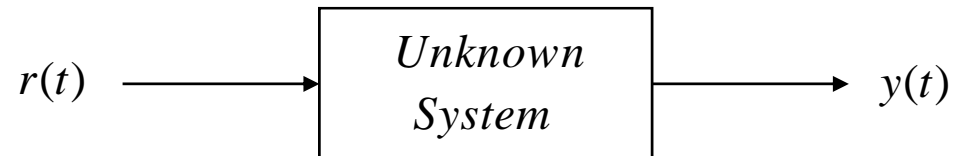
Control System Design in Frequency Response

1. large ω_m → ⟦fast time response⟧

2. M_m, M_p : ⟦function of damping ratio ζ⟧

   large M_m → large M_p

3. good damping characteristics $1 < M_m < 1.4$

4. minimum effect of any undesirable noise

Experimental Determination of Transfer Function

$$r(t) \longrightarrow \boxed{\begin{array}{c} \textit{Unknown} \\ \textit{System} \end{array}} \longrightarrow y(t)$$
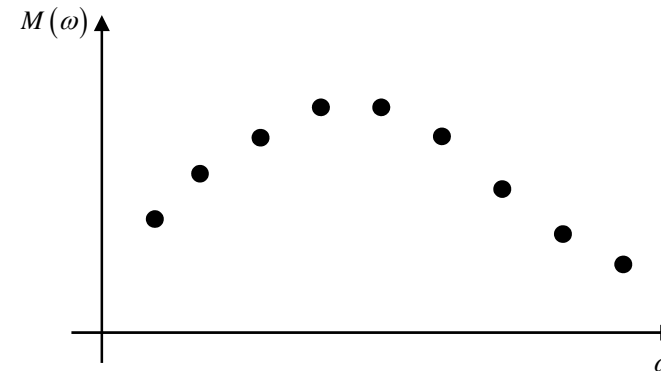
$$r(t) = \sin \omega_i t \quad ; \quad y(t) = M(\omega_i) \sin(\omega_i t - \phi)$$

some frequency $\omega_i \rightarrow$ measure $M(\omega_i)$

steady state

$$G(s) = G(j\omega)$$

$$M(\omega) = |G(j\omega)|$$

Transfer Functions and Frequency Magnitude Plot

- Bode Plot (Logarithmic Plots)

  i) Log Magnitude $\quad L_m G(j\omega) = \boxed{20\log\left|G(j\omega)\right| dB}$

  ii) dB : decibel, Logarithm of the magnitude

  iii) 1 decade : $\quad 1Hz \sim 10Hz \quad$ frequency width

  $\quad\quad\quad\quad\quad 2.5Hz \sim 25Hz$

- Drawing the Bode Plots

  The reason of using logarithm – mathematical operation $\quad \times, \div \rightarrow +, -$

  $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ – typical three types : simple straight line

  $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ asymtotic

  $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ approximations

Transfer Functions and Frequency Magnitude Plot

ex) $$G(s) = \frac{K(1+sT_1)(1+sT_2)}{s^2(1+sT_3)(1+as+bs^2)}$$

$$L_m G(j\omega) = L_m K + L_m(1+j\omega T_1) + L_m(1+j\omega T_2)$$

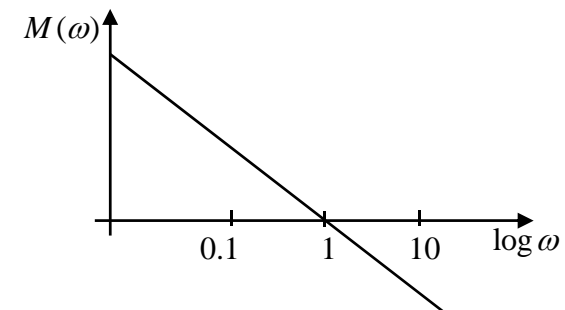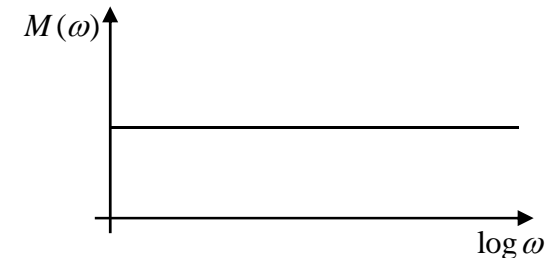$$-2L_m(j\omega) - L_m(1+j\omega T_3) - L_m\left\{1+aj\omega+b(j\omega)^2\right\}$$

