2012년 1학기 공학수학 1 강의자료
(Engneering Mathematics 1 Lecture Note)

서울대학교 조선해양공학과

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이규열

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Naval Architecture & Ocean Engineering

Engineering Mathematics

Chapter 1. Mathematical Modeling

Spring 2012

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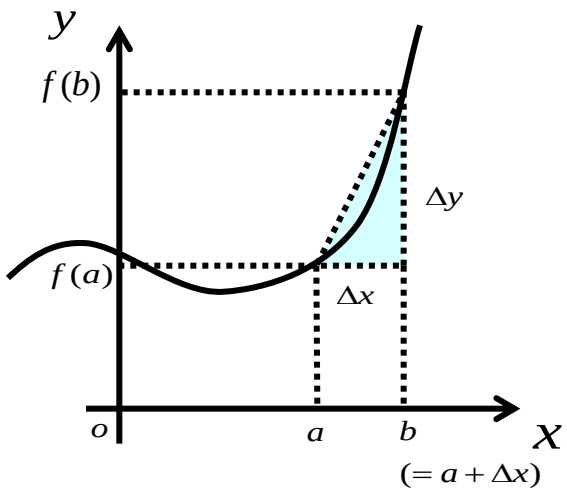
REVIEW : DIFFERENTIATION AND INTEGRATION

Definition of Differential coefficient(미분계수)

☑Average rate of change

$$y = f(x)$$

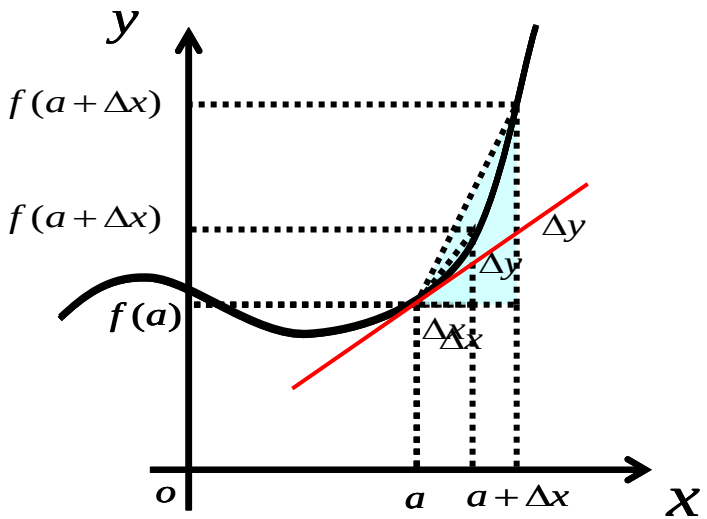
$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a} = \frac{f(a + \Delta x) - f(a)}{\Delta x}$$



- Instantaneous rate of change

$$y = f(x)$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$



홍성대, 수학의 정석-미분과 적분, 성지출판㈜

Definition of Derivative

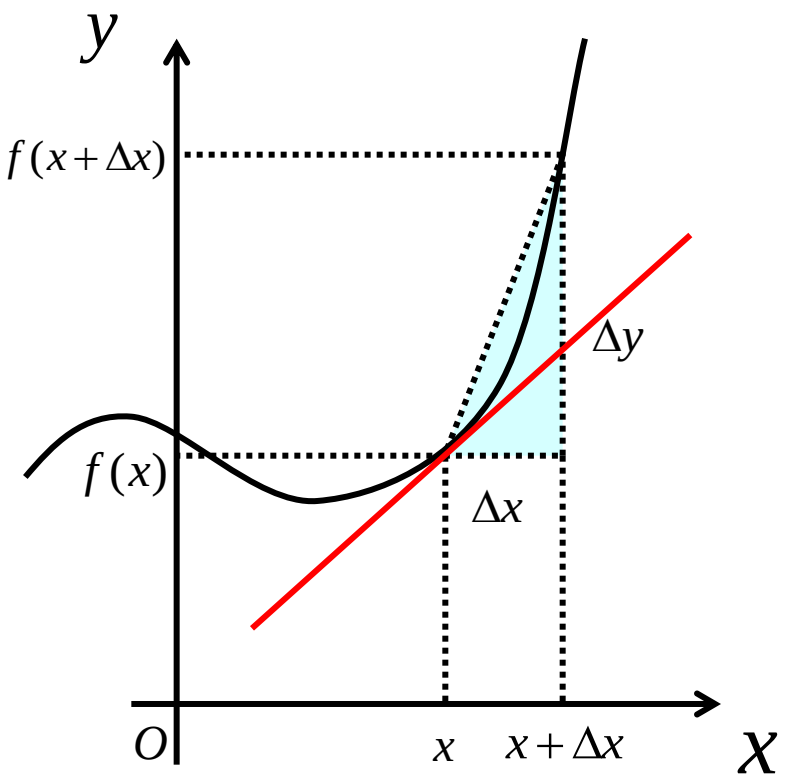
☑derivative

$$y = f(x)$$

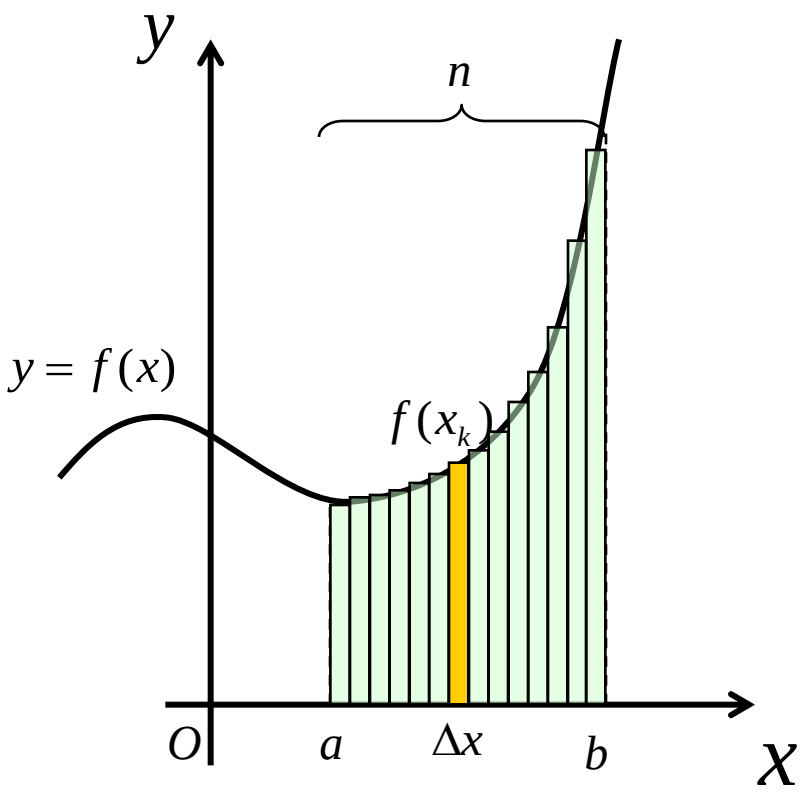
Instantaneous rate of change on arbitrary point, x. (differential coefficient)

$$\lim_{x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{dy}{dx} = y' = f'(x)$$



Definition of integration



S : the area surrounded by $y=f(x)$, $x=a$, $x=b$ and $y=0$

n : the number of rectangles

$$\Delta x = \frac{b-a}{n}$$ **: base line**

S_n : the total area of rectangles

$$S_n = \sum_{k=1}^{n} f(x_k)\Delta x$$

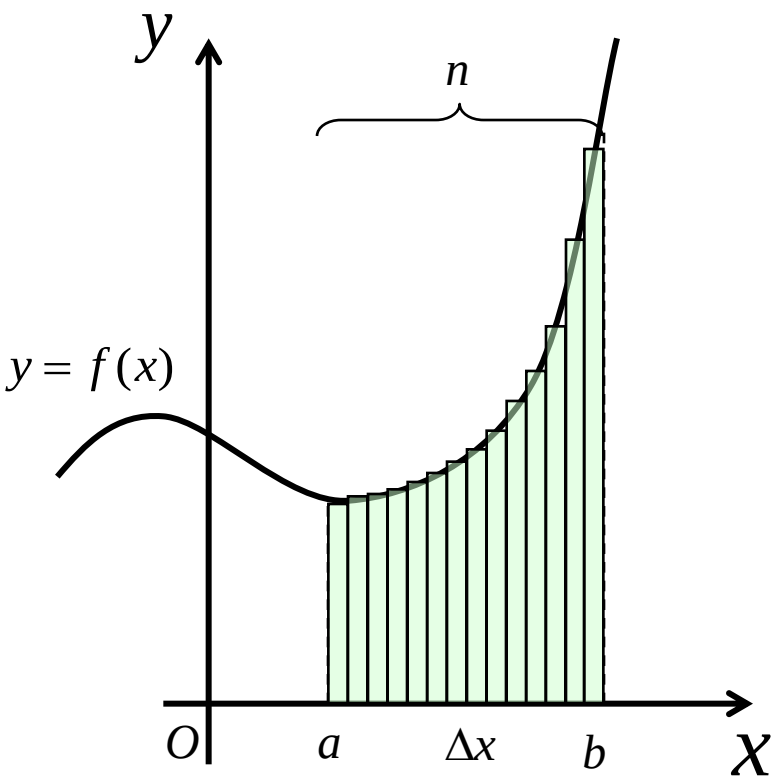
If n $\to\infty$, then

$$S_n = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{\Delta x\to 0}\sum_{k=1}^{\infty} f(x_k)\Delta x$$

$$\int_a^b f(x)dx = \lim_{\Delta x\to 0}\sum_{k=1}^{\infty} f(x_k)\Delta x$$

$$S_n = \int_a^b f(x)dx = S$$

Definition of integration



$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(x_k)\Delta x$$

Integration

Interval

Variable of integration

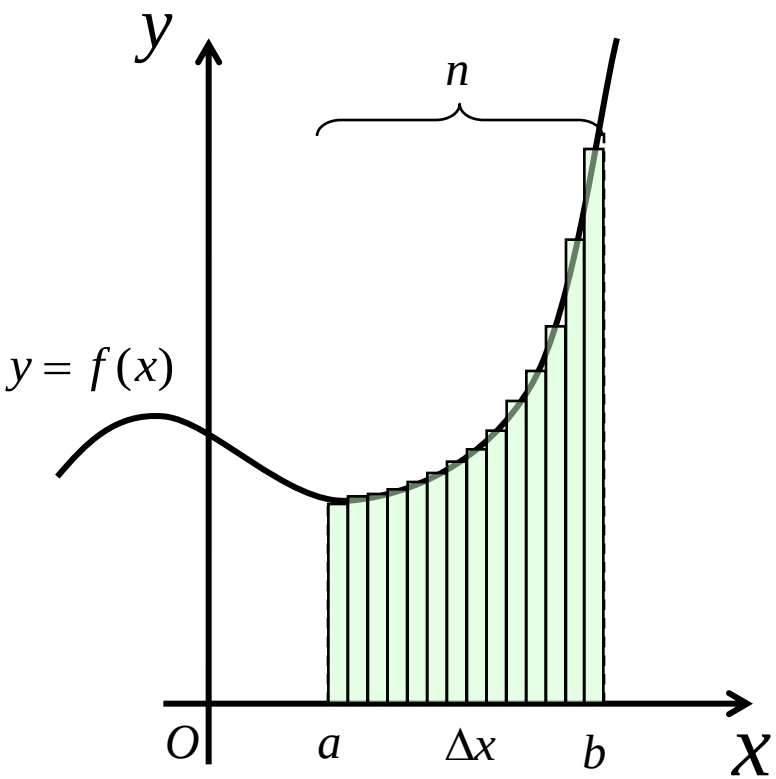
$$\int_a^b f(x)dx$$

Integral

Integrand

Infinitesimal quantity of x

Definition of integration



$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(x_k)\Delta x$$

If we divide the integral by the interval(*b*-*a*),

it becomes the average value($\bar{y}$) of the *y*=*f*(*x*).

$$\bar{y} = \frac{1}{b-a}\int_a^b f(x)dx$$

Example of Integration

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(x_k)\Delta x$$

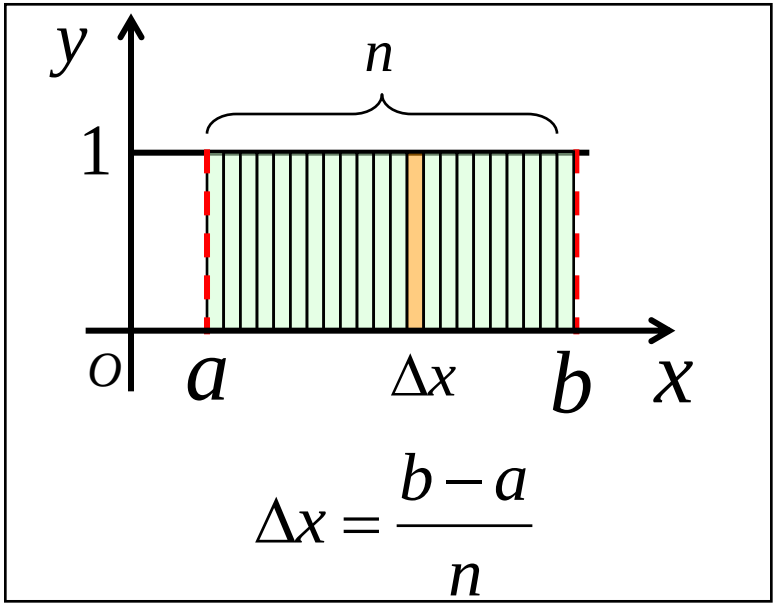
$$\int_a^b dx = ?$$

$$\int_a^b dx = \int_a^b 1 \cdot dx$$

$$f(x) = 1$$

$$\int_a^b 1dx = \lim_{\Delta x \to 0} \sum_{k=1}^{n} 1 \cdot \Delta x$$

$$= b - a$$



$$\Delta x = \frac{b-a}{n}$$

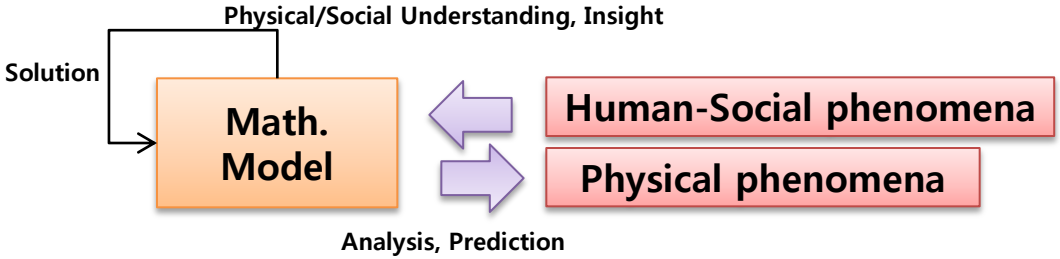
$$\int 1dx = x + C$$

Integration constant

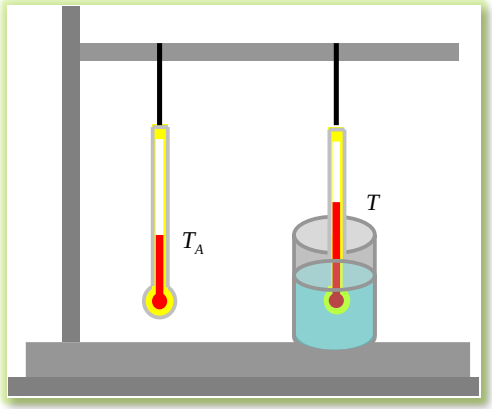
When a function is differentiated , it looses it's constant. So, when we integrate certain function, we have to consider integration constant.

1-1. Why Mathematics?

Why Mathematics?



How long does it take to cool down the hot water



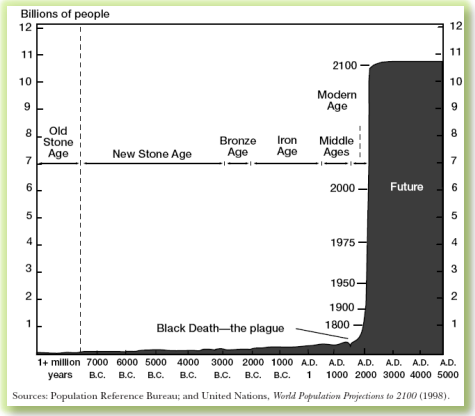
Changing **rate** of water temperature $\propto$ Temperature difference between water and outside

$$\frac{dT(t)}{dt} = k(T - T_A), k < 0$$

T : Water temperature
T_A :Outside temperature (constant)

<Newton's law of cooling>

Increase of population in the future



Increasing **rate** of population $\propto$ Present Population

$$\frac{dy(t)}{dt} = k \cdot y(t)$$

y : population
t : time
k : proportional constant

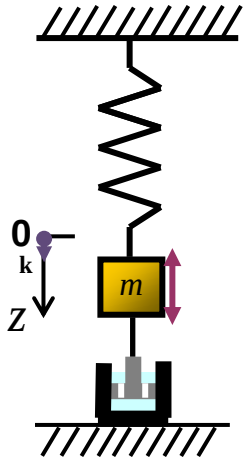
<Malthus's population dynamics >

Why do we need to study "Engineering Mathematics"?

Real World

Idealization or Simplification*

-Insight
-Constitutive Relations



Linearization

-Taylor Series

Explain or Predict

Problem-Solving Abilities

Solution

$$z(t) = e^{-\alpha t}(A\cos\omega t + B\sin\omega t) + C\cos\omega_0 t$$

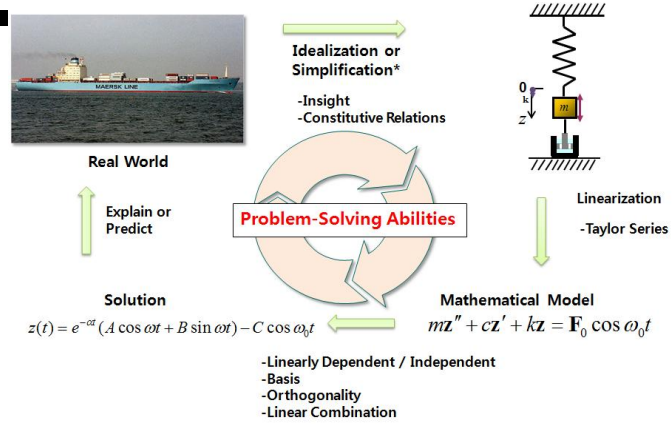
Mathematical Model

$$m\mathbf{z}'' + c\mathbf{z}' + k\mathbf{z} = \mathbf{F}_0\cos\omega_0 t$$

-Linearly Dependent / Independent
-Basis
-Orthogonality
-Linear Combination

* Keener, J.P., Principles of Applied Mathematics, Westview Press, 2000, p.xi : ...there is the goal to explain or predict the behavior of some physical situation. One begins by constructing a mathematical model which captures the essential features of the problem without ,asking its content with overwhelming detail

Modeling of Engineering Problems



1. Statics
    1) Deflection Curve of a Beam
    2) Buckling of Column
2. Law of cooling/ Population dynamics
3. Newton's Dynamics
    1) Unforced Mass-Spring System
    2) Forced Mass-Spring-Damper System
    3) Motions of a Ship (Coupled Heave / Roll / Pitch Motions)
    4) Rotation String
4. Fluid Mechanics
    1) Cauchy Eq., Navier-Stokes Eq., Euler Eq., Bernoulli Eq.
    2) Ship Hydrostatic & Hydrodynamic Forces
5. Basic PDEs
    1) Wave Equation
    2) Heat Diffusion Equation
    3) Laplace Equation

Differential Equation

☑ **Ordinary differential equation** is an equation that contains one or several derivatives of an unknown function, $y(x)$

$$y' = 2x \ , \quad y' = \cos x + y$$

$$y'' + 4y = 0 \ , \quad x^2 y''' y' + 2e^x y'' = (x^2 + 2)y^2$$

First-order differential equations contain only y' and may contain y and given functions of x.

$$F(x, y, y') = 0$$

$$y' = f(x, y)$$

- **Partial differential equations** involve an unknown function of two or more variables and its partial derivatives, for example, $y(x_1, x_2)$.

Differential Equation

Definition 1.1*

Differential Equation

An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a Differential Equation(DE)

$$\frac{dy(x)}{dx} = 0.2xy$$

How to solve a Differential Equation?

*Zill D.G., Cullen M.R., Advanced Engineering Mathematics 3rd Edition, Johns and Bartlett, 2006, p5

Examples of ODEs

- **Falling object** $y''(t) = g$
- **Malthus's law** $y'(t) = ky(t)$
- **Radiocarbon Dating** $y'(t) = ky(t)$
- **Newton's law of cooling** $T'(t) = k(T - T_A)$
- **Vibrating mass on a spring** $mz'' + cz' + kz = F_{ext}$
- **Current I in an RLC circuit** $LI'' + RI' + \frac{1}{C} I = E'(t)$
- **Deformation of beam** $EIy'''' = f(x)$
- **Legendre's equation** $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$
- **Bessel's equation** $x^2y'' + xy' + (x^2 - \nu^2)y = 0$
- **Sturm-Liouville problems** $[p(x)y']' + [q(x) + \lambda r(x)]y = 0$

The methods to solve ODEs

- Integration
- Separating variables
- Exact ODEs
- $y = e^{\lambda x}$
- $y = x^m$
- Reduction of order
- Undetermined coefficient
- Variation of parameter
- Power series method

ODE & PDE Map

I.V.P : Initial Value Problem
B.V.P : Boundary Value Problem
IC: Initial Condition
BC: Boundary Condition

	Second Order ODE	Power series	PDE	Model
Introduction	General Solution, Linearly independent, Basis, Superposition	Power series	Linear PDE, Separable Variables	
Linear	Homogeneous $ay''+by'+cy=0$ I.V.P $y=c_1e^{m_1x}+c_2e^{m_2x}$ $y=c_1e^{m_1x}+c_2xe^{m_1x}$ $y=e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)$ Reduction of order Nonhomogeneous $ay''+by'+cy=g(x)$ $y_p=$ liner combination of $a_2x^2+a_1x+a_0, e^{\alpha x}, e^{\alpha x}\cos\beta x, e^{\alpha x}\sin\beta x$ If (a basis of y_p) $=y_c$ Variation of parameters $y_p=u_1(x)y_1+u_2(x)y_2$	Sturm-Liouville B.V.P $[r(x)y']'+[q(x)+\lambda p(x)]y=0$ Special function Bessel Function $x^2y''+xy'-(x^2+v^2)y=0$ Legendre Function $(1-x^2)y''-2xy'+n(n+1)y=0$ Fourier Series $y''+\lambda y=0,\ y(\pi)=y(-\pi),\ y'(\pi)=y'(-\pi)$ Euler-Cauchy Eq. $ax^2y''+bxy'+cy=0$	Fourier Series $u(x,t)=F(x)G(t)$ B.V.P. I.V.P. • $u_t=c^2u_{xx}$ Heat Eq. IC(1),BC(2) • $u_{tt}=c^2u_{xx}$ Wave Eq. IC(2),BC(2) steady $u(x,y)=F(x)G(y)$ B.V.P. • $u_{xx}+u_{yy}=0$ Laplace Eq. BC(4) -Polar Coord. Euler-Cauchy -Cylindrical Coord. Bessel -Spherical Coord. Legendre	I.V.P. Free undamped/damped Forced undamped/damped $\frac{d^2x}{dt^2}+2\lambda\frac{dx}{dt}+\varpi^2x=F_0\sin\gamma t$ B.V.P. Deflection of a Beam $EI\frac{d^4y}{dx^4}=w(x)$ Buckling of a Column $EI\frac{d^4y}{dx^4}+py=0$
Nonlinear	Nonlinear Eq. -reduction of order $F(x,y,y'')\rightarrow F(y,y',y'')$ or $F(x,y',y'')$ $u=y'$ -use of Taylor series -numerical solver			Nonlinear Spring $m\frac{d^2x}{dt^2}+F(x)=0, F(x)\neq kx$ Pendulum $\frac{d^2\theta}{dt^2}+\frac{g}{l}\sin\theta=0$ Telephone wires/catenary (analytic solvable) $\frac{d^2y}{dx^2}=\frac{\rho}{T}\sqrt{1+(\frac{dy}{dx})^2}$ Variable Mass $x\frac{d^2x}{dt^2}+\frac{dx}{dt}+a_0x=b_0$

ODE Map

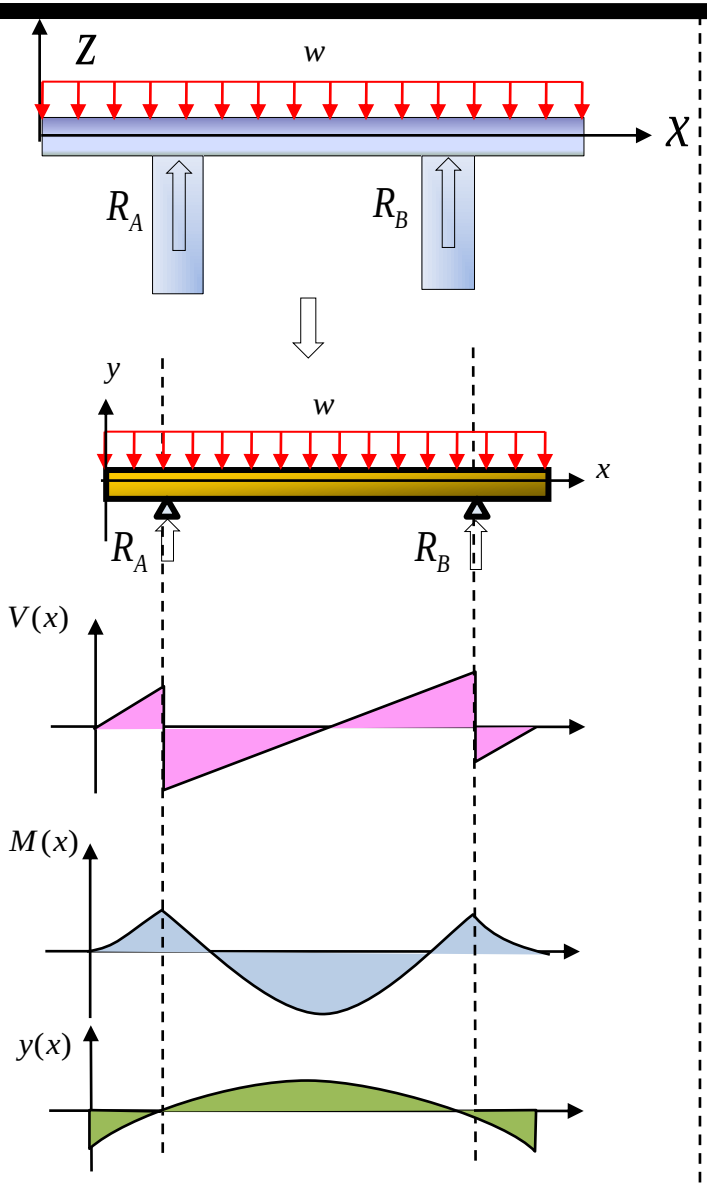
	First-Order	Second- Order	Systems	Numerical Solution
Introduction				
Linear	Integration $\frac{dy}{dx} = g(x)$ Linear Equation $\frac{dy}{dx} + p(x)y = f(x)$ Variation of parameters $y_p = u_1(x)y_1$ Integrating factor $e^{\int p(x)dx}$	Cauchy-Euler Homogeneous $ax^2y'' + bxy' + cy = 0$ $y = x^m$ Reduction of order $y = c_1x^{m_1} + c_2x^{m_2}$ $y = c_1x^{m_1} + c_2x^{m_1}\ln x$ $y = x^{\alpha}[c_1\cos(\beta\ln x) + c_2\sin(\beta\ln x)]$ Nonhomogeneous $y'' + p(x)y' + q(x)y = f(x)$ Variation of parameters $y_p = u_1(x)y_1 + u_2(x)y_2$	System of Linear D.E.	
Nonlinear	Separable Variables $\frac{dy}{dx} = g(x)h(y)$ Exact Equation $M(x,y)dx + N(x,y)dy = 0$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Nonexact Equation $\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$ Integrating factor $\mu(x) = e^{\int \frac{M_y - N_x}{N}dx}, \mu(y) = e^{\int \frac{N_x - M_y}{M}dy}$ By substitutions Homogeneous $M(1,u)dx + N(1,u)dy = 0$ $y = ux$ Bernoulli's Equation $\frac{dy}{dx} + p(x)y = f(x)y^n$ $u = y^{1-n}$		System of Nonlinear D.E.	**Numerical Solutions** Euler Method Improved Euler Method Runge-Kutta Adams-Bashforth-Moulton Higher-Order reduction to first order systems

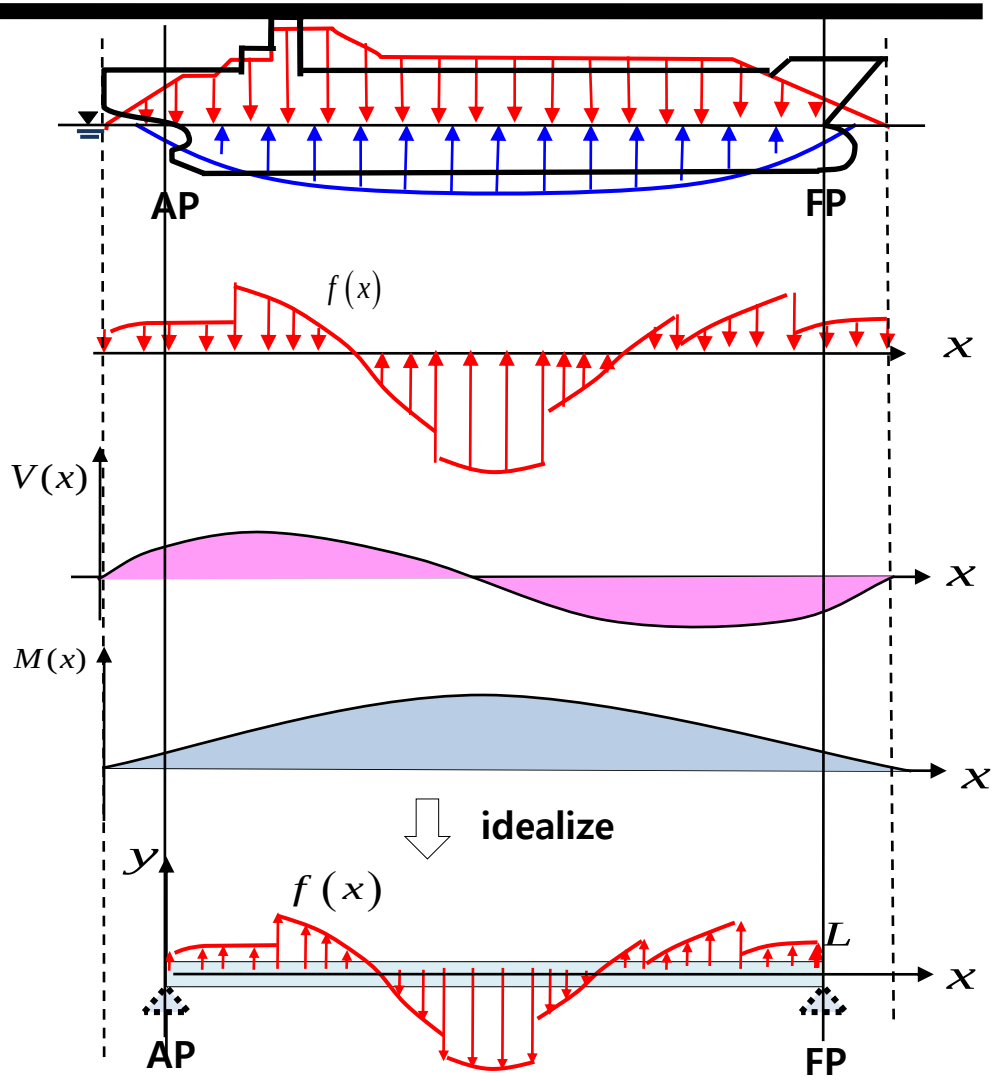
means $M(x,y) = x^{\alpha}M(1,u)$, $N(x,y) = x^{\alpha}N(1,u)$

*Zill, D.G. and Cullen, M.R., Advanced Engineering Mathematics, Third edition, Johns and Bartlett, 2006

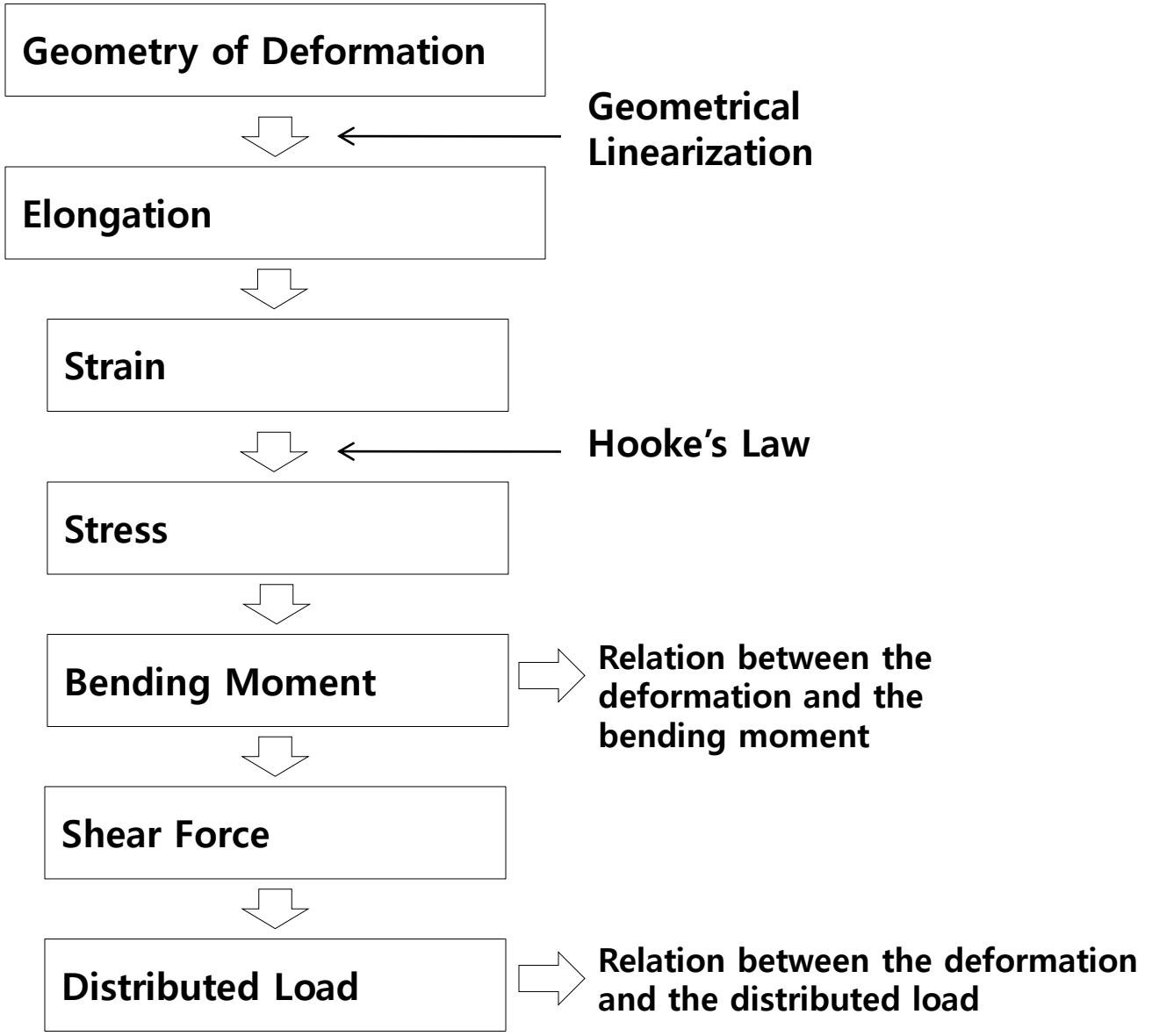
1-2. Modeling of Beam:
Derivation of Deflection Curve of a Beam

Applying beam theory on a ship

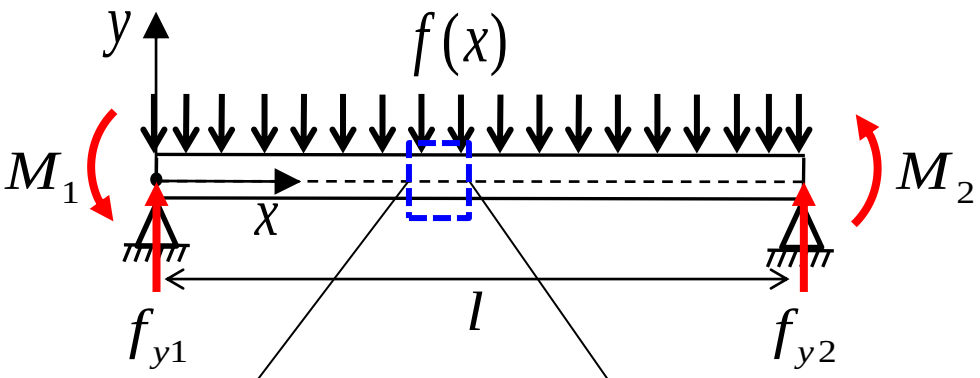


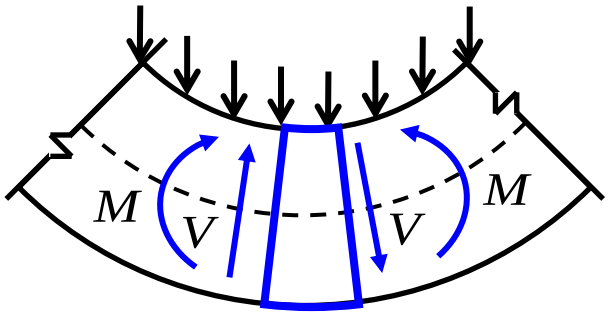


Overview of Derivation Procedure



Geometry of Deformation





This figure is represented in very exaggerated way to recognize the deformation of a beam!

1) Concentrated forces f_{y1} and f_{y2} are exerted on the ends of the beam.
2) Moment M_1 and M_2 are exerted on the ends of the beam.

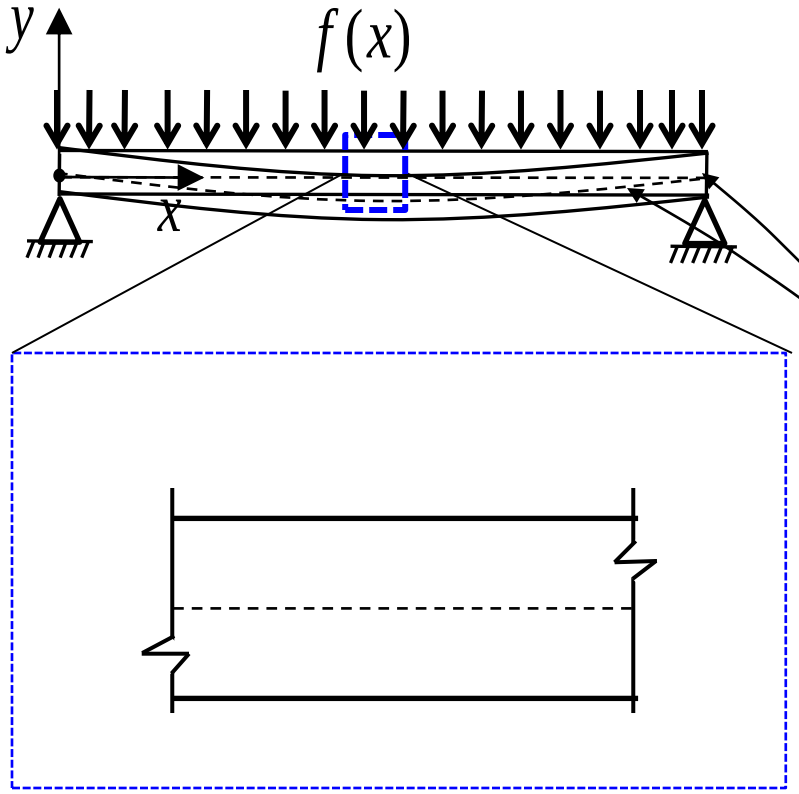
3) Distributed force f(x) is applied to the element

An infinitesimal element will be introduced for the derivation of deflection curve of a beam. Here, the bending moment M and the shear force V are stress resultants, and the positive directions of the stress resultants are shown in left figure.

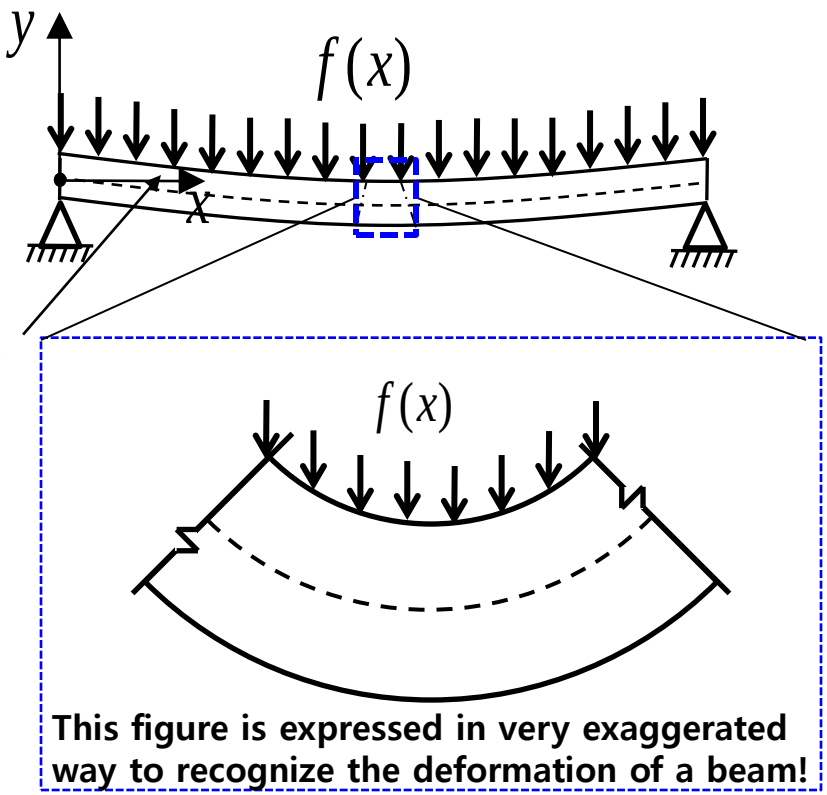
Geometry of Deformation

Geometry of a beam

-before deformation



-after deformation



This figure is expressed in very exaggerated way to recognize the deformation of a beam!

The x axis is a line along the neutral surface of the undeformed beam. Of course, when the beam deflects, the neutral surface moves with the beam, but the x axis remains fixed in position.

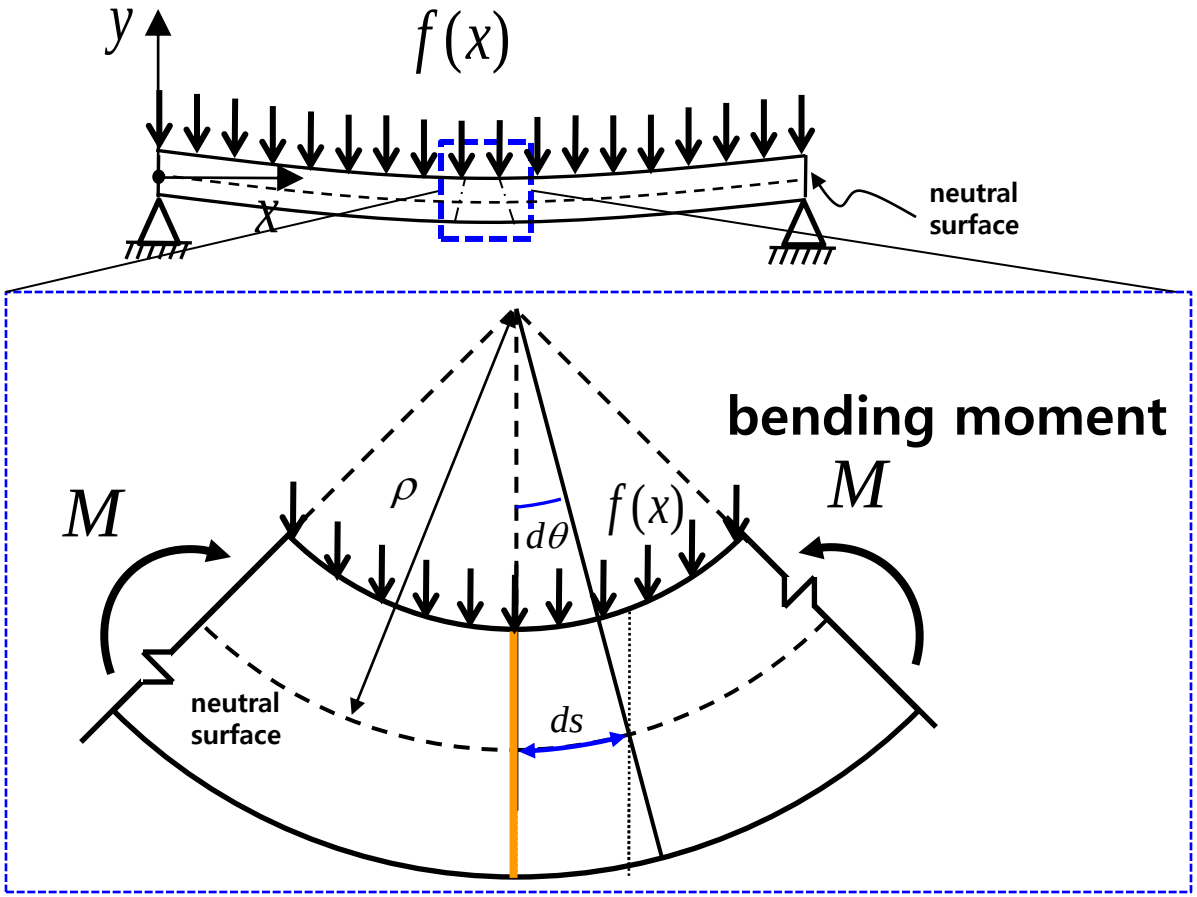
* neutral surface : Longitudinal lines on the lower part of the beam are elongated, whereas those on the upper part are shortened. Thus the lower part of the beam is in tension and the upper part is in compression. Somewhere between the top and bottom of the beam is a surface in which longitudinal lines do not change in length. This surface is called neutral surface

Geometry of Deformation

Geometry of a beam -after deformation

$$\rho \cdot d\theta = ds$$

$$\kappa \equiv \frac{1}{\rho} = \frac{d\theta}{ds}$$



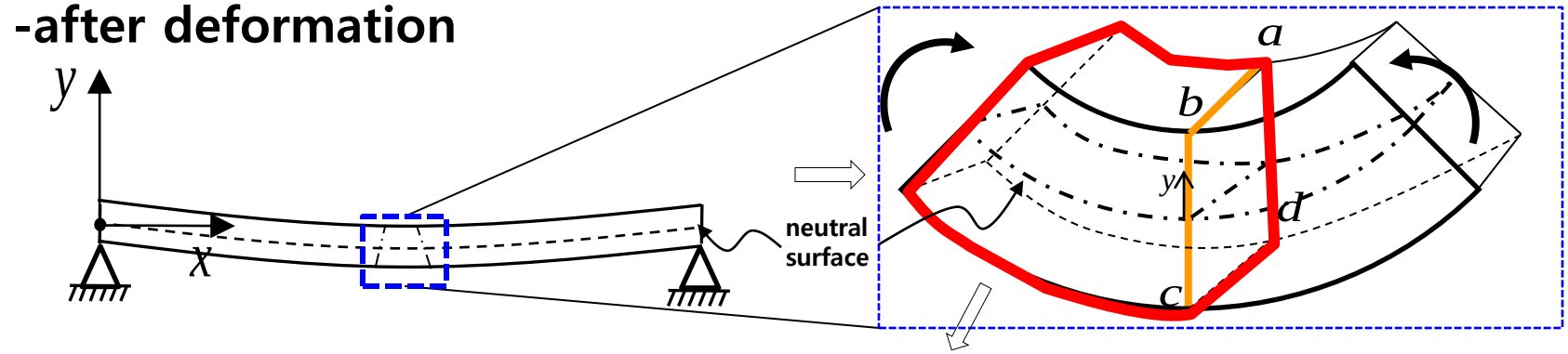
ρ: radius of curvature, κ : curvature

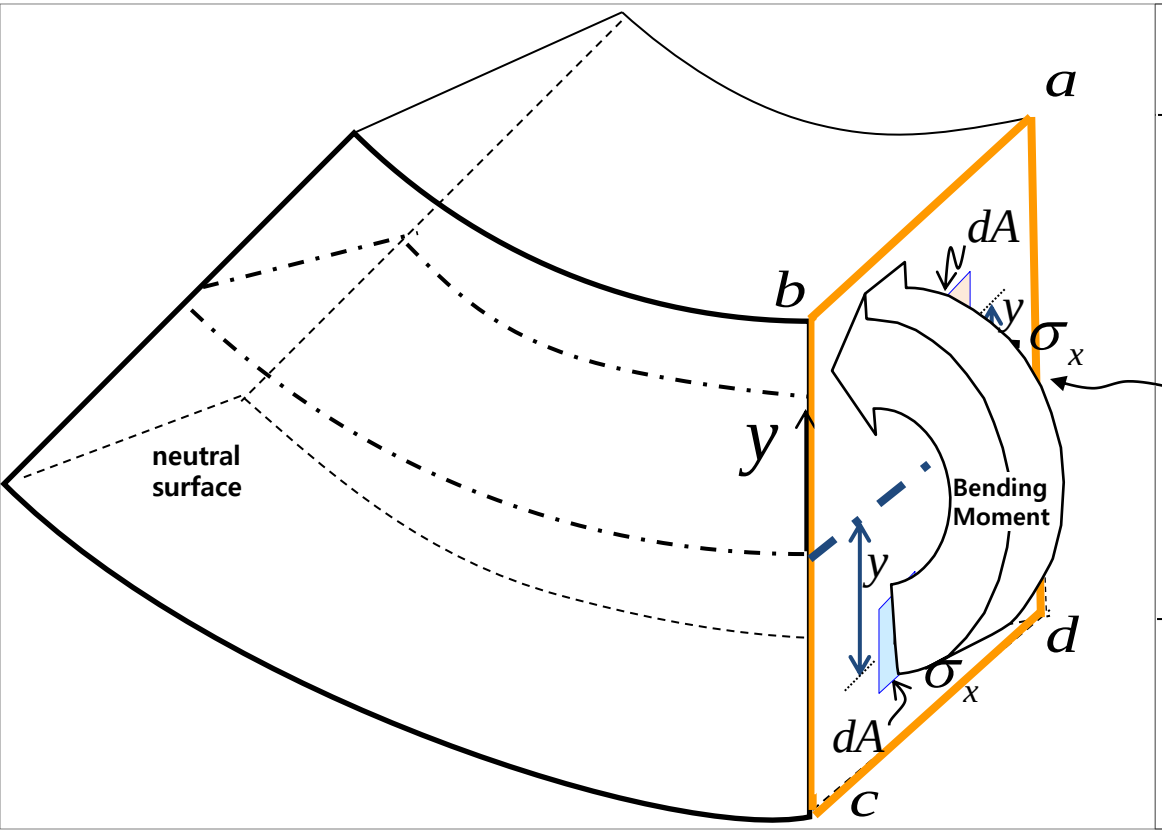
Geometry of Deformation

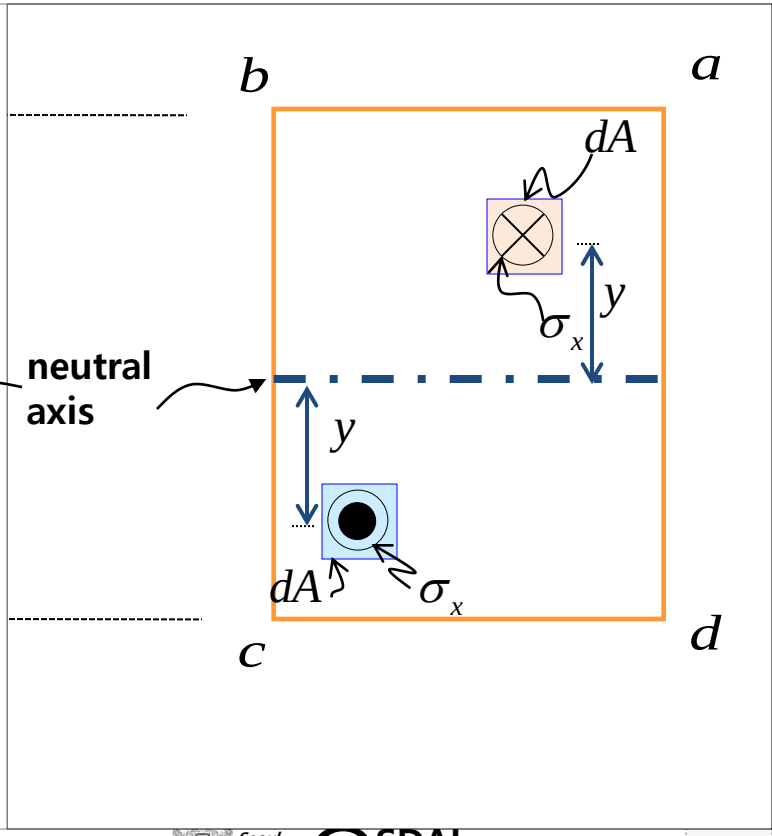
$$dM = y\,dF$$

$$dM = y\,\sigma_x\,dA$$

-after deformation

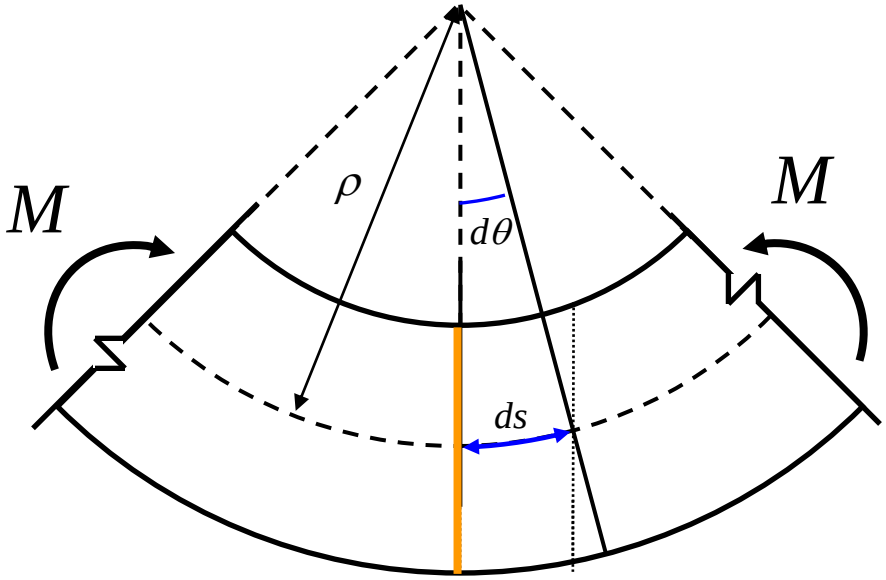






Geometry of Deformation

$$\kappa \equiv \frac{1}{\rho} = \frac{d\theta}{ds}$$



The differential equation for the deflection curve of a beam is supposed to be expressed based on the Cartesian coordinate system.