i) $L_m \cdot K = \boxed{20\log K \; dB}$

    straight line

ii) $L_m\left(\dfrac{1}{j\omega}\right) = \boxed{20\log\left|\dfrac{1}{j\omega}\right| = -20\log\omega}$
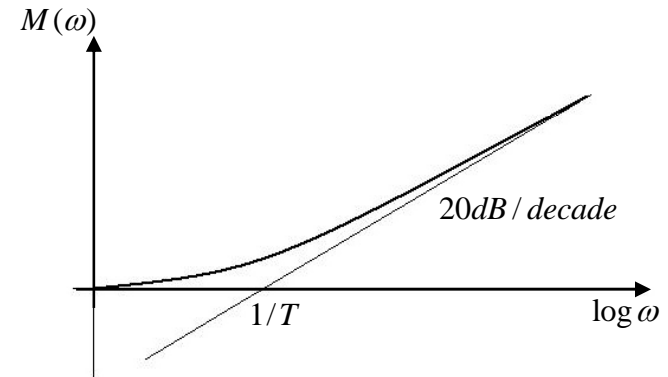
    straight line

    With negative slope   −20dB/decade

Transfer Functions and Frequency Magnitude Plot

$iii)$ $\left(1 + j\omega T\right)$

$$L_m\left(1 + j\omega T\right) = \boxed{20\log\left|1 + j\omega T\right|} = \boxed{20\log\sqrt{1 + \omega^2 T^2}}$$

$$\omega T \ll 1 \quad \rightarrow \quad L_m\left(1 + j\omega T\right) \cong 20\log 1 = 0 dB$$

$$\omega T \gg 1 \quad \rightarrow \quad L_m\left(1 + j\omega T\right) \cong 20\log\omega T$$

$$= 20\log\omega + 20\log T$$

$iv)$ $\dfrac{1}{1 + j\omega T}$

$$L_m\left(1 + j\omega T\right)^{-1} = \boxed{-20\log\left|1 + j\omega T\right| = -20\log\sqrt{1 + \omega^2 T^2}}$$

$$\omega T \ll 1 \quad \rightarrow \quad L_m\left(1 + j\omega T\right)^{-1} \cong -20\log 1 = 0 dB$$

$$\omega T \gg 1 \quad \rightarrow \quad L_m\left(1 + j\omega T\right)^{-1} \cong -20\log\omega T$$

$$= -20\log\omega - 20\log T$$
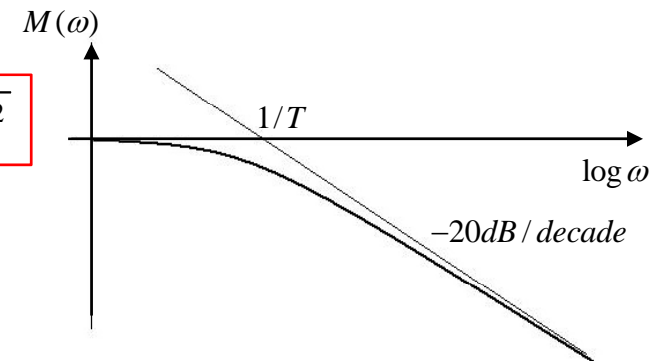
Transfer Functions and Frequency Magnitude Plot

$v)$

$$\frac{1}{\left(1+\frac{2\zeta}{\omega_n}s+\frac{1}{\omega_n^2}s^2\right)} \Rightarrow \frac{1}{\left[1+\frac{2\zeta}{\omega_n}j\omega+\frac{1}{\omega_n^2}(j\omega)^2\right]}$$
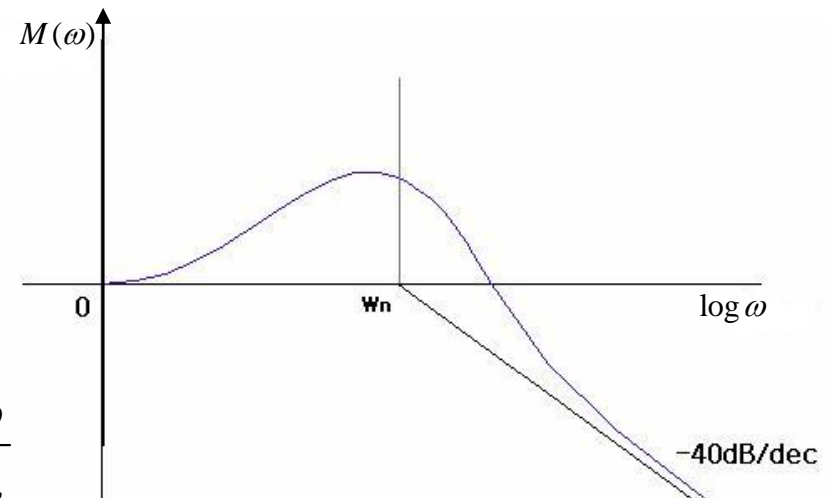
$$L_m\frac{1}{\left[1+\frac{2\zeta}{\omega_n}j\omega+\frac{1}{\omega_n^2}(j\omega)^2\right]} = 20\log\frac{1}{\left[1+\frac{2\zeta}{\omega_n}j\omega+\frac{1}{\omega_n^2}(j\omega)^2\right]} = -20\log\left[\left(1-\frac{\omega^2}{\omega_n^2}\right)+\left(\frac{2\zeta\omega}{\omega_n}\right)^2\right]^{1/2}$$

$$\frac{\omega}{\omega_n} \ll 1$$

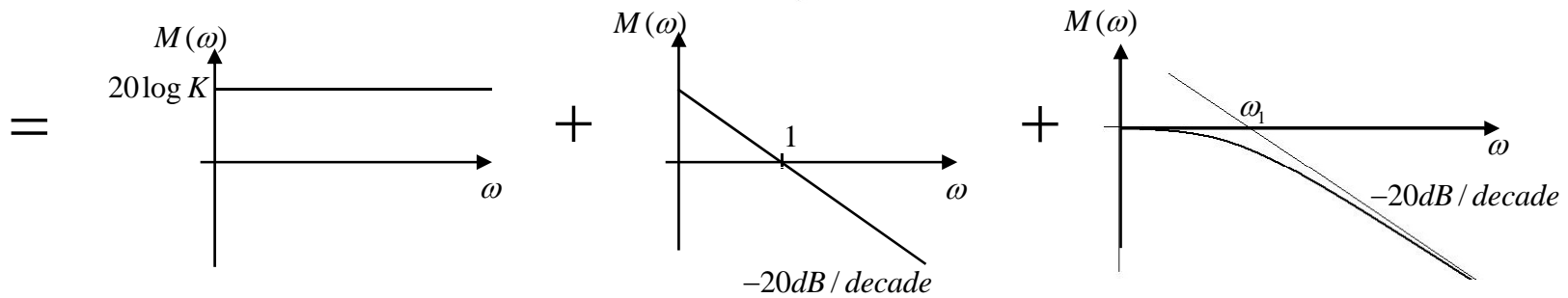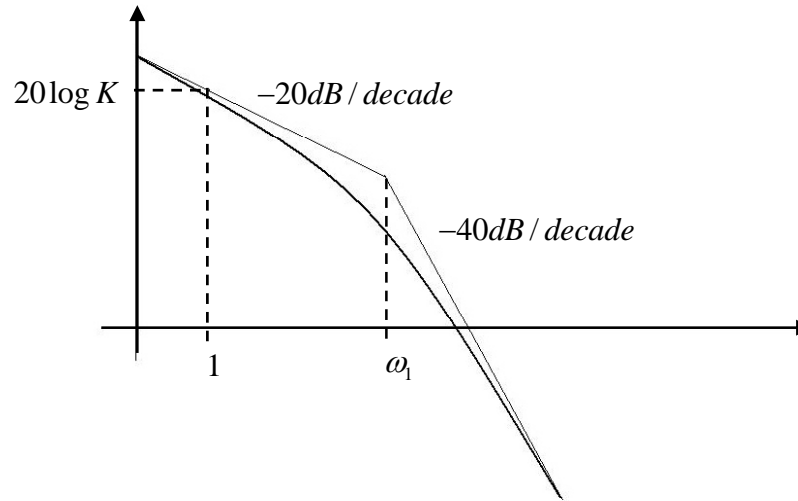$$\rightarrow \quad L_m[G(j\omega)] \cong -20\log 1 = 0dB$$

$$\frac{\omega}{\omega_n} \gg 1$$

$$\rightarrow \quad L_m[G(j\omega)] \cong -20\log\frac{\omega^2}{\omega_n^2} = -40\log\frac{\omega}{\omega_n}$$
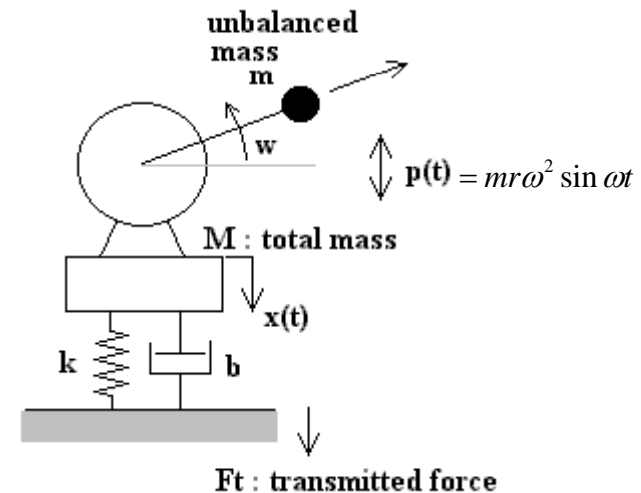
Transfer Functions and Frequency Magnitude Plot

ex)