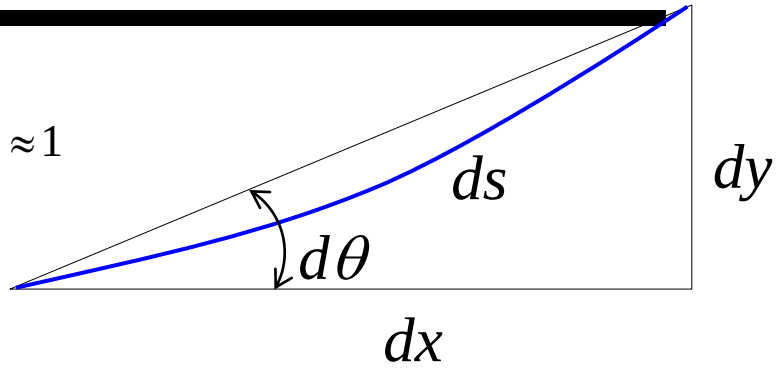
How can we express the geometry with dx and dy instead of ds and $d\theta$

Geometry of Deformation : Linearization

IF WE ASSUME, $\theta << 1$

$$\therefore f(z) = \sqrt{1+z} \approx 1$$

$$\therefore ds \approx dx$$



$$ds^2 = dx^2 + dy^2$$

$$\rightarrow ds = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$let, z = \left(\frac{dy}{dx}\right)^2 \; then, \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+z}$$

$$f(z) = \sqrt{1+z} \xrightarrow{\text{Taylor series Expansion}} f(z) = 1 + \frac{1}{2}z + \frac{1}{2}\left(-\frac{1}{4}\right)z^2 + \ldots$$

, since

$$f(0) = 1$$

$$f'(0) = \frac{1}{2}(1+z)^{-\frac{1}{2}}\Big|_{z=0} = \frac{1}{2}$$

$$f''(0) = -\frac{1}{4}(1+z)^{-\frac{3}{2}}\Big|_{z=0} = -\frac{1}{4}$$

$$\rightarrow ds = dx\left[1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2 + \frac{1}{2}\left(-\frac{1}{4}\right)\left(\frac{dy}{dx}\right)^4 + \cdots\right]$$

$$if, \; \theta << 1$$

$$ds \approx dx$$

Geometry of Deformation : Linearization

IF WE ASSUME, $\theta \ll 1$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots$$

$$\theta \approx \tan(\theta)$$

$$\therefore \theta \approx \frac{dy}{dx}$$

Elongation

$ds \approx dx$

- at the neutral surface

No elongation

length of AB :

$$ds \approx dx$$

length of $A'B'$:

$$ds' = (\rho - y)d\theta$$

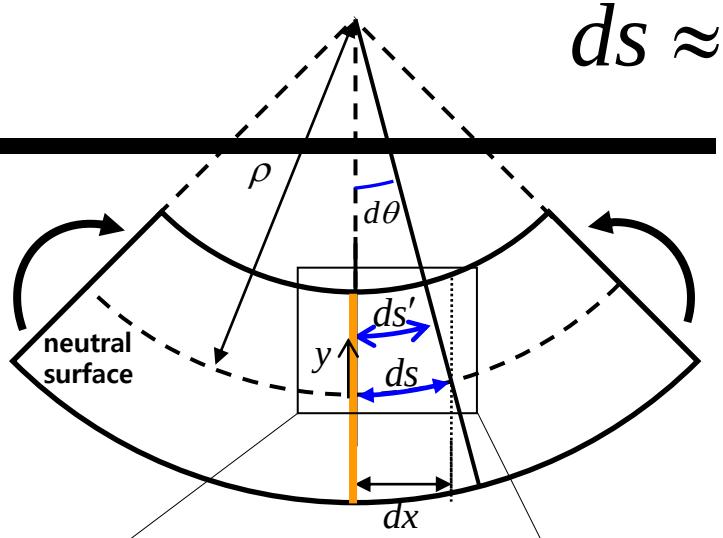
$$\Downarrow (ds' \approx dx')$$

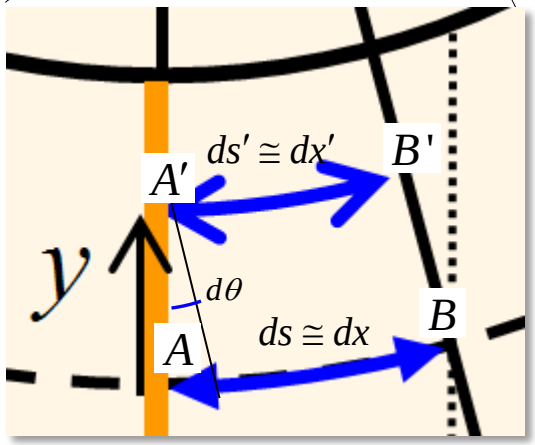
$$dx' = (\rho - y)d\theta$$

$$dx' = \rho d\theta - y d\theta$$

$$dx' = dx - y d\theta$$

$$dx' = dx - y\frac{dx}{\rho}$$





$$\rho d\theta = ds \approx dx$$

$$\therefore d\theta = \frac{dx}{\rho}$$

Strain

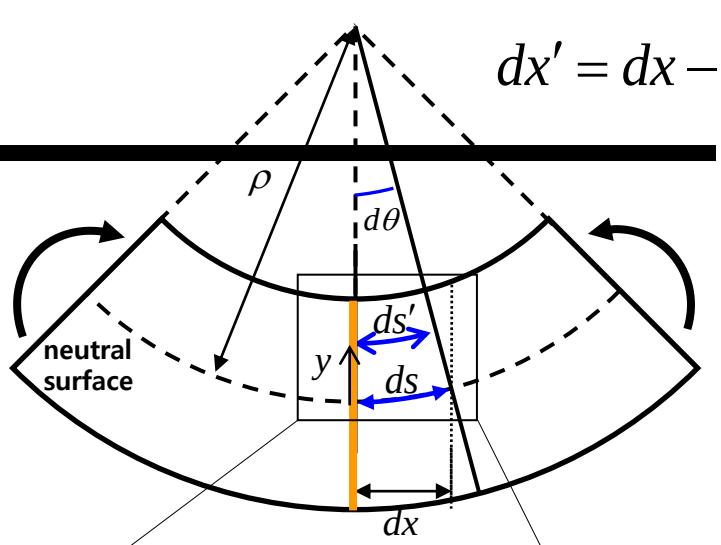
$$dx' = dx - y\frac{dx}{\rho}$$

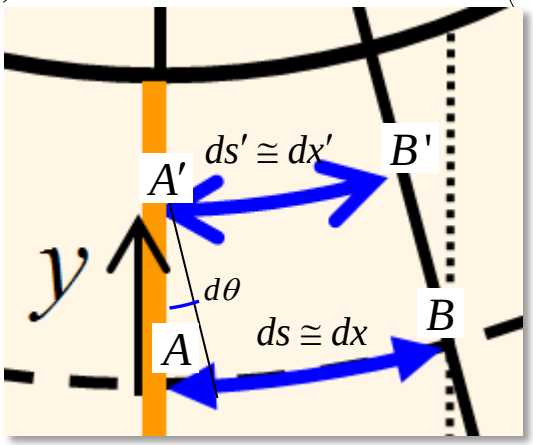
$$dx' - dx = -y\frac{dx}{\rho}$$

$$\frac{dx' - dx}{dx} = -\frac{y}{\rho}$$

→ **Definition of strain**

$$\therefore \varepsilon_x = -\frac{y}{\rho}$$





Stress

$$\varepsilon_x = -\frac{y}{\rho}$$

Hooke's Law

$$\sigma_x = E\varepsilon_x$$

$$\therefore \sigma_x = -E\frac{y}{\rho}$$

Bending Moment

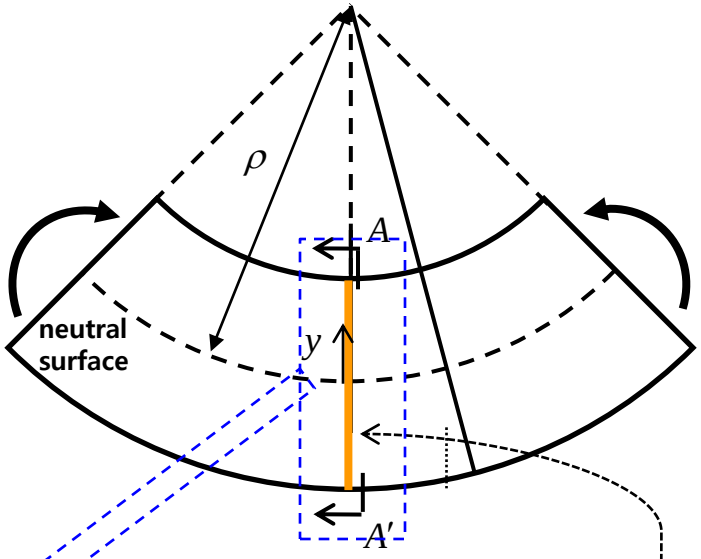
$$\sigma_x = -E\frac{y}{\rho}$$

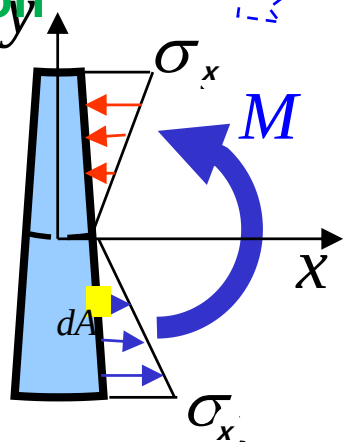
Bending moment about the neutral axis due to the normal stress acting on an infinitesimal area dA

$$dM = y\,\sigma_x\,dA$$

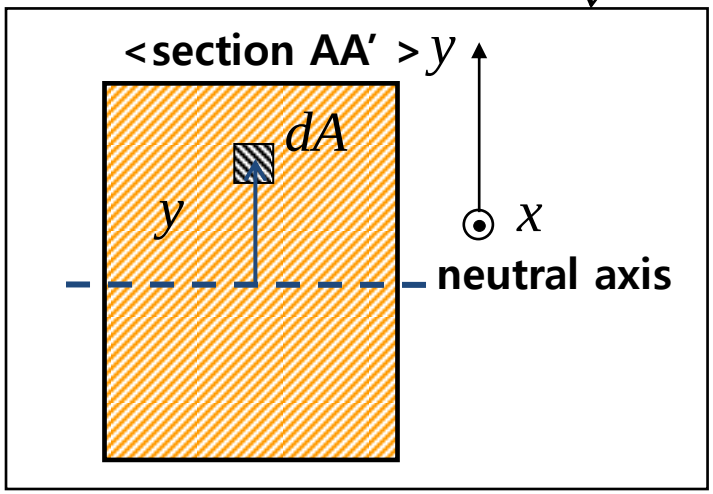
Considering the sign convention, we need to add 'minus sign' for the bending moment. (Ref: To see this in detail, refer to the lecture on "the sign convention")

$$\Rightarrow dM = -y\,\sigma_x\,dA$$





<Elevation view>



<Section view>

Bending Moment

$$dM = -\sigma_x y dA \quad , \sigma_x = -E\frac{y}{\rho}$$

The total Bending moment about the neutral axis due to the normal stress acting on the sectional area

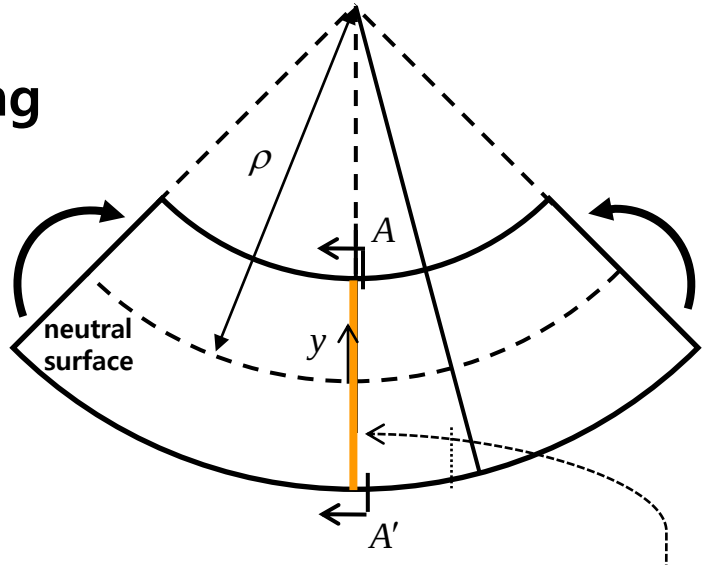
$$M = \int_A dM$$

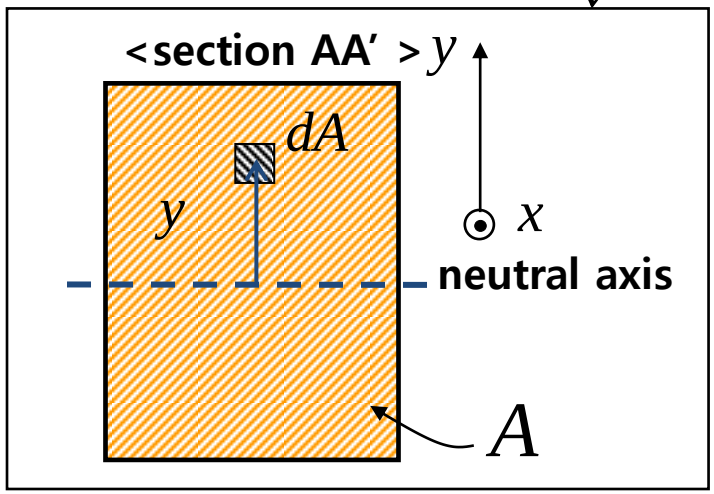
$$M = -\int_A y\,\sigma_x dA$$

$$= -\int_A y\,(-E\frac{y}{\rho})\,dA$$

$$= \frac{E}{\rho}\int_A y^2 dA$$

$$\therefore M = \frac{E}{\rho} I \qquad , I = \int_A y^2 dA$$



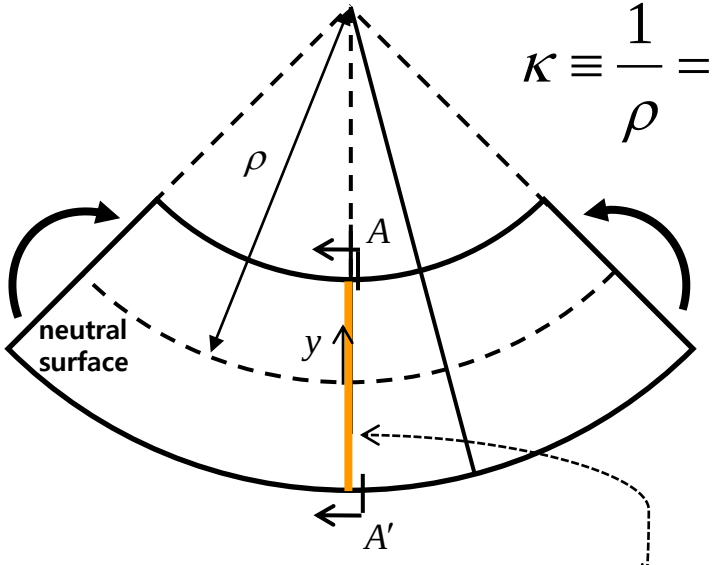


<Section view>

Relation between the deformation and the bending moment

$$M = \frac{E}{\rho} I \quad \Rightarrow \quad \frac{M}{EI} = \frac{1}{\rho} \quad \text{or} \quad \frac{M}{EI} = \kappa$$

$$\kappa \equiv \frac{1}{\rho} = \frac{d\theta}{ds}$$



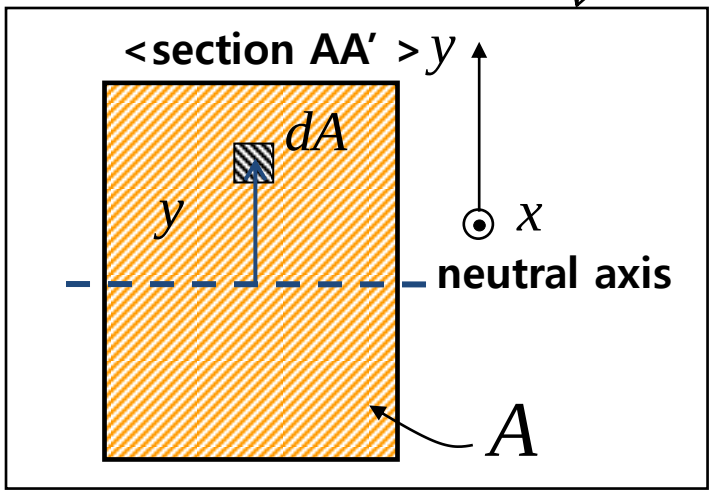
By performing linearization of the deformed geometry for small deflection

$$\kappa = \frac{d\theta}{ds}$$

$$\approx \frac{d}{ds}\tan(\theta) = \frac{d}{ds}\left(\frac{dy}{dx}\right)$$

$$\approx \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

$$\therefore \frac{M}{EI} = \frac{d^2y}{dx^2}$$

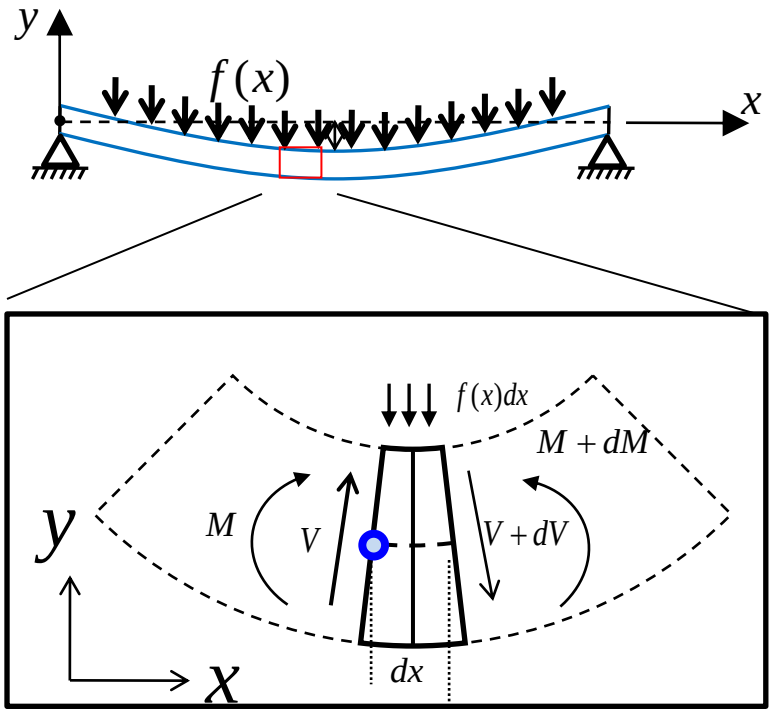


<Section view>

$$I = \int_A y^2 dA$$

Relation between bending moment and shear force

Let us consider the distributed load acting on a beam



From the moment equilibrium about z-axis through the point o , we obtain :

$$-M+(M+dM)-(V+dV)dx-\frac{1}{2}dx\cdot f(x)dx=0$$

$$dM-Vdx-dV\cdot dx-\frac{1}{2}(dx)^2\cdot f(x)=0$$

neglecting the high order terms

$$\therefore \frac{dM}{dx}=V(x)$$

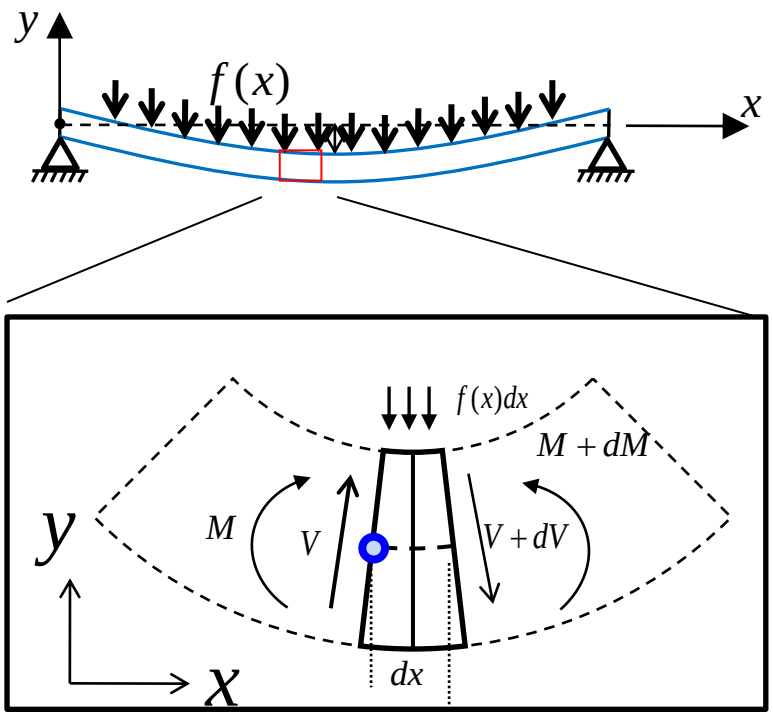
Ref: To see the direction of the shear forces and bending moments in the Fig. in detail, refer to the lecture on "the sign convention"

Relation between shear force and distributed load

From the force equilibrium, we obtain

$$V-(V+dV)-f(x)dx=0$$

$$\therefore \frac{dV}{dx}=-f(x)$$



Relation between the deformation and the distributed load

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \rightarrow EI\frac{d^2y}{dx^2} = M$$

Differentiation with respect to *x*

$$EI\frac{d^3y}{dx^3} = V$$

Differentiation with respect to *x*

$$EI\frac{d^4y}{dx^4} = -f(x)$$

"Deflection Curve of a Beam"

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{dM}{dx} = V(x)$$

$$\frac{dV}{dx} = -f(x)$$

What if the distributed load is not applied?

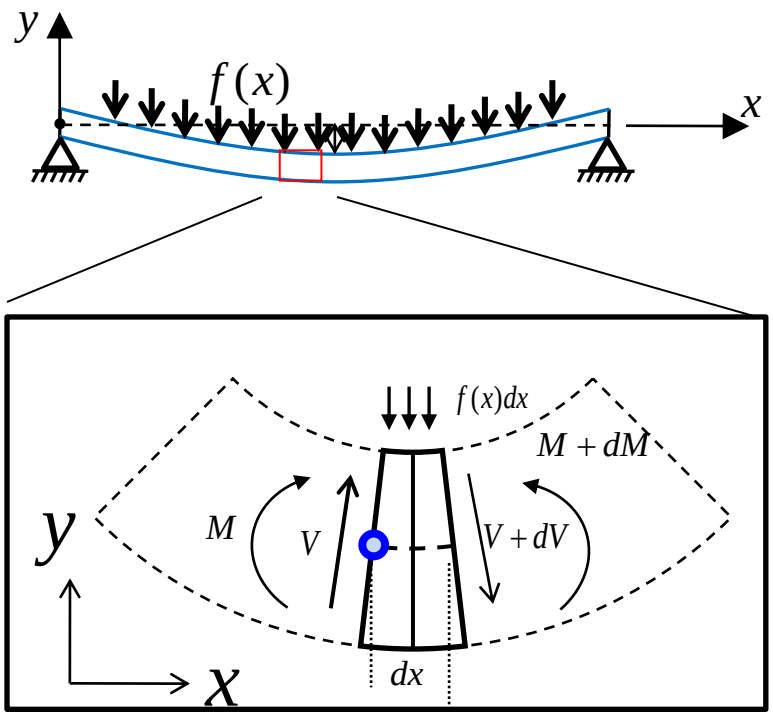
1) From the moment equilibrium about z-axis through tho point , we obtain :

$$-M+(M+dM)-(V+dV)dx-\frac{1}{2}dx\cdot f(x)dx=0$$

$$dM-Vdx-dV\cdot dx-\frac{1}{2}(dx)^2\cdot f(x)=0$$

neglecting the high order terms, and $f(x)=0$

$$\frac{dM}{dx}=V(x) \quad \Longrightarrow \quad \boxed{dM=V(x)dx}$$



2) From the force equilibrium, we obtain $V-(V+dV)-f(x)dx=0$

Since $f(x)=0$

$$\boxed{dV=0}$$

If the distributed load is not applied to the beam, the shear force V is constant, but the bending moment M is not.

Ship Structural Design

● Ship Structural Design

what is designer's **major** interest?

● **Safety :**

Won't 'IT' fail under the load?

- a ship, a stiffener, a plate → **global / local**
- a ship

Load : $f(x)$

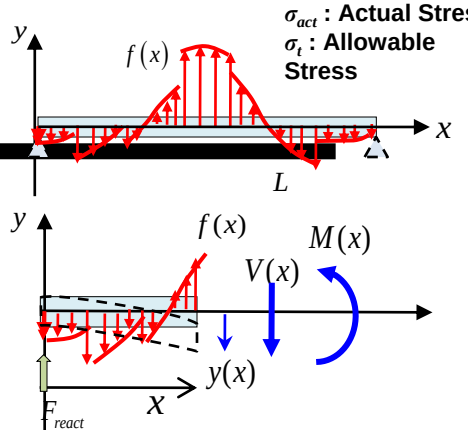
cause

Shear Force : $V(x)$
Bending Moment : $M(x)$
Deflection : $y(x)$

'relations' of load, S.F., B.M., and deflection

$$\frac{dV(x)}{dx} = -f(x) \;,\; \frac{dM(x)}{dx} = V(x)$$
$$,\; EI\frac{d^2y(x)}{dx^2} = M(x)$$

σ_{act} : Actual Stress
σ_t : Allowable Stress



what is our interest?

● **Safety :**
Won't it fail under the load?

Stress should meet :

$$\sigma_{act} \le \sigma_l \quad , \text{where } \sigma_{act} = \frac{M}{I_{\bar{y}}/\bar{y}_i} = \frac{M}{Z}$$

● **Geometry :**
How much it would be bent under the load?

Differential equations of the defection curve

$$EI\frac{d^4y(x)}{dx^4} = -f(x)$$

what kinds of load f cause hull girder moment?

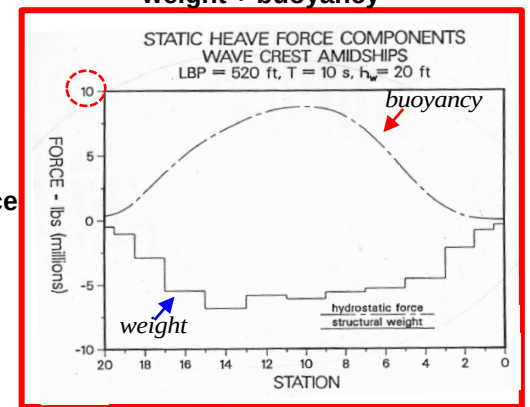
$$\sigma_{act.} \le \sigma_l \;,\; \sigma_{act.} = \frac{M_S + M_W}{Z_{mid}}$$

, M_S = Still water bending moment
M_W = Vertical wave bending moment

$$f(x) = f_S(x) + f_W(x)$$
$$V(x) = V_S(x) + V_W(x)$$
$$M(x) = M_S(x) + M_W(x)$$

Hydrostatics

$f_S(x)$: load in still water
= weight + buoyancy



$f_S(x)$: load in still water

$$V_S(x) = \int_0^x f_S(x)dx$$

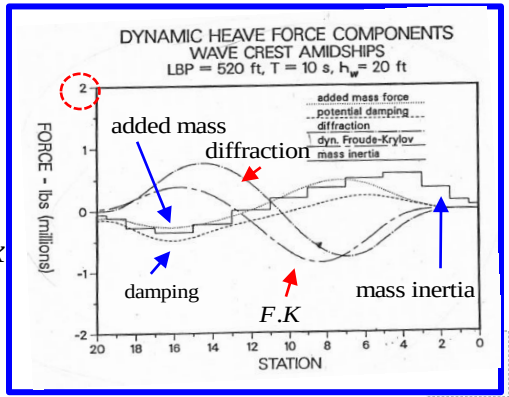
$V_S(x)$: still water shear force

$$M_S(x) = \int_0^x V_S(x)dx$$

$M_S(x)$: still water bending moment

Hydrodynamics

$f_W(x)$: load in wave
= added mass + diffraction
+ damping + Froude-Krylov + mass inertia



$f_W(x)$: load in wave

$$V_W(x) = \int_0^x f_W(x)dx$$

$V_W(x)$: wave shear force

$$M_W(x) = \int_0^x V_W(x)dx$$

$M_W(x)$: vertical wave bending moment

Ship Structural Design

● Ship Structural Design

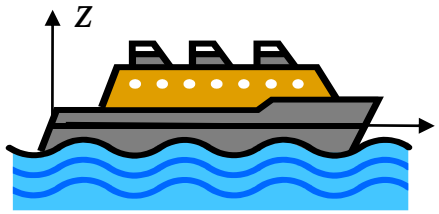
what is designer's **major** interest?

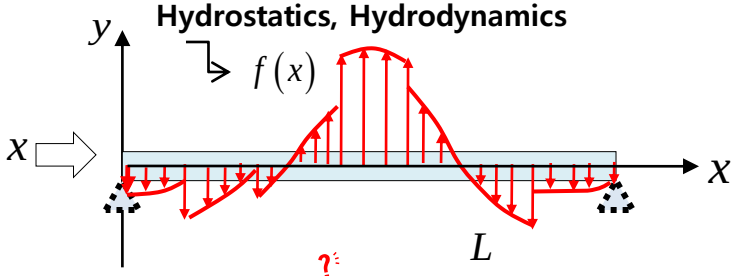
● **Safety :**

Won't 'IT' fail under the load?

- a ship — global
- a stiffener — local
- a plate — local

a ship





Hydrostatics, Hydrodynamics

Actual stress on midship section should be less than allowable stress

$$\sigma_{act.} \le \sigma_{allow}$$

$$, \sigma_{act.} = \frac{M_{mid}}{Z_{mid}} = \frac{M_s + M_w}{I_{ship,N.A}/\bar{y}_i}$$

Allowable stress by Rule : (for example)

$$, \sigma_{allow} = 175 f_1 \ [N/mm^2]$$

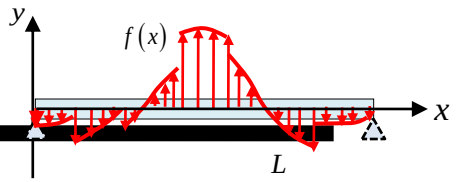
how we can meet the rule?

'Midship Design' is to arrange the structural members and fix the thickness of them to secure enough section modulus to the rule.

Load : $f(x)$

cause

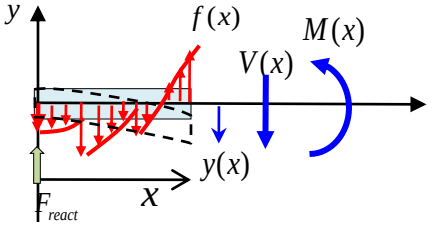
Shear Force : $V(x)$
Bending Moment : $M(x)$
Deflection : $y(x)$



'relations' of load, S.F., B.M., and deflection

$$\frac{dV(x)}{dx} = -f(x) \ , \frac{dM(x)}{dx} = V(x)$$

$$, EI\frac{d^2y(x)}{dx^2} = M(x)$$



what is our interest?

● **Safety :**
Won't it fail under the load?

Stress should meet :

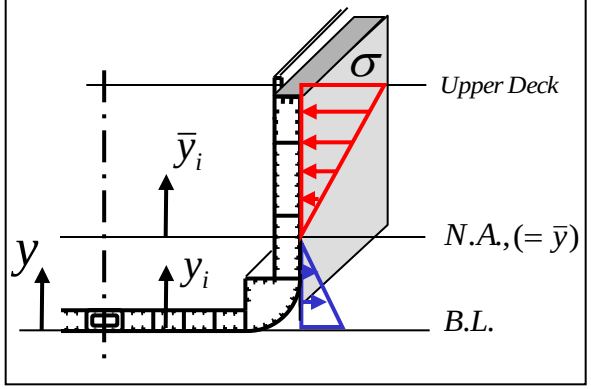
$$\sigma_{act} \le \sigma_{allow} \ , \text{ where } \sigma_{act} = \frac{M}{I_{\bar{y}}/\bar{y}_i} = \frac{M}{Z}$$

● **Geometry :**
How much it would be bent under the load?

Differential equations of the defection curve

$$EI\frac{d^4y(x)}{dx^4} = -f(x)$$

<Midship section>



$, M_w$: vertical wave bending moment

$, M_s$: still water bending moment

$, I_{ship,N.A.}$: moment of inertia from N.A. of Midship section

Solving for ODE of Deflection Curve of a Beam

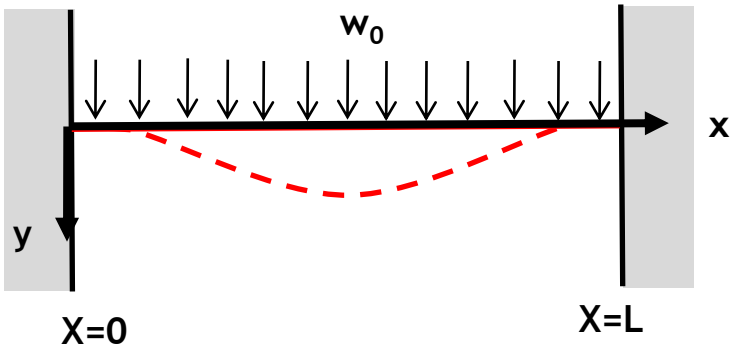
Simple Integration

How to solve a Differential Equation?

↳ Integration!

☑ Example 1 An Embedded Beam

Find the deflection of a Beam if a constant load w_0 is uniformly distributed along its length



ODE of Deflection of a Beam

(Ref: Strength of Materials, Gere)

$$EI\frac{d^4y(x)}{dx^4} = w_0(x)$$

We need 4 initial conditions to determine 4 integral coefficients

There is no vertical deflection at both end points :

$$y(0)=0 \quad y(L)=0$$

$$y'(0)=0 \quad y'(L)=0$$

(4 boundary conditions)

$$EI\frac{d^4y(x)}{dx^4}=w_0(x)$$

After integrating four times,

$$y(x)=c_1+c_2x+c_3x^2+c_4x^3+\frac{w_0}{24EI}x^4$$

$$y'(x)=c_2+2c_3x+3c_4x^2+\frac{w_0}{6EI}x^3$$

$y(0)=0$ **and** $y'(0)=0$ **give**

$$c_1=0,\ c_2=0$$

$y(L)=0$ **and** $y'(L)=0$ **give**

$$c_3L^2+c_4L^3+\frac{w_0}{24EI}L^4=0$$

$$3c_3L+3c_4L^2+\frac{w_0}{6EI}L^3=0$$

$$c_3=\frac{w_0L^2}{24EI},\ c_4=-\frac{w_0L}{12EI}$$

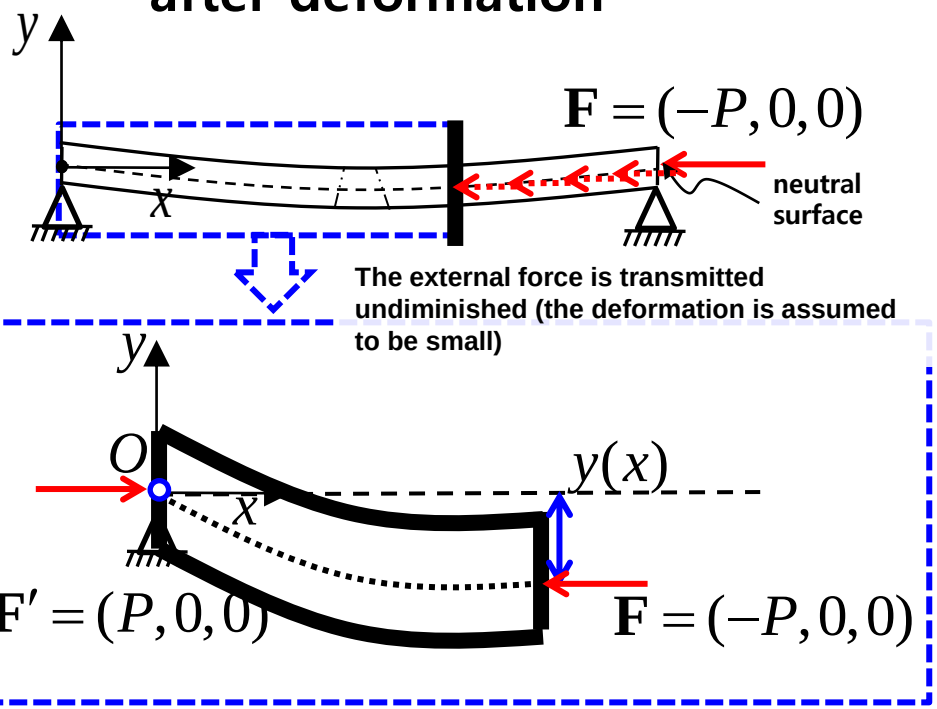
$$\therefore y(x)=\frac{w_0L^2}{24EI}x^2-\frac{w_0L}{12EI}x^3+\frac{w_0}{24EI}x^4$$

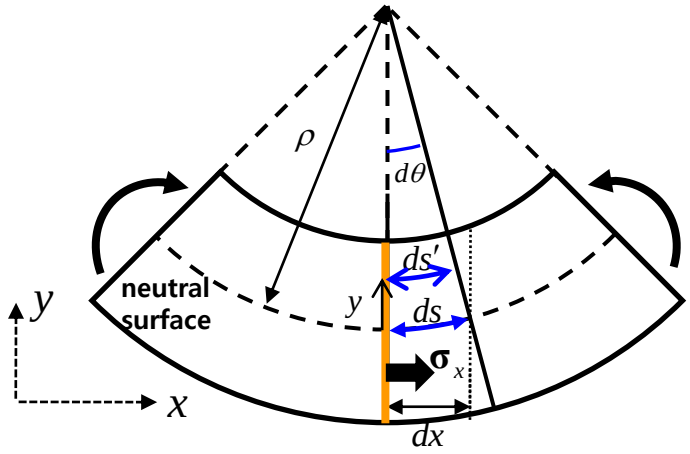
or $$y(x)=\frac{w_0}{24EI}x^2(x-L)^2$$

1-3. Modeling of Buckling of a Thin Vertical Column

Deflection of a Beam with Vector Notation

-after deformation





Equilibrium of forces:

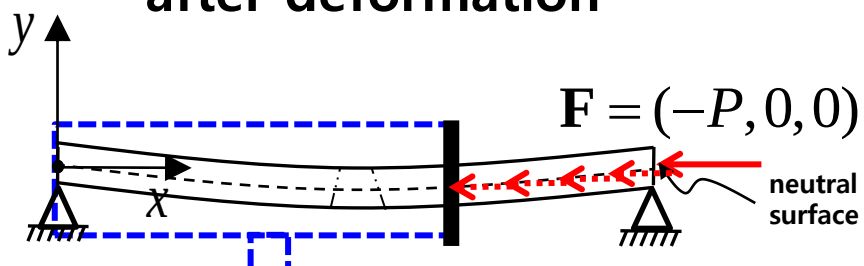
$$\mathbf{F}' + \mathbf{F} = \mathbf{0}$$

$$(P, 0, 0) + (-P, 0, 0) = \mathbf{0}$$

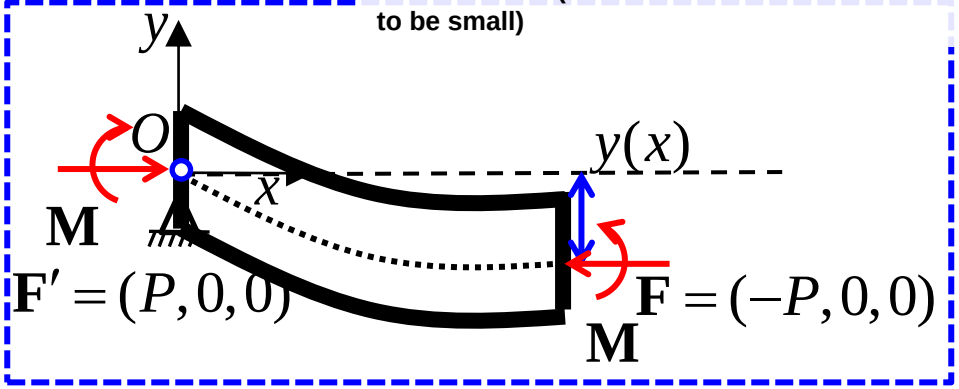
- **Differential Equation of the deflection curve of a beam:** $M = EI \dfrac{d^2 y}{dx^2}$

F′: reaction force

-after deformation



The external force is transmitted undiminished (the deformation is assumed to be small)

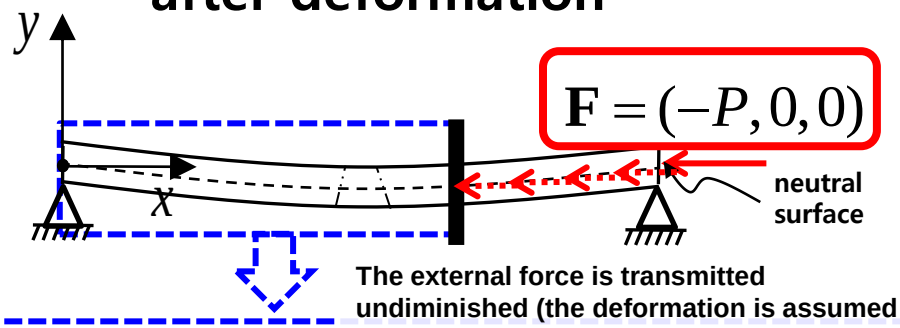


Equilibrium of forces: $\mathbf{F}' + \mathbf{F} = \mathbf{0}$

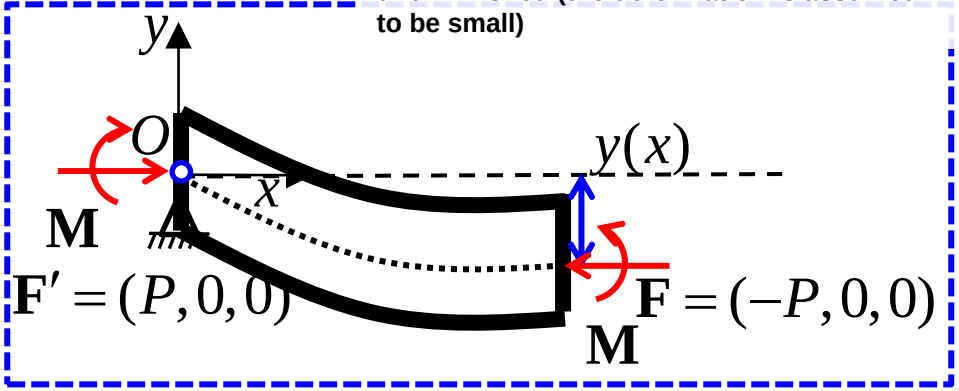
Equilibrium of moments about point O:

F′: reaction force

-after deformation



The external force is transmitted undiminished (the deformation is assumed to be small)



Equilibrium of forces $\mathbf{F}' + \mathbf{F} = \mathbf{0}$

Equilibrium of moments about point O:

$$\mathbf{i} \cdot M_x + \mathbf{i}(y \cdot F_z - z \cdot F_y)$$
$$+\mathbf{j} \cdot M_y + \mathbf{j}(-x \cdot F_z + z \cdot F_x) = 0$$
$$+\mathbf{k} \cdot M_z + \mathbf{k}(x \cdot F_y - y \cdot F_x)$$

F′: reaction force

$$\mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} \quad \begin{matrix} \mathbf{i}(y \cdot F_z - z \cdot F_y) \\ = +\mathbf{j}(-x \cdot F_z + z \cdot F_x) \\ +\mathbf{k}(x \cdot F_y - y \cdot F_x) \end{matrix}$$

$$\mathbf{k} \cdot M_z + \mathbf{k}(-y \cdot F_x) = 0$$

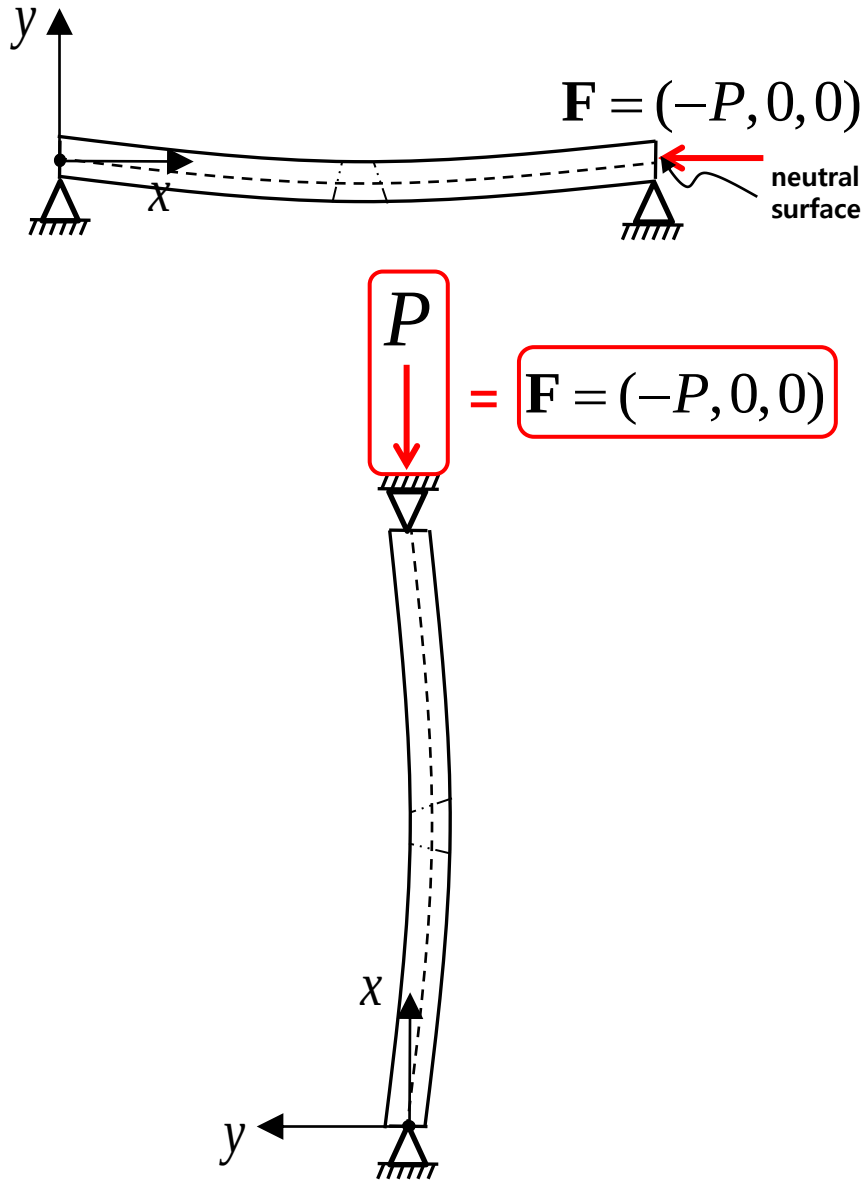
$$\downarrow \quad F_x = -P$$

$$\mathbf{k} \cdot M_z + \mathbf{k}(y \cdot P) = 0$$

$$\downarrow \quad \textbf{Drop the vector k}$$

$$M + y \cdot P = 0$$

Buckling of a Thin Vertical Column



- **Differential Equation of the deflection curve of a beam:**

$$M = EI\frac{d^2y}{dx^2} \longrightarrow EI\frac{d^2y}{dx^2} - M = 0$$

- **Equilibrium of moments about point A:**

$$M + y \cdot P = 0$$

$$EI\frac{d^2y}{dx^2} + y \cdot P = 0$$

Column Buckling

James M. Gere, Mechanics of Materials 6th Edition, Thomson, pp. 748~762

- **Differential equation for column buckling:** $EIy'' + Py = 0$

$$\frac{P}{EI} = k^2, k = \frac{n\pi}{L}$$

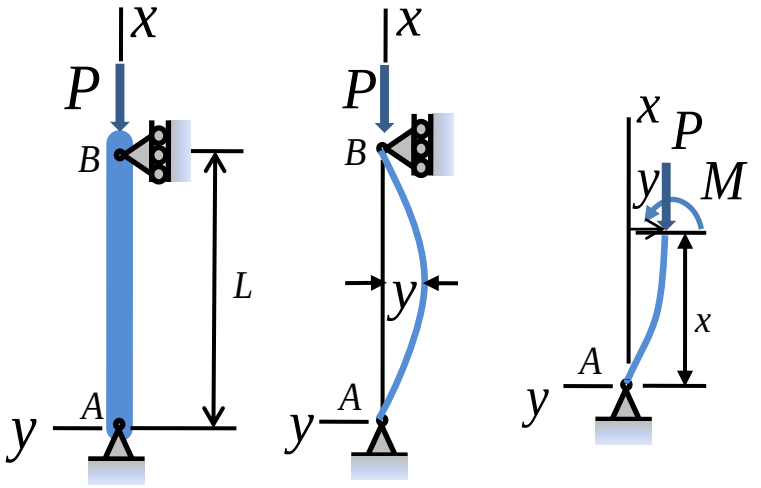
Using the notation $k^2 = \frac{P}{EI}$, $y'' + k^2 y = 0$

General solution of the equation: $y = C_1 \sin kx + C_2 \cos kx$

Boundary conditions: $y(0) = 0,\ y(L) = 0$

$$y(0) = C_2 = 0$$

$$y(L) = C_1 \sin kL = 0$$



E **= modulus of elasticity**
I **= moment of inertia of section area**
EI **= flexural rigidity**
P **= axial load**
v **= deflection of column**
L **= length of column**

1) If C_1**=0,** y**=0(trivial solution).**

2) If $\sin kl$**=0, (**$\sin kl$**=0: buckling equation)**

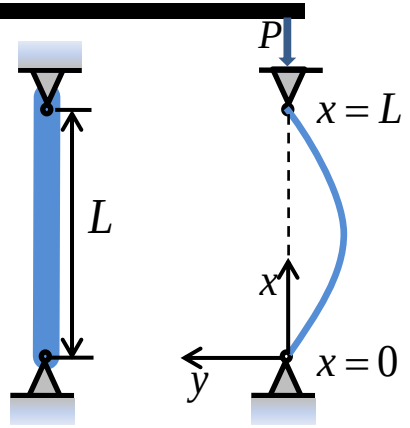
① If kl**=0,** y**=0(trivial solution).**

② If $kl = n\pi$ **(n=1,2,3) or** $P = \left(\frac{n\pi}{L}\right)^2 EI$**, it is nontrivial solution.**

$$\therefore y = C_1 \sin kx = C_1 \sin \frac{n\pi x}{L}, n = 1, 2, 3...$$

Buckling of a Thin Vertical Column: Critical Loads

$$y'' + k^2 y = 0 \quad , y(0) = 0 \quad , y(L) = 0 \quad , where\ k^2 = P / EI$$



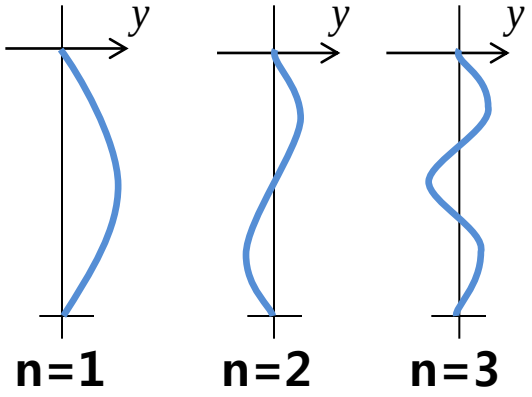
The deflection curves are $y_n(x) = c_1 \sin(n\pi x / L)$ **, corresponding to the eigenvalues** $P_n = n^2\pi^2 EI / L^2, n = 1, 2, 3...$

Physically this means that the column will buckle or deflect only when the compressive force is one of the values

$$P_n = n^2\pi^2 EI / L^2, n = 1, 2, 3...$$: **Critical loads.**

The smallest critical load $P_1 = \pi^2 EI / L^2$ **called 'Euler load'**

The deflection curves corresponding to n=1, n=2, and n=3 are shown in right figure.



First-Order Differential Equation : Solution by Separating Variables

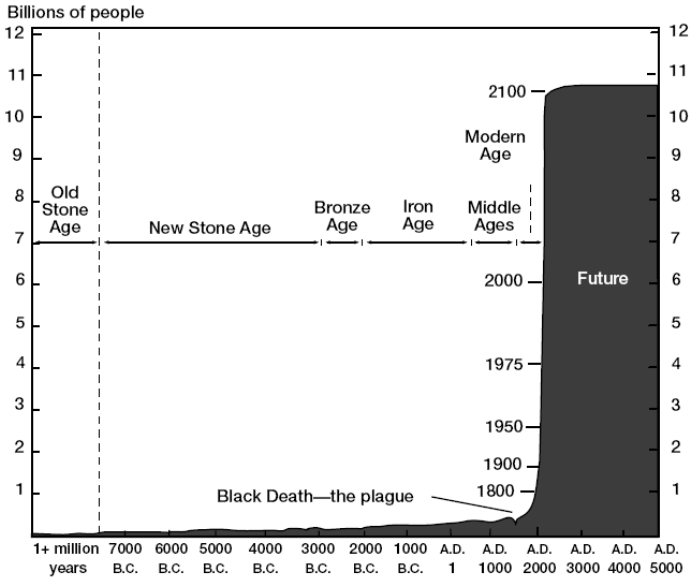
☑Population Dynamics

- Proposed by English economist Thomas Malthus in 1798

① Population varies with time P=P(t)

② Population of country grows at a certain time is proportional to the total population

$$\frac{dP}{dt} \propto P \;\; -> \;\; \frac{dP}{dt} = kP$$



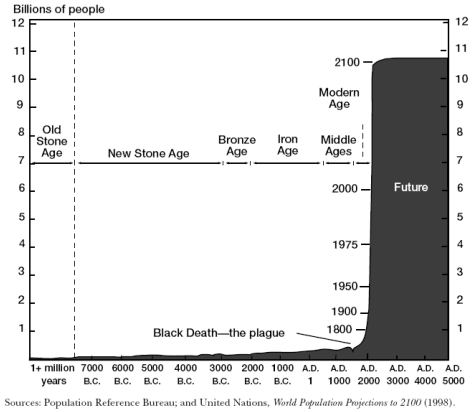
Sources: Population Reference Bureau; and United Nations, *World Population Projections to 2100* (1998).

Can we always solve ODE by simple Integration ?

How to solve a Differential Equation?

Ex) Population dynamics

$$\frac{dP(x)}{dx} = kP(x)$$



Sources: Population Reference Bureau; and United Nations, *World Population Projections to 2100* (1998).

Integration!!

L.H.S:

$$\int \frac{dP(x)}{dx} dx = P(x) + C$$

R.H.S:

$$\int kP(x)dx = k\int P(x)dx$$

$$\therefore \boxed{P(x)} + C = k\int \boxed{P(x)}dx$$

Can we solve?

Then, how?

1-4. Solving Ordinary Differential Equations(ODEs) by Separation of Variables

$$\therefore P(x) + C = k\int P(x)dx$$

Can we solve?

$$\frac{dP(x)}{dx} = kP(x)$$

transform

$$\frac{dP(x)}{P} = kdx$$

Separation of Variables

L.H.S:

$$\int \frac{dP(x)}{p} = \ln|P(x)| + c$$

R.H.S:

$$\int kdx = kx$$

$$\therefore \ln|P(x)| + c = kx$$

First-Order Differential Equation : Solution by Separating Variables

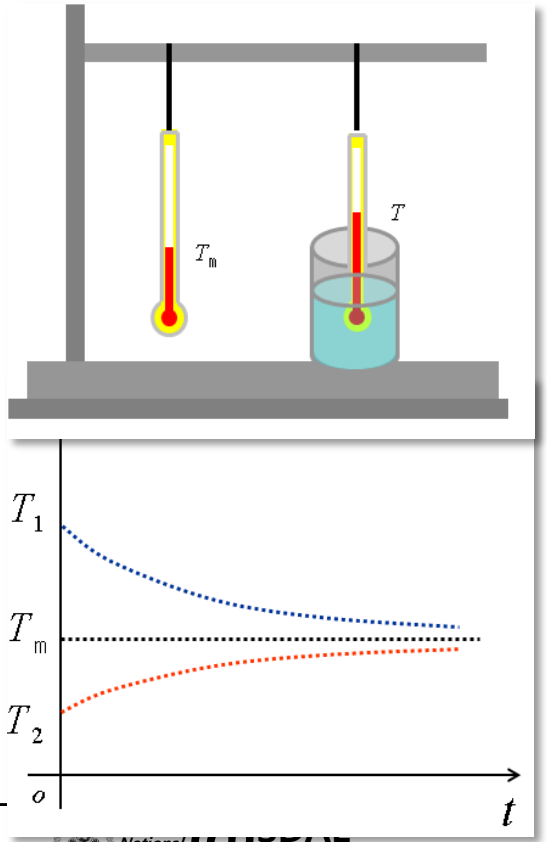
☑ Newton's Law of Cooling

- The rate at which the temperature change of a body is proportional to the difference between temperature of the body and the temperature of the surrounding medium

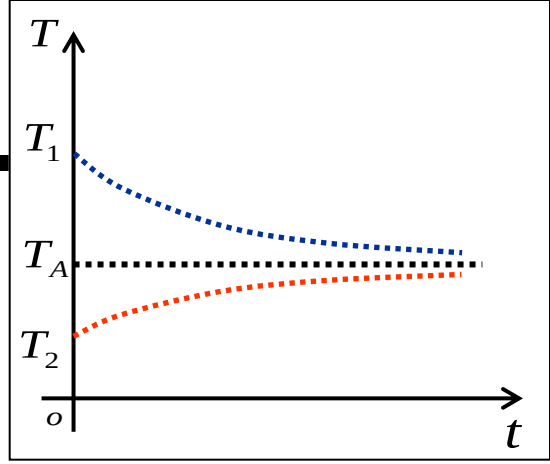
$$\frac{dT}{dt} \propto (T - T_A) \; -> \; \frac{dT}{dt} = k(T - T_A)$$

T : body temperature

T_A : Surronding medium temperature

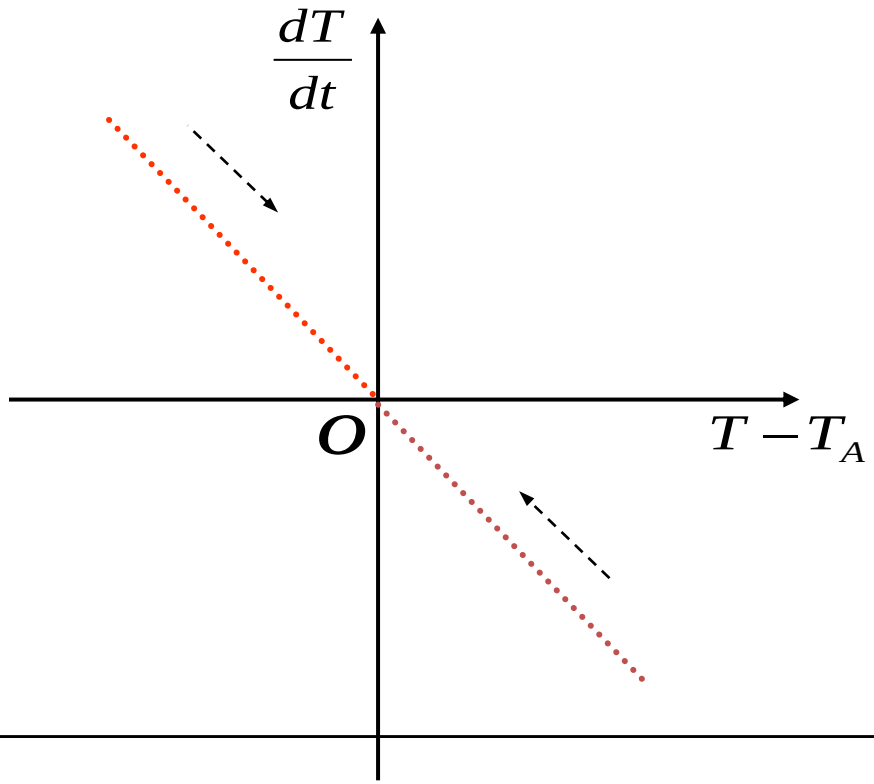


Newton's law of cooling(2)



Great Idea !!

Relation between $\frac{dT(t)}{dt}$ **and** $T - T_A$.



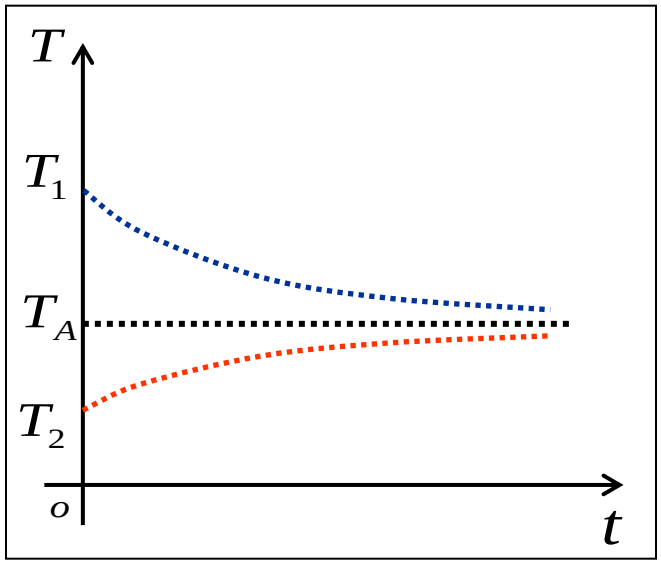
$$\frac{dT}{dt} \propto T - T_A$$

$$\frac{dT}{dt} = k(T - T_A)$$

$$,(k < 0)$$

Newton's Law of Cooling : Solution by Separating Variables

Ex.)Newton's law of cooling



T : body temperature

T_A : outside temperature(constant)

T_1, T_2 : Initial Body temperatures

$$T_1 > T \ , \ T_2 < T$$

Great Idea !!

Relation between $\dfrac{dT(t)}{dt}$ **and** $(T - T_A)$

$$\frac{dT}{dt} = k(T - T_A), \quad k < 0$$

$$\frac{dT}{dt} = k(T - T_A)$$

$$\frac{dT}{T - T_A} = k \cdot dt$$

Ex.) Newton's law of cooling

$$\frac{dT}{dt} = k(T - T_A), \quad k < 0$$

$$\frac{dT}{T - T_A} = k \cdot dt$$

$$Y = T - T_A$$

$$\frac{dY}{dT} = 1$$

$$dY = dT$$

$$\frac{dT}{T - T_A} = k \cdot dt$$

$$\frac{dY}{Y} = k \cdot dt$$

$$\int \frac{dY}{Y} = \int k \cdot dt$$

$$\ln|Y| + c_L = kt + c_R$$

$$\ln|Y| = kt + c_R - c_L$$

$$= kt + c$$

①

$$\frac{dT}{dt} = k(T - T_A),$$
$$k < 0$$

$$\ln|Y| = kt + c$$

$$e^{\ln|Y|} = e^{kt+c}$$

$$|Y| = e^{kt+c}$$

②

$$|Y| = \tilde{c}e^{kt}$$

If $Y \geq 0$,

$$Y = \tilde{c}e^{kt}$$

If $Y < 0$,

$$-Y = \tilde{c}e^{kt}$$

$$Y = -\tilde{c}e^{kt}$$

③

$$Y = Ce^{kt}$$

$$,(C = \text{sgn}(Y = T - T_A) \cdot \tilde{c})$$

$$T - T_A = Ce^{kt}$$

$$T = Ce^{kt} + T_A$$

Solution by Separating Variables

$$\frac{dy}{dx} = f(x,y) \Rightarrow \frac{dy}{dx} = g(x)h(y) \rightarrow \int \frac{dy}{h(y)} = \int g(x)dx + c$$

Definition 2.1*

Separable Equation

A first order differential equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is said to be separable or to have separable variables

*Dennis G. Zill, Michael R. Cullen, Advanced Engineering Mathematics 3rd Edition, Johns and Bartlett, 2006, p45

Examples : Separating Variables

☑ Example 1

Solve $(1+x)dy - ydx = 0$

Dividing by $(1+x)y$

$$\frac{dy}{y} = \frac{dx}{(1+x)}$$

$$\int \frac{dy}{y} = \int \frac{dx}{(1+x)}$$

$$\ln|y| = \ln|1+x| + c_1$$

$$y = e^{\ln|1+x|+c_1} = e^{\ln|1+x|} \cdot e^{c_1}$$

$$= |1+x|\, e^{c_1}$$

$$= \pm e^{c_1}(1+x)$$

$$y = c(1+x)$$

Separable Variables

☑ Example 2

Solve the IVP

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(4) = 3$$

$$ydy = -xdx$$

$$\int ydy = -\int xdx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c_1$$

$$x^2 + y^2 = c^2$$

Substitute x=4 and y=-3 for satisfying initial condition

$$4^2 + (-3)^2 = c^2$$

$$c^2 = 25$$

$$\therefore x^2 + y^2 = 25$$

Separable Variables

☑ Example 3
Losing a Solution

Solve $\frac{dy}{dx} = y^2 - 4$

$$\frac{dy}{y^2-4} = dx \quad (y \neq \pm 2)$$

$$\frac{1}{4}\left(\frac{1}{y-2} - \frac{1}{y+2}\right)dy = dx$$

$$\frac{1}{4}\int \left(\frac{1}{y-2} - \frac{1}{y+2}\right)dy = \int dx$$

$$\frac{1}{4}(\ln|y-2| - \ln|y+2|) = x + c_1$$

$$\ln\left|\frac{y-2}{y+2}\right| = 4x + c_2$$

$$\frac{y-2}{y+2} = e^{4x+c_2} = ce^{4x}$$

$$y = 2\frac{1+ce^{4x}}{1-ce^{4x}}, \; where \; (y \neq \pm 2)$$

$y = \pm 2$ **is constant solution, but cannot be obtained from above solution**

First-Order Linear Differential Equation

Definition 2.2 **Linear Equation**

A first order differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

is said to be Linear equation in the dependant variable y.

☑ Standard form

$$\frac{dy}{dx} + P(x)y = f(x) \xrightarrow{\text{solution}} y = y_c + y_p$$

$$\frac{d}{dx}[y_c + y_p] + P(x)[y_c + y_p]$$

$$= \left[\frac{d}{dx}y_c + P(x)y_c\right] + \left[\frac{d}{dx}y_p + P(x)y_p\right] = f(x)$$

☑ Homogeneous

$$f(x) = 0 \xrightarrow[y_c]{\text{solution}} \frac{dy_c}{dx} + P(x)y_c = 0$$

☑ Nonhomogeneous

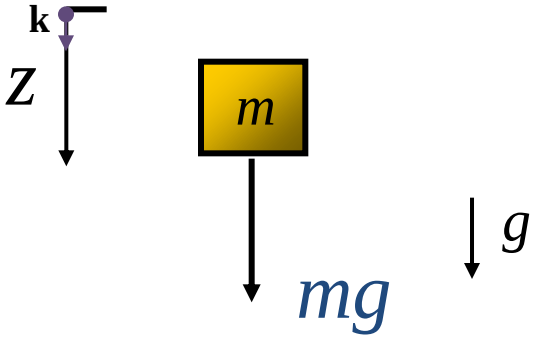
$$f(x) \neq 0 \xrightarrow[y_p]{\text{solution}} \frac{dy_p}{dx} + P(x)y_p = f(x)$$

1-5. Modeling of Mass-Spring-Damping System

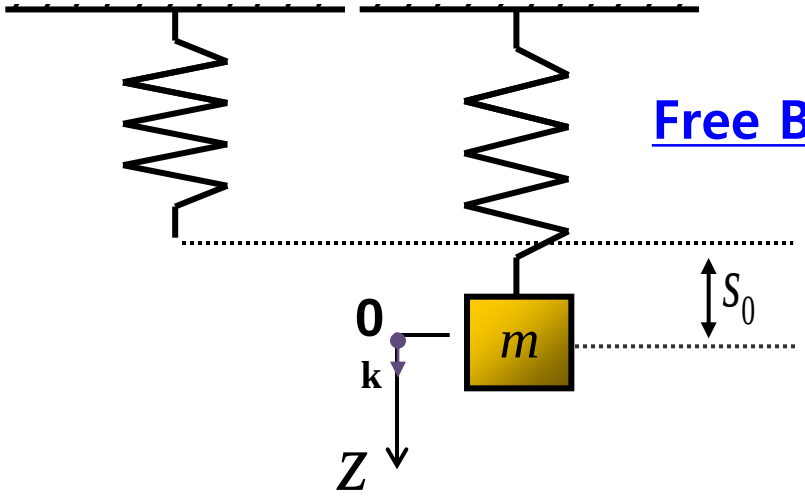
Modeling : Mass-Spring Systems: Driven Motion

Step 1: An object with mass of m ✓ **Step 2**: The object is connected with a spring having spring constant k

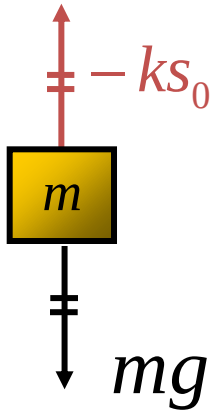
Free Body Diagram



:Gravity force



Free Body Diagram



By **Newton's 2nd law**,

$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= mg\mathbf{k}$$

where $\mathbf{z} = z\mathbf{k}$ $\quad z = z(t)$

$$\ddot{\mathbf{z}} = \frac{d^2\mathbf{z}}{dt^2} = \frac{d^2z}{dt^2}\mathbf{k} = \ddot{z}\,\mathbf{k}$$

Nonlinearity of spring

$$\mathbf{F}(z) = -k\mathbf{z} - k_1\mathbf{z}^3$$

linearize

Hooke's law

$$F \propto z$$
$$\mathbf{F}_{spring} = -k\mathbf{z}$$

k : spring constant

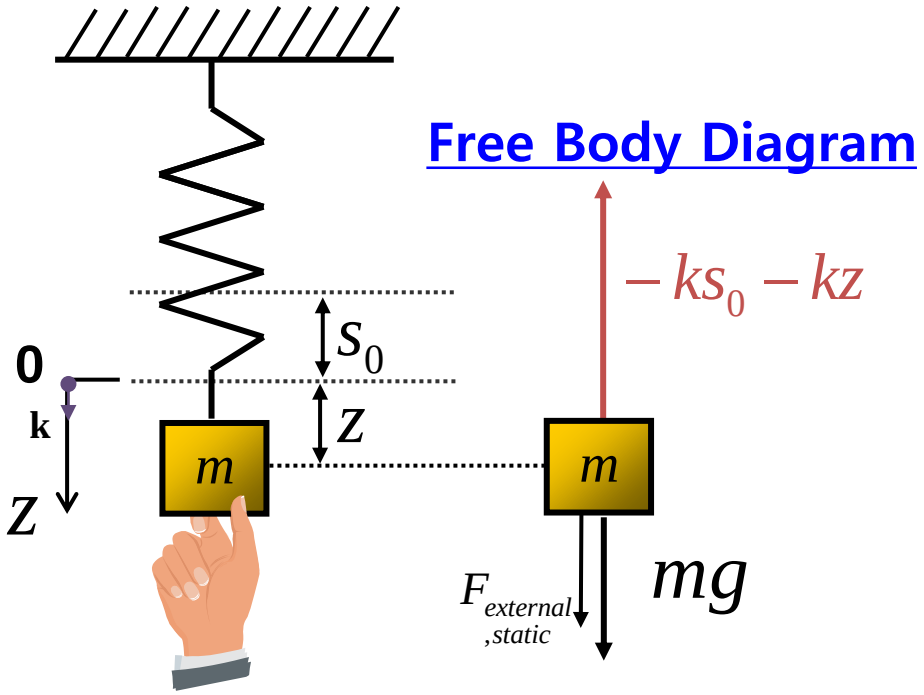
○ opposite to the direction of displacement

$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k}$$
$$= 0 \ (\because \ddot{\mathbf{z}} = 0)$$

: **static equilibrium**

Suppose that an external force is applied on the object

Step 3: An external force, $F_{external,static}$, is applied on the object



Free Body Diagram

$$m\ddot{\mathbf{z}} = \mathbf{F}$$

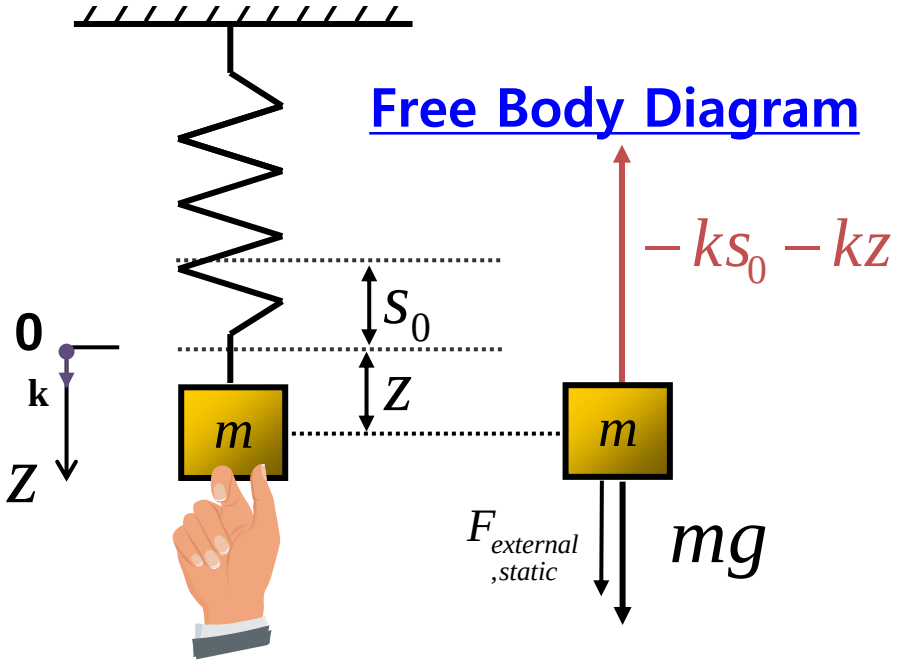
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} + \mathbf{F}_{external,static}$$

$$= -kz\mathbf{k} + \mathbf{F}_{external,static}$$

$$= 0 \quad (\because \ddot{\mathbf{z}} = 0)$$

Shortly after, the external force is removed. What will happen ?

✓ Step 4: Shortly after, the external force is removed.



$$m\ddot{\mathbf{z}} = \mathbf{F}$$

$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} + \mathbf{F}_{external,static}$$

$$= -kz\mathbf{k} + \mathbf{F}_{external,static}$$

Restoring force

Physical Phenomenon

Mathematical Model

$$m\ddot{z} + kz = 0$$

Oscillation due to the restoring force

$$m\ddot{\mathbf{z}} = -kz\mathbf{k}$$

where $\mathbf{z} = z\mathbf{k}$

$$\ddot{\mathbf{z}} = \frac{d^2\mathbf{z}}{dt^2} = \frac{d^2z}{dt^2}\mathbf{k} = \ddot{z}\,\mathbf{k}$$

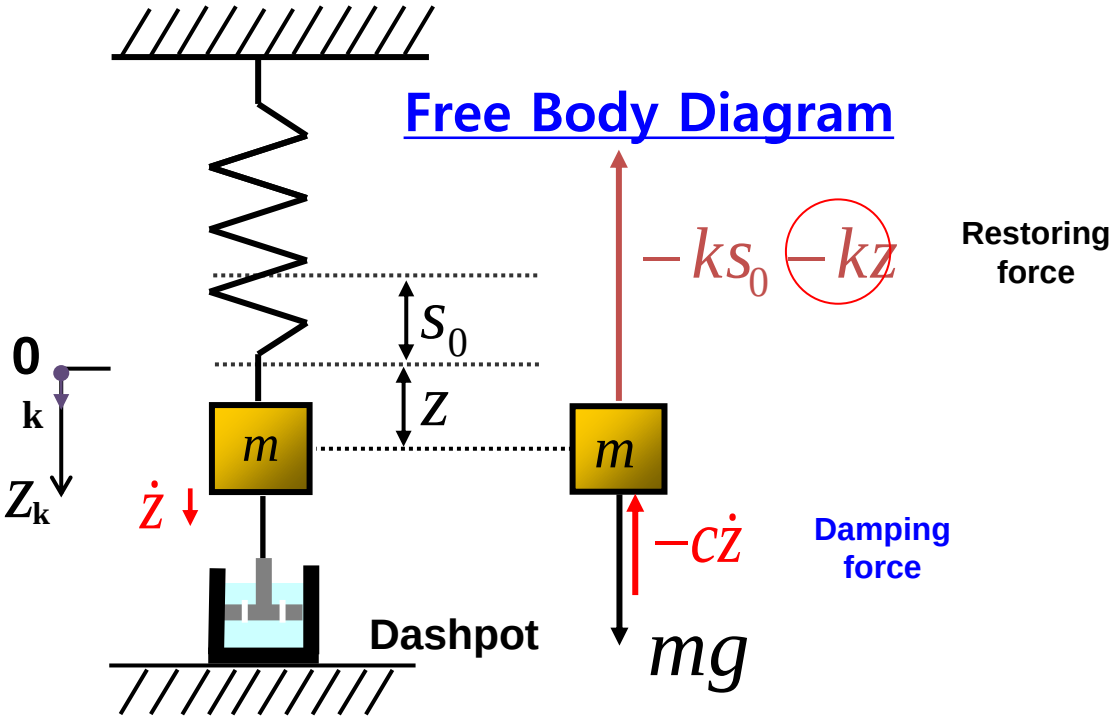
$$m\ddot{z}\mathbf{k} = -kz\mathbf{k}$$

$$m\ddot{z} = -kz$$

$$m\ddot{z} + kz = 0$$

$z = z(t)\,, \quad z'' = \dfrac{d^2 z}{dt^2}$

✓ **Step 5**: Frictional force(damping force) opposite to the velocity of the object

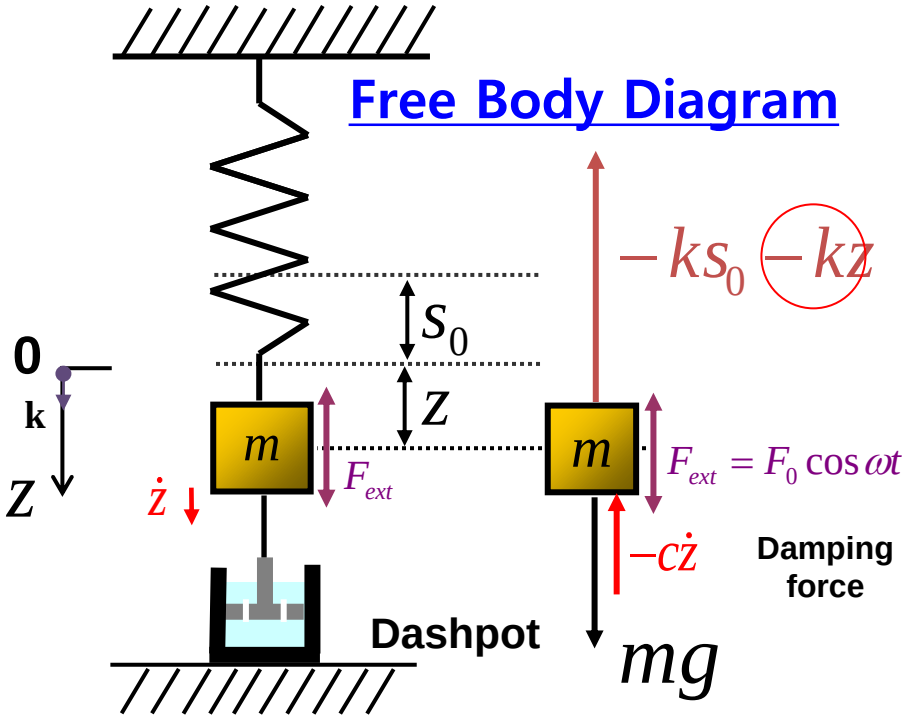


$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - c\dot{z}\mathbf{k}$$
$$= -kz\mathbf{k} - c\dot{z}\mathbf{k}$$

$$m\ddot{z} + c\dot{z} + kz = 0$$

Consider an oscillatory external force (Nonhomogeneous term)

Step 6: Consider an oscillatory external force



$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - c\dot{z}\mathbf{k} + \mathbf{F}_0 \cos \omega t$$
$$= -kz\mathbf{k} - c\dot{z}\mathbf{k} + \mathbf{F}_0 \cos \omega t$$

$$m\ddot{z} + c\dot{z} + kz = \mathbf{F}_0 \cos \omega t$$

1-6. Second Order Homogeneous ODEs

Definition 3.3 **Fundamental Set of Solutions**

Any set $y_1, y_2, \ldots, y_n$ of n **linearly independent** solutions of the homogeneous linear *n*th-order D.E. on an interval I is said to be a fundamental set of solutions on the interval.

'Basis'

"Any solution of an *n*th-order homogeneous linear differential equation on an interval I can be expressed as a linear combination of n basis solutions. I

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

Linear Combination

Basis Solutions (Linearly Independent)

Linear Independence

Definition 3.1

Linear Dependence / Independence

A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is said to be 'linearly dependent' on an interval I if there exist constant $c_1, c_2, \ldots c_n$, not all zero such that $c_1 f_1(x) + c_2 f_2(x) + \cdots c_n f_n(x) = 0$ for every x in the interval.

If the set of functions is not linearly dependent on the interval, it is said to be 'linearly independent'

In other words, a set of functions is 'linearly independent' if the only constants for

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

are $c_1 = c_2 = \cdots = c_n = 0$

"two functions are linearly independent when neither is a constant multiple of the other"

Free Oscillation of Spring-Mass-damping System

Linear Differential Equation

$$m\ddot{\mathbf{z}} + c\dot{\mathbf{z}} + k\mathbf{z} = F_0 \cos \omega t$$

- **Step1. homogeneous solution**

$$m\ddot{\mathbf{z}} + c\dot{\mathbf{z}} + k\mathbf{z} = 0$$

Try: $z = e^{\lambda t}$

$$(m\lambda^2 + c\lambda + k)e^{\lambda t} = 0$$

$$m\lambda^2 + c\lambda + k = 0 \quad \leftarrow \textbf{Auxiliary equation}$$

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$$z_1 = e^{\lambda_1 t}$$
$$z_2 = e^{\lambda_2 t} \qquad where,\ \lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

Z_1, Z_2 are the **fundamental set of solution**

so, the general solution for the homogeneous equation will be…

$$z = c_1 z_1 + c_2 z_2 \left(= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\right)$$

But.. are you sure Z_1, Z_2 are linearly independent?

There could be three cases depending on the condition

cases	$c^2 - 4mk$	Root
Case I	$c^2 - 4mk > 0$	Distinct Real Roots λ_1, λ_2
Case II	$c^2 - 4mk = 0$	Repeated Real Roots λ_1
Case III	$c^2 - 4mk < 0$	Conjugated Complex Roots $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$

Case 1: $c^2 - 4mk > 0$

cases	$c^2 - 4mk$	Root
Case I	$c^2 - 4mk > 0$	Distinct Real Roots $\lambda_1 \neq \lambda_2$

Solution
$z = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

$$\lambda_1 = \frac{-c + \sqrt{c^2 - 4mk}}{2m} \Rightarrow z = e^{\lambda_1 t}$$

$$\lambda_2 = \frac{-c - \sqrt{c^2 - 4mk}}{2m} \Rightarrow z = e^{\lambda_2 t}$$

Linearly independent

Linear combination

Recall the example

$c_1 e^t + c_2 e^{2t} = 0$

Satisfied only when $c_1 = c_2 = 0$

Linearly Independent

Case 2: $c^2 - 4mk = 0$

cases	$c^2 - 4mk$	Root
Case II	$c^2 - 4mk = 0$	Repeated Real Roots λ_1

Solution

$$\lambda_1 = \frac{-c + \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \Rightarrow z_1 = e^{\lambda_1 t}$$

To create another linearly independent solution , multiply t to z_1 $\Rightarrow$ $z_2 = te^{\lambda_1 t}$

Cf.) Reduction of order

cases	$c^2 - 4mk$	Root
Case II	$c^2 - 4mk = 0$	Repeated Real Roots λ_1

Solution
$z = c_1 e^{\lambda_1 t} + c_2 te^{\lambda_1 t}$

$$\lambda_1 = \frac{-c + \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \Rightarrow$$

$z_1 = e^{\lambda_1 t}$

$z_2 = te^{\lambda_1 t}$

Linearly independent

Linear combination

Case 3: $c^2 - 4mk < 0$

cases	$c^2 - 4mk$	Root	
Case III	$c^2 - 4mk < 0$	Conjugated Complex Roots	$\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$

Solution

$$\lambda_1 = \frac{-c}{2m} + \frac{\sqrt{c^2 - 4mk}}{2a} = \frac{-c}{2m} + i\frac{\sqrt{4mk - c^2}}{2m} = \alpha + i\beta$$

$$\lambda_2 = \frac{-c}{2m} - \frac{\sqrt{c^2 - 4mk}}{2m} = \underbrace{\frac{-c}{2m}}_{\alpha} - i\underbrace{\frac{\sqrt{4mk - c^2}}{2m}}_{\beta} = \alpha - i\beta$$

$$z_1 = e^{(\alpha+i\beta)t}$$

$$z_2 = e^{(\alpha-i\beta)t}$$

Linearly independent → Linear combination

$$z = C_1 z_1 + C_2 z_2$$

Linear combination

$$z = C_1 e^{(\alpha+i\beta)t} + C_2 e^{(\alpha-i\beta)t}$$

Two solutions by the choices

$$C_1 = 1, C_2 = 1$$

$$z_1 = e^{(\alpha+i\beta)t} + e^{(\alpha-i\beta)t}$$

$$C_1 = 1, C_2 = -1$$

$$z_2 = e^{(\alpha+i\beta)t} - e^{(\alpha-i\beta)t}$$

Euler's formula

$$e^{it} = \cos t + i \sin t$$

$$z_1 = e^{(\alpha+i\beta)t} + e^{(\alpha-i\beta)t} = e^{\alpha t}(\cos \beta t + i \sin \beta t) + e^{\alpha t}(\cos \beta t - i \sin \beta t) = 2e^{\alpha t} \cos \beta t$$

$$z_2 = e^{(\alpha+i\beta)t} - e^{(\alpha-i\beta)t} = e^{\alpha t}(\cos \beta t - i \sin \beta t) + e^{\alpha t}(\cos \beta t - i \sin \beta t) = 2e^{\alpha t} \sin \alpha t$$

New fundamental set of solution

cases	$c^2 - 4mk$	Root	
Case III	$c^2 - 4mk < 0$	Conjugated Complex Roots	$\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$

Solution
$y = e^{\alpha t}(c_1 \cos \beta t + c_2 \cos \beta t)$

$z_1 = 2e^{\alpha t} \cos \beta t$ — We mulitply the result z_1 by $\frac{1}{2}$ → $e^{\alpha t} \cos \beta t$

$z_2 = 2ie^{\alpha t} \sin \beta t$ — We mulitply the result z_2 by $\frac{1}{2i}$ → $e^{\alpha t} \sin \beta t$

Linearly independent → **Linear combination**

a basis of real solutions.

This results obtained by addition and multiplication by constants are again solutions.

Free Oscillation of Spring-Mass-damping System

Linear Differential Equation

$$mz'' + cz' + kz = F_0 \cos \omega t$$

c : constant
m : constant
m : constant

- **Step1. homogeneous solution**

$$mz'' + cz' + kz = 0 \qquad \text{Try: } z = e^{\lambda t} \rightarrow m\lambda^2 + c\lambda + k = 0$$

Case 1 $c^2 - 4mk > 0$

$$z_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \left(\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} \right), (\lambda_1, \lambda_2 < 0)$$

Case 2 $c^2 - 4mk = 0$

$$z_h = (c_1 + c_2 \cdot t) \cdot e^{\frac{-c}{2m} \cdot t} \quad , \left(\frac{-c}{2m} < 0 \right)$$

Case 3 $c^2 - 4mk < 0$

$$z_h = e^{\frac{-c}{2m} t} \cdot \left(A \cdot \cos(\omega_0 t) + B \cdot \sin(\omega_0 t) \right) \quad , \left(\omega_0 = \sqrt{\left(\frac{k}{m}\right) - \frac{1}{4} \cdot \left(\frac{c}{m}\right)^2} = \frac{1}{2m} \sqrt{4mk - c^2} \right)$$

Homogeneous Linear Equation with Constant Coefficients

☑ Example 1

Second-Order DEs

Solve the following differential equations

(a) $2y'' - 5y' - 3y = 0$

(b) $y'' - 10y' + 25y = 0$

(c) $y'' + 4y' + 7y = 0$

(a) **Substitute** $y = e^{mx}$

$$2m^2 - 5m - 3 = (2m+1)(m-3) = 0$$

$$m_1 = \frac{-1}{2},\ m_2 = 3$$ **Distinct Real Root**

$$y = c_1 e^{-\frac{1}{2}x} + c_2 e^{3x}$$

(b) **Substitute** $y = e^{mx}$

$$m^2 - 10m + 25 = (m-5)^2 = 0$$

$$m_1 = m_2 = 5$$ **Repeated Real Root**

$$y = c_1 e^{5x} + c_2 x e^{5x}$$

Homogeneous Linear Equation with Constant Coefficients

(c) Substitute $y = e^{mx}$

$$m^2 + 4m + 7 = 0$$

$$m_1 = -2 + \sqrt{3}i,\ m_2 = -2 - \sqrt{3}i$$

Conjugate Complex Root

$$y = e^{-2x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x)$$

Homogeneous Linear Equation with Constant Coefficients

☑ Example 2 An Initial-Value Problem

Solve the IVP

$$4y'' + 4y' + 17y = 0$$

$$y(0) = -1,\ y'(0) = 2$$

Substitute $y = e^{mx}$

$$4m^2 + 4m + 17 = 0$$

$$m_1 = -\frac{1}{2} + 2i,\ m_2 = -\frac{1}{2} - 2i$$

$$y = e^{-\frac{1}{2}x}(c_1 \cos 2x + c_2 \sin 2x)$$

$$y' = -\frac{1}{2}e^{-\frac{1}{2}x}(c_1 \cos 2x + c_2 \sin 2x) + e^{-\frac{1}{2}x}(-2c_1 \sin 2x + 2c_2 \cos 2x)$$

$$-1 = 1(c_1 + 0),\ c_1 = -1$$

$$2 = -\frac{1}{2}c_1 + 2c_2 = \frac{1}{2} + 2c_2,\ c_2 = \frac{3}{4}$$

$$\therefore y = e^{-\frac{1}{2}x}(-\cos 2x + \frac{3}{4}\sin 2x)$$

Homogeneous Linear Equation with Constant Coefficients

☑ Example 3

Third-Oder DE

Solve

$$y''' + 3y'' - 4y = 0$$

Substitute $y = e^{mx}$

$$m^3 + 3m^2 - 4 =$$

$$(m-1)(m^2 + 4m + 4) = 0$$

$$m_1 = 1,\ m_2 = m_3 = -2$$

$$\therefore y = c_1 e^{x} + c_2 e^{-2x} + c_3 x e^{-2x}$$

Homogeneous Linear Equation with Constant Coefficients

☑ Example 4
Fourth-Oder DE

Solve

$$y^{(4)} + 2y'' + y = 0$$

Substitute $y = e^{mx}$

$$m^4 + 2m^2 + 1 = (m^2 + 1)^2 = 0$$

$$m_1 = m_3 = i, \quad m_2 = m_4 = -i$$

$$y = C_1 e^{ix} + C_2 e^{-ix} + C_3 x e^{ix} + C_4 x e^{-ix}$$

$$y = C_1 e^{ix} + C_2 e^{-ix}$$

$$= c_1 \cos x + c_2 \sin x$$

$$\therefore y = c_1 \cos x + c_2 \sin x$$

$$+ c_3 x \cos x + c_4 x \sin x$$

1-7. Second Order Nonhomogeneous ODEs

Forced Oscillation of Spring-Mass-damping System

Second Order Nonhomogeneous ODE

$$mz'' + cz' + kz = F_0 \cos \omega t$$

- **Step2. particular solution**

By the method of Undetermined Coefficient, we choose that

$$z_p(t) = a\cos\omega t + b\sin\omega t$$

$$z_p'(t) = -a\omega\sin\omega t + b\omega\cos\omega t$$

$$z_p''(t) = -a\omega^2\cos\omega t - b\omega^2\sin\omega t$$

$y'' + ay' + by = r(x)$

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n (n = 0,1,\cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k\cos\omega x$ $k\sin\omega x$	$K\cos\omega x + M\sin\omega x$
$ke^{\alpha x}\cos\omega x$ $ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(K\cos\omega x + M\sin\omega x)$

$$m\left(-\omega^2 a\cos\omega t - \omega^2 b\sin\omega t\right) + c\left(-\omega a\sin\omega t + \omega b\cos\omega t\right) + k\left(a\cos\omega t + b\sin\omega t\right) = F_0\cos\omega t$$

Linear Differential Equation

$$mz'' + cz' + kz = F_0 \cos \omega t$$

$$z_p(t) = a\cos \omega t + b \sin \omega t$$

$$m\left(-\omega^2 a\cos \omega t - \omega^2 b \sin \omega t\right) + c\left(-\omega a \sin \omega t + \omega b \cos \omega t\right) + k\left(a\cos \omega t + b \sin \omega t\right) = F_0 \cos \omega t$$

$$[(k - m\omega^2)a + \omega cb]\cos \omega t + [-\omega ca + (k - m\omega^2)b]\sin \omega t = F_0 \cos \omega t$$

$$\begin{cases} (k - m\omega^2)a + \omega cb = F_0 \\ -\omega ca + (k - m\omega^2)b = 0 \end{cases}$$

$$\therefore a = F_0 \frac{m(\omega^2{}_0 - \omega^2)}{m^2(\omega^2{}_0 - \omega^2)^2 + \omega^2 c^2}, \quad b = F_0 \frac{\omega c}{m^2(\omega^2{}_0 - \omega^2)^2 + \omega^2 c^2}$$

$$z_p(t) = a\cos \omega t + b \sin \omega t$$

Particular Solution – Another Superposition

Theorem 3.7

Superposition Principle– Nonhomogeneous Equations

Let $y_{p1}, y_{p2}, \ldots, y_{pk}$ **be** k **particular solution of the nonhomogeneous linear *n*th-order differential equation on an interval** I **corresponding, in turn, to** k **distinct Functions** $g_1, g_2, \ldots, g_k$ **. That is,** y_{pi} **denotes a particular solution of the corresponding differential equation**

$$a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y' + a_0(x)y = g_i(x)$$

Where $i = 1, 2, \cdots, k$**.Then**

$$y_p = y_{p_1}(x) + y_{p_2}(x) + \cdots + y_{p_k}(x)$$

is a particular solution of

$$a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1y' + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x)$$

Particular Solution – Another Superposition

Ex) Superposition-nonhomogenous DE

$$y_{p_1} = -4x^2$$

$$y_{p2} = e^{2x}$$

$$y_{p3} = xe^x$$

is a particular solution of each DE

$$y'' - 3y' + 4y = \underbrace{-16x^2 + 24x - 8}_{g_1(x)}$$

$$y'' - 3y' + 4y = \underbrace{2e^{2x}}_{g_2(x)}$$

$$y'' - 3y' + 4y = \underbrace{2xe^x - e^x}_{g_3(x)}$$

$$y'' - 3y' + 4y$$
$$= -16x^2 + 24x - 8 + 2e^{2x} + 2xe^x - e^x$$

$$y_p = y_{p1} + y_{p2} + y_{p3}$$
$$= -4x^2 + e^{2x} + xe^x$$

is a solution of

$$y'' - 3y' + 4y$$
$$= \underbrace{-16x^2 + 24x - 8}_{g_1(x)} + \underbrace{2e^{2x}}_{g_2(x)}$$
$$+ \underbrace{2xe^x - e^x}_{g_3(x)}$$

Forced Oscillation of Spring-Mass-damping System

$$mz'' + cz' + kz = F_0 \cos \omega t$$ **Linear Differential Equation**

- **Step3. general solution**

$$z(t) = z_h(t) + z_p(t)$$

$$z_h(t) = \begin{cases} c_1 e^{-(\frac{c}{2m} - \frac{1}{2m}\sqrt{c^2-4mk})t} + c_2 e^{-(\frac{c}{2m} + \frac{1}{2m}\sqrt{c^2-4mk})t} & ,(c^2 - 4mk > 0) \\ (c_1 + c_2 t) e^{-\frac{c}{2m}t} & ,(c^2 - 4mk = 0) \\ e^{-\frac{c}{2m}t}(A\cos\omega_0 t + B\sin\omega_0 t) = Ce^{-\frac{c}{2m}t}\cos(\omega_0 t - \delta) & ,(c^2 - 4mk < 0) \end{cases}$$

$$z_p(t) = a\cos\omega t + b\sin\omega t \quad , \left(a = F_0 \frac{m(\omega^2{}_0 - \omega^2)}{m^2(\omega^2{}_0 - \omega^2)^2 + \omega^2 c^2} , \quad b = F_0 \frac{\omega c}{m^2(\omega^2{}_0 - \omega^2)^2 + \omega^2 c^2} \right)$$

$$\text{as } t \to \infty,\ z_h(t) \to 0, \text{ so } \ z(t) \to z_p(t)$$

Forced Oscillation of Spring-Mass-damping System

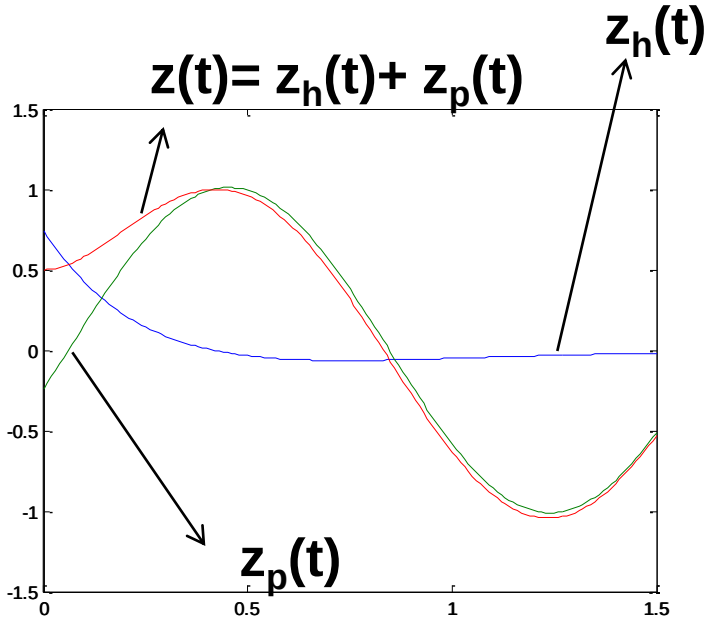
☑Transient and Steady-State Terms

$$mz'' + cz' + kz = F(t) = F_0 \cos \omega t$$

■ When *F* is periodic function, general solution have nonperiodic function $z_h(t)$ and periodic function $z_p(t)$

$z_h(t)$: transient solution

$z_p(t)$: steady-state solution



Nonlinear Model : Nonlinear Spring

Linear spring mass system

$$mz'' + cz' + kz = F_0 \cos \omega t$$

But in realistic, damping and restoring force are not expressed as linear term. For example, it is expressed as

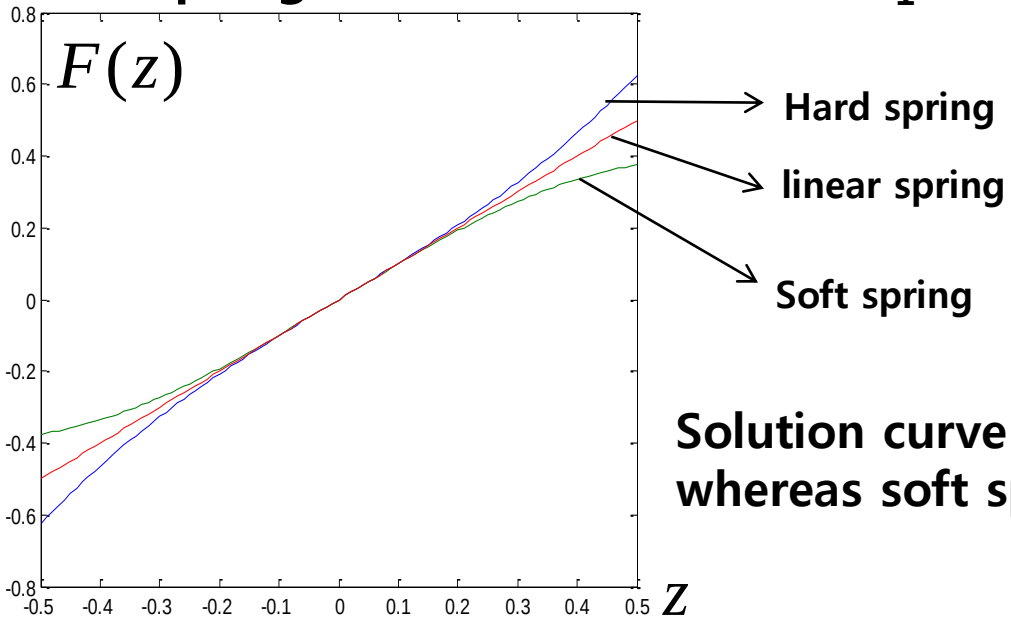
$$m\frac{d^2z}{dt^2} + \beta\left|\frac{dz}{dt}\right|\frac{dz}{dt} + kz + k_1 z^3 = 0$$

Nonlinear Spring: Hard & Soft Spring

When restoring force is given by $F(z) = kz + k_1 z^3$
Model of system would be expressed as

$$m\frac{d^2z}{dt^2} + kz + k_1 z^3 = 0$$

The spring is said to be *hard* if $k_1>0$, and *soft* if $k_1<0$



Solution curve of hard spring is oscillatory, whereas soft spring is not oscillatory

1-8. Linear/nonlinear model(equation)

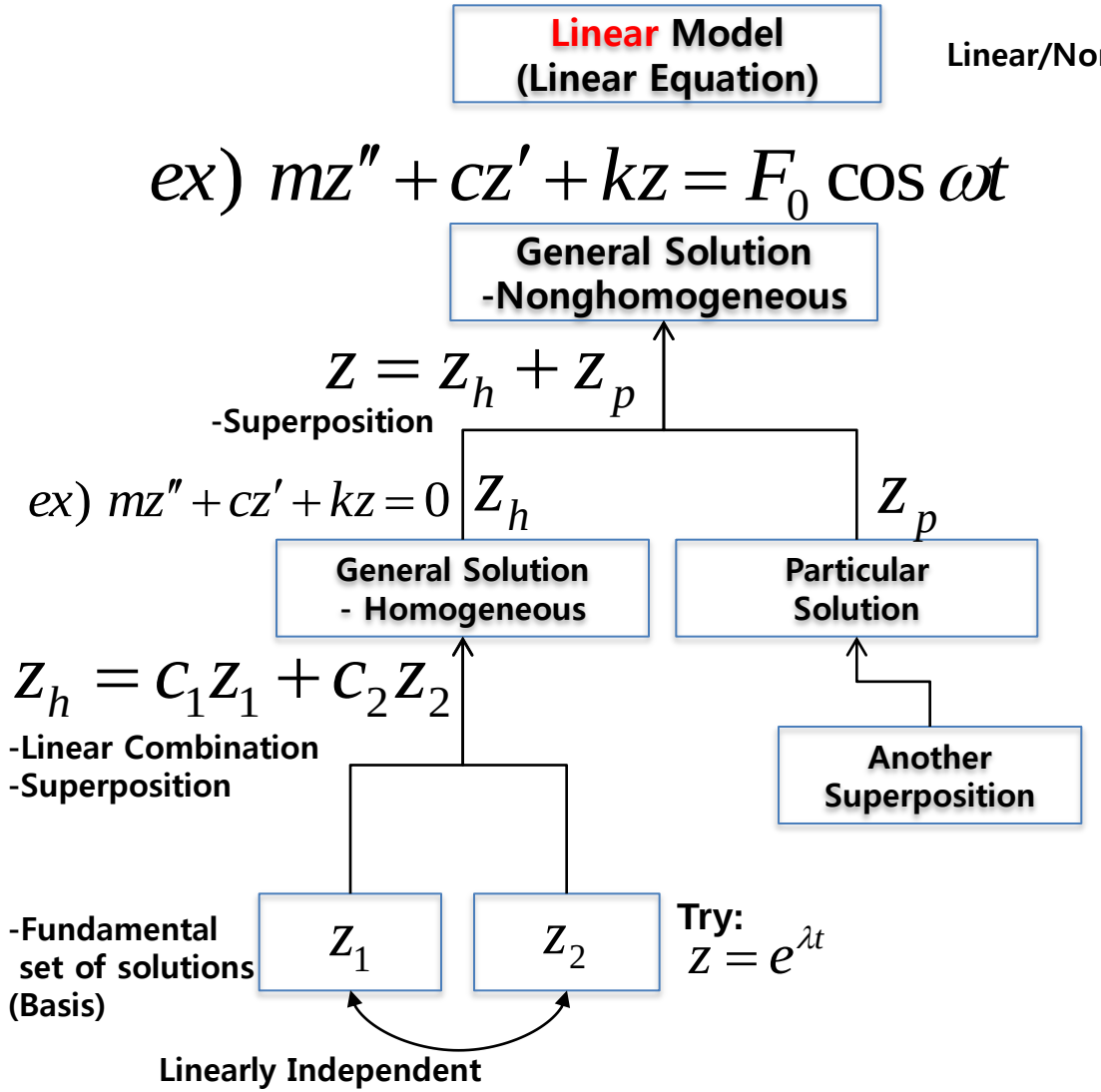
Linearly independent

Basis

Linear Combination

Superposition

Overview



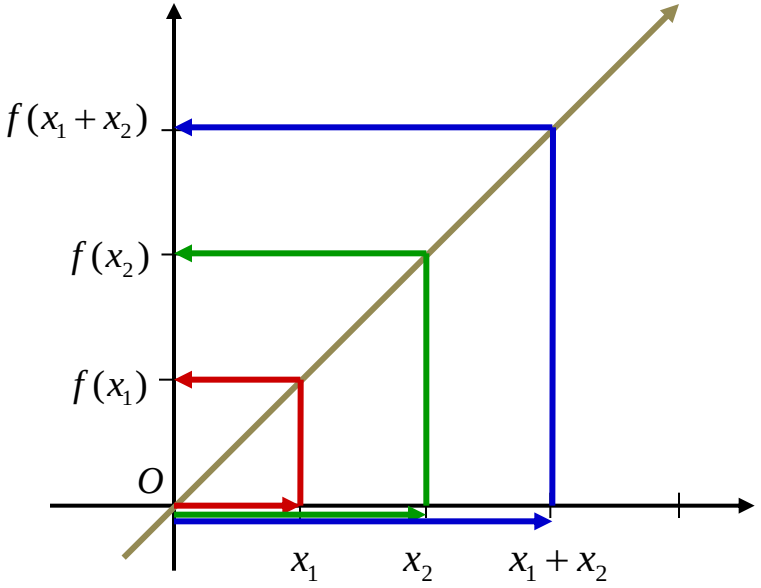
$ex)\ mz'' + kz + k_1 z^3 = 0$

$ex)\ (y'')^2 - y^2 = 0$

-superposition ?