$20\log K$ $-20dB/decade$

$-40dB/decade$

1 ω_1

$=$

$M(\omega)$

$20\log K$

ω

$+$

$M(\omega)$

1

ω

$-20dB/decade$

$+$

$M(\omega)$

ω_1

ω

$-20dB/decade$

$=$ $\dfrac{K}{(j\omega)\left(1+\dfrac{j\omega}{\omega_1}\right)}$ $\Rightarrow$ $\dfrac{K}{s\left(1+\dfrac{1}{\omega_1}s\right)}$

Vibration Isolation in Rotating Systems

Vibration due to rotating unbalance

(1) $M\ddot{x} + b\dot{x} + kx = p(t)$

$$= mr\omega^2 \sin\omega t$$

$$\frac{X(s)}{P(s)} = \frac{1}{Ms^2 + bs + k} = G(s)$$

unbalanced
mass
m

w

$p(t) = mr\omega^2 \sin\omega t$

M : total mass

x(t)

k   b

Ft : transmitted force
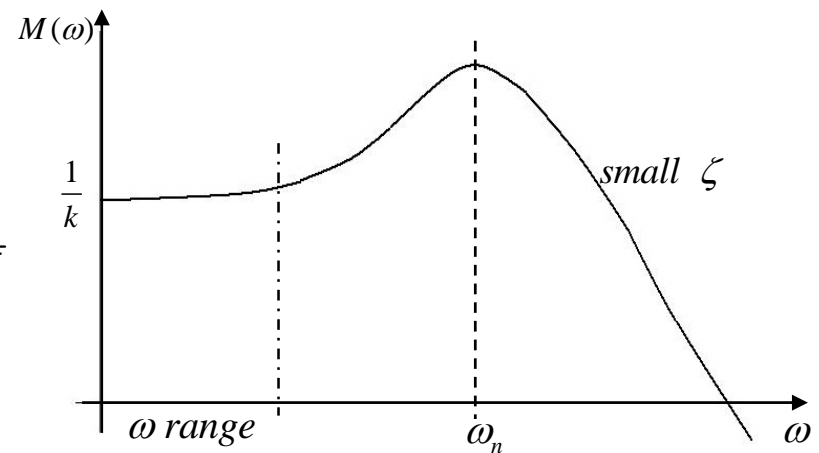
Vibration Isolation in Rotating Systems

(2) **Frequency response**

$$\frac{x(j\omega)}{p(j\omega)} = \frac{1}{-M\omega^2 + bj\omega + k} = G(j\omega)$$

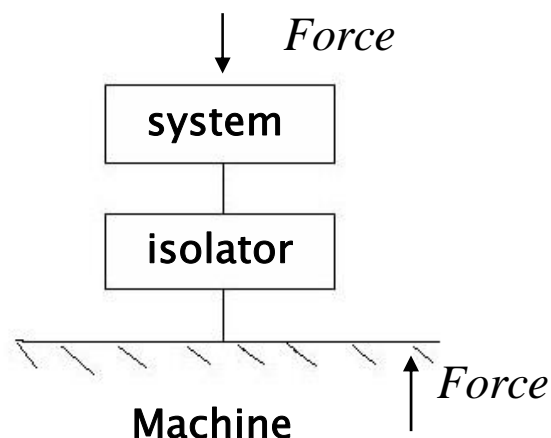$$x(t) = X\sin(\omega t + \phi) = |G(j\omega)|\sin(\omega t + \phi) \cdot mr\omega^2$$

$$\phi = -\tan^{-1}\frac{b\omega}{k - M\omega^2} = -\tan^{-1}\frac{2\zeta\omega/\omega_n}{1 - \omega^2/\omega_n^2}$$

$$|G(j\omega)| = \frac{1}{\sqrt{\left(k - M\omega^2\right)^2 + b^2\omega^2}}$$

$$= \frac{1/k}{\sqrt{\left(1 - \omega^2/\omega_n^2\right)^2 + \left(2\zeta\omega/\omega_n\right)^2}}$$
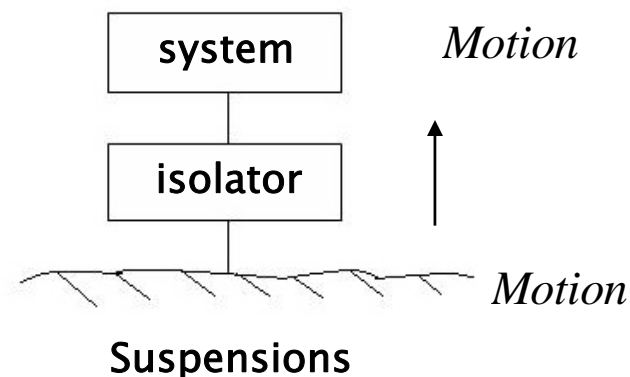
Vibration Isolation in Rotating Systems