Linearity, Superposition

$$f(x) = mx$$

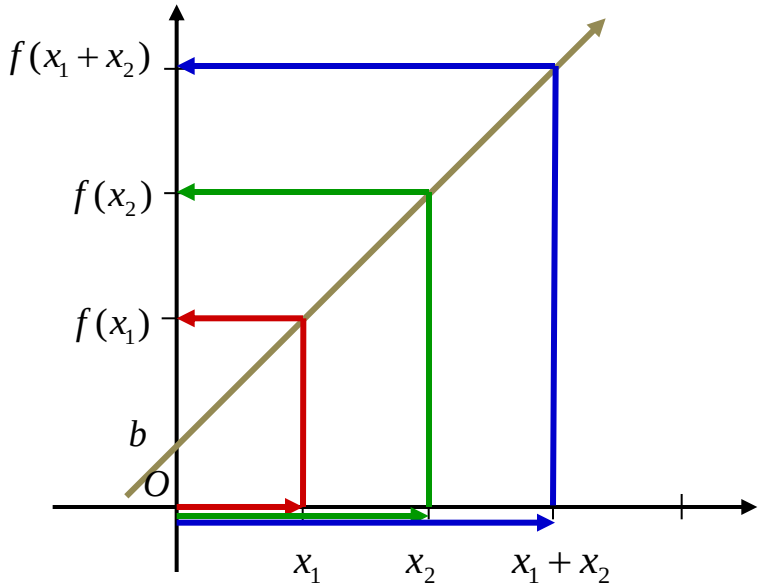


$$f(x_1) = mx_1, \quad f(x_2) = mx_2$$

$$f(x_1 + x_2) = m(x_1 + x_2)$$
$$= mx_1 + mx_2$$
$$= f(x_1) + f(x_2)$$

Linearity, Superposition

$$f(x) = mx + b$$



$$f(x_1) = mx_1 + b, \quad f(x_2) = mx_2 + b$$

$$f(x_1 + x_2) = m(x_1 + x_2) + b$$
$$= mx_1 + mx_2 + b$$
$$\neq f(x_1) + f(x_2)$$

Linearity, Superposition

If a function $f(x)$ has linearity,

the function $f(x)$ has these two properties:

$$f(ax_1 + bx_2) = af(x_1) + bf(x_2)$$

Homogeneity

$$(b = 0)$$

$$f(ax_1) = af(x_1)$$

Additivity

$$(a = b = 1)$$

$$f(x_1 + x_2) = f(x_1) + f(x_2)$$

x_1, x_2 : independent variable

Linear O.D.E

$$f(ax_1 + bx_2) = af(x_1) + bf(x_2)$$

- **Linear O.D.E.**

The dependent variable y and all its derivatives y′, y″,…,$y^{(n)}$ are of the first degree, that is the power of each term involving y is 1.
The coefficients a_0 ,…,a_n of y′, y″,…,$y^{(n)}$ depend at most on the independent variable x.

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

$$ex)\ my'' + cy' + ky = f(x)$$

$$y'' + \alpha y = 0$$

$$x^2y'' + xy' - \alpha y = 0$$

$$xy'' + y' + \alpha^2 y = 0$$

$$(1-x^2)y'' - 2xy + n(n+1)y = 0$$

$$where,\ y = y(x),$$

$$m, c, k = \text{constant}$$

$$n = 0,1,2\ldots$$

Nonlinear O.D.E

$$f(ax_1 + bx_2) \neq af(x_1) + bf(x_2)$$

- **linear O.D.E.**

The dependent variable y and all its derivatives y′, y″,…,$y^{(n)}$ are of the first degree, that is the power of each term involving y is 1
The coefficients a_0 ,…,a_n of y′, y″,…,$y^{(n)}$ depend at most on the independent variable x

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

- **Nonlinear O.D.E.**

$$(1-y)y' + 2y = e^x \qquad y'' + \sin y = e^x \qquad y^{(4)} + y^2 = e^x$$

Nonlinear ODE is *difficult to solve analytically* → Linearization or Numerical Method .

Linear Independence

$$\begin{cases} f_1(x) = \sin 2x \\ f_2(x) = \sin x \cos x \end{cases} \quad on(-\infty, \infty)$$

Linearly Independent?

$$f_2(x) = \sin x \cos x = \frac{1}{2} \sin 2x$$

$$\therefore f_2(x) = \frac{1}{2} f_1(x)$$

Linearly Dependent

Linear Independence

Definition 3.1

Linear Dependence / Independence

A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is said to be 'linearly dependent' on an interval I if there exist constant $c_1, c_2, \ldots c_n$, not all zero such that $c_1 f_1(x) + c_2 f_2(x) + \cdots c_n f_n(x) = 0$ for every x in the interval.

If the set of functions is not linearly dependent on the interval, it is said to be 'linearly independent'

In other words, a set of functions is 'linearly independent' if the only constants for

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

are $c_1 = c_2 = \cdots = c_n = 0$

"two functions are linearly independent when neither is a constant multiple of the other"

Linear Independence

a set of functions is 'linearly independent' if the only constants for

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

are $c_1 = c_2 = \cdots = c_n = 0$

"two functions are linearly independent when neither is a constant multiple of the other"

$$\begin{cases} f_1(t) = e^t \\ f_2(t) = e^{2t} \end{cases} \quad on(-\infty, \infty)$$

Linearly Independent ?

$c_1 e^t + c_2 e^{2t} = 0$ Satisfied only when $c_1 = c_2 = 0$ on the interval

Linearly Independent

Fundamental Set of Solutions 'Basis'

Definition 3.3 **Fundamental Set of Solutions**

Any set $y_1, y_2, \ldots, y_n$ of n linearly independent solutions of the homogeneous linear *n*th-order D.E. on an interval I is said to be a fundamental set of solutions on the interval.

'Basis'

Theorem 3.4 **Existence of a Fundamental Set**

There exists a fundamental set of solutions for the homogeneous linear *n*th-order differential equation on an interval I

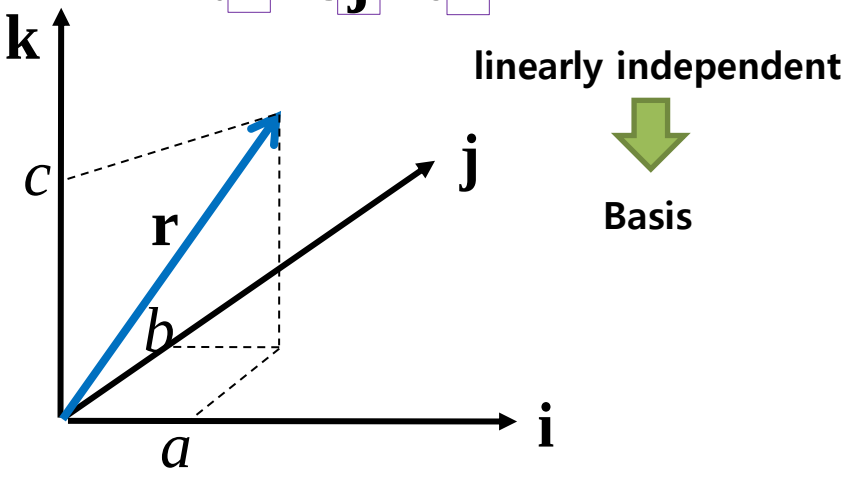
Definition 3.3 **Fundamental Set of Solutions**

Any set $y_1, y_2, \ldots, y_n$ of n **linearly independent** solutions of the homogeneous linear *n*th-order D.E. on an interval I is said to be a fundamental set of solutions on the interval.

'Basis'

"Analogous to the fact that any vector in three dimensions can be expressed as a linear combination of the linearly independent vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ "

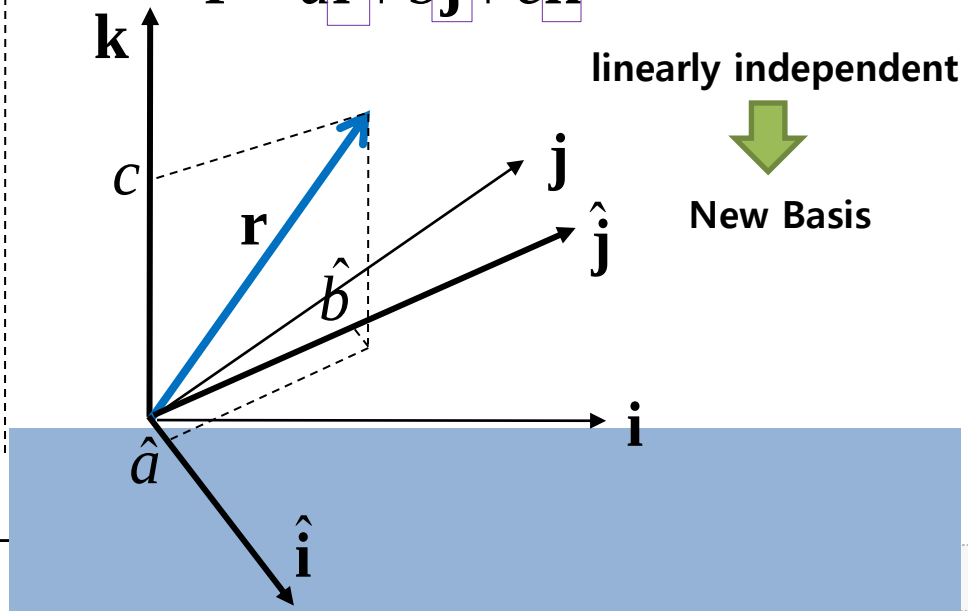
$$\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$



linearly independent

Basis

$$\mathbf{r} = \hat{a}\hat{\mathbf{i}} + \hat{b}\hat{\mathbf{j}} + c\mathbf{k}$$



linearly independent

New Basis

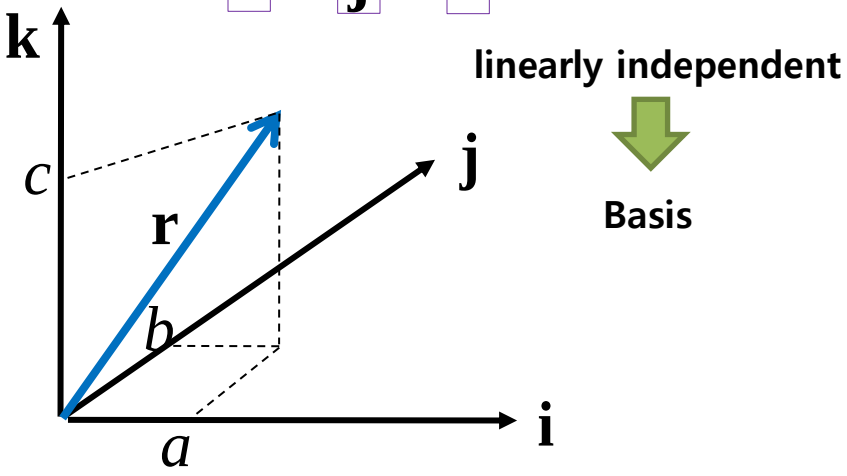
Definition 3.3 **Fundamental Set of Solutions**

Any set $y_1, y_2, \ldots, y_n$ of n **linearly independent** solutions of the homogeneous linear *n*th-order D.E. on an interval I is said to be a fundamental set of solutions on the interval.

'Basis'

"Analogous to the fact that any vector in three dimensions can be expressed as a linear combination of the linearly independent vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ "

$$\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

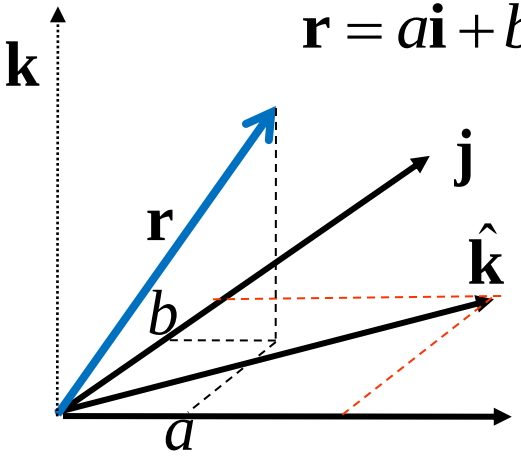


linearly independent

Basis

$\{\mathbf{i}, \mathbf{j}, \hat{\mathbf{k}}\}$ can be another new basis?

$$\mathbf{r} = a\mathbf{i} + b\mathbf{j} + \hat{c}\hat{\mathbf{k}} \; ??$$



$\hat{\mathbf{k}} = \mathbf{i} + \mathbf{j}$

Linearly dependent

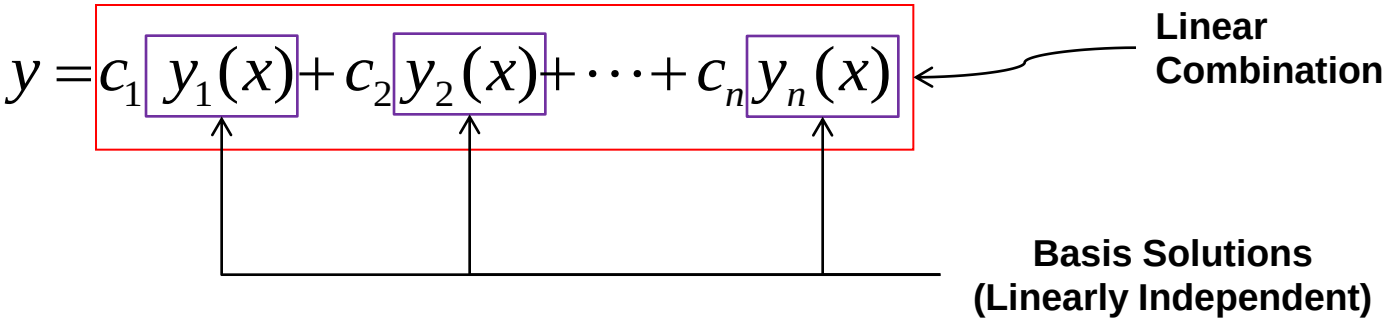
~~New Basis~~

Definition 3.3 **Fundamental Set of Solutions**

Any set $y_1, y_2, \ldots, y_n$ of n linearly independent solutions of the homogeneous linear *n*th-order D.E. on an interval I is said to be a fundamental set of solutions on the interval.

'Basis'

"Any solution of an *n*th-order homogeneous linear differential equation on an interval I can be expressed as a linear combination of n linearly independent solutions on I .



General Solution – Linear Combination

Linear Differential Equation

$$mz'' + cz' + kz = F_0 \cos \omega t$$

homogeneous solution $mz'' + cz' + kz = 0$

If there are two basic solutions z_1, z_2 for the homogeneous equation, then what is the <u>general</u> solution?

$$mz_1'' + cz_1' + kz_1 = 0$$
$$mz_2'' + cz_2' + kz_2 = 0$$

Linear Differential Equation

$$mz'' + cz' + kz = F_0 \cos \omega t$$

homogeneous solution $mz'' + cz' + kz = 0$

If z_1 **is a solution for the homogenous equation, then** $c_1 z_1$ **is a solution too.** $c_1 = \text{nonzero constant}$

L.H.S: $mz'' + cz' + kz$

$$= m(c_1 z_1)'' + c(c_1 z_1)' + k(c_1 z_1)$$

$$= c_1 (mz_1'' + cz_1' + kz_1)$$

$$= c_1 \cdot 0 = 0$$

R.H.S: 0

Linear Differential Equation

$$mz'' + cz' + kz = F_0 \cos \omega t$$

homogeneous solution $mz'' + cz' + kz = 0$

If z_2 **is a solution for the equation,**
then $c_2 z_2$ **is a solution too.** $c_2 = \text{nonzero constant}$

L.H.S: $mz'' + cz' + kz$

$$= m(c_2 z_2)'' + c(c_2 z_2)' + k(c_2 z_2)$$

$$= c_2 (mz_2'' + cz_2' + kz_2)$$

$$= c_2 \cdot 0 = 0$$

R.H.S: 0

Linear Differential Equation

$$mz'' + cz' + kz = F_0 \cos \omega t$$

homogeneous solution $mz'' + cz' + kz = 0$

If $c_1 z_1$ and $c_2 z_2$ are solutions for the equation, then $c_1 z_1 + c_2 z_2$ is a solution too.

$$c_1, c_2 = \text{nonzero constant}$$

$$mz_1'' + cz_1' + kz_1 = 0$$
$$mz_2'' + cz_2' + kz_2 = 0$$

L.H.S: $mz'' + cz' + kz$

$$= m(c_1 z_1 + c_2 z_2)'' + c(c_1 z_1 + c_2 z_2)' + k(c_1 z_1 + c_2 z_2)$$
$$= m(c_1 z_1)'' + m(c_2 z_2)'' + c(c_1 z_1)' + c(c_2 z_2)' + k(c_1 z_1) + k(c_2 z_2)$$
$$= m(c_1 z_1)'' + c(c_1 z_1)' + k(c_1 z_1) + m(c_2 z_2)'' + c(c_2 z_2)' + k(c_2 z_2)$$
$$= m(c_1 z_1)'' + c(c_1 z_1)' + k(c_1 z_1) + m(c_2 z_2)'' + c(c_2 z_2)' + k(c_2 z_2)$$
$$= c_1(mz_1'' + cz_1' + kz_1) + c_2(mz_1'' + cz_1' + kz_1)$$

Linear Differential Equation

$$mz'' + cz' + kz = F_0 \cos \omega t$$

homogeneous solution $mz'' + cz' + kz = 0$

$c_1, c_2 = \text{nonzero constant}$

$z_1, z_2 : \text{linearly independent}$

$f(z) = mz'' + cz' + kz$ ⇨ if $f(z_1) = 0, f(z_2) = 0$ then,

$$f(c_1 z_1 + c_2 z_2) = c_1 f(z_1) + c_2 f(z_2) = 0$$

Ex) $z'' - 9z' + 14z = 0$ **Basis solutions:** $z_1 = e^{7x}, z_2 = e^{2x}$

$$z = c_1 z_1 + c_2 z_2 = c_1 e^{7x} + c_2 e^{2x}$$

L.H.S: $z'' - 9z' + 14z$

$$= (c_1 e^{7x} + c_2 e^{2x})'' - 9(c_1 e^{7x} + c_2 e^{2x})' + 14(c_1 e^{7x} + c_2 e^{2x})$$

$$= (49c_1 e^{7x} + 4c_2 e^{2x}) - 9(7c_1 e^{7x} + 2c_2 e^{2x}) + 14(c_1 e^{7x} + c_2 e^{2x})$$

$$= (49 - 63 + 14)c_1 e^{7x} + (4 - 18 + 14)c_2 e^{2x} = 0$$

R.H.S: 0

General Solution – Linear Combination

Linear Differential Equation

$$mz'' + cz' + kz = F_0 \cos \omega t$$

homogeneous solution $mz'' + cz' + kz = 0$

If there are two basic solutions z_1, z_2 for the homogeneous equation, then what is the general solution?

$$mz_1'' + cz_1' + kz_1 = 0$$
$$mz_2'' + cz_2' + kz_2 = 0$$

$c_1, c_2 = \text{nonzero constant}$

General Solution: $z = c_1 z_1 + c_2 z_2$

Linearly Independent

Linear Combination

So, solving the homogeneous equation means finding basic solutions which are linearly independent!

Nonlinear Equation : Some Differences

☑Some Differences

■ Nonlinear Equation *do not have superposition property*

Even though y_1 and y_2 are solutions of nonlinear DE,

$y = c_1 y_1 + c_2 y_2$ is not a solution of nonlinear DE

Ex) $(y'')^2 - y^2 = 0$

These two are linearly independent solutions

$y = e^x$ ------> L.H.S.: $(e^x)^2 - e^{2x} = 0$ R.H.S.: 0

$y = \cos x$ ------> L.H.S.: $(-\cos x)^2 - \cos^2 x = 0$ R.H.S.: 0

Is the linearly combined solution also a solution too?

$y = c_1 e^x + c_2 \cos x$,where c_1, c_2 are nonzero

Nonlinear Equation : Some Differences

L.H.S.: $(y'')^2 - y^2$

$$= (c_1 e^x - c_2 \cos x)^2 - (c_1 e^x + c_2 \cos x)^2$$

$$= -4 c_1 c_2 e^x \cos x$$

R.H.S.: 0

L.H.S. ≠R.H.S.

linearly independent solutions

$y = \cos x$
$y = e^x$

$y = c_1 e^x + c_2 \cos x$ (Linear combination) → Not a solution **Nonlinear**

1-9. Modeling of Rotating String

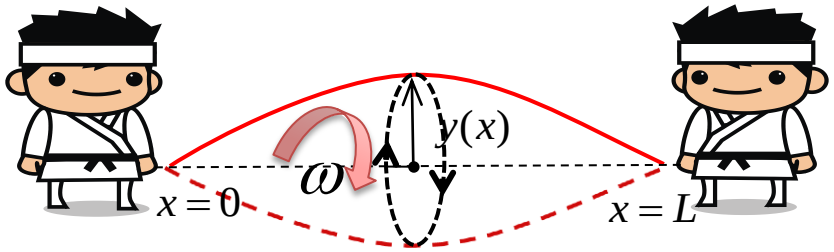
Modeling : Rotating string

Nonlinearity of the nature → Nonlinear Mathematical Model → Numerical Method

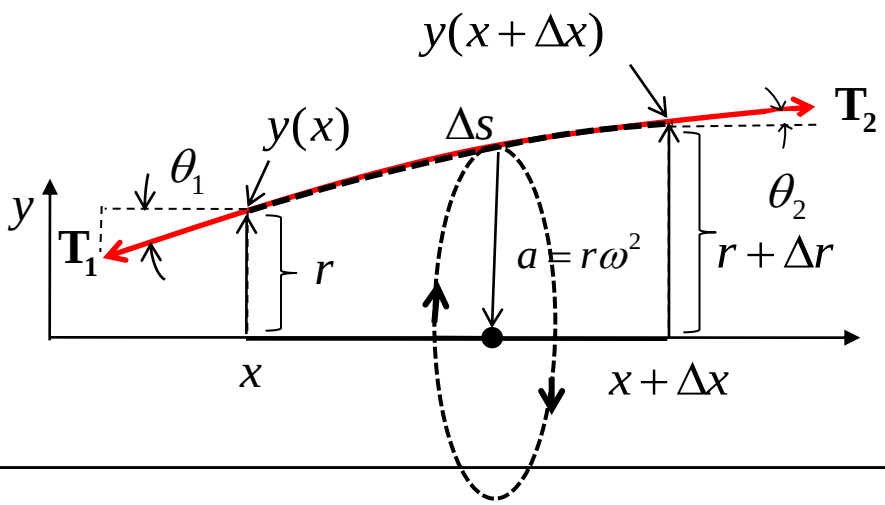
Linearization

Linear Mathematical Model → Analytic Solution

Ex) Rotating String



ρ : string density
ω : string angular velocity
T : magnitude of tension



$$\sum F_x = \mathbf{T}_2 \cos\theta_2 - \mathbf{T}_1 \cos\theta_1 = 0$$

$$\mathbf{T}_1 \cos\theta_1 = \mathbf{T}_2 \cos\theta_2 = T$$

$$\sum F_y = \mathbf{T}_2 \sin\theta_2 - \mathbf{T}_1 \sin\theta_1$$

$$= T\frac{\sin\theta_2}{\cos\theta_2} - T\frac{\sin\theta_1}{\cos\theta_1}$$

$$= T\tan\theta_2 - T\tan\theta_1$$

$$= T[y'(x+\Delta x) - y'(x)]$$

$$\because \tan\theta_1 = \left.\frac{dy}{dx}\right|_x, \tan\theta_2 = \left.\frac{dy}{dx}\right|_{x+\Delta x}$$

$$\Delta x y' \approx 0$$

$$\sum F_y = T[y'(x+\Delta x) - y'(x)]$$

$\because \Delta x \ll 1, \Delta s \approx \Delta x$ means y'(=dy/dx) is small too

assum.: $\Delta x \ll 1$ linearization

Acceleration points in the direction opposite to the positive y direction

Mass: $m = \rho \Delta s \approx \rho \Delta x$

Centripetal acceleration: $a = \ominus r\omega^2$

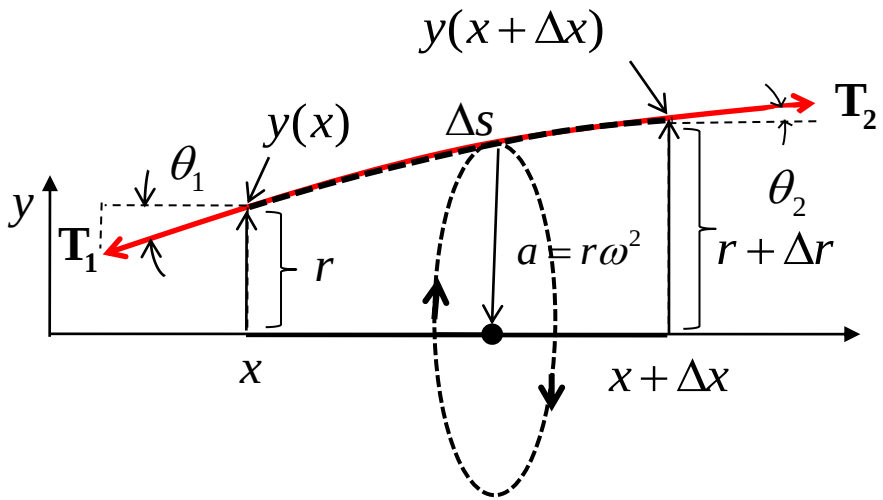
When Δx is small,

$$r + \Delta r = r + r'\Delta x + \cdots = y + y'\Delta x + \cdots$$

linearization

$$r + \Delta r \approx r = y \,, a = -r\omega^2 = -y\omega^2$$

$$\therefore \sum F_y = ma \approx -(\rho \Delta x) y \omega^2$$



$$\sum F_y = T[y'(x+\Delta x) - y'(x)]$$

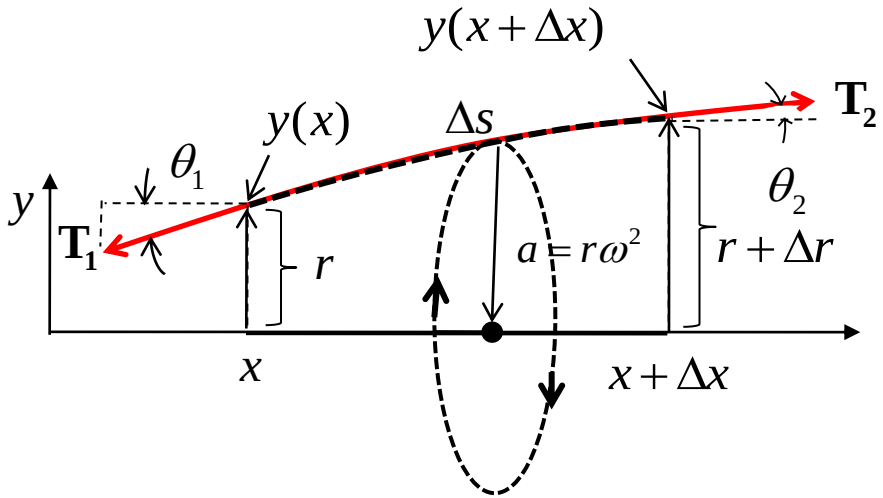
$$\sum F_y = ma \approx -(\rho \Delta x) y \omega^2$$

$$\therefore T[y'(x+\Delta x) - y'(x)] = -(\rho \Delta x) y \omega^2$$

$$T \frac{y'(x+\Delta x) - y'(x)}{\Delta x} + \rho \omega^2 y = 0$$

$$\frac{y'(x+\Delta x) - y'(x)}{\Delta x} \approx \frac{d^2 y}{dx^2}$$

$$\therefore T \frac{d^2 y}{dx^2} + \rho \omega^2 y = 0$$



What if ..

$$r + \Delta r = r + r'\Delta x + \cdots = y + y'\Delta x + \cdots$$

$$then, a = -(r + \Delta r)\omega^2 = -(y + y'\Delta x)\omega^2$$

$$\therefore T[y'(x + \Delta x) - y'(x)] = -(\rho \Delta x)(y + y'\Delta x)\omega^2$$

$$\Rightarrow T\frac{d^2y}{dx^2} + \rho\omega^2 \underline{\Delta x y'} + \rho\omega^2 y = 0$$

Not a form of $\frac{dy}{dx}$ or polynomial of x

$\Delta x y' \approx 0 \because \Delta x \ll 1, \Delta s \approx \Delta x$ means y'(=dy/dx) is small too

$$\therefore T\frac{d^2y}{dx^2} + \rho\omega^2 y = 0$$

1-10. TAYLOR SERIES EXPANSION

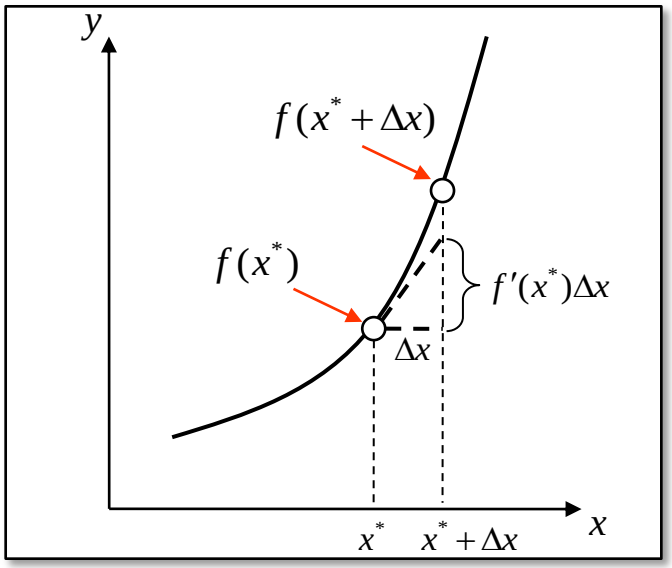
Taylor series expansion

Given : $x^*,\ f(x^*),$ *up to the i-th* degree derivatives at x^* $\left(\dfrac{\partial^i f(x^*)}{\partial x^i}\right)$

Find : $f(x^* + \Delta x)$

※ Taylor series expansion for a function of single variable

$$f(x^* + \Delta x) = f(x^*) + f'(x^*)\Delta x + \frac{1}{2} f''(x^*)\Delta x^2 + \ldots$$



Maclaurin series

let $x^* \to 0,\ \Delta x \to x$

$$f(x) = f(0) + f'(0)x + \frac{1}{2} f''(0)x^2 + \ldots$$

Taylor series expansion of trigonometric functions

$$f(x^* + \Delta x) = f(x^*) + f'(x^*)\Delta x + \frac{1}{2} f''(x^*)\Delta x^2 + \ldots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots = \sum_{n=0}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1}$$

Taylor series expansion for a function of two variables

Given : $\left(x_1^*, x_2^*\right), f(x_1^*, x_2^*)$, **Derivatives at** x^* $\left(\dfrac{\partial^{i+j} f(x_1^*, x_2^*)}{\partial x_1{}^i \partial x_2{}^j}\right)$

Find : $f(x_1^* + \Delta x_1, x_2^* + \Delta x_2)$

Taylor series expansion for a function f of two variables, x1, x2

$$f(x_1^* + \Delta x_1, x_2^* + \Delta x_2) = f(x_1^*, x_2^*) + \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2$$

$$+ \frac{1}{2}\left(\frac{\partial^2 f}{\partial x_1^2}\Delta x_1{}^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\Delta x_1 \Delta x_2 + \frac{\partial^2 f}{\partial x_2^2}\Delta x_2{}^2\right) +$$

$$+ \frac{1}{3}\left(\frac{\partial f^3}{\partial x_1^3}\Delta x_1^3 + 3\frac{\partial f^3}{\partial x_1^2 \partial x_2}\Delta x_1^2 \Delta x_2 + 3\frac{\partial f^3}{\partial x_1 \partial x_2^2}\Delta x_1 \Delta x_2^2 + \frac{\partial f^3}{\partial x_2^3}\Delta x_2^3\right)$$

$$+ \ldots$$

$$\nabla f(\mathbf{x}^*)^T(\mathbf{x}-\mathbf{x}^*) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}^T \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

$$f(x_1^* + \Delta x_1, x_2^* + \Delta x_2) = f(x_1^*, x_2^*) + \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x_1^2}\Delta x_1^{\,2} + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\Delta x_1 \Delta x_2 + \frac{\partial^2 f}{\partial x_2^2}\Delta x_2^{\,2} \right) + \ldots$$

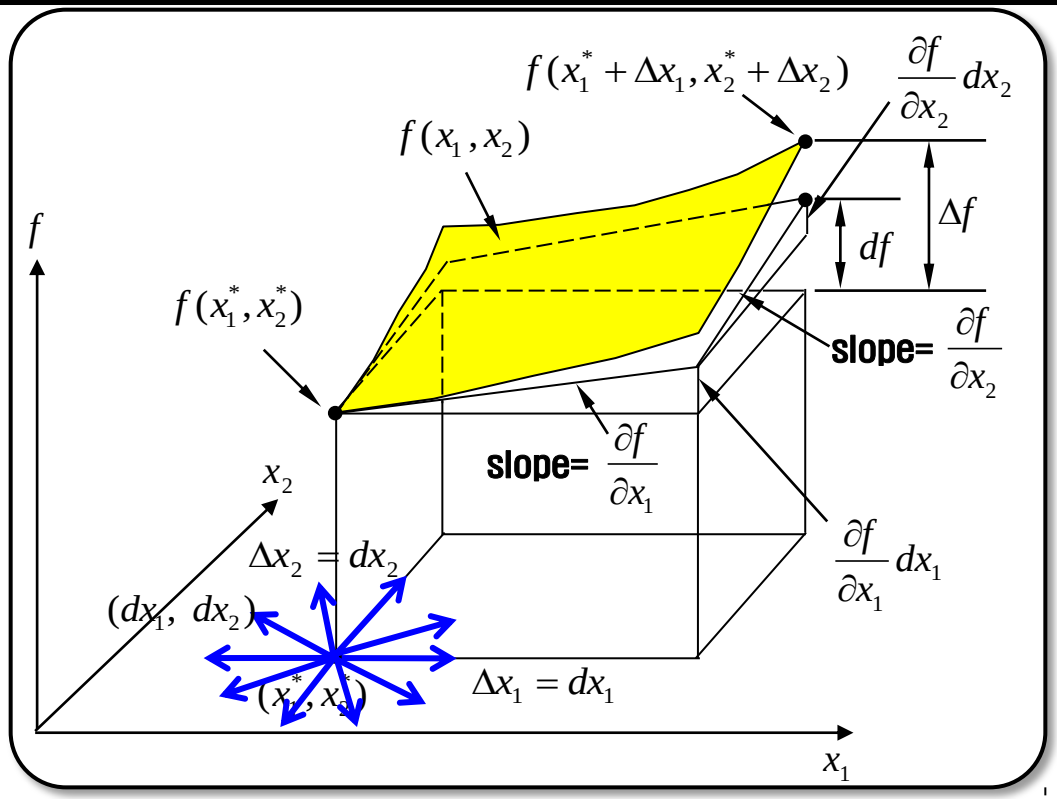
$$\frac{1}{2}(\Delta\mathbf{x})^T \mathbf{H}(\mathbf{x}^*)(\Delta\mathbf{x}) = \frac{1}{2}\begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(\Delta x_1) + \frac{\partial^2 f}{\partial x_2 \partial x_1}(\Delta x_2) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(\Delta x_1) + \frac{\partial^2 f}{\partial x_2^2}(\Delta x_2) \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} \Delta x_1 & \Delta x_2 \end{bmatrix} \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

$$\therefore f(\mathbf{x}) = f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^T(\Delta\mathbf{x}) + \frac{1}{2}(\Delta\mathbf{x})^T \mathbf{H}(\mathbf{x}^*)(\Delta\mathbf{x}) + R$$

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

Total differentials for a function of two variables x1, x2



The symbol *"d"* refers to the infinitesimal change. In accordance with the notation we write the change of the function f as follows

the change of the function in x_2 direction

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2$$

the change of the function in x_1 direction

The change of the function is defined as follows

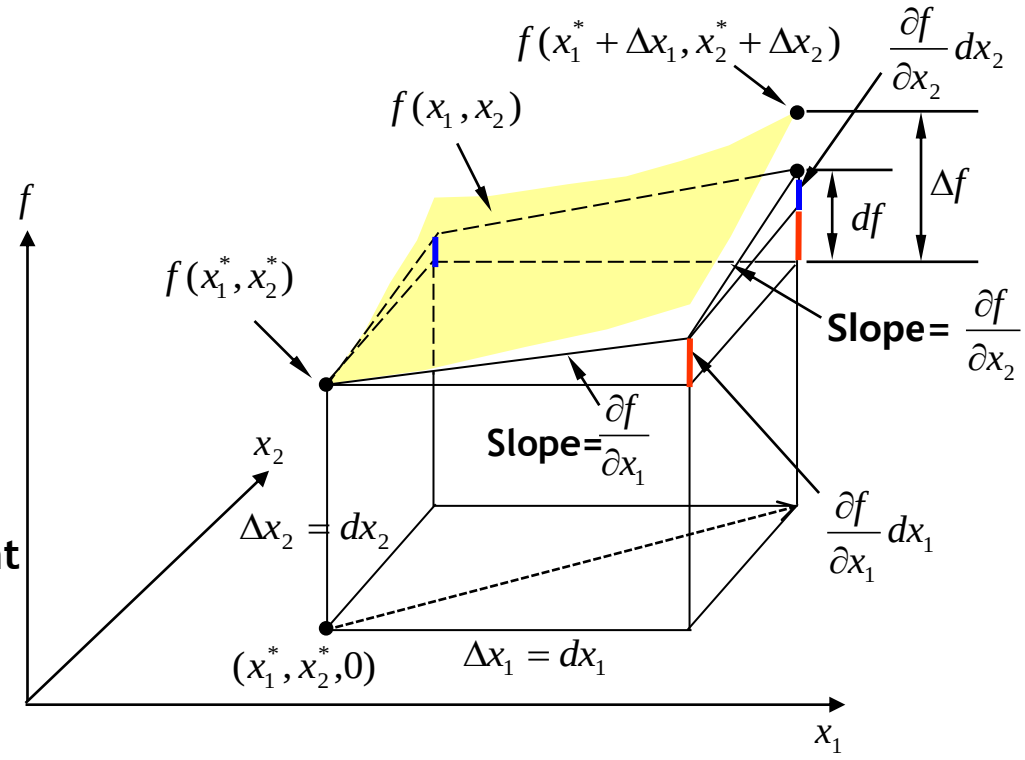
$$\Delta f = \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x_1^2}\Delta x_1^{\,2} + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\Delta x_1 \Delta x_2 + \frac{\partial^2 f}{\partial x_2^2}\Delta x_2^{\,2}\right) + R$$

If $\Delta x_1 \to 0, \Delta x_2 \to 0$, the first-order term $\frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2$ dominates other terms.

Therefore Δf can be approximated as $\Delta f \approx \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2$

Total differentials for a function of two variables x1,x2

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2$$



$f = \mathbf{V}$, $x_1 = t$, $x_2 = x$

velocity time displacement

$$d\mathbf{V} = \frac{\partial \mathbf{V}}{\partial t} dt + \frac{\partial \mathbf{V}}{\partial x} dx$$

For constant time, the change of the velocity to the displacement

For constant displacement, the change of the velocity to the time

1-11. Motion of a Fluid Element

Pressure and Force acting on a Fluid Element

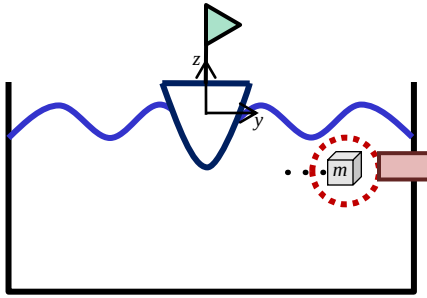
$\mathbf{r}$: displacement of a fluid particle with respect to the time

$$\mathbf{V} = \frac{d\mathbf{r}}{dt}, \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}$$

$F_{F.K}$: Froude- krylov force
F_D: Diffraction force
F_R: Radiation force

1) RTT : Reynold Transport Theorem
2) SWBM : Still Water Bending Moment
3) VWBM : Vertical Wave Bending Moment

✓ Assumption
① Newtonian fluid*
② Stokes Assumption**
③ inviscid fluid
④ irrotational flow ⑤ incompressible flow



$$\mathbf{r} = [\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6]^T$$

ξ_1 : surge , ξ_4 : roll
ξ_2 : sway , ξ_5 : pitch
ξ_3 : heave , ξ_6 : yaw

✓Equations of motion of a Fluid Element

Lagrangian & Eulerian Description

Newton's 2nd Law

$$m\ddot{\mathbf{r}} = \sum \mathbf{F} = (\text{Body force} + \text{Surface force})$$

Shear force

Cauchy equation → Navier-Stokes equation (①②) → Euler equation (③) → Bernoulli equation (④⑤)

Curl & Rotation

Microscopic/ Macroscopic Derivation(RTT[1])

Mass Conservation Law (④⑤) → Laplace Equation

$$\nabla^2\Phi = 0 \quad (\mathbf{V} = \nabla\Phi)$$

$$\rho\frac{\partial\Phi}{\partial t} + P + \frac{1}{2}\rho|\nabla\Phi|^2 + \rho g z = 0$$

* A Newtonian fluid : fluid whose stress versus strain rate curve is linear.
**Definition of viscosity coefficient(μ,λ) due to linear deformation and isometric expansion

Derivation of Buoyant Force
- Equations of Motion of a Fluid Element(Cauchy eq. ~ Bernoulli eq.)[1)]

1) Kundu,P.K., Cohen,I.M., Fluid Mechanics 5th, Academic Press,2012

* A Newtonian fluid : fluid whose stress versus strain rate curve is linear.

**Definition of viscosity coefficient(μ,λ) due to linear deformation and isometric expansion

Cauchy Equation : $\rho\frac{d\mathbf{V}}{dt}=\rho\mathbf{g}+\nabla\bullet\sigma \quad ,\left(\mathbf{V}=[u,v,w]^T\right)$

↓ ① Newtonian fluid* ② Stokes assumption**

Navier-Stokes Equation : (in general form) $\rho\frac{d\mathbf{V}}{dt}=\rho\mathbf{g}-\nabla P+\mu\left(\frac{1}{3}\nabla(\nabla\cdot\mathbf{V})+\nabla^2\mathbf{V}\right)$

$(\mu=0)$ ↓ ③ Inviscid fluid

Euler Equation : $\rho\frac{d\mathbf{V}}{dt}=\rho\mathbf{g}-\nabla P$

$\rho=\rho(P)$ ↓ ④ barotropic flow

Euler Equation : (another form) $\frac{\partial\mathbf{V}}{\partial t}+\nabla B=\mathbf{V}\times\boldsymbol{\omega} \quad ,\left(B=\frac{1}{2}q^2+gz+\int\frac{dP}{\rho}\ ,\ q^2=u^2+v^2+w^2\right)$

$\left(q^2=|\nabla\Phi|^2\right)$

⑤ Steady flow $\left(\frac{\partial\mathbf{V}}{\partial t}=0\right)$ ↓ along the streamlines and vortex lines

Bernoulli equation (case1) $B=\text{Constant}$ $\left(\frac{1}{2}q^2+gz+\int\frac{dP}{\rho}=C\right)$

⑥ unsteady, irrotational flow ↓

Bernoulli equation (case2) $\frac{\partial\Phi}{\partial t}+\frac{1}{2}|\nabla\Phi|^2+gz+\int\frac{dP}{\rho}=F(t)$

$(\rho=constant)$ ↓ ⑦ incompressible flow

Bernoulli equation (case3) $\frac{\partial\Phi}{\partial t}+\frac{1}{2}|\nabla\Phi|^2+gz+\frac{P}{\rho}=F(t)$

Continuity Equation $\frac{\partial\rho}{\partial t}+\nabla\bullet\rho\mathbf{V}=0$

↓ ⑦ incompressible flow $\rho=constant,\left(\frac{\partial\rho}{\partial t}=0\right)$

$\nabla\bullet\mathbf{V}=0$

↓ ⑥ irrotational flow $(\mathbf{V}=\nabla\Phi)$

Laplace Equation $\nabla^2\Phi=0$

$\boldsymbol{\omega}=\nabla\times\mathbf{V}$

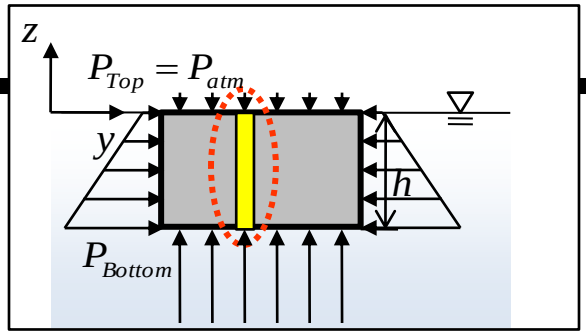
If $\boldsymbol{\omega}=0$ (irrotational flow) then $\nabla\times\mathbf{V}=0$

If $\nabla\times\mathbf{V}=0$ then $\mathbf{V}=\nabla\Phi$

Newtonian fluid
Stokes assumption
inviscid fluid
unsteady flow
irrotational flow
incompressible flow

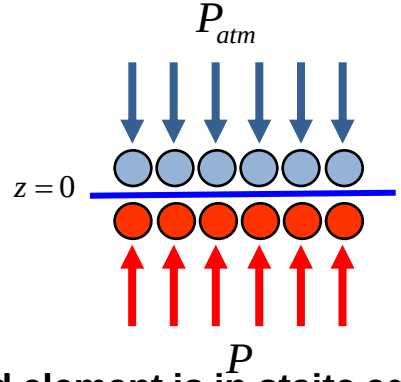
Derivation of Buoyant Force
-Meaning of F(t) in Bernoulli Equation and Gauge Pressure

1) Gauge pressure : The net pressure of the difference of the total pressure and atmospheric pressure



Bernoulli Equation

$$\frac{\partial \Phi}{\partial t} + \frac{P}{\rho} + \frac{1}{2}\left|\nabla \Phi\right|^2 + g\,z = F(t)$$



If a fluid element is in staitc equilibrium state on the free surface (z=0), then

$$\frac{\partial \Phi}{\partial t} = 0,\ \nabla \Phi = 0,\quad P = P_{atm}$$

$$\frac{\partial \Phi}{\partial t} + \frac{P}{\rho} + \frac{1}{2}\left|\nabla \Phi\right|^2 + g z = F(t) \longrightarrow \frac{P_{atm}}{\rho} = F(t)$$

(Atmospheric pressure(P_{atm})) = (Pressure at *z=0*)

$$\therefore \frac{\partial \Phi}{\partial t} + \frac{P}{\rho} + \frac{1}{2}\left|\nabla \Phi\right|^2 + g\,z = \frac{P_{atm}}{\rho}$$

✓**What is the pressure on the bottom of an object ?**

$$\frac{\partial \Phi}{\partial t} + \frac{P_{Bottom}}{\rho} + \frac{1}{2}\left|\nabla \Phi\right|^2 + gz = \frac{P_{atm}}{\rho}$$

$$\frac{\partial \Phi}{\partial t} + \frac{P_{atm} + P_{Fluid}}{\rho} + \frac{1}{2}\left|\nabla \Phi\right|^2 + gz = \frac{P_{atm}}{\rho}$$

$$\therefore \frac{\partial \Phi}{\partial t} + \frac{P_{Fluid}}{\rho} + \frac{1}{2}\left|\nabla \Phi\right|^2 + gz = 0$$

'gauge pressure'

※ In case that R.H.S of Bernoulli equation is expressed by zero, pressure P means the pressure due to the fluid which excludes the atmospheric pressure.

If the motion of fluid is small, square term could be neglected.

$$\frac{\partial \Phi}{\partial t} + \frac{P_{Fluid}}{\rho} + \frac{1}{2}\left|\nabla \Phi\right|^2 + gz = 0$$

$$P_{Fluid} = \underbrace{-\rho \frac{\partial \Phi}{\partial t}}_{P_{dynamic}} \underbrace{- \rho g z}_{P_{static}} = 0$$

'Linearized Bernoulli Equation'

Pressure and Force acting on a Fluid Element

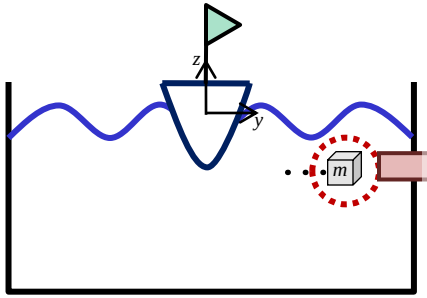
$\mathbf{r}$: displacement of particle with respect to time

$\mathbf{V} = \frac{d\mathbf{r}}{dt}, \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}$

$F_{F.K}$: Froude- krylov force
F_D: Diffraction force
F_R: Radiation force

1) RTT : Reynold Transport Theorem
2) SWBM : Still Water Bending Moment
3) VWBM : Vertical Wave Bendidng Moment

✓ Assumption
① Newtonian fluid*
② Stokes Assumption**
③ invicid fluid
⑤ Incompressible flow



$\mathbf{r} = [\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6]^T$
ξ_1 : surge , ξ_4 : roll
ξ_2 : sway , ξ_5 : pitch
ξ_3 : heave , ξ_6 : yaw

✓Equations of motions of Fluid Particles

Lagrangian & Eulerian Description

Newton's 2nd Law

$m\ddot{\mathbf{r}} = \sum \mathbf{F} = (\text{Body force} + \text{Surface force})$

Shear force

Cauchy equation → Navier-Stokes equation (①②) → Euler equation (③) → Bernoulli equation (④⑤)

Curl & Rotation

Microscopic/ Macroscopic Derivation(RTT[1])

Mass Conservation Law (④⑤) → Laplace Equation

$\nabla^2\Phi = 0 \quad (\mathbf{V} = \nabla\Phi)$

$\rho\frac{\partial\Phi}{\partial t} + P + \frac{1}{2}\rho|\nabla\Phi|^2 + \rho g z = 0$

Velocity potential Φ

$\Phi = \Phi_I$ (Incident wave potential)
$+\Phi_D$ (Diffraction potential)
$+\Phi_R$ (Radiation potential)

✓Calculation of Fluid Force

Linearization $\left(\frac{1}{2}\rho|\nabla\Phi|^2 = 0\right)$

$P = -\rho g z - \rho\frac{\partial\Phi}{\partial t}$

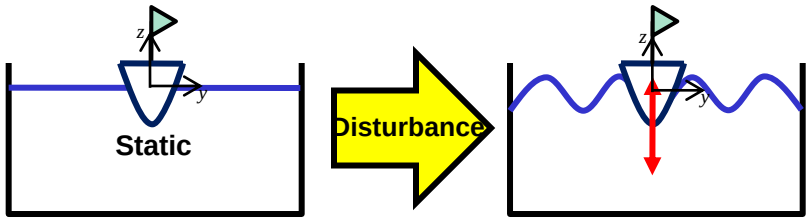
$\mathbf{F}_{Fluid}(\mathbf{r},\dot{\mathbf{r}},\ddot{\mathbf{r}}) = \iint_{S_B} P\mathbf{n}dS = \mathbf{F}_{Buoyancy}(\mathbf{r}) + \mathbf{F}_{F.K}(\mathbf{r}) + \mathbf{F}_D(\mathbf{r}) + \mathbf{F}_R(\mathbf{r},\dot{\mathbf{r}},\ddot{\mathbf{r}})$

(S_B: wetted surface Area)

* A Newtonian fluid : fluid whose stress versus strain rate curve is linear.
**Definition of viscosity coefficient(μ,λ) due to linear deformation and isometric expans

Pressure and Force acting on a Fluid Element - Fluid Force acting on the ship

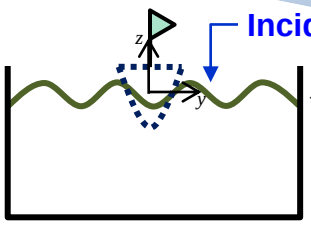
$$\rho\frac{\partial\Phi}{\partial t}+P+\frac{1}{2}\rho|\nabla\Phi|^2+\rho g\,z=0$$



✓ **Pressure due to the fluid Elements around the ship in wave**

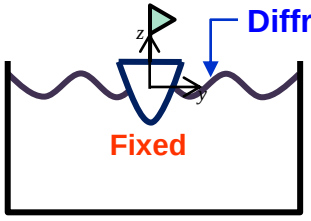
: Velocity, acceleration, pressure of the fluid elements are changed due to the motion of fluid, then the pressure of fluid elements acting on the ship is changed.

Linearization



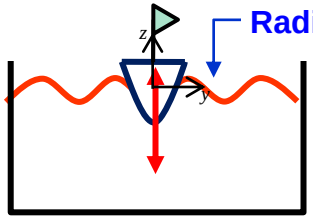
Incident wave velocity potential (Φ_I)

✓ Velocity potential of an incoming wave that is independent of the body motion



Diffraction wave velocity potential (Φ_D)

✓ Velocity potential of the disturbance of the incident wave by the body that is fixed in position[1)]



Radiation wave velocity potential (Φ_R)

✓ Velocity potential of the wave that is induced by the body motions, in the absence of the incident wave.[1)]

✓ **Total Velocity Potential**

$$\Phi_T=\Phi_I+\Phi_D+\Phi_R$$

Superposition Theorem

For homogeneous linear PDE, the superposed solution is also a solution of the linear PDE[2)]

$$P=-\rho gz-\rho\frac{\partial\Phi_T}{\partial t}$$

$$\mathbf{F}_{Fluid}=\iint_{S_B}P\mathbf{n}dS$$

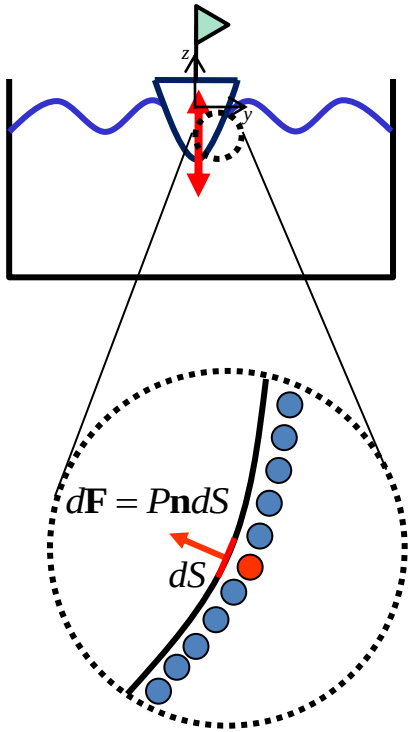
$$=\mathbf{F}_{static}+\mathbf{F}_{F.K}+\mathbf{F}_D+\mathbf{F}_R$$

1) Newman, J.N. , Marine Hydrodynamics, The MIT Press, Cambridge, 1997, pp 287

Φ_I : Incident wave velocity potential

Φ_D : Diffraction potential

Φ_R : Radiation potential



✓ Bernoulli Equation

$$\rho\frac{\partial\Phi}{\partial t} + P + \frac{1}{2}\rho|\nabla\Phi|^2 + \rho gz = 0$$

Linearization

✓ Laplace Equation

$$\nabla^2\Phi = 0$$

$$\Phi = \Phi_I + \Phi_D + \Phi_R$$

Linear combination of the Basis solutions

Basis solutions

$$P_{Fluid}(\mathbf{r},\dot{\mathbf{r}},\ddot{\mathbf{r}}) = -\rho gz - \rho\frac{\partial\Phi}{\partial t} = -\rho gz - \rho\left(\frac{\partial\Phi_I}{\partial t} + \frac{\partial\Phi_D}{\partial t} + \frac{\partial\Phi_R}{\partial t}\right)$$

Pressure of the fluid elements acting on the ship

$$= P_{Buoyancy}(\mathbf{r}) + \underbrace{P_{F.K}(\mathbf{r}) + P_D(\mathbf{r}) + P_R(\mathbf{r},\dot{\mathbf{r}},\ddot{\mathbf{r}})}_{P_{dynamic}}$$

$$P_{Fluid} = \mathbf{F}_{Buoyancy}(\mathbf{r}) + \mathbf{F}_{F.K}(\mathbf{r}) + \mathbf{F}_D(\mathbf{r}) + \mathbf{F}_R(\mathbf{r},\dot{\mathbf{r}},\ddot{\mathbf{r}})$$

(S_B: wetted surface)

$d\mathbf{F}$: Differential force of the fluid elements acting on the ship

dS : Differential Area

$\mathbf{n}$: Normal vector of the differential Area

$\mathbf{r} = [\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6]^T$

ξ_1 : surge , ξ_4 : roll

ξ_2 : sway , ξ_5 : pitch

ξ_3 : heave , ξ_6 : yaw

Integration over the wetted surface area of the ship

(Forces and moments acting on the ship due to the fluid elements)

$$\mathbf{F}_{Fluid}(\mathbf{r},\dot{\mathbf{r}},\ddot{\mathbf{r}}) = \iint_{S_B} \quad \mathbf{n}dS$$

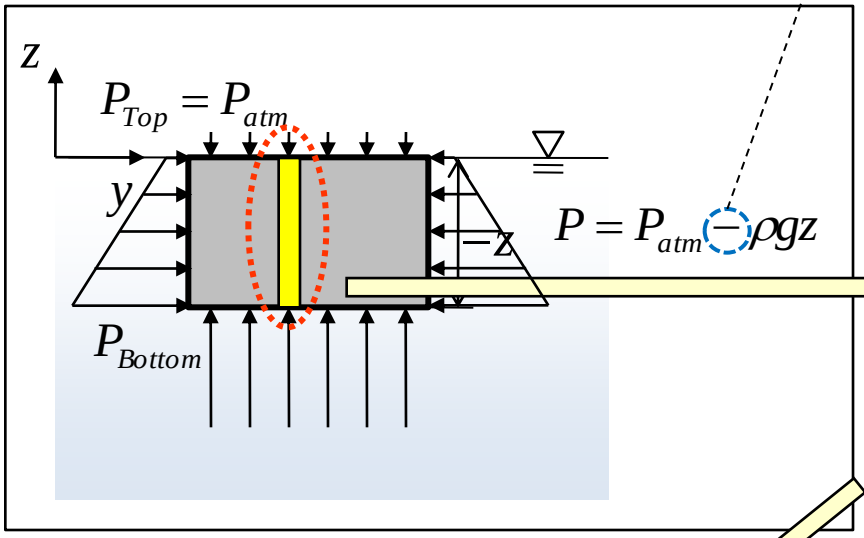
Derivation of Buoyant Force:

Hydrostatic Pressure and Buoyant Force acting on a Ship

※ Pressure : force per unit area applied in a direction perpendicular to the surface of an object.
To calculate force, we should multiply pressure by area and natormal vector of the area.

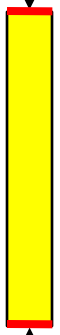
According to the reference frame, (-) sign is added because the value of z is (-).

✓ What is the force acting on the bottom of an object?



: Force acting on the upper differential area

$$d\mathbf{F}_{Top} = P_{Top} \cdot \mathbf{n}_1 dS \quad \left(P_{Top} = P_{atm} - \rho g \cdot 0,\ \mathbf{n}_1 = -\mathbf{k} \right)$$



$\mathbf{n}_1$: Normal vector

dS : Surface area

$$d\mathbf{F}_{Bottom} = P_{Bottom} \cdot \mathbf{n}_2 dS \quad \left(P_{Bottom} = P_{atm} - \rho g z,\ \mathbf{n}_2 = \mathbf{k} \right)$$

: Force acting on the lower differential area

$$\begin{aligned} d\mathbf{F} &= d\mathbf{F}_{Top} + d\mathbf{F}_{Bottom} \\ &= P_{Top} \cdot \mathbf{n}_1 dS + P_{Bottom} \cdot \mathbf{n}_2 dS \\ &= P_{atm}(-\mathbf{k})dS + (P_{atm} - \rho g z)\mathbf{k} dS \\ &= -\rho g z \mathbf{k} dS = \mathbf{k}(-\rho g z \cdot dS) \end{aligned}$$

: Force due to the atmospheric pressure is vanished.

$$\mathbf{F} = \int d\mathbf{F} = \iint_{S_B} P \mathbf{n} dS \quad , (P = P_{static} = -\rho g z)$$

Static fluid pressure excluding the atmospheric pressure.

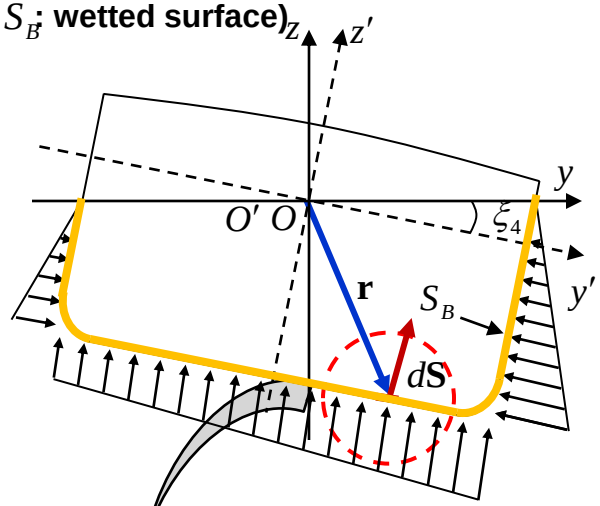
$$= -\rho g \iint_{S_B} \mathbf{n} z dS$$

Cf) Linearized Bernoulli e

$$P = \underbrace{-\rho g z}_{P_{static}} \underbrace{- \rho \frac{\partial \Phi}{\partial t}}_{P_{dynamic}}$$

In case that ship is inclined about x- axis
(Front view)

(S_B: wetted surface)



(Hydrostatic force acting on the differential area)

$$d\mathbf{F} = Pd\mathbf{S} = P\mathbf{n}dS$$

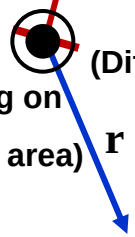
$$P = P_{static}$$

$$= -\rho gz\mathbf{n}dS$$

$$d\mathbf{M} = \mathbf{r} \times d\mathbf{F}$$

(Differential area)

(Moment acting on the differential area)



- **Hydrostatic force (Surface force) is calculated by integrating**
- **the differential force over the wetted surface area.**

✓ **Hydrostatic force acting on the differential area :**

$$d\mathbf{F} = P \cdot d\mathbf{S} = P \cdot \mathbf{n}dS$$

P **is hydrostatic pressure, P_{static}.**

$$P = P_{static} = -\rho gz$$

$$d\mathbf{F} = P_{static} \cdot \mathbf{n}dS = -\rho gz \cdot \mathbf{n}dS$$

✓ **Total force :** (S_B: wetted surface area)

$$\mathbf{F} = \iint_{S_B} P\mathbf{n}dS \quad \Rightarrow \quad \mathbf{F} = -\rho g \iint_{S_B} z\mathbf{n}dS$$

- **Hydrostatic Moment : (Moment)=(Position vector) X (Force)**

✓ **Moment acting on the differential area :**

$$d\mathbf{M} = \mathbf{r} \times d\mathbf{F} = \mathbf{r} \times P\mathbf{n}dS = P(\mathbf{r} \times \mathbf{n})dS$$

✓ **Total moment :**

$$\mathbf{M} = \iint_{S_B} P(\mathbf{r} \times \mathbf{n})dS \quad \Rightarrow \quad \mathbf{M} = -\rho g \iint_{S_B} z(\mathbf{r} \times \mathbf{n})dS$$

Buoyant Force

1) Erwin Kreyszig, Advanced Engineering Mathematics 9th ,Wiley,Ch10.7(p458~46
2) Erwin Kreyszig, Advanced Engineering Mathematics 9th ,Wiley,Ch9.9(p414~417

✓ **Hydrostatic force**

$$\mathbf{F} = -\rho g \iint_{S_B} z\mathbf{n}\,dS$$ (S wetted surface area)

By divergence theorem[1] ,

$$\left(\iint_S f \cdot \mathbf{n}\,dA = \iiint_V \nabla f\,dV \right)$$

$$\mathbf{F} = \rho g \iiint_V \nabla z\,dV \qquad \left(\nabla z \overset{2)}{=} \frac{\partial z}{\partial x}\mathbf{i} + \frac{\partial z}{\partial y}\mathbf{j} + \frac{\partial z}{\partial z}\mathbf{k} = \mathbf{k} \right)$$

$$= \mathbf{k}\rho g \iiint_V dV$$

$$= \mathbf{k}\rho g V(t)$$

When ship moves, the displacement volume(*V*) of the ship is changed with time.
That means *V* is the function of time, *V*(*t*).

: The buoyant force on an immersed body has the same magnitude as the weight of the fluid displaced by the body1). And the direction of the buoyant force is opposite to the gravity (Archimedes' Principle)

※ The reason why (-) sign is disappeared

: Divergence theorem is based on the outer unit vector of the surface.

Normal vector for the calculation of the buoyant force is based on the inner unit vector of the surface, so (-)sign is added, and then divergence theorem is applied

Hydrostatic Moment

1) Erwin Kreyszig, Advanced Engineering Mathematics 9th ,Wiley,Ch10.7(pp.458~46
2) Erwin Kreyszig, Advanced Engineering Mathematics 9th ,Wiley,Ch9.9(pp.414~417

✓ **Hydrostatic moment**

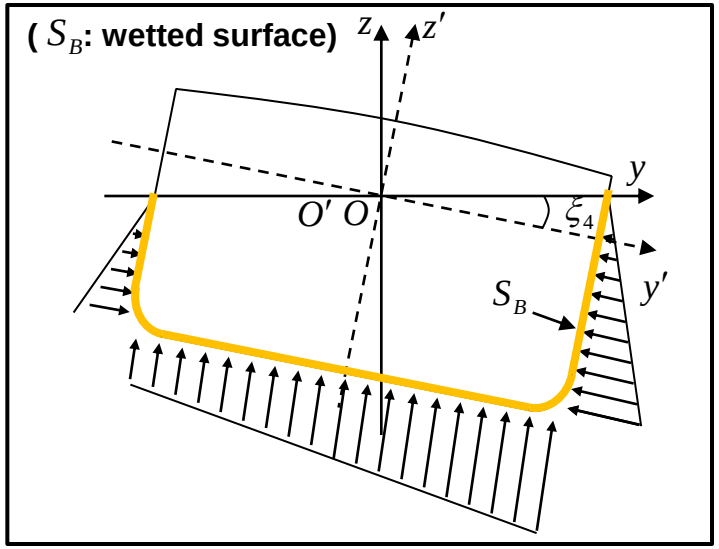
$$\mathbf{M} = -\rho g \iint_{S_B} (\mathbf{r}\times\mathbf{n}) z dS = \rho g \iint_{S_B} (\mathbf{n}\times\mathbf{r}) z dS$$

By divergence theorem[1] ,

$$\left(\iiint_V \nabla\times\mathbf{F} dV = \iint_S \mathbf{n}\times\mathbf{F} dA \right)$$

$$\mathbf{M} = -\rho g \iiint_V (\nabla\times\mathbf{r}) z dV$$

Because direction of normal vector is opposite, (-) sign is added



[2)]

$$\nabla\times\mathbf{r}\, z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & yz & z^2 \end{vmatrix} = \mathbf{i}\left(\frac{\partial}{\partial y} z^2 - \frac{\partial}{\partial z} yz\right) + \mathbf{j}\left(\frac{\partial}{\partial z} xz - \frac{\partial}{\partial x} z^2\right) + \mathbf{k}\left(\frac{\partial}{\partial x} yz - \frac{\partial}{\partial y} xz\right) = -\mathbf{i}y + \mathbf{j}x$$

$$\therefore \mathbf{M} = -\rho g \iiint_V [-\mathbf{i}y + \mathbf{j}x] dV$$

1-12. Modeling of Heave Motion of a Ship in Ocean Waves

6 D.O.F Equations of ship motion

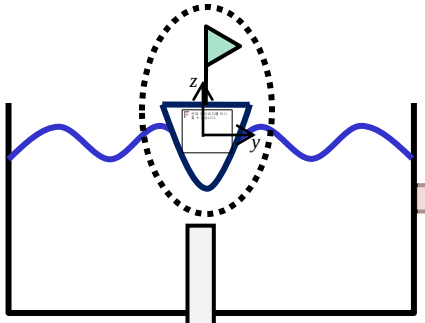
$\mathbf{r}$: displacement of particle with respect to time

$\mathbf{V} = \frac{d\mathbf{r}}{dt}, \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}$

$F_{F.K}$: Froude- krylov force
F_D: Diffraction force
F_R: Radiation force

1) RTT : Reynold Transport Theorem
2) SWBM : Still Water Bending Moment
3) VWBM : Vertical Wave Bending Moment

✓ Assumption
① Newtonian fluid*
② Stokes Assumption**
③ inviscid fluid
④ Irrotational flow
⑤ Incompressible flow



✓Equations of motions of Fluid Particles

Lagrangian & Eulerian Description

Newton's 2nd Law

$m\ddot{\mathbf{r}} = \sum \mathbf{F} = (\text{Body force} + \text{Surface force})$

Shear force

Cauchy equation ① ② → Navier-Stokes equation ③ → Euler equation ④ ⑤ → Bernoulli equation

Curl & Rotation

Microscopic/ Macroscopic Derivation(RTT[1])

Mass Conservation Law ④ ⑤ → Laplace Equation

$\nabla^2\Phi = 0 \quad (\mathbf{V} = \nabla\Phi)$

$\rho\frac{\partial\Phi}{\partial t} + P + \frac{1}{2}\rho|\nabla\Phi|^2 + \rho g z = 0$

✓6 D.O.F equations of motion

① Coordinate system(Reference frame)
(Water surface-fixed & Body-fixed frame)

② Newton's 2nd Law

$$\mathbf{M}\ddot{\mathbf{r}} = \sum \mathbf{F} = (Body\ Force) + (Surface\ Force)$$

$$= \mathbf{F}_{gravity}(\mathbf{r}) + \mathbf{F}_{Fluid}(\mathbf{r},\dot{\mathbf{r}},\ddot{\mathbf{r}})$$

$$= \mathbf{F}_{gravity} + \mathbf{F}_{Buoyancy}(\mathbf{r}) + \mathbf{F}_{F.K}(\mathbf{r}) + \mathbf{F}_{D}(\mathbf{r}) + \mathbf{F}_{R,Damping}(\mathbf{r},\dot{\mathbf{r}}) + \mathbf{F}_{R,Mass}(\mathbf{r},\ddot{\mathbf{r}})$$

Nonlinear terms → Nonlinear equation
→ Difficulty of getting analytic solution

Numerical Method

Velocity potential Φ

$\Phi = \Phi_I$ (Incident wave potential)
$+\Phi_D$ (Diffraction potential)
$+\Phi_R$ (Radiation potential)

Linearization $\left(\frac{1}{2}\rho|\nabla\Phi|^2 = 0\right)$

$$P = -\rho g z - \rho\frac{\partial\Phi}{\partial t}$$

✓Calculation of Fluid Force

$$\mathbf{F}_{Fluid}(\mathbf{r},\dot{\mathbf{r}},\ddot{\mathbf{r}}) = \iint_{S_B} P\mathbf{n}dS = \mathbf{F}_{Buoyancy}(\mathbf{r}) + \mathbf{F}_{F.K}(\mathbf{r}) + \mathbf{F}_{D}(\mathbf{r}) + \mathbf{F}_{R}(\mathbf{r},\dot{\mathbf{r}},\ddot{\mathbf{r}})$$

(S_B: wetted surface)

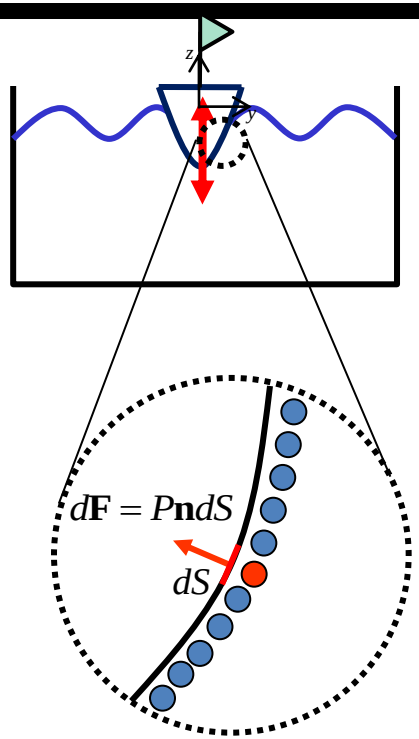
(displacement : $\mathbf{r} = [\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6]^T$)

ξ_1 : surge ξ_4 : roll
ξ_2 : sway ξ_5 : pitch
ξ_3 : heave ξ_6 : yaw

Derivation of 6 D.O.F Equations of ship motion

$F_{F.K}$: Froude- krylov force
F_D: Diffraction force
F_R: Radiation force

Φ_I : Incident wave velocity potential
Φ_D : Diffraction potential
Φ_R : Radiation potential



$d\mathbf{F}$: Force of fluid elements acting on the differential surface of a ship

dS : Differential surface area

$\mathbf{n}$: Normal vector of the differential surface area

$$\mathbf{r} = \left[\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6\right]^T$$

ξ_1 : surge
ξ_2 : sway
ξ_3 : heave

$\mathbf{M}_A$: 6×6 added mass matrix
$\mathbf{B}$: 6×6 damping coeff. matrix
$\mathbf{C}$: 6×6 restoring coeff. matrix

G : Gravity Constant

✓ Surface forces: Fluid forces acting on a ship

$$\mathbf{F}_{Fluid}(\mathbf{r},\dot{\mathbf{r}},\ddot{\mathbf{r}}) = \iint_{S_B} P\mathbf{n}dS = \mathbf{F}_{Buoyancy}(\mathbf{r},\dot{\mathbf{r}},\ddot{\mathbf{r}}) + \mathbf{F}_{F.K}(\mathbf{r},\dot{\mathbf{r}},\ddot{\mathbf{r}}) + \mathbf{F}_D(\mathbf{r},\dot{\mathbf{r}},\ddot{\mathbf{r}}) + \mathbf{F}_R(\mathbf{r},\dot{\mathbf{r}},\ddot{\mathbf{r}})$$

: Fluid forces are obtained by integrating the fluid hydrostatic and hydrodynamic pressure over the wetted surface of a ship.

✓ 6 D.O.F equations of motion

Newton's 2nd Law

$$M\ddot{\mathbf{r}} = \sum \mathbf{F} = (\text{Body Force}) + (\text{Surface Force})$$

$$= \mathbf{F}_{gravity} + \mathbf{F}_{Fluid} + \mathbf{F}_{external}$$

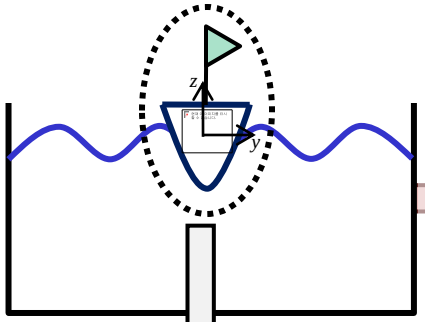
Body force Surface force

$$M\ddot{\mathbf{r}} = \mathbf{F}_{Gravity}(\mathbf{r}) + \mathbf{F}_{Buoyancy}(\mathbf{r},\dot{\mathbf{r}},\ddot{\mathbf{r}}) + \mathbf{F}_{Hydrodynamic}(\mathbf{r},\dot{\mathbf{r}},\ddot{\mathbf{r}}) + \mathbf{F}_{external,\,dynamic} + \mathbf{F}_{external,\,static}$$

Assume that forces are constant or proportional to the displacement, velocity and acceleration of the ship.

$$M\ddot{\mathbf{r}} = \mathbf{F}_{gravity} + \mathbf{F}_{Buoyancy}(\mathbf{r}) + \mathbf{F}_{F.K}(\mathbf{r}) + \mathbf{F}_D(\mathbf{r}) + \mathbf{F}_{R,Damping}(\mathbf{r},\dot{\mathbf{r}}) + \mathbf{F}_{R,Mass}(\mathbf{r},\ddot{\mathbf{r}}) + \mathbf{F}_{ext,\,dynamic} + \mathbf{F}_{ext,\,static}$$

Shear Force and Bending Moment



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Lagrangian & Eulerian Description

Shear force

Curl & Rotation

Newton's 2nd Law

$m\ddot{\mathbf{r}} = \sum \mathbf{F} = (\text{Body force} + \text{Surface force})$

Cauchy equation ① ② → Navier-Stokes equation ③ → Euler equation ④ ⑤ → Bernoulli equation

Microscopic/ Macroscopic Derivation(RTT[1])

Mass Conservation Law ④ ⑤ → Laplace Equation

$\nabla^2 \Phi = 0 \quad (\mathbf{V} = \nabla \Phi)$

$$\rho \frac{\partial \Phi}{\partial t} + P + \frac{1}{2}\rho |\nabla \Phi|^2 + \rho g z = 0$$

✓6 D.O.F equations of motions

① Coordinate system
(Waterplane Fixed & Body-fixed frame)

② Newton's 2nd Law

Non-linear terms → Non-linear equation
→ Difficulty of getting analytic solution

Numerical Method

Velocity potential Φ

$\Phi = \Phi_I$ (Incident wave potential)

$+\Phi_D$ (Diffraction potential)

$+\Phi_R$ (Radiation potential)

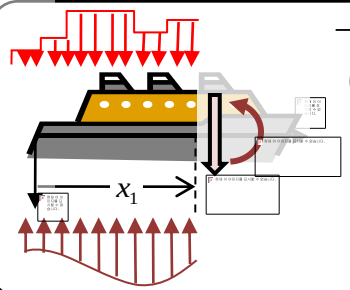
✓Calculation of Fluid Force

Linearization $\left(\frac{1}{2}\rho|\nabla\Phi|^2 = 0\right)$

$$P = -\rho g z - \rho \frac{\partial \Phi}{\partial t}$$

$$\mathbf{F}_{Fluid}(\mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}) = \iint_{S_B} P\mathbf{n}dS = \mathbf{F}_{Buoyancy}(\mathbf{r}) + \mathbf{F}_{F.K}(\mathbf{r}) + \mathbf{F}_D(\mathbf{r}) + \mathbf{F}_R(\mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}})$$

(S_B: wetted surface)



$-m(x)a_z$, $f_{Gravity}$

(a_z : Acceleration induced by heave & pitch motion)

$f_{F.K.}$, f_D, $-a_{33}a_z$, $-b_{33}v_z$, f_{Static}

✓Shear force(S.F.) & Bending moment(B.M.)

Shear force(*S.F.*)

⬇ Integration

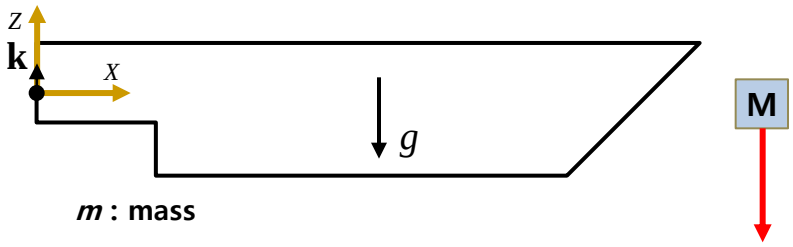
Bending moment(*B.M.*)

Modeling : Heave motion of a ship

Nonlinearity of the nature → Nonlinear Mathematical Model → Numerical Method

Nonlinear Mathematical Model → (Linearization) → Linear Mathematical Model → Analytic Solution

Step 1: Ship in Air

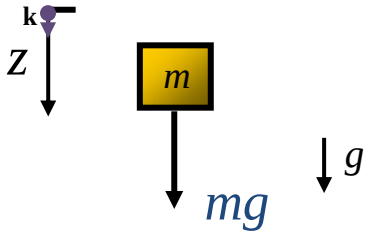


$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= \mathbf{F}_{gravity}$$
$$= -mg\mathbf{k}$$

where $\mathbf{z} = z\mathbf{k}$

$$\ddot{\mathbf{z}} = \frac{d^2\mathbf{z}}{dt^2} = \frac{d^2z}{dt^2}\mathbf{k} = \ddot{z}\,\mathbf{k}$$

✓ Mass in Air



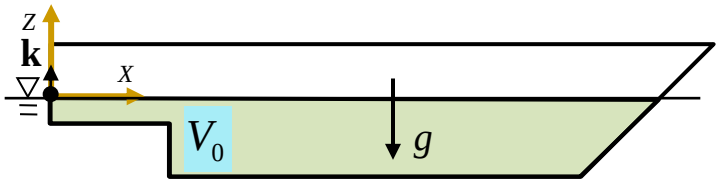
By Newton's 2nd law,

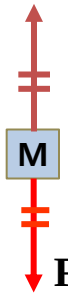
$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= mg\mathbf{k}$$

Step 2 : Ship in Water

$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS = \rho g V_0 \mathbf{k}$$

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0 \mathbf{k}$

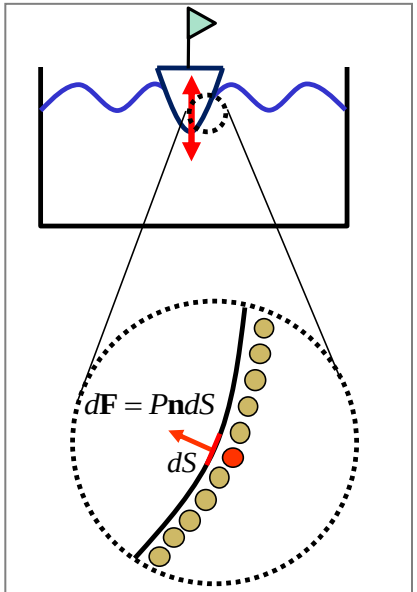




$= -mg\mathbf{k}$

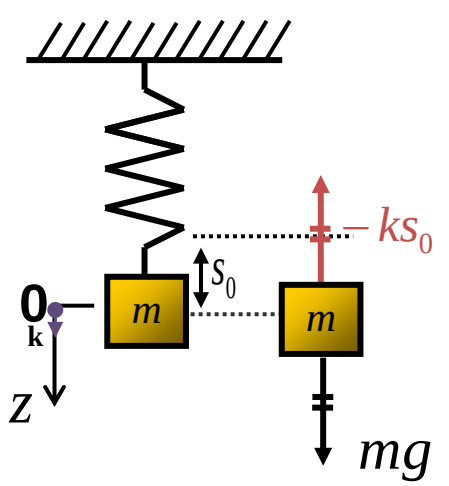
m : mass $\quad$ ρ : density of sea water
V_0 : submerged volume
S_B : wetted surface area

$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= \mathbf{F}_{gravity} + \mathbf{F}_{static}$$
$$= -mg\mathbf{k} + \rho g V_0 \mathbf{k}$$
$$= 0 \quad (\because \ddot{\mathbf{z}} = 0)$$: static equilibrium



dS : infinitesimal submerged surface area
$d\mathbf{F}$: force exerted by the infinitesimal fluid element on dS
$\mathbf{n}$: normal vector of dS

✓ Mass-Spring system



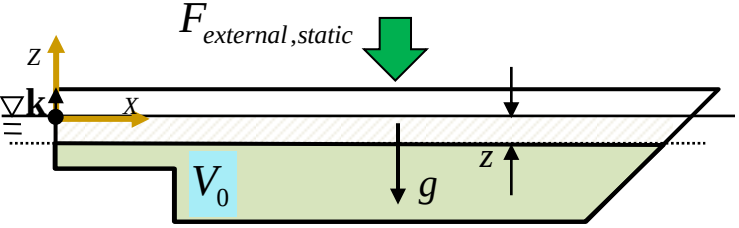
$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k}$$
$$= 0 \quad (\because \ddot{\mathbf{z}} = 0)$$
: static equilibrium

where $\mathbf{z} = z\mathbf{k}$

$$\ddot{\mathbf{z}} = \frac{d^2\mathbf{z}}{dt^2} = \frac{d^2 z}{dt^2}\mathbf{k} = \ddot{z}\,\mathbf{k}$$

Suppose that an external force is applied, and a ship is in static equilibrium state.

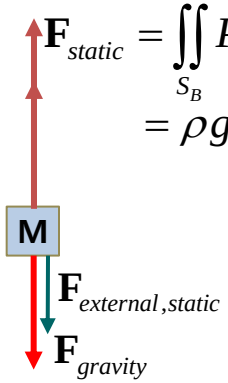
Step 3: External force is applied on the ship



m : mass ρ : density of sea water
V_0 : submerged volume
S_B : submerged surface area
Awp : waterplane area

$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS = \rho g V_0 \mathbf{k} + \mathbf{F}_{changed\ bouyancy}$$

changed buoyancy due to the change in displacement z



$$= -mg\mathbf{k}$$

Linearized Restoring Force

if, z is small

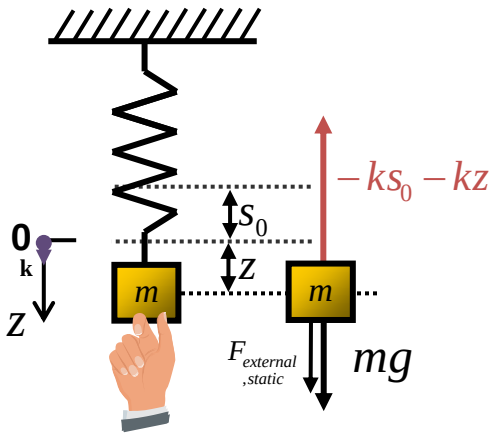
$$\mathbf{F}_{changed\ bouyancy} = -\rho g A_{WP} z\,\mathbf{k}$$

$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= \mathbf{F}_{gravity} + \mathbf{F}_{static} + \mathbf{F}_{external,static}$$
$$= -mg\mathbf{k} + \rho g V_0 \mathbf{k} - \rho g A_{wp}\, z\,\mathbf{k} + \mathbf{F}_{external,static}$$
$$= -\rho g A_{wp}\, z\,\mathbf{k} + \mathbf{F}_{external,static}$$
$$= 0\ (\because \ddot{\mathbf{z}} = 0) \quad : \text{static equilibrium}$$

$$\boxed{m\ddot{z} + \rho g A_{wp}\, z\,\mathbf{k} = 0}$$

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0 \mathbf{k}$

✓ Mass-Spring system



$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} + \mathbf{F}_{external,static}$$
$$= -kz\mathbf{k} + \mathbf{F}_{external,static}$$
$$= 0$$

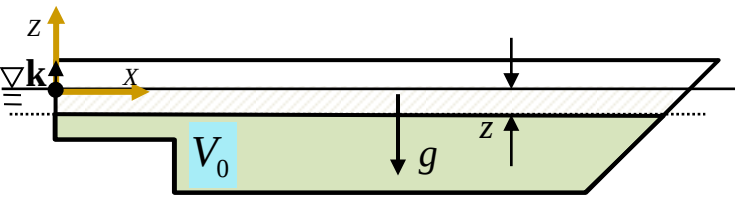
$$\boxed{m\ddot{z} + kz = 0}$$ Oscillation by the restoring force

What will happen, when shortly after, the external force is removed?

Ex) External force is removed: step 4

Oscillation due to the restoring force

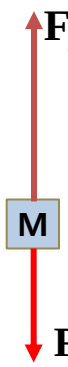
Step 4: The ship will be oscillated due to the restoring force



m : mass ρ : density of sea water
V_0 : submerged volume
S_B : submerged surface area
Awp : waterplane area

$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS$$

$$= \rho g V_0 \mathbf{k} - \rho g A_{WP}\, z\,\mathbf{k}$$



$$\mathbf{F}_{gravity} = -mg\mathbf{k}$$

$$m\ddot{\mathbf{z}} = \mathbf{F}$$

$$= \mathbf{F}_{gravity} + \mathbf{F}_{static}$$

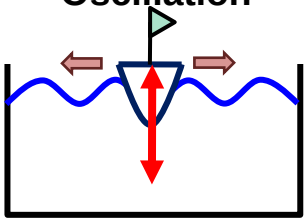
$$= -mg\mathbf{k} + \rho g V_0 \mathbf{k} - \rho g A_{wp}\, z\,\mathbf{k}$$

$$= -\rho g A_{wp}\, z\,\mathbf{k}$$

Does the ship oscillate forever?

No ! Energy is dissipated by radiating waves

Forced Oscillation



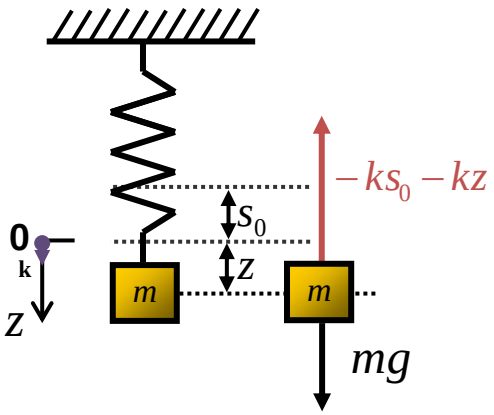
Radiation Force

$$\mathbf{F}_{radiation} = \iint_{S_B} P_{radiation}\mathbf{n}dS$$

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0 \mathbf{k}$

✓ Mass-Spring system

④



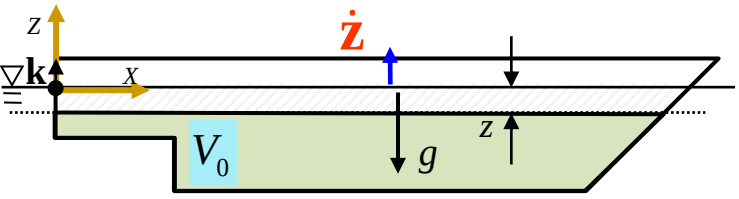
$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k}$$
$$= -kz\mathbf{k}$$

$$m\ddot{z} + kz = 0$$

Oscillation by the restoring force

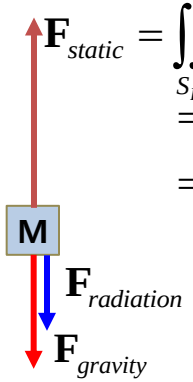
Damping force due to the radiated waves

step 5: Damping force



m : mass ρ : density of sea water
V_0 : submerged volume
S_B : submerged surface area
Awp : waterplane area

$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= \mathbf{F}_{gravity} + \mathbf{F}_{static} + \mathbf{F}_{radiation}$$
$$= -mg\mathbf{k} + \rho g V_0 \mathbf{k} - \rho g A_{wp} z\mathbf{k} - c\dot{z}\mathbf{k}$$
$$= -\rho g A_{wp} z\mathbf{k} - c\dot{z}\mathbf{k}$$



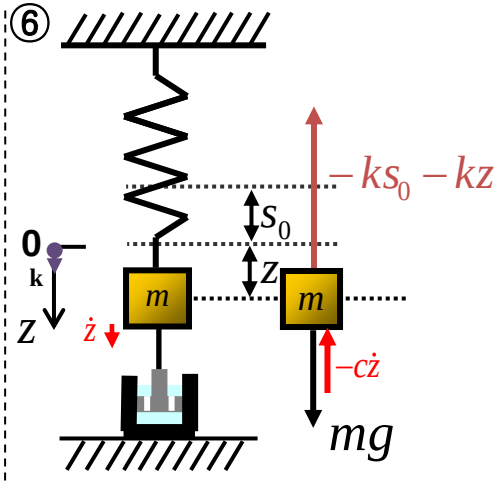
$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS$$
$$= \rho g V_0 \mathbf{k} - \rho g A_{WP}\mathbf{z}$$
$$= \rho g V_0 \mathbf{k} - k\mathbf{z}$$

$$\mathbf{F}_{radiation} = -c\dot{\mathbf{z}}$$

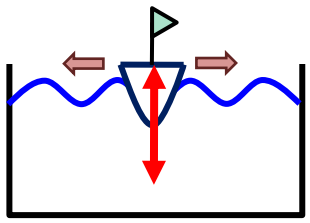
$$\mathbf{F}_{gravity} = -mg\mathbf{k}$$

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0 \mathbf{k}$

✓ Mass-Spring-Damping system



$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - c\dot{z}\mathbf{k}$$
$$= -kz\mathbf{k} - c\dot{z}\mathbf{k}$$



Radiation Force

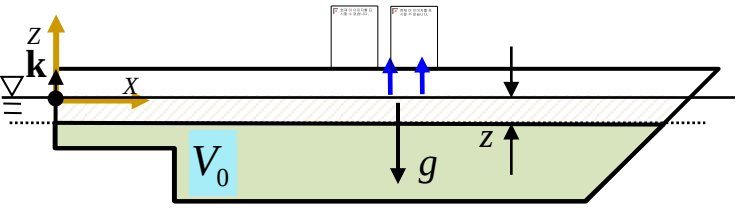
$$\mathbf{F}_{radiation} = \iint_{S_B} P_{radiation}\mathbf{n}dS$$
$$= -c\dot{\mathbf{z}}$$

opposite to the velocity of the ship

c : damping coefficient

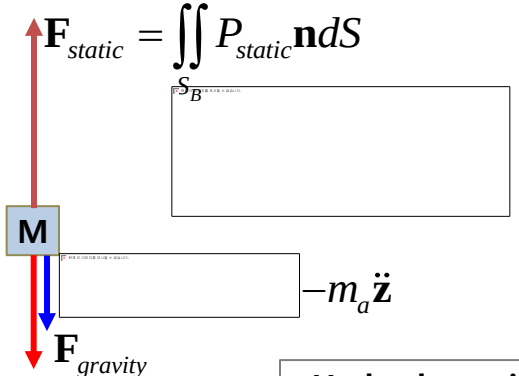
Added Mass

step 6: Added mass



m : mass ρ : density of sea water
V_0 : submerged volume
S_B : submerged surface area
A_{WP} : waterplane area

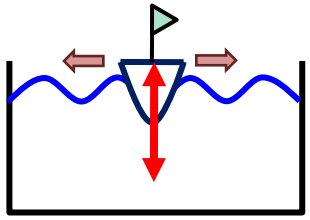
$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS$$



$$= -mg\mathbf{k}$$

$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= \mathbf{F}_{gravity} + \mathbf{F}_{static} + \mathbf{F}_{radiation}$$
$$= -mg\mathbf{k} + \rho g V_0\mathbf{k}$$

Hydrodynamic force due to the forced oscillation



Radiation Force

$$\mathbf{F}_{radiation} = \iint_{S_B} P_{radiation}\mathbf{n}dS$$

$-m_a\ddot{\mathbf{z}}$

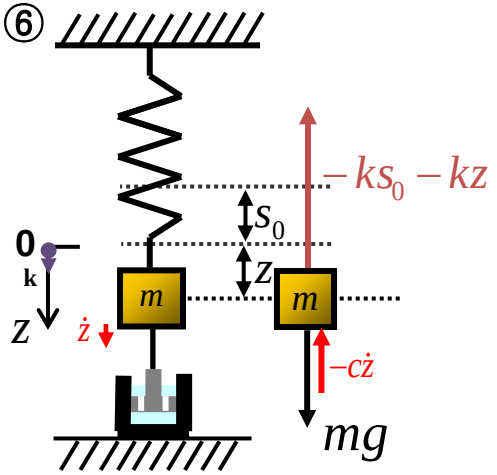
opposite to the velocity of the ship

opposite to the acceleration of the ship

c : damping coefficient
m_a : added mass

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0\mathbf{k}$

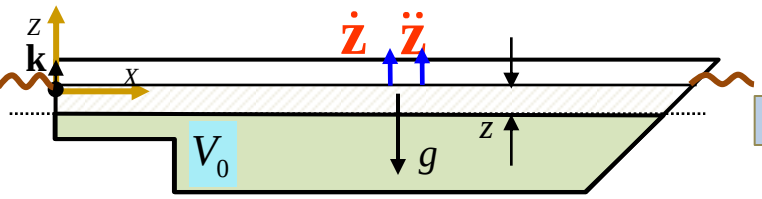
✓ Mass-Spring-Damping system



$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - c\dot{z}\mathbf{k}$$
$$= -kz\mathbf{k} - c\dot{z}\mathbf{k}$$

Forced Heave Motion due to the Wave Exciting Forces Induced by the Incident Waves

step 7: Wave Exciting Forces



m : mass ρ : density of sea water
V_0 : submerged volume
S_B : submerged surface area
Awp : waterplane area

$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS = \rho g V_0\mathbf{k} - \rho g A_{WP}\mathbf{z} = \rho g V_0\mathbf{k} - k\mathbf{z}$$

$$\mathbf{F}_{radiation} = -c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}}$$

$$\mathbf{F}_{gravity} = -mg\mathbf{k}$$

$$m\ddot{\mathbf{z}} = \mathbf{F}$$

$$= \mathbf{F}_{gravity} + \mathbf{F}_{static} + \mathbf{F}_{radiation} + \mathbf{F}_{hydrodynamicexcitingforce(wave)}$$

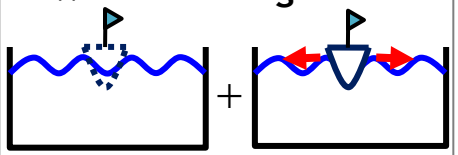
$$= -mg\mathbf{k} + \rho g V_0\mathbf{k} - \rho g A_{wp} z\mathbf{k} - c\dot{z}\mathbf{k} - m_a\ddot{z}\mathbf{k} + \mathbf{F}_{wave}$$

$$= -\rho g A_{wp} z\mathbf{k} - c\dot{z}\mathbf{k} - m_a\ddot{z}\mathbf{k} + F_{wave}\mathbf{k}$$

$$= -kz\mathbf{k} - c\dot{z}\mathbf{k} - m_a\ddot{z}\mathbf{k} + F_{wave}\mathbf{k}$$

$$(m + m_a)\ddot{z} + c\dot{z} + kz = F_{wave}$$

Wave exciting forces

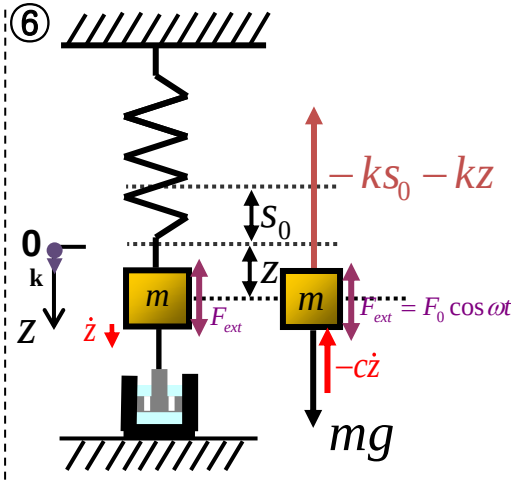


Froude-Krylov Force Diffraction Force

$$\mathbf{F}_{wave} = \iint_{S_B} P_{wave}\mathbf{n}dS$$

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0\mathbf{k}$

✓ Mass-Spring-Damping system



$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - c\dot{z}\mathbf{k} + \mathbf{F}_0\cos\omega t$$
$$= -kz\mathbf{k} - c\dot{z}\mathbf{k} + F_0\cos\omega t\mathbf{k}$$

$$m\ddot{z} + c\dot{z} + kz = F_0\cos\omega t$$

Non-homogeneous Second order Linear Ordinary Differential Equation

1-13 Modeling of Roll Motion of a Ship in Ocean Waves

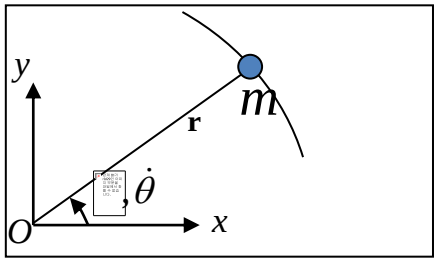
Modeling : Roll motion of a ship

[Dynamics for roll motion]

Angular momentum defined : $\mathbf{L} = I\dot{\theta}\,\mathbf{i}$

$where, I = m\mathbf{r}^2$:moment of inertia

$\phi = \phi(t)$



Rate of change of Angular momentum :

$$\frac{d\mathbf{L}}{dt} = I\frac{d(\dot{\theta})}{dt}\mathbf{i} = I\ddot{\theta}\,\mathbf{i} \quad (I : \text{constant})$$

Euler's Equation: $\dfrac{d\mathbf{L}}{dt} = I\ddot{\phi}\mathbf{i} = \boldsymbol{\tau}_{net}$

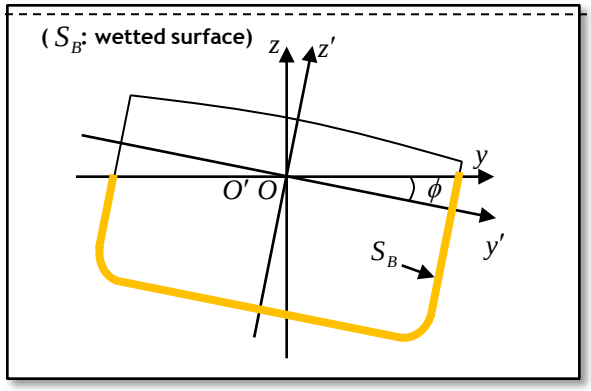
Euler's Equation for roll motion:

$$I\ddot{\phi}\mathbf{i} = \sum \mathbf{M}$$

$$I\ddot{\phi}\mathbf{i} = \mathbf{M}_{body} + \mathbf{M}_{surface}$$

$$I\ddot{\phi}\mathbf{i} = M_{body}\mathbf{i} + M_{surface}\mathbf{i}$$

$$I\ddot{\phi} = M_{body} + M_{surface}$$



Equation of Roll motion of a ship

$$I\ddot{\phi}\mathbf{i} = \sum \mathbf{M}$$

$$I\ddot{\phi}\mathbf{i} = M_{body}\mathbf{i} + M_{surface}\mathbf{i}$$

$$= M_{body}\mathbf{i} + \iint P_{static}(\mathbf{r}\times\mathbf{n})ds + \iint P_{dynamic}(\mathbf{r}\times\mathbf{n})ds + M_{external}\mathbf{i}$$

When "r" and "n" are the vectors on the y-z plane,

$$\mathbf{r}\times\mathbf{n} = |\mathbf{r}||\mathbf{n}|\sin\theta\ \mathbf{i}$$

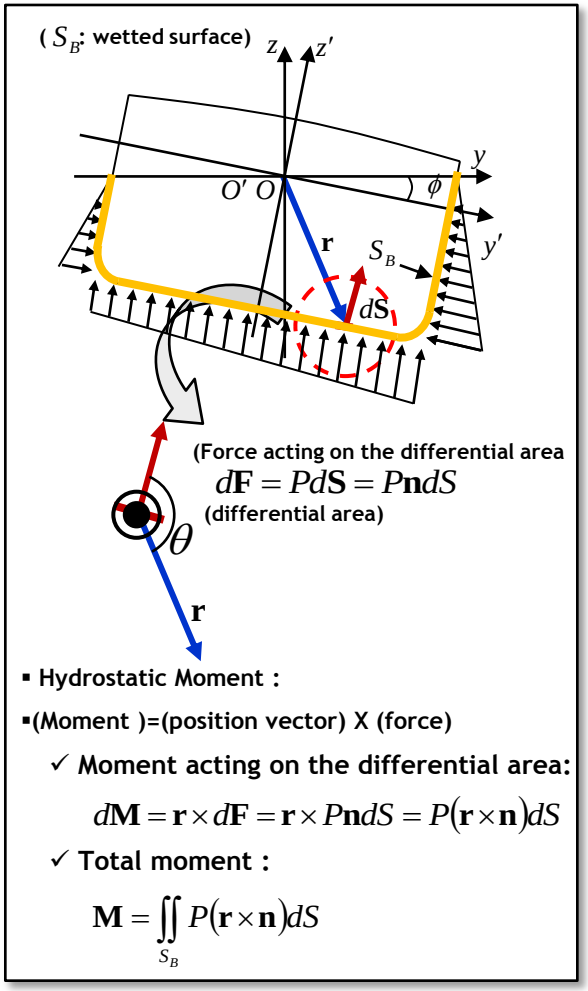
$$= M_{body}\mathbf{i} + \iint P_{static}(|\mathbf{r}||\mathbf{n}|\sin\theta\ \mathbf{i})ds + \iint P_{dynamic}(|\mathbf{r}||\mathbf{n}|\sin\theta\ \mathbf{i})ds + M_{external}\mathbf{i}$$

$$= M_{body}\mathbf{i} + \iint P_{static}(|\mathbf{r}||\mathbf{n}|\sin\theta)ds\ \mathbf{i} + \iint P_{dynamic}(|\mathbf{r}||\mathbf{n}|\sin\theta)ds\ \mathbf{i} + M_{external}\mathbf{i}$$

$$= M_{gravity}\mathbf{i} + M_{buoyancy}\mathbf{i} + M_{damping}\mathbf{i} + M_{added}\mathbf{i} + M_{external}\mathbf{i}$$

Therefore

$$I\ddot{\phi} = M_{gravity} + M_{buoyancy} + M_{damping} + M_{added} + M_{external}$$



- Hydrostatic Moment :
- (Moment)=(position vector) X (force)
  - ✓ Moment acting on the differential area:

$$d\mathbf{M} = \mathbf{r}\times d\mathbf{F} = \mathbf{r}\times P\mathbf{n}dS = P(\mathbf{r}\times\mathbf{n})dS$$

  - ✓ Total moment :

$$\mathbf{M} = \iint_{S_B} P(\mathbf{r}\times\mathbf{n})dS$$

Equation of Roll motion of a ship

$$I\ddot{\phi} = M_{body} + M_{surface}$$
$$= M_{gravity} + M_{buoyancy} + M_{damping} + M_{added} + M_{wave}$$

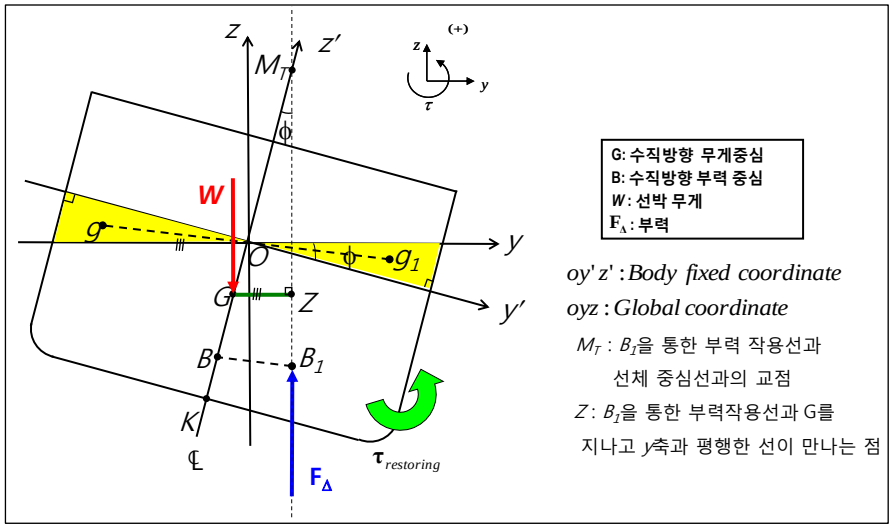
1) $M_{gravity} = (-W) \times 0 = 0$

2) $M_{buoyancy} = -\Delta\overline{GZ}$

3) $M_{damping} = -b\dot{\phi}$, b: damping coeff.