Vibration isolators

Machine

Reduce the magnitude of force transmitted from a machine to its foundation

Suspensions

Reduce the magnitude of motion transmitted from a vibratory foundation to a system

Isolator   i) – load supporting elements, spring.
          – energy–dissipating elements, damper.
        ii) – Synthetic rubber : both load supporting and energy dissipating

Transmissibility

Transmissibility :

A measure of the reduction of transmitted force or motion afforded by an isolator.

$$TR(machine) = \frac{the\ force\ amplitude\ transmitted\ to\ the\ foundation}{the\ amplitude\ of\ the\ exciting\ force}$$

$$TR(suspensions) = \frac{the\ vibration\ amplitude\ of\ the\ system}{the\ vibration\ amplitude\ of\ the\ foundation}$$

Transmissibility for Force Excitation

$$p(t) = mr\omega^2 \sin \omega t = F_0 \sin \omega t$$

$$f(t) = b\dot{x} + kx = F_t \sin(\omega t + \phi)$$

$$X(s) = \frac{1}{Ms^2 + bs + k} P(s)$$

$$F_t(s) = (bs + k) X(s) = \frac{bs + k}{Ms^2 + bs + k} \cdot P(s)$$

Frequency response

$$\frac{F_t(j\omega)}{p(j\omega)} = \frac{bj\omega + k}{-M\omega^2 + bj\omega + k} = \frac{(b/M)j\omega + (k/M)}{-\omega^2 + (b/M)j\omega + k/M}$$
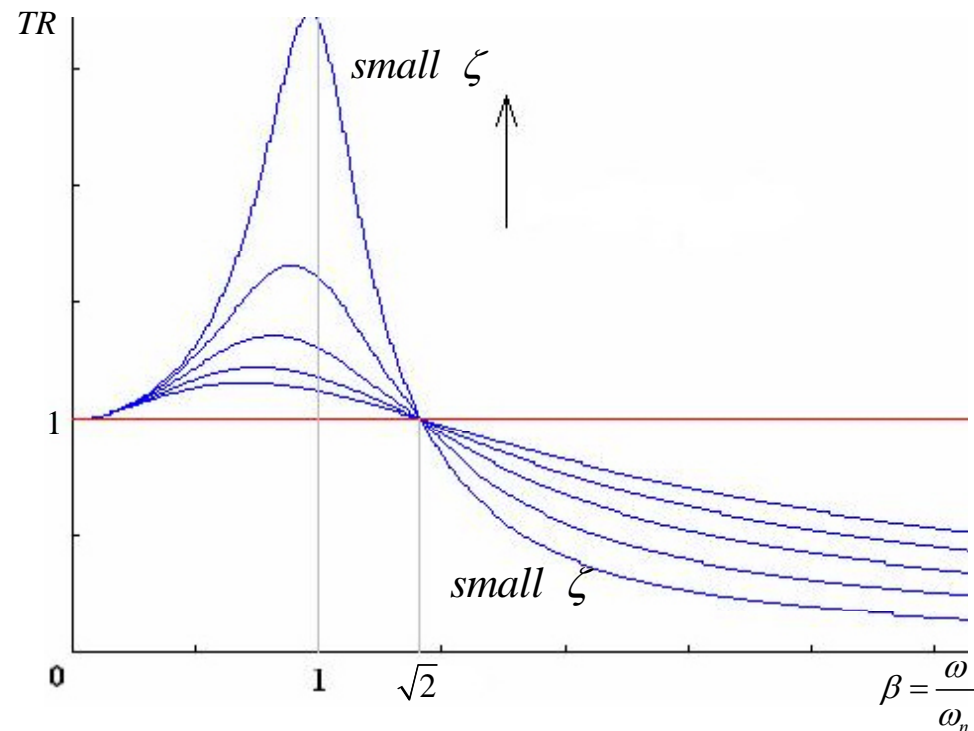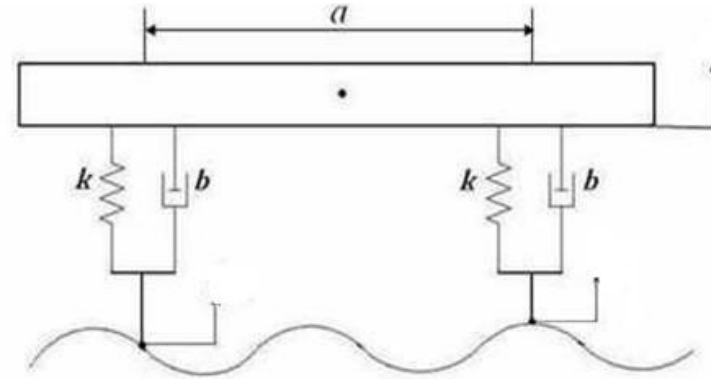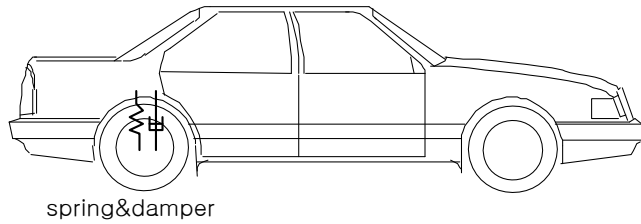
Transmissibility for Force Excitation