4) $M_{added} = -I_{add}\ddot{\phi}$, I_{add} : added Mass moment of inertia

5) M_{wave} : **Wave exciting Moment**



$$\therefore I\ddot{\phi} = (-W) \times 0 + (-\Delta\overline{GZ}) + (-b\dot{\phi}) + (-I_{add}\ddot{\phi}) + M_{wave}$$

Equation of Roll motion of a ship

$$I\ddot{\phi} = (-W)\times 0 + (-\Delta\overline{GZ}) + (-b\dot{\phi}) + (-I_{add}\ddot{\phi}) + M_{wave}$$

$$\therefore (I + I_{add})\ddot{\phi} + b\dot{\phi} + \Delta\overline{GZ} = M_{wave}$$

✓ **Archimedes' Principle**

$$W = \Delta$$

Consider restoring moment

$$\tau_{restoring} = \Delta\overline{GZ}$$

For small angle $\phi\ (<10^{\circ})$

$$\overline{GZ} \approx \overline{GM_T}\cdot\sin\phi \approx \overline{GM_T}\cdot\phi$$

$\overline{GM}_T$: **Meta center Height**

For small ϕ $\quad \sin\phi = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \cdots$

$$\therefore (I + I_{add})\ddot{\phi} + b\dot{\phi} + \Delta\overline{GM_T}\,\phi = M_{wave} \longleftarrow$$

Non-homogeneous Second order Linear Ordinary Differential Equation

CHAPTER 2. ORDINARY DIFFERENTIAL EQUATIONS

First-Order Linear Differential Equation

Definition 2.2 **Linear Equation**

A first order differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

is said to be Linear equation in the dependant variable y.

☑ Standard form

$$\frac{dy}{dx} + P(x)y = f(x) \xrightarrow{\text{solution}} y = y_c + y_p$$

$$\frac{d}{dx}[y_c + y_p] + P(x)[y_c + y_p]$$
$$= \left[\frac{d}{dx}y_c + P(x)y_c\right] + \left[\frac{d}{dx}y_p + P(x)y_p\right] = f(x)$$

☑ Homogeneous

$$f(x) = 0 \xrightarrow[y_c]{\text{solution}} \frac{dy_c}{dx} + P(x)y_c = 0$$

☑ Nonhomogeneous

$$f(x) \neq 0 \xrightarrow[y_p]{\text{solution}} \frac{dy_p}{dx} + P(x)y_p = f(x)$$

2-1. Variation of Parameters

Variation of Parameters

☑Variation of Parameters

■ Homogeneous solution

$$\frac{dy_c}{dx} + P(x)y_c = 0 \Rightarrow \frac{dy_c}{y_c} = -P(x)dx \longrightarrow y_c = ce^{-\int P(x)dx}$$

$$= cy_1(x)$$

■ Nonhomogeneous solution

Assumption: $y_p = u(x)y_1(x)$ $\quad c \rightarrow u(x)$ **Variation of Parameters**

Find : $\xrightarrow{u(x)}$ **Find :** $y_p(x)$

$$y_p(x) = u(x)y_1(x) \xrightarrow{\text{substituting}} \frac{dy_p}{dx} + P(x)y_p = f(x)$$

☑Variation of Parameters

$$\frac{dy_p}{dx} + P(x)y_p = f(x)$$

$y_p = u(x)y_1(x)$ substituting

L.H.S:

$$\frac{dy_p}{dx} + P(x)y_p = \frac{d(uy_1)}{dx} + P(x)(uy_1)$$

$$= y_1\frac{du}{dx} + u\frac{dy_1}{dx} + P(x)uy_1$$

$$= y_1\frac{du}{dx} + u\left[\frac{dy_1}{dx} + P(x)y_1\right] = y_1\frac{du}{dx}$$

Since y_1 is **homogeneous solution**.

$$\left[\frac{dy_1}{dx} + P(x)y_1\right] = 0$$

R.H.S: $f(x)$

$$\therefore y_1\frac{du}{dx} = f(x)$$

☑Variation of Parameters

$$y_1 \frac{du}{dx} = f(x) \xrightarrow[\text{and integrating}]{\text{Separable variables}} u = \int \frac{f(x)}{y_1(x)} dx$$

$$\therefore y_p = u(x) y_1(x) = \left(\int \frac{f(x)}{y_1(x)} dx \right) y_1(x)$$

Recall, $y_c = c e^{-\int P(x)dx} = c y_1(x)$

$$= \left(\int e^{\int P(x)dx} f(x) dx \right) e^{-\int P(x)dx}$$

Now , we get

$$\begin{cases} y_c = c e^{-\int P(x)dx} \\ y_p = \underbrace{e^{-\int P(x)dx}}_{y_1(x)} \underbrace{\int e^{\int P(x)dx} f(x) dx}_{u(x)} \end{cases}$$

Integrating factor

$$\frac{dy}{dx} + P(x)y = f(x) \xrightarrow{\text{solution}} y = y_c + y_p \qquad \begin{cases} y_c = ce^{-\int P(x)dx} \\ y_p = e^{-\int P(x)dx} \int e^{\int P(x)dx} f(x)dx \end{cases}$$

☑Integrating factor

$$y = ce^{-\int P(x)dx} + e^{-\int P(x)dx} \int e^{\int P(x)dx} f(x)dx$$

Multiplied by $e^{\int P(x)dx}$

$$e^{\int P(x)dx} y = c + \int e^{\int P(x)dx} f(x)dx$$

☑Integrating factor

$$\frac{dy}{dx} + P(x)y = f(x)$$ **Linear Equation Standard Form**

$$y = ce^{-\int P(x)dx} + e^{-\int P(x)dx} \int e^{\int P(x)dx} f(x)dx$$

Multiplied by $e^{\int P(x)dx}$

$$e^{\int P(x)dx} y = c + \int e^{\int P(x)dx} f(x)dx$$

divided by $e^{\int P(x)dx}$

is differentiated

$$\frac{d}{dx}\left[e^{\int P(x)dx} y \right] = e^{\int P(x)dx} f(x)$$

$$e^{\int P(x)dx} \frac{dy}{dx} + P(x)e^{\int P(x)dx} y = e^{\int P(x)dx} f(x)$$

☑Integrating factor

$$\frac{dy}{dx} + P(x)y = f(x)$$

Linear Equation Standard Form

$$y = ce^{-\int P(x)dx} + e^{-\int P(x)dx} \int e^{\int P(x)dx} f(x)dx$$

Multiplied by $e^{\int P(x)dx}$

Solution!!!

$$e^{\int P(x)dx} y = c + \int e^{\int P(x)dx} f(x)dx$$

divided by $e^{\int P(x)dx}$

is integrated

is differentiated

$$\frac{d}{dx}\left[e^{\int P(x)dx} y \right] = e^{\int P(x)dx} f(x)$$

$$e^{\int P(x)dx} \frac{dy}{dx} + P(x)e^{\int P(x)dx} y = e^{\int P(x)dx} f(x)$$

☑ Example 1
Solving a Linear DE

Solve

$$\frac{dy}{dx} - 3y = 6$$

This equation is in the standard form

$$\frac{dy}{dx} + p(x)y = f(x)$$

Integrating factor is

$$e^{\int (-3)dx} = e^{-3x}$$

Multiplying integrating factor both side

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 6e^{-3x}$$

$$\frac{d}{dx}[e^{-3x}y] = 6e^{-3x}$$

Integrating both side

$$e^{-3x}y = -2e^{-3x} + c$$

$$y = -2 + ce^{3x}, \quad -\infty < x < \infty$$

☑ Example 2

General Solution

Solve

$$x\frac{dy}{dx} - 4y = x^6 e^x$$

Dividing by x gives the standard form

$$\frac{dy}{dx} - \frac{4}{x}y = x^5 e^x$$

$P(x) = -4/x$ and $f(x) = x^5 e^x$ **in standard form are continuous on (0,∞)**

Integrating factor is

$$e^{-4\int (\frac{1}{x})dx} = e^{-4\ln x} = e^{\ln x^{-4}} = x^{-4}$$

Multiplying integrating factor both side

$$x^{-4}\frac{dy}{dx} - 4x^{-5}y = xe^x$$

$$\frac{d}{dx}[x^{-4}y] = xe^x$$

Integrating both side

$$x^{-4}y = xe^x - e^x + c$$

$$y = x^5 e^x - x^4 e^x + cx^4, \quad 0 < x < \infty$$

☑ Example 3

General Solution

Find a general solution of

$$(x^2-9)\frac{dy}{dx}+xy=0$$

Write DE in standard form

$$\frac{dy}{dx}+\frac{x}{x^2-9}y=0$$

$$P(x)=x/(x^2-9)$$

is continuous on (-∞,-3), (-3,3), (3,∞)

On (-∞,-3), (3,∞) integrating factor is

$$e^{\int(\frac{x}{x^2-9})dx}=e^{\frac{1}{2}\int(\frac{2x}{x^2-9})dx}=e^{\frac{1}{2}\ln|x^2-9|}=\sqrt{x^2-9}$$

Multiplying integrating factor both side

$$\sqrt{x^2-9}\frac{dy}{dx}+\frac{x}{\sqrt{x^2-9}}y=0$$

$$\frac{d}{dx}[\sqrt{x^2-9}y]=0$$

Integrating both side

$$\sqrt{x^2-9}y=c$$

$$y=\frac{c}{\sqrt{x^2-9}},\quad x>3\ or\ x<-3$$

☑ Example 4

Initial-Value Problem

Solve the IVP

$$\frac{dy}{dx} + y = x,\ y(0) = 4$$

Integrating factor is

$$e^{\int (1)dx} = e^{x}$$

Multiplying integrating factor both side and rewrite

$$\frac{d}{dx}[e^{x}y] = xe^{x}$$

$$e^{x}y = xe^{x} - e^{x} + c$$

$$y = x - 1 + ce^{-x}$$

Substitute x=0 and y=4 to satisfy initial condition $y(0) = 4$

$$4 = 0 - 1 + c,\ \ c = 5$$

$$y = x - 1 + 5e^{-x},\ -\infty < x < \infty$$

☑ Example 5

Initial-Value Problem

Solve the IVP

$$\frac{dy}{dx} + y = f(x),\ y(0) = 0$$

$$\text{where } f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$$

Integrating factor is

$$e^{\int (1)dx} = e^{x}$$

(a) For 0≤x≤1

Multiplying integrating factor both side and rewrite

$$\frac{d}{dx}[e^{x}y] = e^{x}$$

$$e^{x}y = e^{x} + c$$

$$y = 1 + ce^{-x}$$

Substitute x=0 and y=0 to satisfy initial condition $y(0) = 0$

$$0 = 1 + c,\ c = -1$$

$$y = 1 - e^{-x}$$

Continued…

(b) For x>1 $f(x)=0$

Multiplying integrating factor both side and rewrite

$$\frac{d}{dx}[e^x y]=0$$

$$e^x y = c$$

$$y = ce^{-x}$$

If y(x) is continuous at x=1,

$$\lim_{x\to 1^+} y(x) = y(1)$$

$$ce^{-1} = 1-e^{-1},\ c = e-1$$

$$y=\begin{cases}1-e^{-x}, & 0\le x\le 1\\ (e-1)e^{-x}, & x>1\end{cases}$$

Error Function

☑ Example 6

The Error Function

Solve the IVP

$$\frac{dy}{dx} - 2xy = x,\ y(0) = 1$$

The integrating factor is

$$e^{\int (-2x)dx} = e^{-x^2}$$

Rearrange the equation

$$\frac{d}{dx}[e^{-x^2} y] = 2e^{-x^2}$$

$$y = 2e^{x^2} \int_0^x e^{-t^2} dt + ce^{x^2}$$

$y(0) = 1$ **gives c=1 , So**

$$y = 2e^{x^2} \int_0^x e^{-t^2} dt + e^{x^2}$$

$$= e^{x^2}[1 + \sqrt{\pi} erf(x)]$$

$$\left(erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt\right)$$

2-2. EXACT EQUATIONS

Exact Equations

$$z = f(x, y)$$

Total differential

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

If, $f(x, y) = const.$

Exact equation

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

Exact differential to the $f(x, y)$

$$z = f(x, y) = xy$$

$$dz = ydx + xdy$$

If, $xy = c$ Geometrical meaning

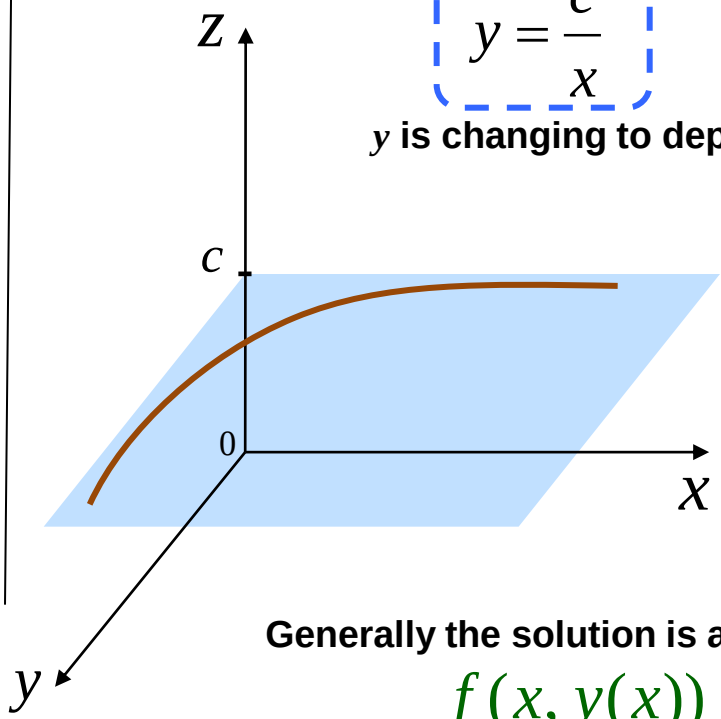
$$ydx + xdy = 0$$

$$z = f(x, y) \quad \text{i.e. } f(x, y) = xy$$

$$xy = c \ , (c = \text{constant})$$

$$y = \frac{c}{x} \qquad y \dashrightarrow y(x)$$

y is changing to dependent variable on x.



Generally the solution is a curve.

$$f(x, y(x)) = xy = c$$

Exact Equations

Definition 2.3 **Exact Equation**

$M(x,y)dx+N(x,y)dy$ is an exact differential if it corresponds to the differential of some function $f(x,y)$.

The first-order differential equation of the form

$$M(x,y)dx+N(x,y)dy=0$$

is said to be exact equation if the expression on the left side is an exact differential .

Theorem 2.1 **Criterion for an Exact Differential**

Let $M(x,y)$ and $N(x,y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a<x<b, c<y<d$.

Then a necessary and sufficient condition that $M(x,y)dx+N(x,y)dy$ be an exact differential is. $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$

Exact Equations

☑Proof of necessity

$$\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$$

$$=M(x,y)dx+N(x,y)dy$$

therefore

$$M(x,y)=\frac{\partial f}{\partial x},\ N(x,y)=\frac{\partial f}{\partial y}$$

and

$$\boxed{\frac{\partial M}{\partial y}}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\boxed{\frac{\partial N}{\partial x}}$$

$$z=f(x,y)$$

Total differential

$$dz=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$$

If,

$$f(x,y)=const.$$

$$\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=0$$

Exact Equations

☑Method of solution

$$M(x,y)dx + N(x,y)dy = 0$$

Recognizing exact equation

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \left(M = \frac{\partial f}{\partial x}, N = \frac{\partial f}{\partial y} \right)$$

Find f by integrating $M(x,y)$ **with respect to** x

$$f(x,y) = \int M(x,y)dx + g(y)$$

differentiate this with respect to y

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\int M(x,y)dx\right) + g'(y) = N(x,y)$$

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\left(\int M(x,y)dx\right)$$

$$z = f(x,y) = xy$$

$$dz = ydx + xdy$$

If, $xy = c$ **Find**

$ydx + xdy = 0$ ← **is integrated**

recognizing the exact equatio

Solution form is

$$\int M(x,y)dx + g(y) = c$$

Exact Equations

☑ Example 1
Solving an Exact DE

Solve $2xydx + (x^2 - 1)dy = 0$

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$$

Thus the equation is exact.
So there exists function such that

$$\frac{\partial f}{\partial x} = 2xy \text{ and } \frac{\partial f}{\partial y} = x^2 - 1 \cdots (1)$$

By integration first term,

$$f(x, y) = x^2 y + g(y)$$

Differentiate f(x,y) to satisfy second term in (1)

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 - 1$$

$$g'(y) = -1$$

$$g(y) = -y + c$$

$$\therefore f(x, y) = x^2 y - y + c = 0$$

$$y = \frac{c}{(1 - x^2)}$$

Exact Equations

☑ Example 2
Solving an Exact DE

Solve

$$(e^{2y} - y\cos xy)dx + (2xe^{2y} - x\cos xy + 2y)dy = 0$$

$$\frac{\partial M}{\partial y} = 2e^{2y} - \cos xy + xy\sin xy = \frac{\partial N}{\partial x}$$

Thus the equation is exact.
So there exist function such that

$$M(x,y) = \frac{\partial f}{\partial x} \text{ and } N(x,y) = \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial y} = 2xe^{2y} - x\cos xy + 2y$$

$$f(x,y) = xe^{2y} - \sin xy + y^2 + h(x)$$

$$\frac{\partial f}{\partial x} = e^{2y} - y\cos xy + h'(x)$$

$$= e^{2y} - y\cos xy$$

$$h'(x) = 0, \quad h(x) = c$$

Hence a family of solution is

$$f(x,y) = xe^{2y} - \sin xy + y^2 + c = 0$$

Exact Equations

☑ Example 3

Initial-Value Problem

Solve IVP

$$\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1-x^2)}, \quad y(0) = 2$$

Rearrange the equation

$$(\cos x \sin x - xy^2)dx + y(1-x^2)dy = 0$$

$$\frac{\partial M}{\partial y} = -2xy = \frac{\partial N}{\partial x}$$

The equation is exact.

Exact equation satisfy

$$M(x,y) = \frac{\partial f}{\partial x} \text{ and } N(x,y) = \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial y} = y(1-x^2)$$

$$f(x,y) = \frac{1}{2}y^2(1-x^2) + h(x)$$

$$\frac{\partial f}{\partial x} = -xy^2 + h'(x)$$

$$= -xy^2 + \cos x \sin x$$

$$h'(x) = \cos x \sin x = \frac{1}{2}\sin 2x$$

$$f(x,y)=\frac{1}{2}y^2(1-x^2)+h(x)$$

$$h'(x)=\cos x\sin x$$

$$h(x)=-\int \cos x(-\sin x)dx=-\frac{1}{2}\cos^2 x$$

$$f(x,y)=\frac{1}{2}y^2(1-x^2)-\frac{1}{2}\cos^2 x+c$$

$$f(0,2)=\frac{1}{2}\cdot 4\cdot 1-\frac{1}{2}\cdot 1+c=0,\ c=-\frac{3}{2}$$

$$f(x,y)=\frac{1}{2}y^2(1-x^2)-\frac{1}{2}\cos^2 x-\frac{3}{2}$$

$$=0$$

$$y^2(1-x^2)-\cos^2 x=3$$

Integrating Factors of Nonexact Equations

☑Integrating factors

$$M(x,y)dx + N(x,y)dy = 0$$

If this is not an exact equation, $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$

Sometimes possible to change it into an exact equation by multiplying $\mu(x)$

$$\mu(x)M(x,y)dx + \mu(x)N(x,y)dy = 0$$

If this is an exact equation,

$$\frac{\partial(\mu(x)M(x,y))}{\partial y} = \frac{\partial(\mu(x)N(x,y))}{\partial x}$$

$$\mu\frac{\partial M}{\partial y} = N\frac{\partial \mu}{\partial x} + \mu\frac{\partial N}{\partial x}$$

$$N\frac{\partial \mu}{\partial x} = \mu\frac{\partial M}{\partial y} - \mu\frac{\partial N}{\partial x}$$

Integrating Factors of Nonexact Equations

☑Integrating factors

$$M(x,y)dx + N(x,y)dy = 0$$

If this is not an exact equation, $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$

Sometimes possible to change it into an exact equation by multiplying $\mu(x)$

$$\mu(x)M(x,y)dx + \mu(x)N(x,y)dy = 0$$

$$N\frac{\partial \mu}{\partial x} = \mu\frac{\partial M}{\partial y} - \mu\frac{\partial N}{\partial x}$$

$$\frac{\partial \mu}{\mu} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\partial x$$

integrating with respect to

$$\ln|\mu| = \int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\partial x$$

$$\therefore \mu(x) = e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx}$$

Integrating factor

If we take $\mu = \mu(y)$

In a similar way

$$\therefore \mu(y) = e^{\int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx}$$

Integrating Factors of Nonexact Equations

☑ Example 4

Nonexact DE Made Exact

Solve

$$xydx + (2x^2 + 3y^2 - 20)dy = 0$$

$$\frac{\partial M}{\partial y} = x \neq \frac{\partial N}{\partial x} = 4x$$

The equation is NOT exact.
So we need integrating factor μ to make the equation exact.

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx} \quad or \quad \mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$$

$$\frac{M_y - N_x}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20}$$

$$= \frac{-3x}{2x^2 + 3y^2 - 20}$$

Depend on x and y

$$\frac{N_x - M_y}{M} = \frac{4x - x}{xy} = \frac{3}{y}$$

Depend only on y

The integrating factor is then

$$\mu(y) = e^{\int (\frac{3}{y})dy} = e^{3\ln y} = e^{\ln y^3} = y^3$$

Multiplying integrating factor

$$xy^4dx+(2x^2y^3+3y^5-20y^3)dy=0$$

$$\frac{\partial M}{\partial y}=4xy^3=\frac{\partial N}{\partial x}$$ **Exact!**

$$\frac{\partial f}{\partial x}=xy^4$$

$$f(x,y)=\frac{1}{2}x^2y^4+g(y)$$

$$\frac{\partial f}{\partial y}=2x^2y^3+g'(y)$$

$$=2x^2y^3+3y^5-20y^3$$

$$g'(y)=3y^5-20y^3$$

$$g(y)=\frac{1}{2}y^6-5y^4+c$$

$$\therefore f(x,y)=\frac{1}{2}x^2y^4+\frac{1}{2}y^6-5y^4+c$$

$$=0$$

Solutions by Substitutions

☑Substitution

$$\frac{dy}{dx} = f(x,y) \Rightarrow y = g(x,u)$$

$$\frac{dy}{dx} = \frac{\partial}{\partial x} g(x,u) + \frac{\partial}{\partial u} g(x,u) \frac{du}{dx}$$

↓ New D.E.

Solving for $\frac{du}{dx}$ → find $\frac{du}{dx} = F(x,u)$ → Find u $u = \phi(x)$

Then, final solution is

$$y = g(x,u) = g(x,\phi(x))$$

Find $\phi(x)$

Solutions by Substitutions

☑ Homogeneous

f is homogeneous function of degree α

If $f(tx,ty) = t^{\alpha} f(x,y)$

$$f(x,y) = x^3 + y^3$$

$$f(tx,ty) = (tx)^3 + (ty)^3 = t^3(x^3 + y^3)$$

$$\therefore f(tx,ty) = t^3 f(x,y)$$

Vs. $f(x,y) = x^3 + y^3 + 1$

Solutions by Substitutions

☑ Homogeneous

$$M(x,y)dx + N(x,y)dy = 0 \quad \cdots(1)$$

Equation is homogeneous if

$$M(tx,ty) = t^{\alpha} M(x,y) \;, N(tx,ty) = t^{\alpha} N(x,y)$$

○ **Same degree**

☑ Substitution

Then, we can rewrite (1) by substituting $y = ux$

$$x^{\alpha} M(1,u)dx + x^{\alpha} N(1,u)dy = 0$$

or $M(1,u)dx + N(1,u)dy = 0 \cdots(2)$

from $y = ux \longrightarrow dy = udx + xdu$

So (2) is

$$M(1,u)dx + N(1,u)[udx + xdu] = 0$$

Solutions by Substitutions

$$M(1,u)dx + N(1,u)[udx + xdu] = 0$$

$$\downarrow$$

$$(M(1,u) + uN(1,u))dx = -xN(1,u)du$$

$$\downarrow$$

Separable variables

$$\frac{N(1,u)}{(M(1,u) + uN(1,u))}du = -\frac{dx}{x}$$

Solving it $u = \phi(x)$

Then, final solution is

$$y = ux = \phi(x)x$$

If we take $x = vy$

In a similar way

$$\frac{M(v,1)}{(vM(v,1) + N(v,1))}dv = -\frac{dy}{y}$$

$$v = \psi(x)$$

$$y = \frac{v}{x} = \frac{\psi(x)}{x}$$

Solutions by Substitutions

☑First-order linear D.E, standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

☑Bernoulli's Equation

$$if \;\; n = 0, n = 1$$

$$\frac{dy}{dx} + P(x)y = f(x)y^n, n : real \;\; number$$

Solution : try $u = y^{1-n}$

☑Reduction to Separation of Variables

$$\frac{dy}{dx} = f(Ax + By + C), \; where \;\; B \neq 0$$

Solution : try $u = Ax + By + C$

Solutions by Substitutions

☑ Example 1

Solving a Homogeneous DE :

Solve

$$(x^2+y^2)dx+(x^2-xy)dy=0$$

$$M(tx,ty)=t^2(x^2+y^2)$$

: 2 - degree homogenous

$$N(tx,ty)=t^2(x^2-xy)$$

: 2 - degree homogenous

So this equation is homogeneous

Substitute $y=ux$

$$dy=udx+xdu$$

$$(x^2+u^2x^2)dx$$

$$+(x^2-ux^2)[udx+xdu]=0$$

$$(x^2+u^2x^2+ux^2-u^2x^2)dx$$

$$+(x^3-ux^3)du=0$$

$$x^2(1+u)dx+x^3(1-u)du=0$$

$$\frac{1-u}{1+u}du+\frac{dx}{x}=0$$

$$[-1+\frac{2}{1+u}]du+\frac{dx}{x}=0$$

After integration

$$-u+2\ln|1+u|+\ln|x|=\ln|c|$$

$$-\frac{y}{x}+2\ln|1+\frac{y}{x}|+\ln|x|=\ln|c|$$

$$\ln\left|\frac{x(1+\frac{y}{x})^2}{c}\right|=\ln\left|\frac{(x+y)^2}{cx}\right|=\frac{y}{x}$$

$$\frac{(x+y)^2}{cx}=e^{\frac{y}{x}},\ (x+y)^2=cxe^{\frac{y}{x}}$$

Solving a Bernoulli DE by Substitutions

☑ Example 2

Solving a Bernoulli DE:

Solve $x\frac{dy}{dx}+y=x^2y^2$

Rewrite the equation as

$$\frac{dy}{dx}+\frac{y}{x}=xy^2$$

Substitute $y=u^{-1}$

$$\frac{dy}{dx}=-u^{-2}\frac{du}{dx}$$

$$-u^{-2}\frac{du}{dx}+\frac{1}{ux}=\frac{x}{u^2}$$

$$\frac{du}{dx}-\frac{u}{x}=-x$$ **Standard form**

Integrating factor

$$e^{-\int(\frac{1}{x})dx}=e^{-\ln x}=e^{\ln x^{-1}}=x^{-1} \text{ on } (0,\infty)$$

Multiply I.F and rewrite

$$\frac{d}{dx}[\frac{u}{x}]=-1$$

$u=y^{-1}$

$$\frac{u}{x}=-x+c,\ u=-x^2+cx$$

$$y=\frac{1}{(-x^2+cx)}$$

Solve IVP by Substitutions

☑ Example 3

An Initial-Value Problem :

Solve IVP

$$\frac{dy}{dx} = (-2x + y)^2 - 7,\ y(0) = 0$$

Let $u = -2x + y,\quad (y = 2x + u)$

then, $du/dx = -2 + dy/dx$

DE transformed into

$$\frac{du}{dx} + 2 = u^2 - 7,\ \frac{du}{dx} = u^2 - 9$$

$$\frac{1}{u^2 - 9}du = dx$$

$$\frac{1}{6}\left[\frac{1}{u+3} - \frac{1}{u-3}\right]du = dx$$

$$\frac{1}{6}\ln\left|\frac{u+3}{u-3}\right| = x + c_1$$

$$\frac{u-3}{u+3} = c \cdot e^{6x}$$

$$u - 3 = ce^{6x}u + 3ce^{6x}$$

$$u = 3\frac{1+ce^{6x}}{1-ce^{6x}},\quad y = 2 + 3\frac{1+ce^{6x}}{1-ce^{6x}}$$

Substitute x=0 and y=0 for initial condition

$$0 = 0 + 3\frac{1+c}{1-c}$$

$$c = -1$$

$$\therefore y = 2 + 3\frac{1-e^{6x}}{1+e^{6x}}$$

2-3. Reduction of Order

Reduction of Order

☑ For what?

■ When we have only a single solution

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \quad \Rightarrow \quad y_1$$

- A second solution can be constructed based on y_1
- Basic idea : the linear second-order equation can be reduced to a linear first-order D.E. by means of a substitution involving the know solution y_1

$$y'' + P(x)y' + Q(x)y = 0$$ Standard form

$y_1(x)$: known

$$y_2(x) = u(x)y_1(x),$$

Reduction of Order

$$y(x) = u(x)y_1(x)$$

$$y' = u'(x)y_1(x) + u(x)y_1'(x)$$

$$y'' = u''(x)y_1(x) + 2u'(x)y_1'(x) + u(x)y_1''(x)$$

$y_1(x)$: known

$$\therefore y_1'' + P(x)y_1' + Q(x)y_1 = 0$$

Put them into the standard form

$$y'' + P(x)y' + Q(x)y = 0$$

L.H.S: Since y_1 is homogeneous solution. $[y_1''(x) + P(x)y_1'(x) + Q(x)y_1(x)] = 0$

$$[u''(x)y_1(x) + 2u'(x)y_1'(x) + u(x)y_1''(x)] + P(x)[u'(x)y_1(x) + u(x)y_1'(x)] + Q(x)[u(x)y_1(x)]$$

$$= u(x)[y_1''(x) + P(x)y_1'(x) + Q(x)y_1(x)] + u'(x)[2y_1'(x) + P(x)y_1(x)] + u''(x)y_1(x)$$

$$= u'(x)[2y_1'(x) + P(x)y_1(x)] + u''(x)y_1(x)$$

R.H.S: 0 $\left[\frac{dy_1}{dx} + P(x)y_1\right] = 0$

$$\therefore u'(x)[2y_1'(x) + P(x)y_1(x)] + u''(x)y_1(x) = 0$$ From this find $u(x)$!

Reduction of Order

$$u'(x)[2y_1'(x)+P(x)y_1(x)]+u''(x)y_1(x)=0$$

Let $w=u'(x)$ **then** $w(x)[2y_1'(x)+P(x)y_1(x)]+w'(x)y_1(x)=0$

divided by $w(x)y_1(x)$ ..... $[2\frac{y_1'(x)}{y_1(x)}+P(x)]+\frac{w'(x)}{w(x)}=0$

$$2\frac{1}{y_1(x)}\frac{dy_1(x)}{dx}+P(x)+\frac{1}{w(x)}\frac{dw(x)}{dx}=0$$

multiplied by dx ..... $2\frac{dy_1}{y_1(x)}+P(x)dx+\frac{dw}{w(x)}=0$

$$2\ln|y_1(x)|+\int P(x)dx+\ln|w(x)|=c$$

$$\ln|y^2{}_1(x)w(x)|=-\int P(x)dx+c$$

$$y^2{}_1(x)w(x)=e^{-\int P(x)dx+c}$$

Reduction of Order

$$u'(x)[2y_1'(x)+P(x)y_1(x)]+u''(x)y_1(x)=0$$

Let $w=u'(x)$ **then** $w(x)[2y_1'(x)+P(x)y_1(x)]+w'(x)y_1(x)=0$

$$y^2{}_1(x)w(x)=e^{-\int P(x)dx+c} \qquad c_1=e^c\neq 0$$

$$w(x)=\frac{c_1}{y^2{}_1(x)}e^{-\int P(x)dx}$$

since $w(x)=u'(x)$ ..... $u'(x)=\frac{c_1}{y^2{}_1(x)}e^{-\int P(x)dx}$

$$u(x)=c_1\int\frac{e^{-\int P(x)dx}}{y^2{}_1(x)}dx+c_2$$

by choosing $c_1=1, c_2=0$ ..... $y_2=u(x)y_1(x)=y_1(x)\int\frac{e^{-\int P(x)dx}}{y^2{}_1(x)}dx$

$c_1=1, c_2=0 \because$ *concept of basis*

Reduction of Order

☑ General solution

$$y'' + P(x)y' + Q(x)y = 0$$

$$y(x) = \underline{ay_1(x) + by_2(x)} = ay_1(x) + bu(x)y_1(x)$$

$$u(x) = c_1 \int \frac{e^{-\int P(x)dx}}{y^2{}_1(x)} dx + c_2$$

$$y(x) = ay_1(x) + b\left(c_1 \int \frac{e^{-\int P(x)dx}}{y^2{}_1(x)} dx + c_2 \right) y_1(x)$$

$$= (a + bc_2) y_1(x) + bc_1 \left(\int \frac{e^{-\int P(x)dx}}{y^2{}_1(x)} dx \right) y_1(x)$$

$$= \underline{a'y_1(x) + b'y_2(x)}$$

Reduction of Order

Recall reduction of order

$$\boxed{m\lambda^2 + c\lambda + k = 0} \quad \text{in case, } c^2 - 4mk = 0$$

$$\lambda_1 = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \quad \Rightarrow \quad z_1 = e^{\lambda_1 t}$$

$$z_2 = u(t)z_1(t) = z_1(t)\int \frac{e^{-\int P(t)dt}}{z_1^2(t)}\,dt$$

$$\boxed{\begin{aligned} &\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0 \\ &\lambda^2 + P(t)\lambda + G(t) = 0 \end{aligned}} \quad \Rightarrow \quad P(t) = \frac{c}{m} = -2\lambda_1$$

Reduction of Order

Recall reduction of order

$$P(t) = \frac{c}{m} = -2\lambda_1 \qquad z_1 = e^{\lambda_1 t}$$

$$\Rightarrow \quad z_2 = z_1(t)\int \frac{e^{-\int P(t)dt}}{z_1^2(t)}\,dt$$

$$= e^{\lambda_1 t}\int \frac{e^{-\int(-2\lambda_1)dt}}{e^{2\lambda_1 t}}\,dt = e^{\lambda_1 t}\int \frac{e^{2\lambda_1 t}}{e^{2\lambda_1 t}}\,dt = e^{\lambda_1 t}\int dt$$

$$= te^{\lambda_1 t}$$

2-4. Method of undetermined coefficients

Method of Undetermined Coefficients

☑Two things to solve a nonhomogeneous differential equation

$$a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 y' + a_0 y = g(x),\ a_k = const, k = 1,2,..,n,\ \ g(x) \neq 0$$

- **Find the complementary function** y_c
  - **is general solution of associated homogeneous differential equation** $a_n(x) y^n + a_{n-1}(x) y^{n-1} + \cdots + a_1(x) y' + a_0(x) y = 0$
- **Find any particular solution**
  - **Method of undetermined coefficients** y_p
    - **motivated by the kinds of functions that make up the input function** $g(x)$
    - **applicable when** $g(x)$ **is a constant, a polynomial function, exponential function, sine or cosine function or finite sums and products of these functions.**

$$what\ about\ g(x) = \tan x ?$$

Particular Solution Using Undetermined Coefficients

☑ Method of undetermined coefficients

■ By Superposition

- Step 1 : find the solution of the associated homogeneous equation
- Step 2 : superposition of nonhomogeneous equations
- Step 3 : general solution

Particular Solution Using Undetermined Coefficients

☑ Example 1

General Solution Using Undetermined Coefficients

Solve

$$y'' + 4y' - 2y = 2x^2 - 3x + 6$$

 # Step 1

solve homogeneous equation

$$y'' + 4y' - 2y = 0$$

Substitute $y = e^{mx}$

$$m^2 + 4m - 2 = 0$$

$$m_1 = -2 - \sqrt{6}, \quad m_2 = -2 + \sqrt{6}$$

$$y_c = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x}$$

 # Step 2

Assume Particular solution

$$y_P = Ax^2 + Bx + C$$

Then, $y_P' = 2Ax + B \qquad y_P'' = 2A$

$$y_P'' + 4y_p' - 2y_p$$

$$= 2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C$$

$$= -2Ax^2 + (8A - 2B)x + 2A + 4B - 2C$$

Particular Solution Using Undetermined Coefficients

Step 2
Assume Particular solution

$$y_P = Ax^2 + Bx + C$$

$$-2Ax^2 + (8A - 2B)x + 2A + 4B - 2C$$
$$= 2x^2 - 3x + 6$$

$$2A = 2, \quad 8A - 2B = -3,$$
$$2A + 4B - 2C = 6$$

$$A = 1, \quad B = \frac{11}{2}, \quad C = 9$$

$$y_P = x^2 + \frac{11}{2}x + 9$$

Step 3
General solution

$$y = y_c + y_p$$

$$= c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x} + x^2 + \frac{11}{2}x + 9$$

☑ Example 2

Find particular solution of

$$y'' - y' + y = 2\sin 3x$$

Assume Particular solution

$$y_P = A\cos 3x + B\sin 3x$$

Then,

$$y_P' = -3A\sin 3x + 3B\cos 3x$$

$$y_P'' = -9A\cos 3x - 9B\sin 3x$$

$$y_p'' - y_p' + y_p$$

$$= -9Ac - 9Bs + 3As - 3Bc + Ac + Bs$$

$$= (3A - 8B)\sin 3x$$

$$-(8A + 3B)\cos 3x$$

$$= 2\sin 3x + 0\cdot\cos 3x$$

$$3A - 8B = 2$$

$$8A + 3B = 0$$

$$A = \frac{6}{73}, \quad B = -\frac{16}{73}$$

$$\therefore y_P = \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x$$

☑ Example 3
Forming y_p by Superposition

Solve

$$y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$$

 # Step 1
solve homogeneous equation

$$y'' - 2y' - 3y = 0$$

Substitute $y = e^{mx}$

$$m^2 - 2m - 3 = (m-3)(m+1) = 0$$

$$m_1 = -1, \; m_2 = 3$$

$$y_c = c_1 e^{-x} + c_2 e^{3x}$$

 # Step 2
Assume that

$$g(x) = g_1(x) + g_2(x)$$

$$= polynomial + exponetial$$

Correspondingly, $y_p = y_{p_1} + y_{p_2}$

where, $y_{p_1} = Ax + B$

$$y_{P_2} = Cxe^{2x} + Ee^{2x}$$

Step 2

$$y_p = y_{p_1} + y_{p_2}$$
$$= Ax + B + Cxe^{2x} + Ee^{2x}$$

$$y_p' = A + Ce^{2x} + 2Cxe^{2x} + 2Ee^{2x}$$

$$y_p'' = 2Ce^{2x} + 2Ce^{2x} + 4Cxe^{2x} + 4Ee^{2x}$$

$$y_p'' - 2y_p' - 3y_p$$
$$= 2Ce^{2x} + 2Ce^{2x} + 4Cxe^{2x} + 4Ee^{2x}$$
$$- 2A - 2Ce^{2x} - 4Cxe^{2x} - 4Ee^{2x}$$
$$- 3Ax - 3B - 3Cxe^{2x} - 3Ee^{2x}$$
$$= -3Ax - (3B + 2A)$$
$$+ (2C - 3E)e^{2x} - 3Cxe^{2x}$$
$$= 4x - 5 + 0 + 6xe^{2x}$$

$$= 4x - 5 + 0 + 6xe^{2x}$$

$$y_p'' - 2y_p' - 3y_p$$

$$= -3Ax - (3B + 2A)$$

$$+ (2C - 3E)e^{2x} - 3Cxe^{2x}$$

$$= 4x - 5 + 0 + 6xe^{2x}$$

$$-3A = 4, \quad 3B + 2A = 5,$$

$$-3C = 6, \quad 2C - 3E = 0$$

$$A = -\frac{4}{3}, \; B = \frac{23}{9}, \; C = -2, \; E = -\frac{4}{3}$$

$$\therefore y_p = -\frac{4}{3}x + \frac{23}{9} - 2xe^{2x} - \frac{4}{3}e^{2x}$$

Step 3
General solution

$$y_c = c_1 e^{-x} + c_2 e^{3x}$$

$$y_p = -\frac{4}{3}x + \frac{23}{9} - 2xe^{2x} - \frac{4}{3}e^{2x}$$

$$y = y_c + y_p$$

$$= c_1 e^{-x} + c_2 e^{3x} - \frac{4}{3}x + \frac{23}{9} - 2xe^{2x} - \frac{4}{3}e^{2x}$$

※ We can solve this example by solving two simpler problem

$$\begin{cases} y'' - 2y' - 3y = 4x - 5 \\ y'' - 2y' - 3y = 6xe^{2x} \end{cases}$$

$$y_{p_1} = -\frac{4}{3}x + \frac{23}{9}, \quad y_{p_2} = -2xe^{2x} - \frac{4}{3}e^{2x}$$

Are particular solution of each DE

☑ Example 4
Glitch in the Method

Find a particular solution of

$$y'' - 5y' + 4y = 8e^x$$

If we assume particular solution

$$y_p = Ae^x$$

$$y_p'' - 5y_p' + 4y_p$$

$$= (A - 5A + 4A)e^x = 0 \cdot e^x = 0$$

$$0 = 8e^x \;\; ??$$

Let assume particular solution as

$$y_p = Axe^x$$

then,

$$y_p' = Ae^x + Axe^x$$

$$y_p'' = 2Ae^x + Axe^x$$

$$y_p'' - 5y_p' + 4y_p$$

$$= (2A + Ax - 5A - 5Ax + 4Ax)e^x$$

$$= -3Ae^x = 8e^x$$

$$A = -\frac{8}{3} \qquad y_p = -\frac{8}{3}xe^x$$

Undetermined Coefficients

☑ Method of undetermined coefficients

■ Glitch in the method

- Case I : No function in the assumed particular solution is a solution of the associated homogeneous differential equation

- Case II : A function in the assumed particular solution is also a solution of the associated homogeneous differential equation

Undetermined Coefficients

☑ Method of undetermined coefficients

■ Glitch in the method

- **Case I : No function in the assumed particular solution is a solution of the associated homogeneous differential equation**

$g(x)$	Form of y_p
$1(any\ const.)$	A
$5x+7$	$Ax+B$
$3x^2-2$	Ax^2+Bx+C
x^3-x+1	Ax^3+Bx^2+Cx+E
$\sin 4x$	$A\cos 4x+B\sin 4x$
$\cos 4x$	$A\cos 4x+B\sin 4x$

Undetermined Coefficients

☑ Method of undetermined coefficients

■ Glitch in the method

- **Case I : No function in the assumed particular solution is a solution of the associated homogeneous differential equation**

$g(x)$	Form of y_p
e^{5x}	Ae^{5x}
$(9x-2)e^{5x}$	$(Ax+B)e^{5x}$
x^2e^{5x}	$(Ax^2+Bx+C)e^{5x}$
$e^{3x}\sin 4x$	$Ae^{3x}\cos 4x+Be^{3x}\sin 4x$
$5x^2\sin 4x$	$(Ax^2+Bx+C)\cos 4x+(Ex^2+Fx+G)\sin 4x$
$xe^{3x}\cos 4x$	$(Ax+B)e^{3x}\cos 4x+(Cx+E)e^{3x}\sin 4x$

Undetermined Coefficients

☑ Method of undetermined coefficients

■ Glitch in the method

- **Case II : A function in the assumed particular solution is also a solution of the associated homogeneous differential equation**

Multiplication Rule : if any y_p **contains terms that duplicate terms in** y_c **then that** y_{pi} **must multiplied by** x^n **, where *n* is the smallest positive integer that eliminates that duplication.**

☑ Example 5

Forms of Particular Solutions-Case I

Determine the form of a particular solution of

(a) $y'' - 8y' + 25y = 5x^3 e^{-x} - 7e^{-x}$

(b) $y'' + 4y = x\cos x$

(a) $g(x) = (5x^3 - 7)e^{-x}$

So, assume a particular solution

$$y_p = (Ax^3 + Bx^2 + Cx + E)e^{-x}$$

It is not duplicated with y_c

$$y_c = e^{4x}(c_1 \cos 3x + c_2 \sin 3x)$$

(b) $g(x) = x\cos x$

$$y_p = (Ax + B)\cos x + (Cx + E)\sin x$$

It is not duplicated with y_c

$$y_c = c_1 \cos 2x + c_2 \sin 2x$$

Forming Particular Solution by Superposition

☑ Example 6

Forming y_p by Superposition-Case I

Determine the form of a particular solution of

$$y'' - 9y' + 14y = 3x^2 - 5\sin 2x + 7xe^{6x}$$

 # Corresponding to $3x^2$ we assume

$$y_{p_1} = Ax^2 + Bx + C$$

 # Corresponding to $-5\sin 2x$ we assume

$$y_{p_2} = E\cos 2x + F\sin 2x$$

 # Corresponding to $7xe^{6x}$ we assume

$$y_{p_3} = (Gx + H)e^{6x}$$

$$y = y_{p_1} + y_{p_2} + y_{p_3}$$

$$= Ax^2 + Bx + C + E\cos 2x + F\sin 2x + (Gx + H)e^{6x}$$

☑ Example 7

Particular Solution-Case II

Find a particular solution of

$$y'' - 2y' + y = e^x$$

Complementary function is

$$y_c = c_1 e^x + c_2 x e^x$$

First, Let's assume particular solution as

$$y_p = Ae^x$$

it is associate with y_c

then, Let's assume particular solution as

$$y_p = Axe^x$$

it is also associate with y_c

So, Let's assume particular solution as

$$y_p = Ax^2 e^x$$

it is ***not*** associate with y_c

$$y_p' = 2Axe^x + Ax^2 e^x$$

$$y_p'' = 2Ae^x + 4Axe^x + Ax^2 e^x$$

$$y_p'' - 2y_p' + y_p$$

$$= 2Ae^x + 4Axe^x + Ax^2e^x$$

$$\quad - 4Axe^x - 2Ax^2e^x + Ax^2e^x$$

$$= 2Ae^x = e^x$$

$$2A = 1,\ A = \frac{1}{2}$$

$$\therefore y_p = \frac{1}{2}x^2e^x$$

☑ Example 8
An Initial-Value Problem

Solve IVP

$$y'' + y = 4x + 10\sin x,$$
$$y(\pi) = 0, \quad y'(\pi) = 2$$

Complementary function is

$$y_c = c_1 \cos x + c_2 \sin x$$

$$g(x) = 4x + 10\sin x$$

Let's assume particular solution as

$$y_p = y_{p_1} + y_{p_2}$$
$$= polynomial + trigonometric$$

$$y_{p_1} = Ax + B$$

$$y_{p_2} = C\cos x + E\sin x$$

y_{p_2} **is associated with y_c so**

$$\text{Let } y_{p_2} = Cx\cos x + Ex\sin x$$

$$\therefore y_p = Ax + B + Cx\cos x + Ex\sin x$$

Assume particular solution as

$$y_p = Ax + B + Cx\cos x + Ex\sin x$$

$$y_p' = A + C\cos x - Cx\sin x + E\sin x + Ex\cos x$$

$$y_p'' = -2C\sin x - Cx\cos x + 2E\cos x - Ex\sin x$$

$$y_p'' + y_p = Ax + B - 2C\sin x + 2E\cos x$$

$$= 4x + 10\sin x$$

$$A = 4,\ B = 0,\ C = -5,\ E = 0$$

$$\therefore y_p = 4x - 5x\cos x$$

General solution is

$$y = c_1 \cos x + c_2 \sin x + 4x - 5x \cos x$$

To satisfy initial condition

$$y(\pi) = 0, \quad y'(\pi) = 2$$

$$0 = c_1 \cos \pi + c_2 \sin \pi + 4\pi - 5\pi \cos \pi$$

$$= c_1 - \pi$$

$$c_1 = \pi$$

$$y' = -c_1 \sin x + c_2 \cos x + 4 - 5\cos x + 5x \sin x$$

$$2 = -c_1 \sin \pi + c_2 \cos \pi + 4 - 5\cos \pi + 5\pi \sin \pi$$

$$= c_2 - 1$$

$$c_2 = 3$$

$$\therefore y = \pi \cos x + 3\sin x + 4x - 5x \cos x$$

☑ Example 9
Using the Multiplication Rule

Solve

$$y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$$

$$y_c = c_1 e^{3x} + c_2 x e^{3x}$$

y_{p_2} **is associated with y_c, So multiply x^2 (because multiple of x also associated with y_c) in that solution**

$$y_p = Ax^2 + Bx + C + Ex^2 e^{3x}$$

particular solution assumed as

$$y_p = \underbrace{Ax^2 + Bx + C}_{y_{p1}} + \underbrace{Ee^{3x}}_{y_{p2}}$$

$$y_p' = 2Ax + B + 2Exe^{3x} + 3Ex^2 e^{3x}$$

$$y_p'' = 2A + 2Ee^{3x} + 12Exe^{3x} + 9Ex^2 e^{3x}$$

$$y_p'' - 6y_p' + 9y_p$$

$$= 9Ax^2 + (-12A + 9B)x + 2A - 6B$$

$$+ 9C + 2Ee^{3x}$$

$$= 6x^2 + 2 - 12e^{3x}$$

$$A = \frac{2}{3},\ B = \frac{8}{9},\ C = \frac{2}{3},\ E = -6$$

$$\therefore y_p = \frac{2}{3}x^2 + \frac{8}{9}x + \frac{2}{3} - 6x^2e^{3x}$$

$$y = y_c + y_p$$

$$= c_1e^{3x} + c_2xe^{3x}$$

$$+ \frac{2}{3}x^2 + \frac{8}{9}x + \frac{2}{3} - 6x^2e^{3x}$$

☑ Example 10

Third-Order DE- Case I :

Solve

$$y''' + y'' = e^x \cos x$$

Substitute $y = e^{mx}$

$$m^3 + m^2 = m^2(m+1) = 0$$

$$m_1 = m_2 = 0 \ and \ m_3 = -1$$

$$y_c = c_1 + c_2 x + c_3 e^{-x}$$

Based on entry 10 in table 3.1*

$$y_p = Ae^x \cos x + Be^x \sin x$$

It is not duplicate with y_c

$$y_p' = (A+B)e^x \cos x + (B-A)e^x \sin x$$

$$y_p'' = (2B)e^x \cos x - 2Ae^x \sin x$$

$$y_p''' = 2(-A+B)e^x \cos x - 2(A+B)e^x \sin x$$

$$y_p''' + y_p''$$

$$= (-2A+3B)e^x \cos x - (4A+2B)e^x \sin x$$

$$= e^x \cos x$$

*Dennis G. Zill, Michael R. Cullen, Advanced Engineering Mathematics 3rd Edition, Johns and Bartlett, 2006, p130

$$\begin{cases} -2A+3B=1 \\ 4A+2B=0 \end{cases}$$

$$A=-\frac{1}{10},\ B=\frac{1}{5}$$

$$\therefore y_p=-\frac{1}{10}e^x\cos x+\frac{1}{5}e^x\sin x$$

$$y=y_c+y_p=c_1+c_2x+c_3e^{-x}$$
$$-\frac{1}{10}e^x\cos x+\frac{1}{5}e^x\sin x$$

☑ Example 11
Fourth-Order DE- Case II

Determine the form of particular solution

$$y^{(4)} + y''' = 1 - x^2 e^{-x}$$

Substitute $y = e^{mx}$

$$m^4 + m^3 = m^3(m+1)$$

$$m_1 = m_2 = m_3 = 0 \text{ and } m_4 = -1$$

$$y_c = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x}$$

$$y_p = \underbrace{A}_{y_{p1}} + \underbrace{Bx^2 e^{-x} + Cxe^{-x} + Ee^{-x}}_{y_{p2}}$$

We see that the duplications between y_c **and** y_p **are eliminated when** y_{p1} **is multiplied by** x^3 **and** y_{p2} **is multiplied by** x **.**

$$y_p = Ax^3 + Bx^3 e^{-x} + Cx^2 e^{-x} + Exe^{-x}$$

2-5. Method of Variation of Parameters

Variation of Parameters

C.f.)Variation of parameters of first-order differential equation

C.f.)Reduction of order second order homogeneous

☑Method of Variation of Parameters

- Has a distinct advantage in that it *always* yields a particular solution y_p provided the associated homogeneous equation can be solved.

- In addition, unlike undetermined coefficients, is *not* limited to cases where the input function is a combination of four types of functions, nor is it limited to differential equations with constant coefficients.

Variation of Parameters- Second-order ODE

☑A linear second-order differential equation in standard form

$$y'' + P(x)y' + Q(x) = f(x)$$

$$y_p'' + P(x)y_p' + Q(x) = f(x)$$ **Find particular solution** $y_p(x)$

Substitution

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

Where $y_1(x)$ **and** $y_2(x)$ **form a fundamental set of solutions on I of the associated homogeneous form**

$$y_p' = u_1'(x)y_1(x) + u_1(x)y_1'(x) + u_2'(x)y_2(x) + u_2(x)y_2'(x)$$

$$y_p'' = u_1''(x)y_1(x) + 2u_1'(x)y_1'(x) + u_1(x)y_1''(x)$$
$$+ u_2''(x)y_2(x) + 2u_2'(x)y_2'(x) + u_2(x)y_2''(x)$$

☑A linear second-order differential equation in standard form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

$$y_p'' + P(x)y_p' + Q(x) = f(x)$$

Although there are several methods that can be tried to find particular solutions of a linear ODE, in this method of variation of parameters we seek to find a particular solution

L.H.S.:

$$[u_1''(x)y_1(x) + 2u_1'(x)y_1'(x) + u_1(x)y_1''(x) + u_2''(x)y_2(x) + 2u_2'(x)y_2'(x) + u_2(x)y_2''(x)] +$$
$$P(x)[u_1'(x)y_1(x) + u_1(x)y_1'(x) + u_2'(x)y_2(x) + u_2(x)y_2'(x)] +$$
$$Q(x)$$

$$= u_1(x)[y_1''(x) + P(x)y_1'(x) + Q(x)y_1(x)] + u_1'(x)[2y_1'(x) + P(x)y_1(x)] + u_1''(x)y_1(x) +$$
$$u_2(x)[y_2''(x) + P(x)y_2'(x) + Q(x)y_2(x)] + u_2'(x)[2y_2'(x) + P(x)y_2(x)] + u_2''(x)y_2(x)$$

$$= u_1'(x)[2y_1'(x) + P(x)y_1(x)] + y_1(x)u_1''(x) +$$
$$u_2'(x)[2y_2'(x) + P(x)y_2(x)] + y_2(x)u_2''(x)$$

☑A linear second-order differential equation in standard form

$$y_p'' + P(x)y_p' + Q(x) = f(x) \qquad y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

L.H.S.:

$$= u_1'(x)[2y_1'(x) + P(x)y_1(x)] + y_1(x)u_1''(x) +$$
$$u_2'(x)[2y_2'(x) + P(x)y_2(x)] + y_2(x)u_2''(x)$$

$$= y_1'(x)u_1'(x) + y_1'(x)u_1'(x) + P(x)y_1(x)u_1'(x) + y_1(x)u_1''(x) +$$
$$y_2'(x)u_2'(x) + y_2'(x)u_2'(x) + P(x)y_2(x)u_2'(x) + y_2(x)u_2''(x)$$

$$= \frac{d}{dx}[y_1(x)u_1'(x)] + y_1'(x)u_1'(x) + P(x)u_1'(x)y_1(x) +$$
$$\frac{d}{dx}[y_2(x)u_2'(x)] + y_2'(x)u_2'(x) + P(x)u_2'(x)y_2(x)$$

☑A linear second-order differential equation in standard form

$$y_p'' + P(x)y_p' + Q(x) = f(x) \qquad y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

L.H.S.:

$$= \frac{d}{dx}[y_1(x)u_1'(x)] + u_1'(x)y_1'(x) + P(x)u_1'(x)y_1(x) +$$

$$\frac{d}{dx}[y_2(x)u_2'(x)] + u_2'(x)y_2'(x) + P(x)u_2'(x)y_2(x)$$

$$= \frac{d}{dx}[y_1(x)u_1'(x)] + \frac{d}{dx}[y_2(x)u_2'(x)] + P(x)[y_1(x)u_1'(x) + y_2(x)u_2'(x)] + y_1'(x)u_1'(x) + y_2'(x)u_2'(x)$$

$$= \frac{d}{dx}[y_1(x)u_1'(x) + y_2(x)u_2'(x)] + P(x)[y_1(x)u_1'(x) + y_2(x)u_2'(x)] + y_1'(x)u_1'(x) + y_2'(x)u_2'(x)$$

R.H.S.: $f(x)$

☑A linear second-order differential equation in standard form

$$y_p'' + P(x)y_p' + Q(x) = f(x) \qquad y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

L.H.S.:

$$\frac{d}{dx}[y_1(x)u_1'(x) + y_2(x)u_2'(x)] + P(x)[y_1(x)u_1'(x) + y_2(x)u_2'(x)] + y_1'(x)u_1'(x) + y_2'(x)u_2'(x)$$

R.H.S.: $f(x)$

Further assumption 0

$$\therefore \frac{d}{dx}[\underline{y_1(x)u_1'(x) + y_2(x)u_2'(x)}] + P(x)[\underline{y_1(x)u_1'(x) + y_2(x)u_2'(x)}] + y_1'(x)u_1'(x) + y_2'(x)u_2'(x) = f(x)$$

($\underline{\quad}$ → 0)

Then,

$$y_1(x)u_1'(x) + y_2(x)u_2'(x) = 0$$
$$y_1'(x)u_1'(x) + y_2'(x)u_2'(x) = f(x)$$

☑A linear second-order differential equation in standard form

$$y_p'' + P(x)y_p' + Q(x) = f(x) \qquad y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

Then,

$$\begin{aligned} y_1(x)u_1'(x) + y_2(x)u_2'(x) &= 0 \\ y_1'(x)u_1'(x) + y_2'(x)u_2'(x) &= f(x) \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} \begin{bmatrix} u_1'(x) \\ u_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

$$\therefore \begin{bmatrix} u_1'(x) \\ u_2'(x) \end{bmatrix} = \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

Complementary function is

$$y_c = c_1 e^{2x} + c_2 x e^{2x}$$

Particular solution is

$$y_p = (-\frac{1}{3}x^2 - \frac{1}{2}x^2)e^{2x} + (\frac{1}{2}x^2 + x)xe^{2x}$$

$$= (\frac{1}{6}x^3 + \frac{1}{2}x^2)e^{2x}$$

then general Solution is

$$y = y_c + y_p$$

$$= c_1 e^{2x} + c_2 x e^{2x} (\frac{1}{6}x^3 + \frac{1}{2}x^2)e^{2x}$$

General Solution Using Variation of Parameters

☑ Example 2

General Solution Using Variation of Parameters

Solve

$$y'' + 9y = \frac{1}{4}\csc 3x$$

$$m^2 + 9 = 0$$

$$m_1 = 3i, m_2 = -3i$$

$$y_c = c_1 \cos 3x + c_2 \sin 3x$$

Assume

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

$$W(\cos 3x, \sin 3x)$$

$$= \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 9$$

$$W_1 = \begin{vmatrix} 0 & \sin 3x \\ \frac{1}{4}\csc 3x & 3\cos 3x \end{vmatrix} = -1$$

$$W_2 = \begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & \frac{1}{4}\csc 3x \end{vmatrix} = \frac{1}{4}\frac{\cos 3x}{\sin 3x}$$

$$y_c = c_1 \cos 3x + c_2 \sin 3x$$

Assume

$$y_p = u_1(x) y_1(x) + u_2(x) y_2(x)$$

$$u_1' = \frac{-1/4}{3} = -\frac{1}{12}$$

$$u_2' = \frac{1}{12}\frac{\cos 3x}{\sin 3x}$$

$$u_1 = -\frac{1}{12} x$$

$$u_2 = \frac{1}{36} \ln|\sin 3x|$$

$$y_p = -\frac{1}{12} x \cos 3x + \frac{1}{36} (\sin 3x) \ln|\sin 3x|$$

Complementary function is

$$y_c = c_1 \cos 3x + c_2 \sin 3x$$

Particular solution is

$$y_p = -\frac{1}{12} x \cos 3x + \frac{1}{36} (\sin 3x) \ln |\sin 3x|$$

then general Solution is

$$y = y_c + y_p$$

$$= c_1 e^{2x} + c_2 x e^{2x} (\frac{1}{6} x^3 + \frac{1}{2} x^2) e^{2x}$$

$$-\frac{1}{12} x \cos 3x + \frac{1}{36} (\sin 3x) \ln |\sin 3x|$$

General Solution Using Variation of Parameters

☑ Example 3

General Solution Using Variation of Parameters

Solve

$$y'' - y = \frac{1}{x}$$

$$m^2 - 1 = 0$$

$$m_1 = 1, m_2 = -1$$

$$y_c = c_1 e^x + c_2 e^{-x}$$

Assume

$$y_p = u_1(x) y_1(x) + u_2(x) y_2(x)$$

$$W(e^x, e^{-x}) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$$

$$W_1 = \begin{vmatrix} 0 & e^{-x} \\ \frac{1}{x} & -e^{-x} \end{vmatrix} = -\frac{e^{-x}}{x} = -\frac{1}{xe^x}$$

$$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & \frac{1}{x} \end{vmatrix} = \frac{e^x}{x}$$

$$y_c = c_1 e^x + c_2 e^{-x}$$

Assume

$$y_p = u_1(x) y_1(x) + u_2(x) y_2(x)$$

$$u_1' = \frac{-1/(xe^x)}{-2} = \frac{e^{-x}}{2x}$$

$$u_2' = \frac{1/(xe^{-x})}{-2} = -\frac{e^x}{2x}$$

$$u_1 = \frac{1}{2}\int \frac{e^t}{t} dt$$

$$u_2 = -\frac{1}{2}\int \frac{e^{-t}}{t} dt$$

$$y_p = \frac{1}{2} e^x \int \frac{e^t}{t} dt - \frac{1}{2} e^{-x} \int \frac{e^{-t}}{t} dt$$

Complementary function is

$$y_c = c_1 e^x + c_2 e^{-x}$$

Particular solution is

$$y_p = \frac{1}{2} e^x \int \frac{e^t}{t} dt - \frac{1}{2} e^{-x} \int \frac{e^{-t}}{t} dt$$

then general Solution is

$$\begin{aligned} y &= y_c + y_p \\ &= c_1 e^x + c_2 e^{-x} \\ &\quad - \frac{1}{2} e^x \int \frac{e^t}{t} dt - \frac{1}{2} e^{-x} \int \frac{e^{-t}}{t} dt \end{aligned}$$

2-6. NONLINEAR EQUATIONS

Nonlinear Equations

☑Some Differences

- **Nonlinear Equation** ***do not have superposition property***

  Even though y_1 and y_2 is a solution of nonlinear DE,

  $y = c_1 y_1 + c_2 y_2$ Is not a solution of nonlinear DEc

- **Higher order nonlinear ODE is *difficult to solve analytically***

- ***Special*** **kind of nonlinear 2nd order ODE can be solved analytically by following method**

Solution of Nonlinear Equations by Reduction of Order

☑Reduction of Order

$$F(x, y', y'') = 0$$

Dependent variable y is missing

$$ex)\ \ y'' = 2x(y')^2$$

$$F(y, y', y'') = 0$$

independent variable x is missing

$$ex)\ \ yy'' = (y')^2$$

When substitute $u = y'$

$$F(x, u, u') = 0$$

$$u' = 2xu^2$$

$$F(y, u, u') = 0$$

$$yu' = (u)^2$$

Changed as 1st order DE and can be Solved by separable variables

Solution of Nonlinear Equations by Reduction of Order

☑ Example 1

Dependent Variable y is missing

Solve

$$y'' = 2x(y')^2$$

Let $y' = u$ then, $du/dx = y''$

$$du/dx = 2xu^2$$

$$\frac{1}{u^2}du = 2xdx$$

$$-\frac{1}{u} = x^2 + c$$

For convenience, let

$$-\frac{1}{u} = x^2 + c_1^2$$

$$\frac{dy}{dx} = -\frac{1}{x^2 + c_1^2}$$

$$y = -\int \frac{1}{x^2 + c_1^2}dx$$

$$= -\frac{1}{c_1}\tan^{-1}\frac{x}{c_1} + c_2$$

Solution of Nonlinear Equations by Reduction of Order

☑ Example 2

Independent Variable x is missing

Solve

$$yy'' = (y')^2$$

Let $y' = u$ then, $du/dx = y''$

$$y'' = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = \frac{du}{dy}y' = \frac{du}{dy}u$$

$$yu\frac{du}{dy} = u^2$$

$$\frac{1}{u}du = \frac{1}{y}dy$$

$$\ln|u| = \ln|y| + c_1$$

$$u = c_2 y$$

$$\frac{dy}{y} = c_2 dx$$

$$\ln|y| = c_2 x + c_3$$

$$y = c_4 e^{c_2 x}$$

Solution of Nonlinear Equations by Taylor Series

☑Use of Taylor Series

$$y'' = F(x, y, y') \qquad y(x_0) = y_0, \quad y'(x_0) = y_1$$

Taylor Expansion centered at x_0

$$y(x) = y(x_0) + \frac{y'(x_0)}{1!}x + \frac{y''(x_0)}{2!}x^2 + \frac{y'''(x_0)}{3!}x^3 + \frac{y^{(4)}(x_0)}{4!}x^4 + \cdots$$

After substitute $y(x_0) = y_0, \quad y'(x_0) = y_1$ **We could get** $y''(x_0)$

Higher derivatives are calculated by successive derivatives of DE

$$y'''(x) = \frac{d}{dx}(y'') = \frac{d}{dx}(F(x, y, y'))$$

$$y^{(4)}(x) = \frac{d}{dx}(y''')$$

$\cdots$

Substitute $x = x_0$

$$y'''(x_0)$$

$$y^{(4)}(x_0)$$

$\cdots$

Solution of Nonlinear Equations by Taylor Series

☑ Example 3

Taylor Series Solution of an IVP

Solve

$$y'' = x + y - y^2$$

$$y(0) = -1, \quad y'(0) = 1$$

Assume that solution exist

$$y'''(x) = \frac{d}{dx}(x + y - y^2) = 1 + y' - 2yy'$$

$$y^{(4)}(x) = \frac{d}{dx}(1 + y' - 2yy')$$

$$= y'' - 2(y')^2 - 2yy''$$

$$y^{(5)}(x) = \frac{d}{dx}(y'' - 2(y')^2 - 2yy'')$$

$$= y^{(4)} - 6y'y'' - 2yy'''$$

$$y(0) = -1,\ y'(0) = 1,\ y'''(0) = 4$$

Taylor series expansion centered at 0 :

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots$$

$$y''(0) = -2,\ y^{(4)}(0) = -8$$

$$y^{(5)}(0) = 24 \quad \textbf{(Near 0 !)}$$

$$\therefore y(x) = -1 + x - x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + \frac{1}{5}x^5 \cdots$$

Solution of Nonlinear Equations by a Numerical Method

☑approximating the solution

$$y' = 0.1\sqrt{y} + 0.4x^2, \; y(2) = 4$$

Nonlinear D.E. → ~~X~~ → Analytically

- Qualitative
  - Direction Field
- Numerically
  - Ex) Euler's Method

$$y' = f(x, y), \; y(x_0) = y_0$$

Linearization of the unknown solution

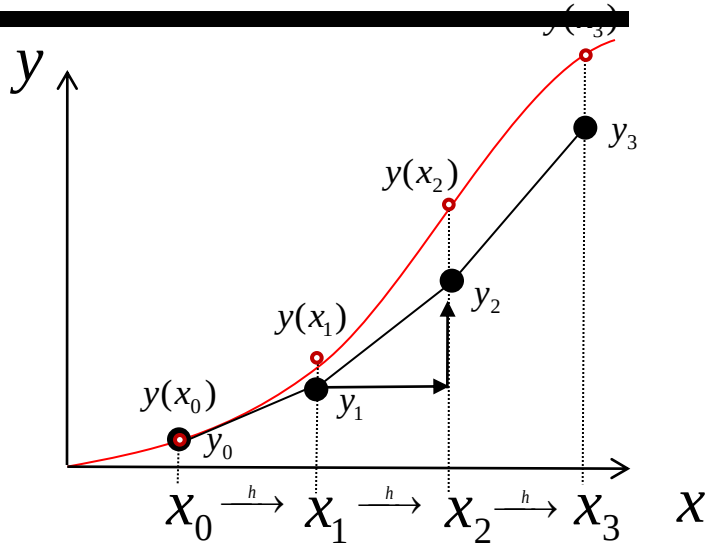
$$y(x_{n+1}) \approx y_{n+1} = L(x_{n+1})$$
$$= f(x_n, y_n)(x_{n+1} - x_n) + y_n$$

Equation of the tangent line at x

Solution of Nonlinear Equations by Euler's Method

☑ Euler's method

$$y' = f(x, y),\ y(x_0) = y_0$$



Linearization of the unknown solution

$$y_{n+1} = L(x_{n+1}) = f(x_n, y_n)(x_{n+1} - x_n) + y_n$$

$x_1 = x_0 + h$

$x_2 = x_1 + h$

$x_3 = x_2 + h$

$$y(x_0) = y_0$$

$$y(x_1) \approx y_1 = L(x_1) = f(x_0, y_0)(x_1 - x_0) + y_0 = y_0 + hf(x_0, y_0)$$

$$y(x_2) \approx y_2 = L(x_2) = f(x_1, y_1)(x_2 - x_1) + y_1 = y_1 + hf(x_1, y_1)$$

$$y(x_3) \approx y_3 = L(x_3) = f(x_2, y_2)(x_3 - x_2) + y_2 = y_2 + hf(x_2, y_2)$$

$\vdots$

Exact solution Approximation

Slope given at each point

Solution of Nonlinear Equations by Euler's Method

☑ Example 1

Euler's Method

Solve IVP

$$y' = 0.1\sqrt{y} + 0.4x^2, \quad y(2) = 4$$

Use Euler's Method using h=0.1 and h=0.05

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$y_{n+1} = y_n + h(0.1\sqrt{y_n} + 0.4x_n{}^2)$$

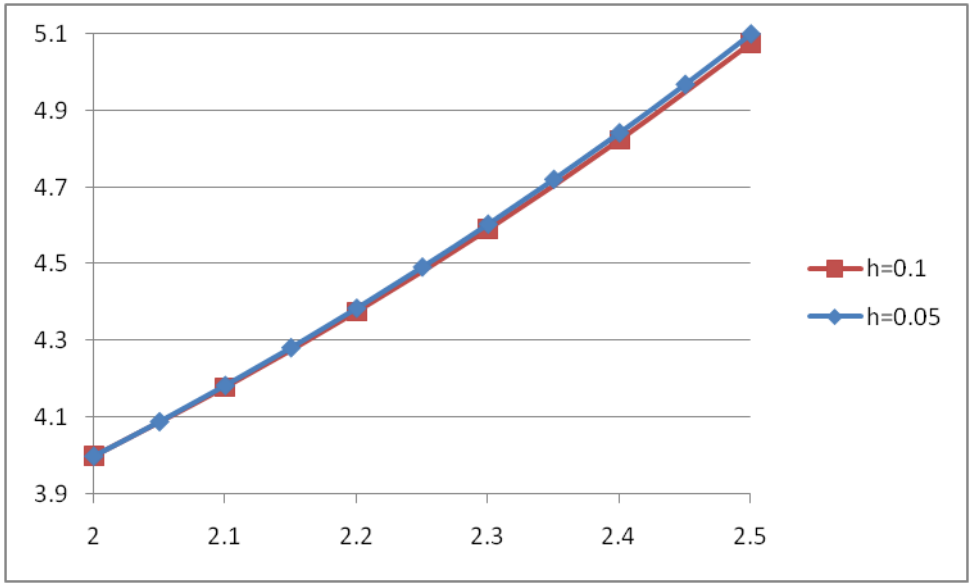
For h=0.1

n	x	y	f(x,y)
0	2	4	1.8
1	2.1	4.18	1.96845
2	2.2	4.376845	2.145209
3	2.3	4.591366	2.330275
4	2.4	4.824393	2.523645
5	2.5	5.076758	2.725317

$$y(2.5) \approx 5.076758$$

For h=0.05

n	x	y	f(x,y)
0	2	4	1.8
1	2.05	4.09	1.883237
2	2.1	4.184162	1.968552
3	2.15	4.282589	2.055944
4	2.2	4.385387	2.145413
5	2.25	4.492657	2.236959
6	2.3	4.604505	2.330581
7	2.35	4.721034	2.426279
8	2.4	4.842348	2.524053
9	2.45	4.968551	2.623902
10	2.5	5.099746	2.725826



For h=0.1 $y(2.5) \approx 5.076758$

$y(2.5) \approx 5.099746$

For h=0.05 $y(2.5) \approx 5.099746$

Euler's Method

☑ Example 2

Solve IVP by using Euler's Method:

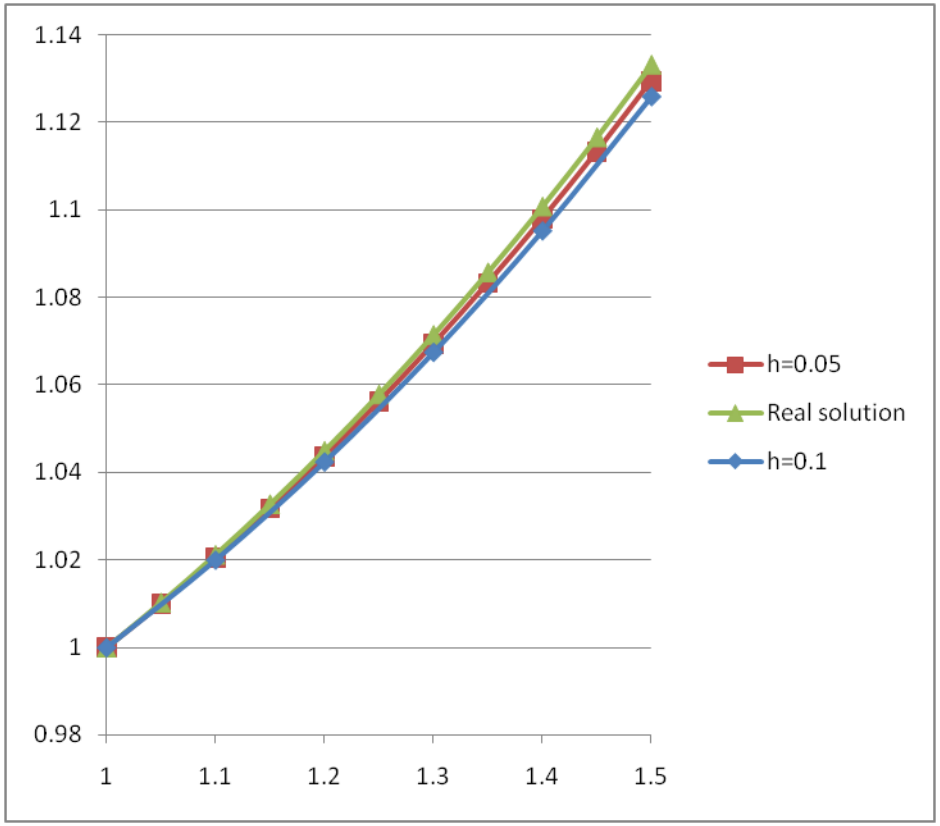
$$y' = 0.2xy, \quad y(1) = 1$$

Use Euler's Method using h=0.1 and h=0.05

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$y_{n+1} = y_n + h(0.2x_n y_n)$$

$$y = e^{0.1(x^2-1)}$$



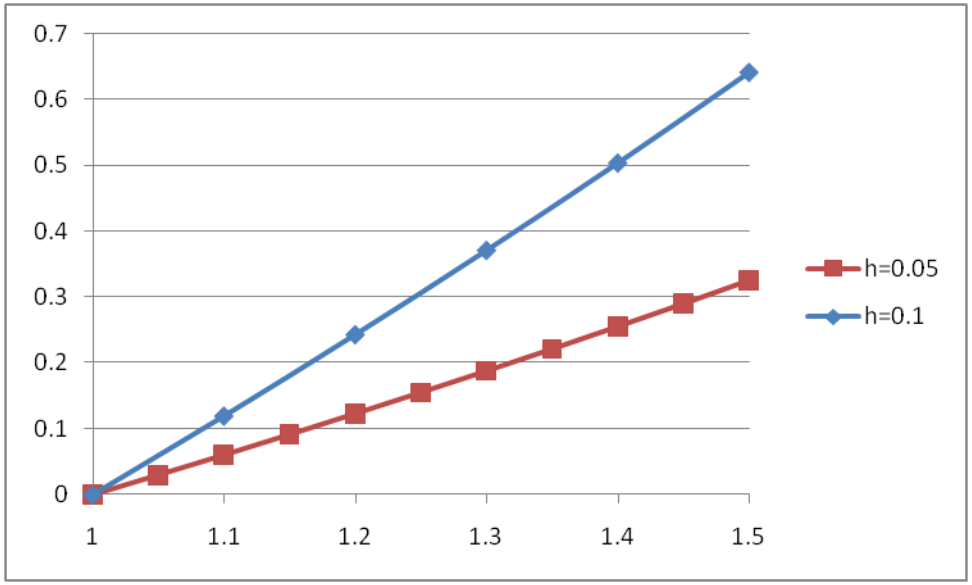
For h=0.1 $y(1.5) \approx 1.125878$

For h=0.05 $y(1.5) \approx 1.129467$

Real Solution $y(1.5) \approx 1.133148$

Comparison of Euler's Method and Exact Values

<Percentage Relative error>



CHAPTER 3. FOURIER SERIES

3.1 FOURIER SERIES

VECTOR SPACE & FUNCTION SPACE

Inner Product of Vectors : Vector Spaces

Inner Product of *vectors of 3-components:*

$$\mathbf{u} = u_1 i + u_2 j + u_3 k,\ \mathbf{v} = v_1 i + v_2 j + v_3 k$$

$$(\mathbf{u}, \mathbf{v}) = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Inner Product (dot product) of vectors of n-components:

$\mathbf{a}, \mathbf{b}$: column vectors

$$\mathbf{a} \bullet \mathbf{b} = a_1 b_1 + \cdots a_n b_n$$

$$\mathbf{a}^{\mathrm{T}} \mathbf{b} = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \sum_{l=1}^{n} a_l b_l = a_1 b_1 + \cdots a_n b_n$$

$$\therefore \mathbf{a} \bullet \mathbf{b} = \mathbf{a}^{\mathrm{T}} \mathbf{b}$$

n-Dimensional Euclidean Space

$$Definition: \mathbf{a} \bullet \mathbf{b} = (\mathbf{a}, \mathbf{b})$$

R^n with the inner product

$$(\mathbf{a}, \mathbf{b}) = \mathbf{a}^{\mathrm{T}}\mathbf{b} = a_1 b_1 + \cdots + a_n b_n$$

$$(\text{where } \mathbf{a}, \mathbf{b} : \text{column vectors})$$

is called the n-dimensional Euclidean space and is denoted by E^n or again simply by R^n.

Euclidean norm or the length of a:

$$\|\mathbf{a}\| = \sqrt{(\mathbf{a}, \mathbf{a})} = \sqrt{\mathbf{a}^{\mathrm{T}}\mathbf{a}} = \sqrt{a_1^2 + \cdots + a_n^2}$$

Inner Product of Functions: Function Space

$$\mathbf{a}^{\mathrm{T}}\mathbf{b} = (\mathbf{a},\mathbf{b}) = \mathbf{a}\bullet\mathbf{b} = \sum_{l=1}^{n} a_l b_l = a_1 b_1 + \cdots a_n b_n$$

The set of all real-valued continuous function $f(x), g(x), \cdots$ on a given interval $\alpha \le x \le \beta$ is a real vector space under the usual addition of functions and multiplication by scalars (real numbers).

On this "function space" we can define an inner product by the integral

$$\begin{aligned}(f,g) &= \int_{\alpha}^{\beta} f(x)g(x)dx \\ &= \sum_{i=1}^{N}\left(f_i(x)\sqrt{dx}\right)\left(g_i(x)\sqrt{dx}\right) \\ &= \sum_{i=1}^{N}\tilde{f}_i(x)\cdot\tilde{g}_i(x) \\ &= \tilde{f}(x)\cdot\tilde{g}(x)\end{aligned}$$

Inner Product of Functions

$$\mathbf{a}^{\mathrm{T}}\mathbf{b} = (\mathbf{a},\mathbf{b}) = \mathbf{a}\bullet\mathbf{b} = \sum_{l=1}^{n} a_l b_l = a_1 b_1 + \cdots a_n b_n$$

$$\|\mathbf{a}\| = \sqrt{(\mathbf{a},\mathbf{a})}$$

$$(f,g) = \int_{\alpha}^{\beta} f(x)g(x)dx$$

$$= \tilde{f}(x)\cdot\tilde{g}(x)$$

norm

$$\|f\| = \sqrt{(f,f)} = \sqrt{\int_{\alpha}^{\beta} f(x)^2 dx}.$$

Our examples give a first impression of the great generality of the abstract concepts of vector space and function space

Functions : Generalization of Vector

☑Introduction

- In certain area of advanced mathematics, a function is considered to be a generalization of a vector

- How the two vector concepts of inner product and orthogonality* of vectors can be extended to functions ?

Orthogonality* [or-thä-gə-'na-lə-tē] [ȯr-'thä-gə-nᵊl] sinánəməs|

In mathematics, orthogonal is synonymous with *perpendicular* when used as a simple adjective that is not part of any longer phrase with a standard definition. It means at right angles. It comes from the Greek *ortho*, meaning *right*, and *gonia*, meaning *angle*. Two streets that cross each other at a right angle are orthogonal to one another.

Properties of Vectors

Inner Product of *vectors*

$$\mathbf{u} = u_1 i + u_2 j + u_3 k,\ \mathbf{v} = v_1 i + v_2 j + v_3 k$$

$$(\mathbf{u}, \mathbf{v}) = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Properties:

$$(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})$$

$$(k\mathbf{u}, \mathbf{v}) = k(\mathbf{v}, \mathbf{u}) \ , k : \text{a scalar}$$

$$(\mathbf{u}, \mathbf{u}) = 0 \ , \ \text{if } \mathbf{u} = 0$$

$$\text{and } (\mathbf{u}, \mathbf{u}) > 0 \ , \ \text{if } \mathbf{u} \neq 0$$

$$(\mathbf{u} + \mathbf{v}, \mathbf{w}) = (\mathbf{u}, \mathbf{w}) + (\mathbf{v}, \mathbf{w})$$

Orthogonal Functions

Definition 12.1 Inner Product of *Functions*

The inner product of two functions f_1 and f_2 on an interval $[a,b]$ is the number

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x) dx$$

Definition 12.2 Orthogonal Functions

Two functions f_1 and f_2 are said to be orthogonal on an interval $[a,b]$ if

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x) dx = 0$$

Ex) $f_1 = x^2 \quad f_2 = x^3 \quad [-1,1]$

$$(f_1, f_2) = \int_{-1}^{1} x^2 x^3 dx = \frac{1}{6} x^6 \Big|_{-1}^{1} = 0$$

→ **orthogonal**

Definition 12.3 **Orthogonal Set**

A set of real-valued functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \cdots\}$ is said to be orthogonal on an interval $[a, b]$ if

$$(\phi_m(x), \phi_n(x)) = \int_a^b (\phi_m(x)\phi_n(x))dx = 0, \quad m \neq n$$

Norm of a function

$$\|\phi_n(x)\| = \sqrt{(\phi_n, \phi_n)} = \sqrt{\int_a^b (\phi_n(x)\phi_n(x))dx}$$

Any orthogonal set of nonzero functions $\{\phi_n(x)\}, n = 0,1,2,\ldots$ can be normalized by dividing by its norm

Orthogonal Functions and Fourier Series

☑ Example 1

Orthogonal Set of Functions

Show that the set $\{1, \cos x, \cos 2x, \cdots\}$

is orthogonal on the interval $[-\pi, \pi]$

Let be $\phi_0(x) = 1, \quad \phi_n(x) = \cos nx$

In the first case :

$$\int_{-\pi}^{\pi} \phi_0(x)\phi_n(x)dx = ? \quad for \quad n \neq 0$$

$$(\phi_0, \phi_n) = \int_{-\pi}^{\pi} \phi_0(x)\phi_n(x)dx = \int_{-\pi}^{\pi} \cos nxdx$$

$$= \frac{1}{n}\sin nx\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{n}[\sin n\pi - \sin(-n\pi)] = 0, \quad n \neq 0$$

In the second case :

$$\int_{-\pi}^{\pi} \phi_m(x)\phi_n(x)dx = ? \quad for \quad m \neq n$$

$$(\phi_m, \phi_n) = \int_{-\pi}^{\pi} \phi_m(x)\phi_n(x)dx = \int_{-\pi}^{\pi} \cos mx \cos nxdx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m+n)x + \cos(m-n)x]dx$$

$$= \frac{1}{2}[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}]_{-\pi}^{\pi} = 0, \quad m \neq n$$

Norm

☑ Example 2 Norms

Find the norms of each function in the orthogonal set given in Example 1

Norm of a function

$$\|\phi_n(x)\| = \sqrt{(\phi_n, \phi_n)} = \sqrt{\int_a^b (\phi_n(x)\phi_n(x))dx}$$

$$\phi_0(x) = 1$$

$$\|\phi_0(x)\|^2 = \int_{-\pi}^{\pi} dx = 2\pi$$

$$\|\phi_0(x)\| = \sqrt{2\pi}$$

$$\phi_n(x) = \cos nx, \quad n > 0$$

$$\|\phi_n(x)\|^2 = \int_{-\pi}^{\pi} \cos^2 nx dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} [1 + \cos 2nx] = \pi$$

Thus for

$$n > 0, \quad \|\phi_n(x)\| = \sqrt{\pi}$$

Orthogonal Series Expansion

Q) Is it possible to express $f(x)$ as a function of orthogonal series expansion defined on interval $[a,b]$?

$$f(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) + \cdots$$

If, yes, how can be the set of coefficients c_n determined?

where $\{\phi_n(x)\}$ is an infinite orthogonal set of functions on interval $[a,b]$

→ **By utilizing the inner product of functions**

$$\int_a^b f(x)\phi_m(x)dx$$

$$= c_0(\phi_0(x),\phi_m(x)) + c_1(\phi_1(x),\phi_m(x)) + \cdots + c_n(\phi_n(x),\phi_m(x)) + \cdots$$

$$= c_0\int_a^b \phi_0(x)\phi_m(x)dx + c_1\int_a^b \phi_1(x)\phi_m(x)dx + \cdots + c_n\int_a^b \phi_n(x)\phi_m(x)dx + \cdots$$

By orthogonality, zero except when $m = n$

$$\int_a^b f(x)\phi_n(x)dx = c_n\int_a^b \phi^2{}_n(x)dx$$

$$\therefore \boxed{c_n} = \frac{\int_a^b f(x)\phi_n(x)dx}{\int_a^b \phi^2{}_n(x)dx}$$

Orthogonal with respect to a weight function

Definition 12.4 — **Orthogonal Set/ Weight Function**

A set of real-valued functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \cdots\}$ is said to be orthogonal with respect to a weight function $w(x)$ on an interval $[a,b]$ if

$$\int_a^b w(x)\phi_m(x)\phi_n(x)dx = 0, \quad m \neq n$$

We express f(x) as a function of orthogonal series expansion defined on interval $[a,b]$.

$$f(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) + \cdots$$

To determine a set of coefficients c_n , utilizing the inner product

$$\int_a^b f(x)w(x)\phi_m(x)dx$$

$$= c_0\int_a^b \phi_0(x)w(x)\phi_m(x)dx + c_1\int_a^b \phi_1(x)w(x)\phi_m(x)dx + \cdots + c_n\int_a^b \phi_n(x)w(x)\phi_m(x)dx + \cdots$$

By orthogonality, zero except when $m = n$

$$\therefore c_n = \frac{\int_a^b f(x)w(x)\phi_n(x)dx}{\int_a^b w(x)\phi^2{}_n(x)dx}$$

Generalized Fourier Series

If $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is a set of real-valued functions that is orthogonal on an interval $[a, b]$ and if $f(x)$ is a function defined on the same interval. Then $f(x)$ can be formally expanded in an orthogonal series:

$$f(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) + \cdots$$

Orthogonal series expansion of $f(x)$ (generalized Fourier Series)

$$f(x) = \sum_{n=0}^{\infty} c_n\phi_n(x) \; where, \; c_n = \frac{\int_a^b f(x)\phi_n(x)dx}{\int_a^b \phi^2{}_n(x)dx} \; or, \; c_n = \frac{\int_a^b f(x)w(x)\phi_n(x)dx}{\int_a^b w(x)\phi^2{}_n(x)dx}$$

Assumption : an orthogonal set is complete.
→ only continuous function *f* orthogonal to each member of the set is zero function

Fourier Series

Definition 12.5

The Fourier series of a function f defined on the interval $(-p, p)$

Is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x \right)$$

where,

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}xdx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}xdx$$

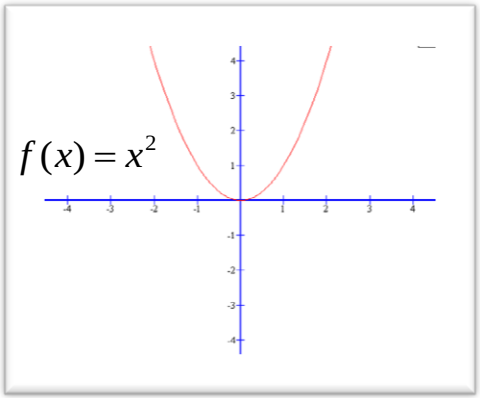
Properties of Even/Odd Functions

Even and Odd functions

f is even if $f(-x) = f(x)$

$$f(x) = x^2 \qquad f(-x) = (-x)^2 = x^2 = f(x)$$

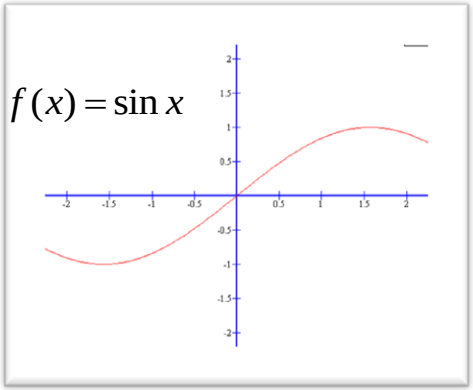
$$f(x) = \cos x \qquad f(-x) = \cos(-x) = \cos x = f(x)$$



f is odd if $f(-x) = -f(x)$

$$f(x) = x^3 \qquad f(-x) = (-x)^3 = -x^3 = -f(x)$$

$$f(x) = \sin x \qquad f(-x) = \sin(-x) = -\sin x = -f(x)$$



Properties of Even/Odd Functions

Theorem 12.2

Properties of Even/Odd Functions

(a) The product of two even functions is even.

(b) The product of two odd functions is even.

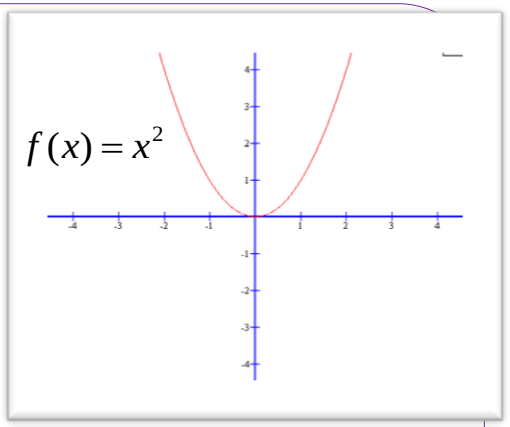
(c) The product of an even function and an odd function is odd.

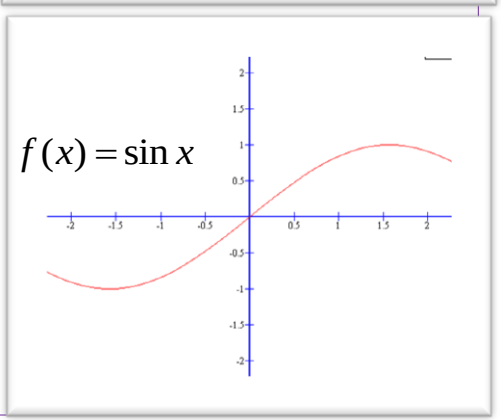
(d) The sum (difference) of two even functions is even.

(e) The sum (difference) of two odd functions is even.

(f) If f **is even, then** $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

(g) If f **is odd, then** $\int_{-a}^{a} f(x)dx = 0$





Proof: The product of two odd functions is even.

$$let\ \ f(-x) = -f(x),\ \ g(-x) = -g(x),$$

$$if\ \ F(x) = f(x)g(x)$$

$$then\ \ F(-x) = f(-x)g(-x) = (-f(x))(-g(x)) = f(x)g(x) = F(x)$$

cos : even
sin : odd
cos · cos : even
sin · sin : even
cos · sin : odd

Fourier Series

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{p}x+b_n\sin\frac{n\pi}{p}x\right)$$

defined on the interval $(-p, p)$

where,

$$a_0=\frac{1}{p}\int_{-p}^{p}f(x)dx$$

$$a_n=\frac{1}{p}\int_{-p}^{p}f(x)\cos\frac{n\pi}{p}xdx$$

$$b_n=\frac{1}{p}\int_{-p}^{p}f(x)\sin\frac{n\pi}{p}xdx$$

Is the set of trigonometric functions

$$\left\{1,\cos\frac{\pi}{p}x,\cos\frac{2\pi}{p}x,\cos\frac{3\pi}{p}x,\ldots,\sin\frac{\pi}{p}x,\sin\frac{2\pi}{p}x,\sin\frac{3\pi}{p}x,\ldots\right\}$$

orthogonal on the interval $(-p, p)$?