$$k/M = \omega_n^2, \quad b/M = 2\zeta\omega_n \quad \Rightarrow \quad \frac{F_t(j\omega)}{p(j\omega)} = \frac{1 + j(2\zeta\omega/\omega_n)}{1 - \omega^2/\omega_n^2 + j(2\zeta\omega/\omega_n)}$$

$$TR = \frac{F_t}{F_0} = \left| \frac{F_t(j\omega)}{p(j\omega)} \right|$$

$$= \frac{\sqrt{1 + (2\zeta\omega/\omega_n)^2}}{\sqrt{(1 - \omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}}$$

$$= \frac{\sqrt{1 + (2\zeta\beta)^2}}{\sqrt{(1 - \beta^2)^2 + (2\zeta\beta)^2}}$$

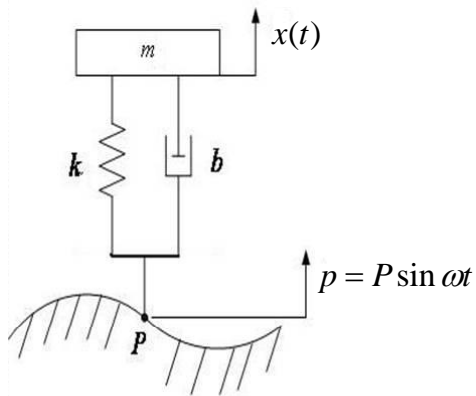*Note,* $\beta = \sqrt{2}$, $TR = 1$

for any ζ

Automobile Suspension Systems

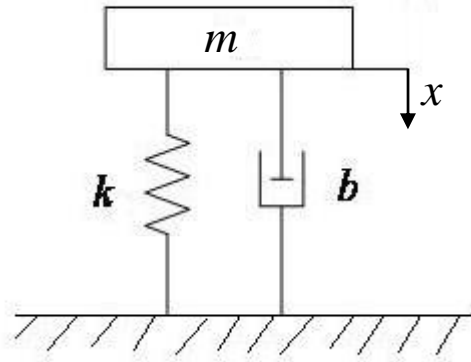**Simplified model**

$x(t)$

k b

$p = P \sin \omega t$

P

$$m\ddot{x} + b(\dot{x} - p) + k(x - p) = 0$$

$$\frac{X(s)}{P(s)} = \frac{bs + k}{ms^2 + bs + k}, \qquad \frac{x(j\omega)}{p(j\omega)} = \frac{bj\omega + k}{-m\omega^2 + bj\omega + k}$$

$$TR = \left| \frac{x(j\omega)}{p(j\omega)} \right| = \frac{\sqrt{b^2\omega^2 + k^2}}{\sqrt{\left(k - m\omega^2\right)^2 + b^2\omega^2}}$$

$$= \frac{\sqrt{1 + \left(2\zeta\beta\right)^2}}{\sqrt{\left(1 - \beta^2\right)^2 + \left(2\zeta\beta\right)^2}}$$

Dynamic Vibration Absorbers
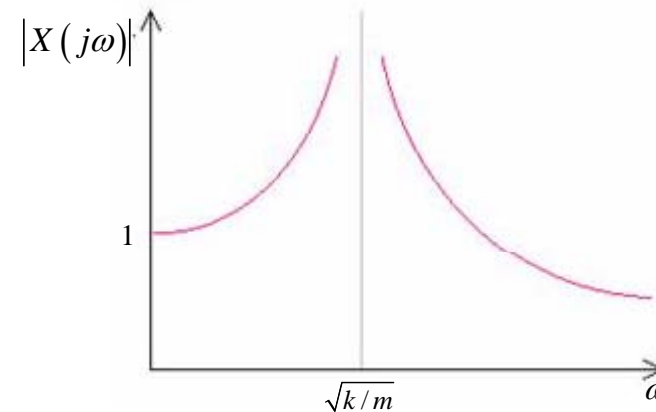
$$P = mr\omega^2$$

$$TR = \frac{\left(mr\omega^2\right)\sqrt{b^2\omega^2 + k^2}}{\sqrt{\left(k - m\omega^2\right)^2 + b^2\omega^2}}$$
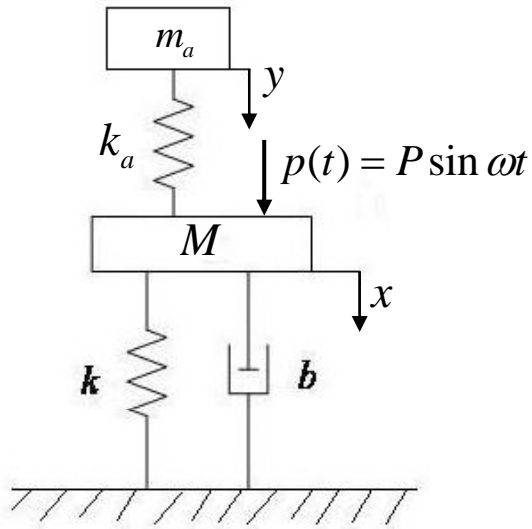
If b is small and $\omega = \omega_n$

Then resonance – excessive vibration

– extremely large force transmitted

ω is very close to ω_n , critical speed,

=> A dynamic vibration absorber

Dynamic Vibration Absorbers

$$M\ddot{x} = -kx - b\dot{x} - k_a(x - y) + p(t)$$

$$m_a\ddot{y} = -k_a(y - x)$$

Laplace Transform

$$\left(Ms^2 + bs + k + k_a\right)X(s) - k_aY(s) = P(s)$$

$$\left(m_as^2 + k_a\right)Y(s) - k_aX(s) = 0$$

$$\Rightarrow \quad \frac{X(s)}{P(s)} = \frac{m_as^2 + k_a}{\left(Ms^2 + bs + k + k_a\right)\left(m_as^2 + k_a\right) - k_a^2}$$

Frequency response

$$\frac{X(j\omega)}{P(j\omega)} = \frac{-m_a\omega^2 + k_a}{\left(-M\omega^2 + k + k_a\right)\left(-m_a\omega^2 + k_a\right) - k_a^2} \qquad \textit{for small } b$$

The transmitted force

$$f(t) = kx + b\dot{x} \cong kx$$

Dynamic Vibration Absorbers

$$\left|X(j\omega)\right| = \left|\frac{k_a - m_a\omega^2}{\left(-M\omega^2 + k + k_a\right)\left(k_a + m_a\omega^2\right) - k_a^2}\right|\left|P(j\omega)\right|$$

$$= \left|\frac{mr\omega^2\left(k_a - m_a\omega^2\right)}{\left(-M\omega^2 + k + k_a\right)\left(k_a + m_a\omega^2\right) - k_a^2}\right|$$

If we choose k_a, m_a so that,

$$k_a - m_a\omega^2 = 0$$

then, $\left|X(j\omega)\right| = 0$, $\left|f(t)\right| = 0$ at this frequency.
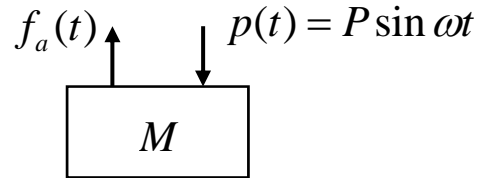
If $\omega_{exc} = \omega_n$

we use dynamic vibration absorbers.
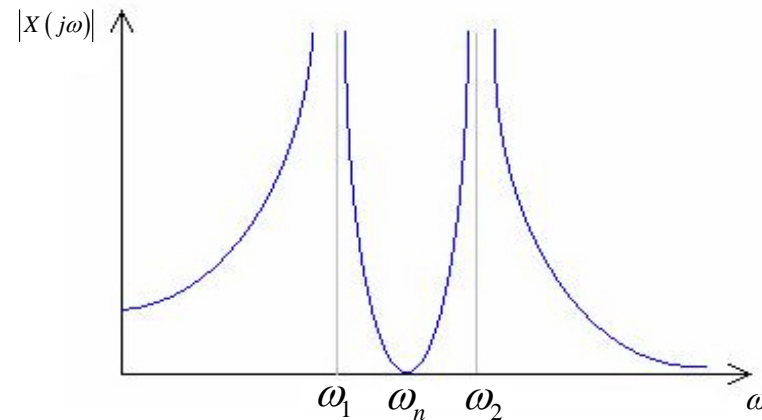
Dynamic Vibration Absorbers

Physically,

$$f_a(t) \qquad p(t) = P\sin\omega t$$

$$\boxed{M}$$

$$f_a(t) = -k_a(x - y) = k_a y = p(t), \qquad \left(at \ \omega_{exc} = \omega_n, \ x \cong 0\right)$$

Spring force $k_a y$ **cancels** $p(t)$.

$$\frac{Y(j\omega)}{P(j\omega)} = \frac{k_a}{\left(-M\omega^2 + k + k_a\right)\left(-m_a\omega^2 + k_a\right) - k_a^2}$$
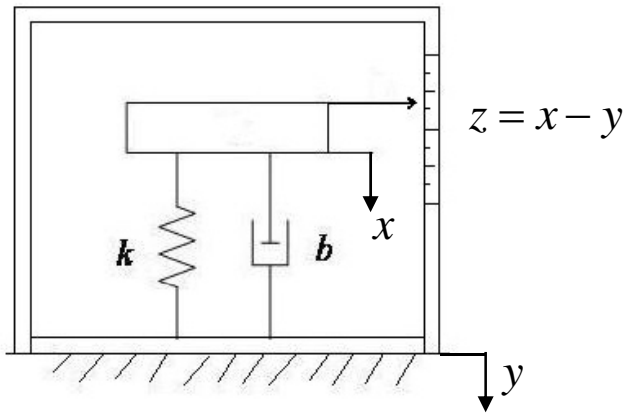
If we choose m_a, k_a **so that,** $k_a - m_a\omega^2 = 0$

then $\dfrac{Y(j\omega)}{P(j\omega)} = -\dfrac{1}{k_a}$ **magnitude ratio** $\left|\dfrac{1}{k_a}\right|$

$$p(t) = \sin\omega t$$

$$y(t) = \frac{1}{k_a}P\sin\left(\omega t - 180°\right) = -\frac{P}{k_a}\sin\omega t \qquad \left(phase \ -180°\right)$$

Seismograph

$z = x - y$

x

y

A device used to measure ground displacement during earthquakes

Equation of motion : $m\ddot{x} = k(y-x) + b(\dot{y}-\dot{x})$

measurement : $z = x - y$

$$\Rightarrow m(\ddot{y}+\ddot{z}) + b\dot{z} + kz = 0$$

$$m\ddot{z} + b\dot{z} + kz = -m\ddot{y}$$

$$\Rightarrow \textit{Transfer Function}: \quad \frac{Z(s)}{Y(s)} = \frac{-ms^2}{ms^2+bs+k} = \frac{-s^2}{s^2+2\zeta\omega_n s + \omega_n^2}$$

$$\left(\frac{k}{m} = \omega_n^2, \quad \frac{b}{m} = 2\zeta\omega_n\right)$$

Seismograph

$$\frac{Z(j\omega)}{Y(j\omega)} = \frac{\omega^2}{-\omega^2 + 2\zeta\omega_n j\omega + \omega_n^2} = \frac{\beta^2}{1 - \beta^2 + 2\zeta\beta j} \qquad (\beta = \omega/\omega_n)$$
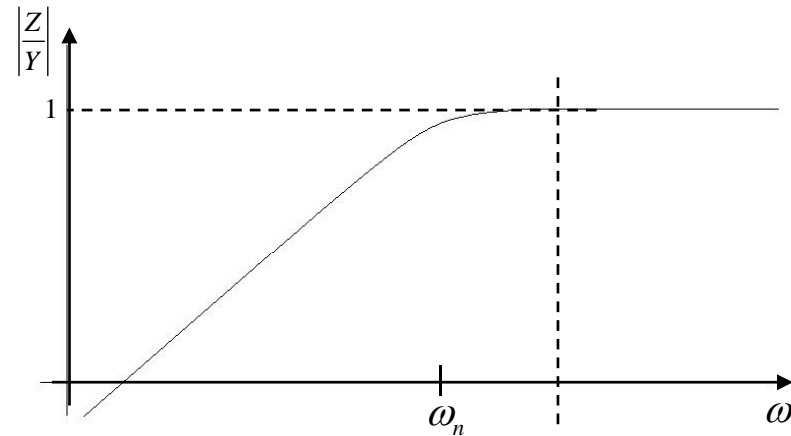
We hope, $\quad z(t) \cong y(t)$

If, $\quad \beta \gg 1, \quad \dfrac{Z(j\omega)}{Y(j\omega)} = -1$

$z(t) \cong y(t) \qquad for \quad \omega \gg \omega_n$

Choose ω_n **as small as possible.**
(large mass, soft spring)

$$\left(\omega_n = \sqrt{\frac{k}{m}} \right)$$

Accelerometer

The system configuration is basically the same as the seismograph, but choice of undamped natural frequency is different.

Equation of motion : the same as the seismograph, $m\ddot{z} + b\dot{z} + kz = -m\ddot{y}$

$$\Rightarrow \textit{Transfer Function} : \quad \frac{Z(s)}{s^2 Y(s)} = \frac{-m}{ms^2 + bs + k} = \frac{-1}{s^2 + 2\zeta\omega_n s + \omega_n^{\,2}}$$

Frequency response

$\omega_n \gg \omega$

$$\frac{Z(s)}{s^2 Y(s)} \cong -\frac{1}{\omega_n}$$