Show the given set of function is orthogonal (step.1)

$$\left\{\sin\frac{n\pi}{p}x\right\}, n=1,2,3\ldots;[0, p]$$

For $m\neq n$:

$$\int_0^p\sin\frac{n\pi}{p}x\sin\frac{m\pi}{p}xdx=\frac{1}{2}\int_0^p\left(\cos\frac{(n-m)\pi}{p}x-\cos\frac{(n+m)\pi}{p}x\right)dx=\frac{p}{2(n-m)\pi}\sin\frac{(n-m)\pi}{p}x\Big|_0^p-\frac{p}{2(n+m)\pi}\sin\frac{(n+m)\pi}{p}x\Big|_0^p=0$$

For $m=n$:

$$\int_0^p\sin^2\frac{n\pi}{p}xdx=\int_0^p\left(\frac{1}{2}-\frac{1}{2}\cos\frac{2n\pi}{p}x\right)dx=\frac{1}{2}x\Big|_0^p-\frac{p}{4n\pi}\sin\frac{2n\pi}{p}x\Big|_0^p=\frac{p}{2}$$

Thus, $$\left\|\sin\frac{n\pi}{p}x\right\|=\sqrt{\frac{p}{2}}$$

Show the given set of function is orthogonal (step.2)

$$\left\{1, \cos\frac{n\pi}{p}x\right\}, n = 1, 2, 3\ldots; [0, p]$$

For $m \neq n$:

$$\int_0^p \cos\frac{n\pi}{p}x\cos\frac{m\pi}{p}xdx = \frac{1}{2}\int_0^p\left(\cos\frac{(n-m)\pi}{p}x + \cos\frac{(n+m)\pi}{p}x\right)dx = \frac{p}{2(n-m)\pi}\sin\frac{(n-m)\pi}{p}x\Big|_0^p + \frac{p}{2(n+m)\pi}\sin\frac{(n+m)\pi}{p}x\Big|_0^p = 0$$

For $m = n$:

$$\int_0^p \cos^2\frac{n\pi}{p}xdx = \int_0^p\left(\frac{1}{2} + \frac{1}{2}\cos\frac{2n\pi}{p}x\right)dx = \frac{1}{2}x\Big|_0^p + \frac{p}{4n\pi}\sin\frac{2n\pi}{p}x\Big|_0^p = \frac{p}{2}$$

Also, $\int_0^p 1\cdot\cos\frac{n\pi}{p}xdx = \frac{p}{n\pi}\sin\frac{n\pi}{p}x\Big|_0^p = 0$ and $\int_0^p 1^2dx = p$ Thus, $\|1\| = \sqrt{p}$, $\left\|\cos\frac{n\pi}{p}x\right\| = \sqrt{\frac{p}{2}}$

If $f(x)$ is even, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

Show the given set of function is orthogonal (step.3)

$$\left\{1, \cos\frac{n\pi}{p}x, \sin\frac{m\pi}{p}x\right\}, n = 1,2,3\ldots, m = 1,2,3\ldots,; [-p, p]$$

for $m \neq n$: $\int_{-p}^{p} \underbrace{\cos\frac{n\pi}{p}}_{\text{even}} x \underbrace{\cos\frac{m\pi}{p}}_{\text{even}} xdx = \underbrace{2\int_{0}^{p}}_{\text{even}} \cos\frac{n\pi}{p}x\cos\frac{m\pi}{p}xdx = 0$

$$\int_{-p}^{p} \underbrace{\sin\frac{n\pi}{p}}_{\text{odd}} x \underbrace{\sin\frac{m\pi}{p}}_{\text{odd}} xdx = \underbrace{2\int_{0}^{p}}_{\text{even}} \sin\frac{n\pi}{p}x\sin\frac{m\pi}{p}xdx = 0$$

also, $\int_{-p}^{p} \sin\frac{n\pi}{p}x\cos\frac{m\pi}{p}xdx = \frac{1}{2}\int_{-p}^{p}\left(\sin\frac{(n-m)\pi}{p}x + \sin\frac{(n+m)\pi}{p}x\right)dx = 0$

$$\int_{-p}^{p} 1\cdot\cos\frac{n\pi}{p}xdx = \frac{p}{n\pi}\sin\frac{n\pi}{p}x\Big|_{-p}^{p} = 0 \quad , \quad \int_{-p}^{p} 1\cdot\sin\frac{n\pi}{p}xdx = -\frac{p}{n\pi}\cos\frac{n\pi}{p}x\Big|_{-p}^{p} = 0$$

$$\int_{-p}^{p} \sin\frac{n\pi}{p}x\cos\frac{n\pi}{p}xdx = \int_{-p}^{p}\frac{1}{2}\sin\frac{2n\pi}{p}xdx = -\frac{p}{4n\pi}\cos\frac{2n\pi}{p}x\Big|_{-p}^{p} = 0$$

for $m = n$: $\int_{-p}^{p} \cos^2\frac{n\pi}{p}xdx = \int_{-p}^{p}\left(\frac{1}{2} + \frac{1}{2}\cos\frac{2n\pi}{p}x\right)dx = p$

$$\int_{-p}^{p} \sin^2\frac{n\pi}{p}xdx = \int_{-p}^{p}\left(\frac{1}{2} - \frac{1}{2}\cos\frac{2n\pi}{p}x\right)dx = p$$

and $\int_{-p}^{p} 1^2 dx = 2p$ so that, $\|1\| = \sqrt{2p}$, $\left\|\cos\frac{n\pi}{p}x\right\| = \sqrt{p}$, $\left\|\sin\frac{n\pi}{p}x\right\| = \sqrt{p}$

Evaluation of a_0 of the Fourier series f(x)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x \right)$$

$$\int_{-p}^{p} 1\cdot\cos\frac{n\pi}{p}x\,dx = \frac{p}{n\pi}\sin\frac{n\pi}{p}x\Big|_{-p}^{p} = 0$$

$$\int_{-p}^{p} 1\cdot\sin\frac{n\pi}{p}x\,dx = -\frac{p}{n\pi}\cos\frac{n\pi}{p}x\Big|_{-p}^{p} = 0$$

Find a_0 :

Integrating both side $[-p, p]$

$$\int_{-p}^{p} f(x)dx = \frac{a_0}{2}\int_{-p}^{p} dx + \sum_{n=1}^{\infty}\left(a_n \int_{-p}^{p} \cos\frac{n\pi}{p}x\,dx + b_n \int_{-p}^{p} \sin\frac{n\pi}{p}x\,dx \right)$$

zero zero

by orthogonality

$$\int_{-p}^{p} f(x)dx = \frac{a_0}{2}(2p)$$

$$\therefore a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx$$

Evaluation of a_n

For $m = n$:

$$\int_0^p \cos^2 \frac{n\pi}{p} x dx = \int_0^p \left(\frac{1}{2} + \frac{1}{2} \cos \frac{2n\pi}{p} x \right) dx = \frac{1}{2} x \Big|_0^p + \frac{p}{4n\pi} \sin \frac{2n\pi}{p} x \Big|_0^p = \frac{p}{2}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

Find a_n :

Multiply by $\cos \frac{m\pi}{p} x$ **and integrating both side** $[-p, p]$

$$\int_{-p}^{p} f(x) \cos \frac{m\pi}{p} x dx$$

by orthogonality

$$= \frac{a_0}{2} \int_{-p}^{p} \cos \frac{m\pi}{p} x dx + \sum_{n=1}^{\infty} \left(a_n \int_{-p}^{p} \cos \frac{n\pi}{p} x \cos \frac{m\pi}{p} x dx + b_n \int_{-p}^{p} \sin \frac{n\pi}{p} x \cos \frac{m\pi}{p} x dx \right)$$

(zero; zero except n=m; zero)

$$\int_{-p}^{p} f(x) \cos \frac{n\pi}{p} x dx = a_n \int_{-p}^{p} \cos \frac{n\pi}{p} x \cos \frac{n\pi}{p} x dx = a_n p$$

$$\therefore a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi}{p} x dx$$

Evaluation of b_n

$$\int_{-p}^{p} \sin^2 \frac{n\pi}{p} x dx = \int_{-p}^{p} \left(\frac{1}{2} - \frac{1}{2} \cos \frac{2n\pi}{p} x \right) dx = p$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

Find b_n :
Multiply by $\sin \frac{m\pi}{p} x$ and integrating both side $[-p, p]$

$$\int_{-p}^{p} f(x) \sin \frac{m\pi}{p} x dx$$

by orthogonality

$$= \frac{a_0}{2} \int_{-p}^{p} \sin \frac{m\pi}{p} x dx + \sum_{n=1}^{\infty} \left(a_n \int_{-p}^{p} \cos \frac{n\pi}{p} x \sin \frac{m\pi}{p} x dx + b_n \int_{-p}^{p} \sin \frac{n\pi}{p} x \sin \frac{m\pi}{p} x dx \right)$$

zero (first term) · zero (second term) · zero except n=m (third term)

$$\int_{-p}^{p} f(x) \sin \frac{n\pi}{p} x dx = b_n \int_{-p}^{p} \sin \frac{n\pi}{p} x \sin \frac{n\pi}{p} x dx = b_n p$$

$$\therefore b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi}{p} x dx$$

Fourier Series :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

where,

$$a_0 = \frac{1}{p} \int_{-p}^{p} f(x) dx$$

$$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi}{p} x dx$$

$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi}{p} x dx$$

Fourier Series -> Fourier Cosine Series of Even function

If f(x) is even function,

$$a_0 = \frac{1}{p}\int_{-p}^{p} \underbrace{f(x)}_{\text{even}}dx = \frac{2}{p}\int_{0}^{p} f(x)dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} \underbrace{\underbrace{f(x)}_{\text{even}}\underbrace{\cos\frac{n\pi}{p}}_{\text{even}}}_{\text{even}}xdx = \frac{2}{p}\int_{0}^{p} f(x)\cos\frac{n\pi}{p}xdx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} \underbrace{\underbrace{f(x)}_{\text{even}}\underbrace{\sin\frac{n\pi}{p}}_{\text{odd}}}_{\text{odd}}xdx = 0$$

Fourier Cosine Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x$$

$$a_0 = \frac{2}{p}\int_{0}^{p} f(x)dx$$

$$a_n = \frac{2}{p}\int_{0}^{p} f(x)\cos\frac{n\pi}{p}dx$$

Fourier Series of an even function on the interval (-*p*, *p*) is the Fourier Cosine Series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x$$

$$a_0 = \frac{2}{p}\int_{0}^{p} f(x)dx$$

$$a_n = \frac{2}{p}\int_{0}^{p} f(x)\cos\frac{n\pi}{p}xdx$$

Fourier Series -> Fourier Sine Series of Odd function

If f(x) is odd function,

$$a_0 = \frac{1}{p}\int_{-p}^{p} \underbrace{f(x)}_{\text{odd}} dx = 0$$

$$a_n = \frac{1}{p}\int_{-p}^{p} \underbrace{\underbrace{f(x)}_{\text{odd}}\underbrace{\cos\frac{n\pi}{p}}_{\text{even}}}_{\text{odd}} x dx = 0$$

$$b_n = \frac{1}{p}\int_{-p}^{p} \underbrace{\underbrace{f(x)}_{\text{odd}}\underbrace{\sin\frac{n\pi}{p}}_{\text{odd}}}_{\text{even}} x dx = \frac{2}{p}\int_{0}^{p} f(x)\sin\frac{n\pi}{p}x dx$$

Fourier Sine Series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x$$

$$b_n = \frac{2}{p}\int_{0}^{p} f(x)\sin\frac{n\pi}{p}dx$$

Fourier Series of an odd function on the interval (-*p*, *p*) is the Fourier Sine Series.

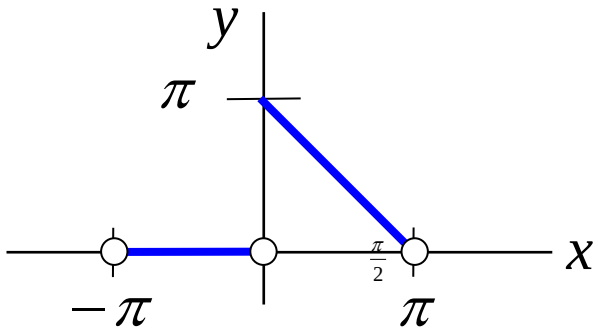
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x \qquad b_n = \frac{2}{p}\int_{0}^{p} f(x)\sin\frac{n\pi}{p}x dx$$

Example 1 : Fourier Series

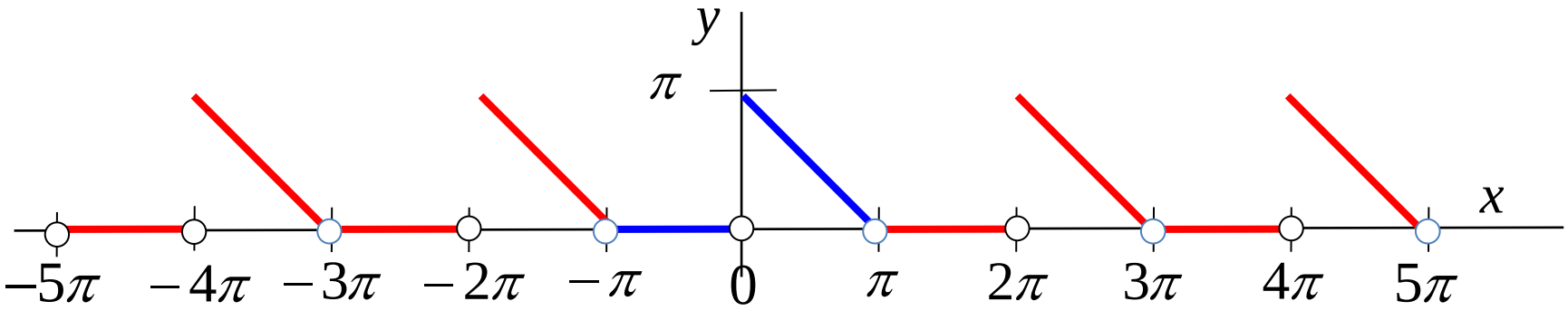
Integration by parts:

$$\int u(x)\, v`(x)\, dx = u(x)\, v(x) - \int u`(x)\, v(x)\, dx$$

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$$



Often in problems
for Fourier series expansion,
you will be given f(x) for only one period;
you should always sketch several periods
so that you see clearly the periodic function
you are expanding.
It is then understood that
f(x) is to be continued periodically
with period 2π outside the interval (-π, π)



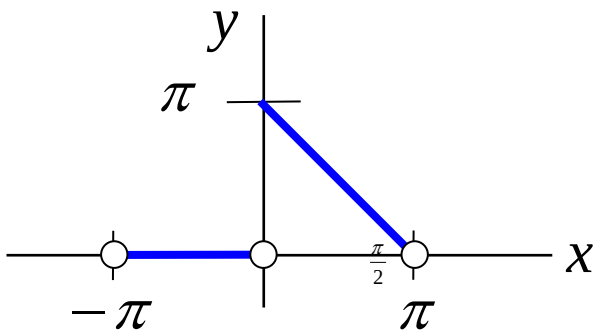
1) Dennis G. zill, Advanced Engineering Mathematics 3rd, Jones and Bartlett, p 660

Example 1 : Fourier Series

Integration by parts:

$$\int u(x)\, v`(x)\, dx = u(x)\, v(x) - \int u`(x)\, v(x)\, dx$$

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$$



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x \right)$$

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx,\ b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}[\int_{-\pi}^{0} 0dx + \int_{0}^{\pi}(\pi - x)dx$$

$$= \frac{1}{\pi}[\pi x - \frac{x^2}{2}]_0^{\pi} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi}(\pi - x)\cos nx\,dx]$$

Using Integration by parts

$$= \frac{1}{\pi}[(\pi - x)\frac{\sin nx}{n}\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}(-1)\sin nx\,dx]$$

(zero)

$$= -\frac{1}{n\pi}\frac{\cos nx}{n}\Big|_0^{\pi} = \frac{-\cos n\pi + 1}{n^2\pi} = \frac{1-(-1)^n}{n^2\pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi}(\pi - x)\sin nx\,dx]$$

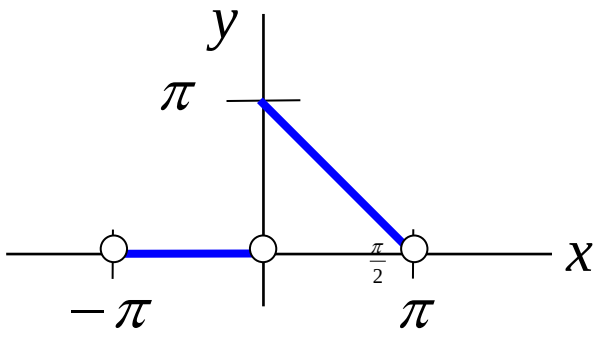
Using integration by parts

$$= \frac{1}{\pi}[-(\pi - x)\frac{\cos nx}{n}\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\cos nx\,dx]$$

$$= \frac{1}{n} - \frac{1}{n^2\pi}\sin nx\Big|_0^{\pi} = \frac{1}{n}$$

Example1 : Fourier Series

$$f(x)=\begin{cases}0, & -\pi<x<0\\ \pi-x, & 0\le x<\pi\end{cases}$$



$$\therefore f(x)=\frac{\pi}{4}+\sum_{n=1}^{\infty}\left\{\frac{1-(-1)^n}{n^2\pi}\cos nx+\frac{1}{n}\sin nx\right\}$$

Fourier Series :

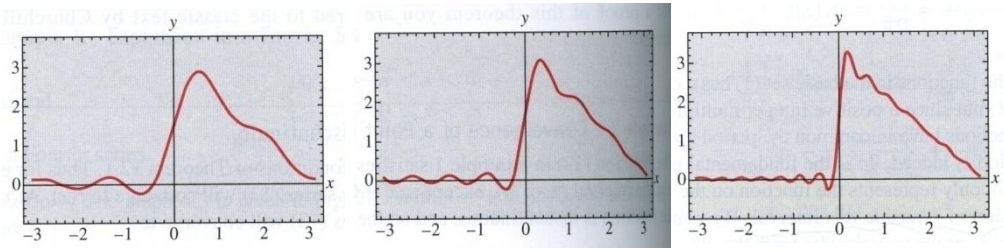
$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{p}x+b_n\sin\frac{n\pi}{p}x\right)$$

where,

$$a_0=\frac{1}{p}\int_{-p}^{p}f(x)dx$$

$$a_n=\frac{1}{p}\int_{-p}^{p}f(x)\cos\frac{n\pi}{p}xdx$$

$$b_n=\frac{1}{p}\int_{-p}^{p}f(x)\sin\frac{n\pi}{p}xdx$$



$S_5(x)\,on(-\pi,\pi)$ $S_8(x)\,on(-\pi,\pi)$ $S_{15}(x)\,on(-\pi,\pi)$

$S_n(x)$ **Subsum to n th- term**

Convergence of a Point of Discontinuity

Theorem 12.1

Conditions for Convergence

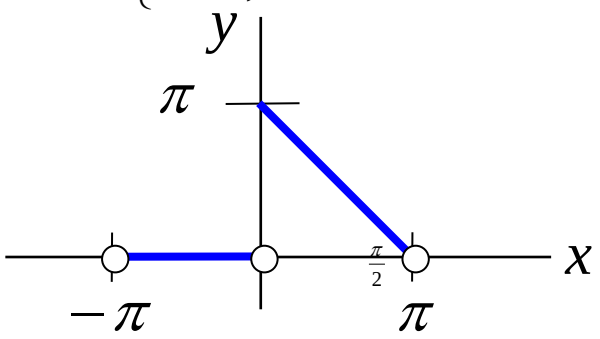
Let $f(x)$ and $f'(x)$ be piecewise continuous on the interval $(-p, p)$; that is, let $f(x)$ and $f'(x)$ be continuous except at a finite number of points in the interval and have only finite discontinuities at these points. Then the Fourier series of $f(x)$ on the interval converges to $f(x)$ at a point of continuity. At a point of discontinuity, the Fourier series converges to the average $\dfrac{f(x+)+f(x-)}{2}$

Where $f(x+)$ and $f(x-)$ denote the limit of f at x from the right and form the left, respectively.

$$f(x+)=\lim_{h\to 0} f(x+h), \quad f(x-)=\lim_{h\to 0} f(x-h), \quad h>0$$

Convergence of a Point of Discontinuity

$$f(x)=\begin{cases}0, & -\pi<x<0\\ \pi-x, & 0\le x<\pi\end{cases}$$



$$f(x)=\frac{\pi}{4}+\sum_{n=1}^{\infty}\left\{\frac{1-(-1)^n}{n^2\pi}\cos nx+\frac{1}{n}\sin nx\right\}$$

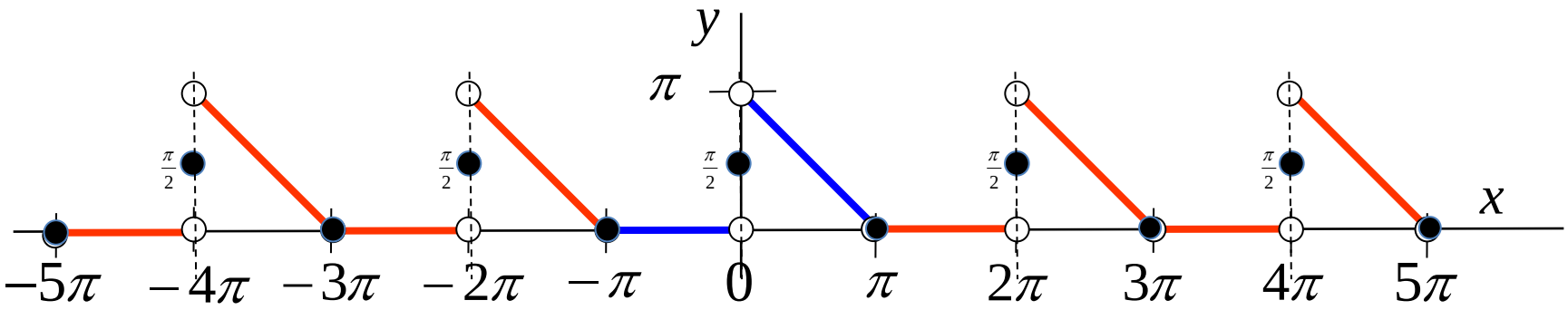
The series will converge to f(x) for every x in the interval $(-\pi,\pi)$, except at $x=0$

at $x=0$ the series will converge to

$$\frac{f(0+)+f(0-)}{2}=\frac{\pi+0}{2}=\frac{\pi}{2}$$

Periodic Extension of Fourier Series

Fourier Series *not only represents the function on the interval*
but also gives *the periodic extension of* f ***outside this interval*** $(-p, p)$



The Fourier series $f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left\{ \frac{1-(-1)^n}{n^2 \pi} \cos nx + \frac{1}{n} \sin nx \right\}$ **converges**

to the periodic extension of $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$ **on the entire x-axis**

Converge to the values $\frac{f(0+) + f(0-)}{2} = \frac{\pi}{2}$, $at\ 0, \pm 2\pi, \ldots$ **and** $\frac{f(\pi+) + f(\pi-)}{2} = 0$, $at\ \pm\pi, \pm 3\pi, \ldots$

Example 2: Fourier Sine Series of Odd Function

$$f(x) = x, \quad -2 < x < 2$$

Given function is odd

$$2p = 2 - (-2) = 4$$

$$p = 2$$

Fourier Sine Series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x$$

$$b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x dx$$

$$= \int_0^2 x \sin \frac{n\pi}{2} x dx = \frac{4(-1)^{n+1}}{n\pi}$$

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{2} x$$

Fourier Sine Series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x$$

$$b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} dx$$

Integration by parts :

$$\int u(x)\, v`(x)\, dx = u(x)\, v(x) - \int u`(x)\, v(x)\, dx$$

Example 3: Fourier Sine Series of Odd Function

$$f(x)=\begin{cases}-1, & -\pi<x<0\\ 1, & 0\le x<\pi\end{cases}$$

Given function is odd

$$2p=\pi-(-\pi)=2\pi$$

$$p=\pi$$

Fourier Sine Series

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi}{p}x$$

$$b_n=\frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}xdx$$

$$=\frac{2}{\pi}\int_0^{\pi}(1)\sin nxdx=\frac{2}{n}\frac{1-(-1)^n}{n}$$

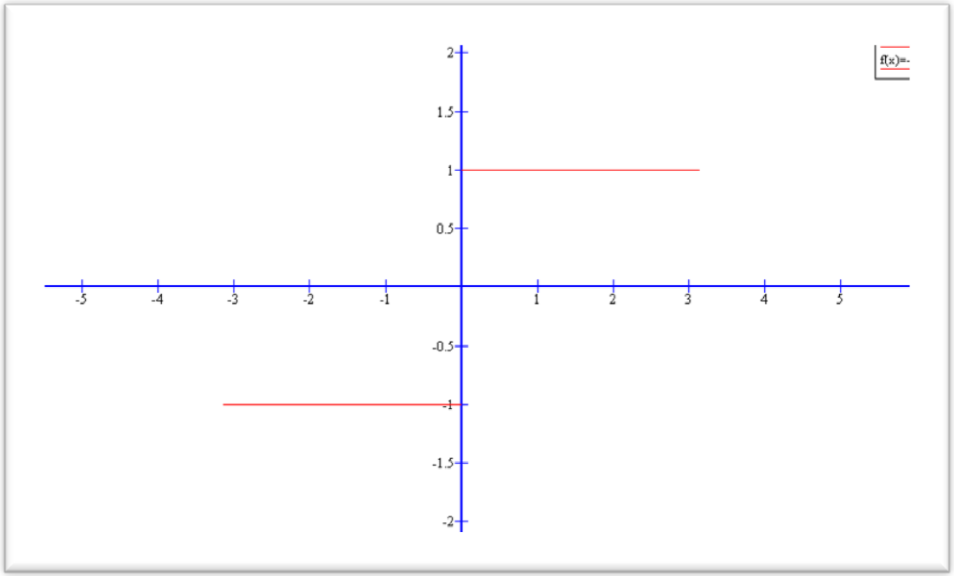
$$f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx$$

Fourier Sine Series

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi}{p}x$$

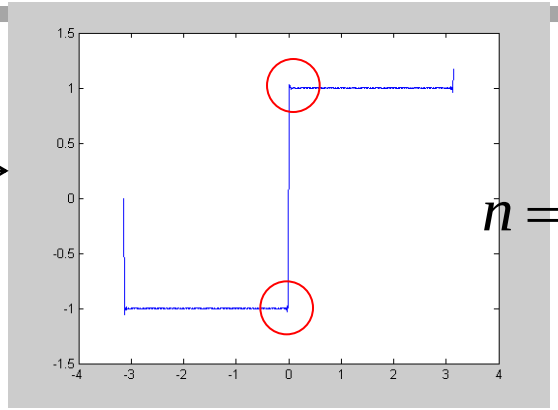
$$b_n=\frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}dx$$

Gibbs Phenomenon : "Overshooting" near a point of discontinuity

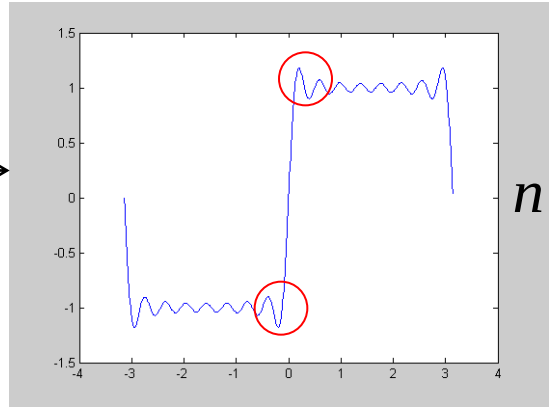


$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx$$

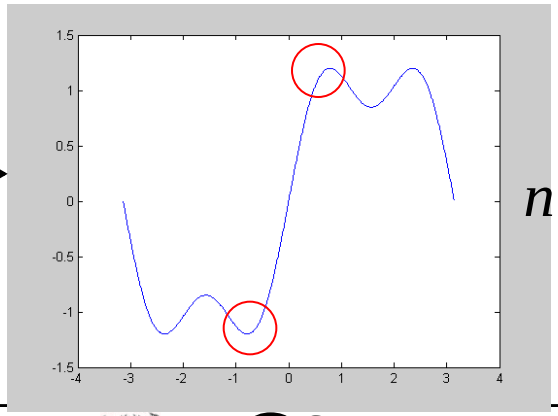
odd function, Fourier sine series



$n = 1000$



$n = 15$

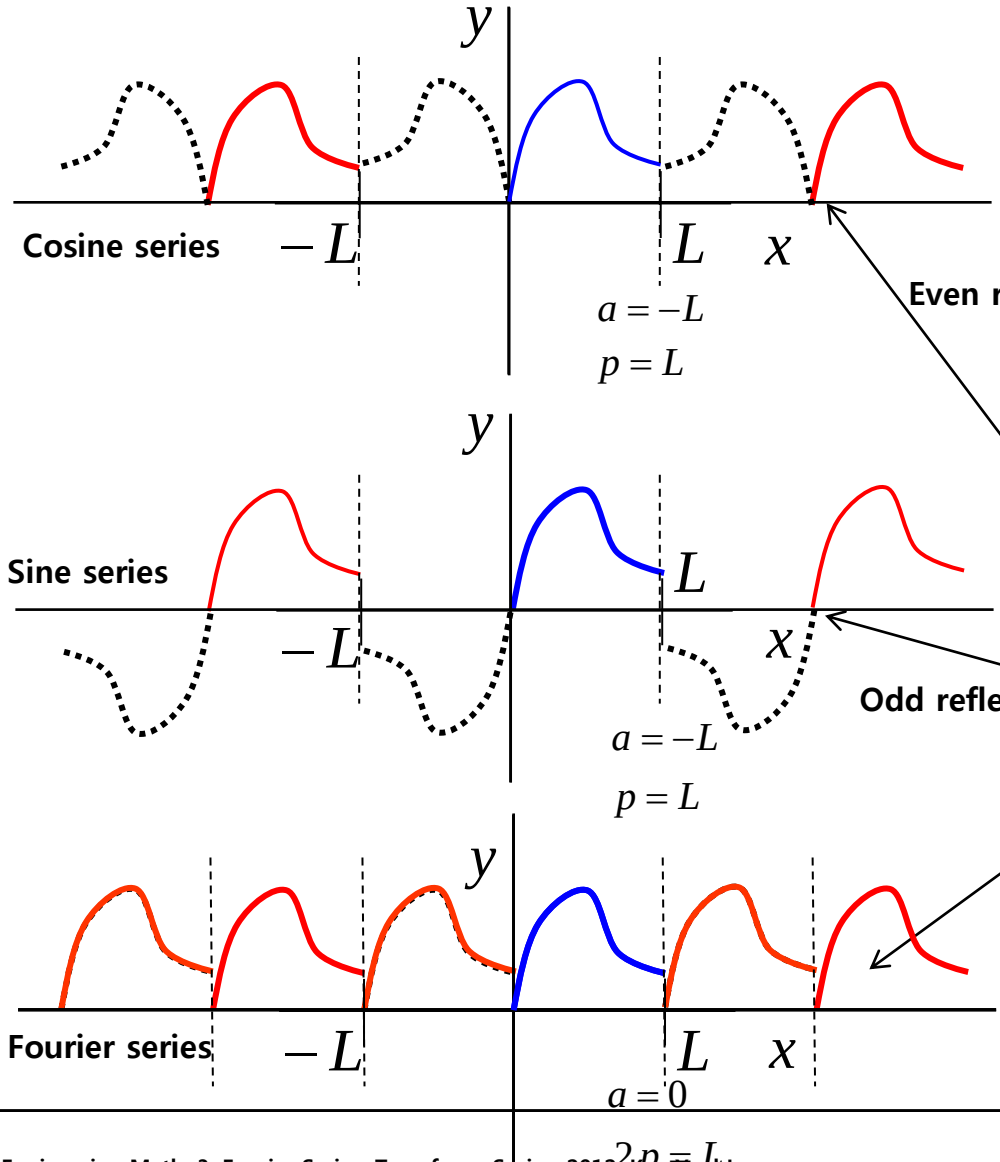


$n = 3$

Half Range Expansions of Fourier Cosine and Sine Series

Half Range Expansions:

Different periodic extension of f



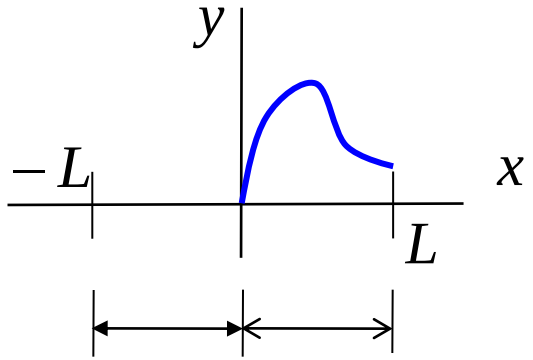
trigonometric system is periodic on $a \le x \le a+2p$ for any real number a.

period $T = a + 2p - a = 2p$

Even reflection

Odd reflection

Identity reflection



Arbitrary definition $-L < x < 0$

our interest $0 < x < L$

Half Range Expansions of Fourier Cosine Series

Fourier Cosine Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x$$

$$a_0 = \frac{2}{p}\int_0^p f(x)dx$$

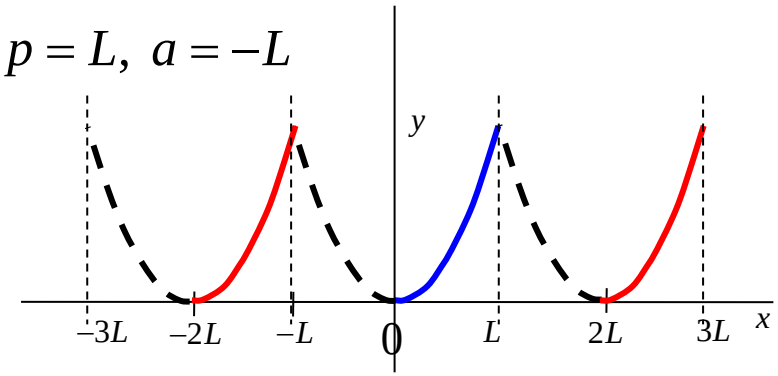
$$a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}xdx$$

$$f(x) = x^2, \quad 0 < x < L$$

Express given function in a cosine series

Fourier Cosine Series $(-p, p) = (-L, L)$

$p = L, \ a = -L$



$$a_0 = \frac{2}{p}\int_0^p f(x)dx = \frac{2}{L}\int_0^L x^2 dx = \frac{2}{3}L^2$$

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}xdx = \frac{2}{L}\int_0^L x^2\cos\frac{n\pi}{L}xdx = \frac{4L^2(-1)^n}{n^2\pi^2}$$

where integration by parts was used twice in the evaluation of a_n

i) We conclude that a Fourier series not only represents the function on the interval but also gives the periodic extension of $f(x)$, or we may assume from the outset that the given function is periodic with period $T = 2p$.

ii) Reflect the graph of the function about the y-axis; the function is now even.

$$f(x) = \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi}{L}x$$

Half Range Expansions of Fourier Sine Series

Fourier Sine Series

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x$$

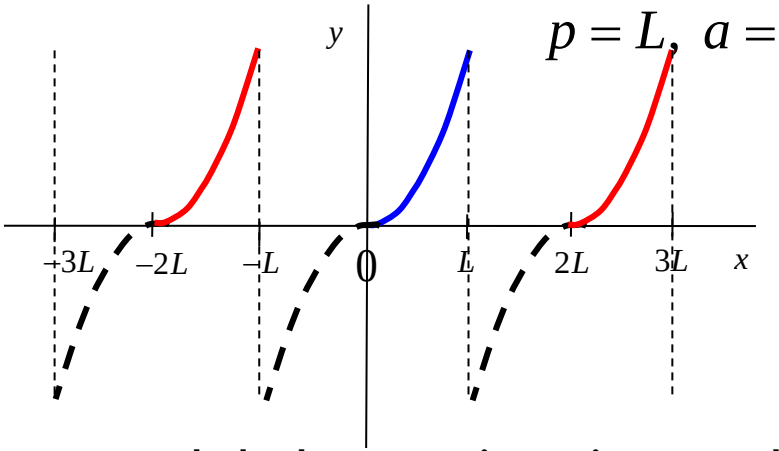
$$b_n=\frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}xdx$$

$$f(x)=x^2,\ 0<x<L$$

Express given function in a sine series

Fourier Sine Series $(-p,p)=(-L,L)$

$p=L,\ a=-L$



i) We conclude that a Fourier series not only represents the function on the interval but also gives the periodic extension of $f(x)$, or we may assume from the outset that the given function is periodic with period $T=2p$.

ii) Reflect the graph of the function through the origin; the function is now odd.

$$b_n=\frac{2}{L}\int_0^L x^2\sin\frac{n\pi}{L}xdx$$

$$b_n=\frac{2L^2(-1)^{n+1}}{n\pi}+\frac{4L^2}{n^3\pi}[(-1)^n-1]$$

where integration by parts was used twice in the evaluation of a_n

Hence

$$f(x)=\frac{2L^2}{\pi}\sum_{n=1}^{\infty}\{\frac{(-1)^{n+1}}{n}+\frac{2}{n^3\pi^2}[(-1)^n-1]\}\sin\frac{n\pi}{L}x$$

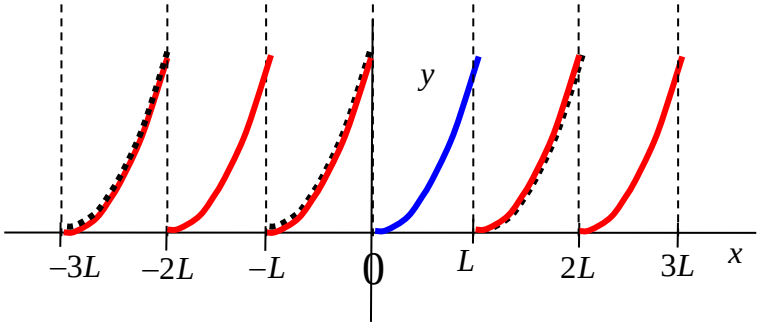
1) Dennis G. zill, Advanced Engineering Mathematics 3rd, Jones and Bartlett, p 661,p666

Half Range Expansions of Fourier Series -> Homework

$$f(x) = x^2, \quad 0 < x < L$$

Express given function in a Fourier series

Fourier Series



$$(-p, p) = (-L, L)$$

$$T = L$$

$$2p = L, \; a = 0$$

i) We conclude that a Fourier series not only represents the function on the interval but also gives the periodic extension of $f(x)$, or we may assume from the outset that the given function is periodic with period $T = 2p$.

ii) Define $f(x)$ by $f(x) = f(x + L)$

Fourier Sine Series

Fourier Sine Series

$$f(t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} t$$

$$b_n = \frac{2}{p} \int_0^p f(t) \sin \frac{n\pi}{p} t dt$$

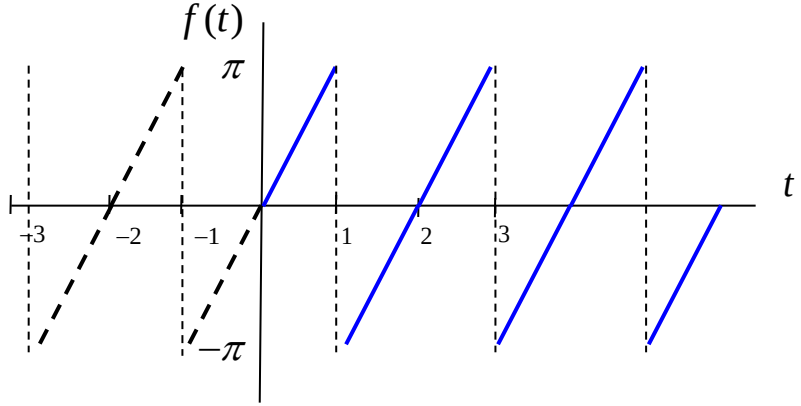
Example 4:

Particular Solution of a DE

- Undamped spring mass system

$$m\ddot{x}(t) + kx(t) = f(t), \quad m = \frac{1}{16}, \; k = 4$$

f(t) is shown in the following figure.(blue line)



If we extend the graph of the function to the negative t-axis, we obtain an odd function.

We need only find the half-range sine expansion of $f(t) = \pi t, \; 0 < t < 1$

$$(-p, p) = (-1, 1) \Rightarrow \quad p = 1$$

$$b_n = 2\int_0^1 \pi t \sin n\,\pi t \, dx = \frac{2(-1)^{n+1}}{n}$$

$$f(t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} t$$

$$= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin n\pi t$$

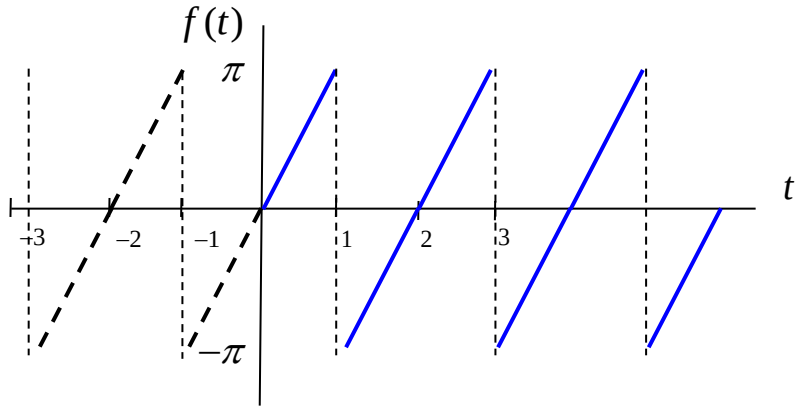
$$\therefore \frac{1}{16}\ddot{x} + 4x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin n\pi t$$

Fourier Sine Series

Example 4:
Particular Solution of a DE
- Undamped spring mass system

$$m\ddot{x}(t)+kx(t)=f(t),\quad m=\frac{1}{16},\ k=4$$

f(t) is shown in the following figure.(blue line)



If we extend the graph of the function to the negative t-axis, we obtain an odd function.

We need only find the half-range sine expansion of $f(t)=\pi t,\ 0<t<1$

$$(-p,\ p)=(-1,1) \ \Rightarrow\ p=1$$

$$\frac{1}{16}\ddot{x}+4x=\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin n\pi t$$

For the simplicity, let's assume particular solution as following

$$x_p(t)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi}{p}t \quad \textbf{,then} \quad \ddot{x}_p(t)=-\sum_{n=1}^{\infty}B_n\left(\frac{n\pi}{p}\right)^2\sin\frac{n\pi}{p}t$$

$$\therefore\ \frac{1}{16}\ddot{x}_p(t)+4x_p(t)=\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin n\pi t$$

$$\frac{1}{16}\left[-\sum_{n=1}^{\infty}B_n\left(\frac{n\pi}{p}\right)^2\sin\frac{n\pi}{p}t\right]+4\sum_{n=1}^{\infty}B_n\sin\frac{n\pi}{p}t=\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin n\pi t$$

$$\sum_{n=1}^{\infty}\left[-\frac{1}{16}B_n\left(\frac{n\pi}{p}\right)^2\sin\frac{n\pi}{p}t+4B_n\sin\frac{n\pi}{p}t\right]=\sum_{n=1}^{\infty}\left[\frac{2(-1)^{n+1}}{n}\sin n\pi t\right]$$

Must be equal

$$-\frac{1}{16}B_n\left(\frac{n\pi}{p}\right)^2\sin\frac{n\pi}{p}t+4B_n\sin\frac{n\pi}{p}t=\frac{2(-1)^{n+1}}{n}\sin n\pi t \qquad p=1$$

$$\left(-\frac{1}{16}n^2\pi^2+4\right)B_n=\frac{2(-1)^{n+1}}{n}$$

$$B_n=\frac{32(-1)^{n+1}}{n(64-n^2\pi^2)}$$

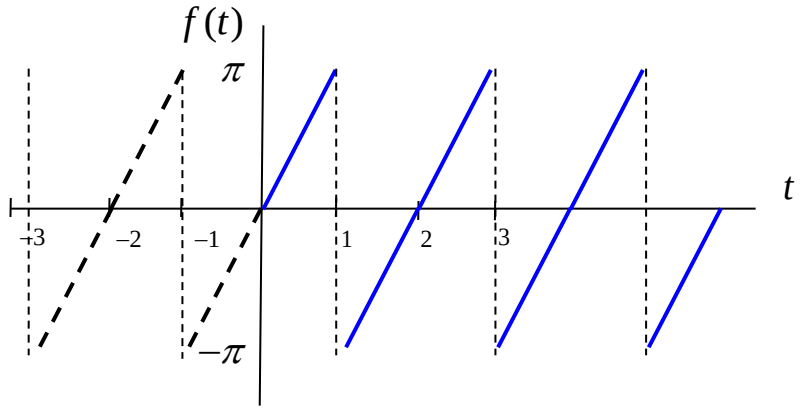
Thus $$x_p(t)=\sum_{n=1}^{\infty}\frac{32(-1)^{n+1}}{n(64-n^2\pi^2)}\sin n\pi t$$

Fourier Sine Series

Example 4:
Particular Solution of a DE
- Undamped spring mass system

$$m\ddot{x}(t)+kx(t)=f(t),\quad m=\frac{1}{16},\ k=4$$

f(t) is shown in the following figure.(blue line)



If we extend the graph of the function to the negative t-axis, we obtain an odd function.

We need only find the half-range sine expansion of $f(t)=\pi t,\ 0<t<1$

$$(-p,\ p)=(-1,1) \ \Rightarrow\ \ p=1$$

$$x_p(t)=\sum_{n=1}^{\infty}\frac{32(-1)^{n+1}}{n(64-n^2\pi^2)}\sin n\pi t$$

⇨ **There is no integer** $n\geq 1$ **for which the denominator** $64-n^2\pi^2$

⇨ **In general, if there is a value of** n,

for which $\frac{n\pi}{p}=\omega_n$ **, where** $\omega_n=\sqrt{k/m}$

, then the system is in a state of pure resonance.

⇨ **In other words, we have pure resonance**

if the Fourier series expansion of the driving force $f(t)$

contains a term $\sin(n\omega)t$ **that has the same**

frequency as the free vibration

Fourier Series (Summary)

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x \right)$$

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}xdx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}xdx$$

Real function

Real valued-functions of a real variable x

Fourier Cosine Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x$$

$$a_0 = \frac{2}{p}\int_{0}^{p} f(x)dx$$

$$a_n = \frac{2}{p}\int_{0}^{p} f(x)\cos\frac{n\pi}{p}dx$$

Fourier Sine Series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x$$

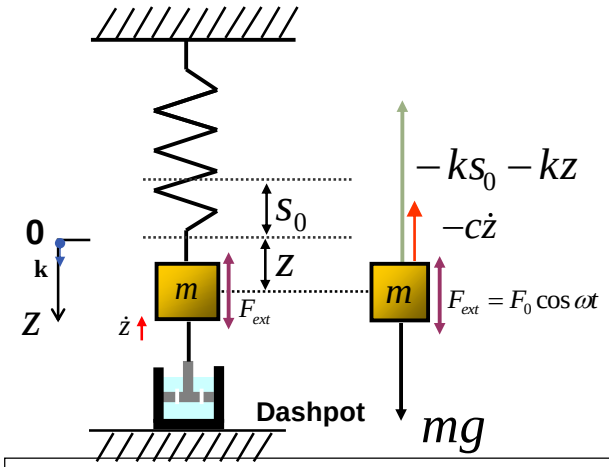
$$b_n = \frac{2}{p}\int_{0}^{p} f(x)\sin\frac{n\pi}{p}dx$$

APPLICATION OF FOURIER SERIES

Application of Fourier Series (1)

✓ Application 1) Express a function which is periodic but not a pure sine or cosine function to a linear combination of sine or cosine functions.

ex) Forced damped mass-spring system



$$m\mathbf{z}'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F}_0 \cos \omega t$$
$$= -kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F}_0 \cos \omega t$$

$$m\mathbf{z}'' + c\mathbf{z}' + k\mathbf{z} = \mathbf{F}_0 \cos \omega t$$

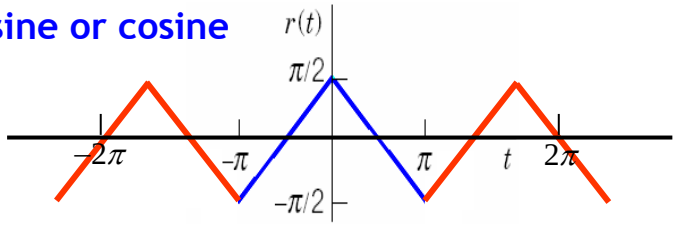
$$mz'' + cz' + kz = F_0 \cos \omega t \quad \text{(z component)}$$

$$m = 1, c = 0.05, k = 25$$

$$\ddot{y} + 0.05\dot{y} + 25y = r(t)$$

$$r(t) = \begin{cases} t + \dfrac{\pi}{2} & \text{if } -\pi < t < 0 \\ -t + \dfrac{\pi}{2} & \text{if } 0 < t < \pi \end{cases}$$
$$r(t + 2\pi) = r(t)$$

periodic but not a sine or cosine



Represent $r(t)$ by a Fourier series

Fourier Cosine Series

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi}{p}x$$

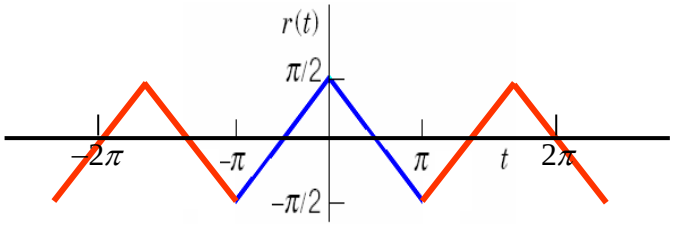
$$a_0=\frac{2}{p}\int_0^p f(x)dx$$

$$a_n=\frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}dx$$

$$\cos n\pi=(-1)^n$$

$$r(t)=\begin{cases} t+\dfrac{\pi}{2} & \text{if } -\pi<t<0, \\ -t+\dfrac{\pi}{2} & \text{if } 0<t<\pi, \end{cases}$$

$$r(t+2\pi)=r(t)$$



if $f(x)$ is an even function its Fourier series reduces to Fourier cosine series.1)

$$a_0=\frac{2}{\pi}\int_0^{\pi}(-t+\frac{\pi}{2})dx=\frac{2}{\pi}\left[\frac{-t^2}{2}+\frac{\pi}{2}t\right]_0^{\pi}$$

$$=0$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}(-t+\frac{\pi}{2})\cos nt\,dt$$

$$=\frac{2}{\pi}\left\{\frac{1}{n}\left(-t+\frac{\pi}{2}\right)\sin nt\Big|_0^{\pi}+\frac{1}{n}\int_0^{\pi}\sin nt\,dt\right\}$$

where integration by parts was used once in the evaluation of a_n

$$=\frac{2}{\pi}\left\{-\frac{1}{n^2}\cos nt\Big|_0^{\pi}\right\}=\frac{2}{\pi}\left\{-\frac{1}{n^2}(\cos n\pi-1)\right\}$$

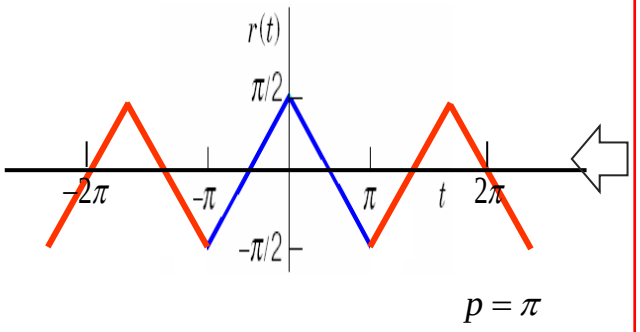
$$=\frac{2}{\pi}\left\{-\frac{1}{n^2}\left[(-1)^n-1\right]\right\}$$

$$=\begin{cases}\dfrac{4}{\pi}\dfrac{1}{n^2} & \text{if } n=odd \\ 0 & \text{if } n=even\end{cases}$$

$$\therefore r(t)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nt,\ n=1,3,\ldots$$

Basic Fourier Transform Analysis

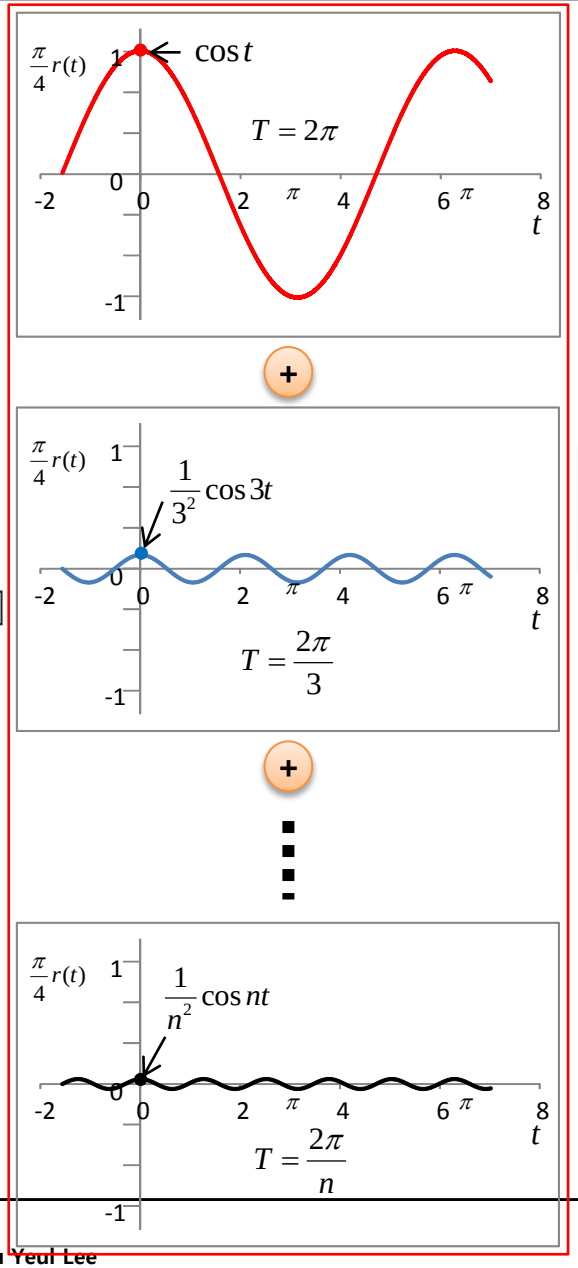
$$r(t)=\begin{cases} t+\dfrac{\pi}{2} & \text{if } -\pi<t<0 \\ -t+\dfrac{\pi}{2} & \text{if } 0<t<\pi \end{cases}$$
$$r(t+2\pi)=r(t)$$



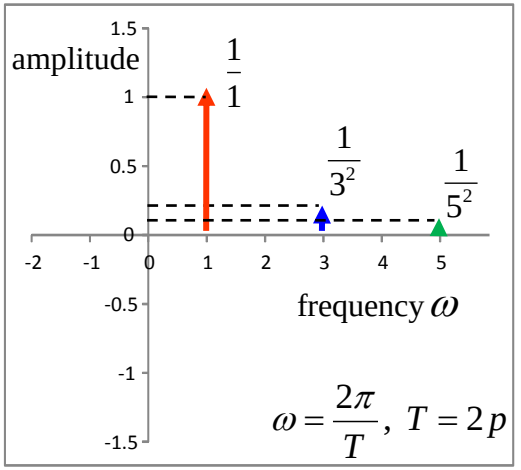
$p=\pi$

1)periodic but not a sine or cosine
2)wave form defined from -∞ to +∞

$$r(t)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nt,\ n=1,3,\ldots$$



Construct a diagram which displays amplitude and frequency of each sinusoid

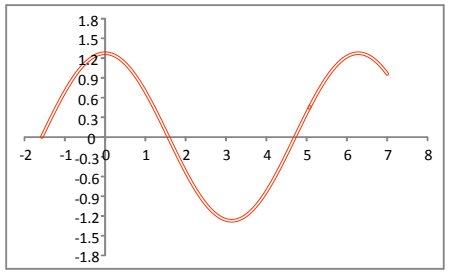


Since T is **fundamental period,**

ω is **fundamental angular frequency.**

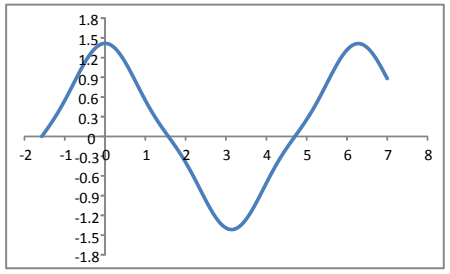
$\omega = 1$ so that $n\omega$ takes on the values $\pm 1, \pm 3, \ldots$

Basic Fourier Transform Analysis



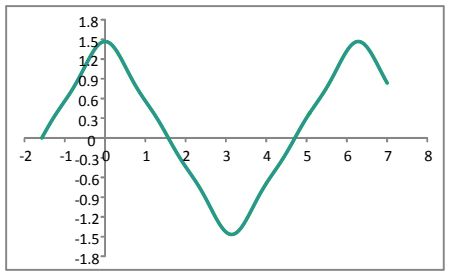
for $n=1$

$$r(0)=\frac{4}{\pi}\sum_{n=1}^{1}\frac{1}{n^2}\cos nt\approx 1.273$$



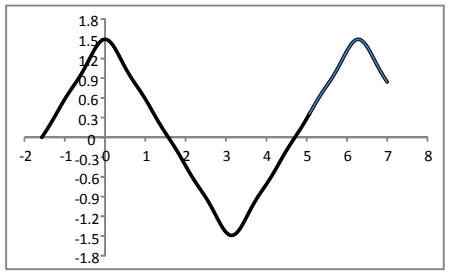
for $n=3$

$$r(0)=\frac{4}{\pi}\sum_{n=1}^{3}\frac{1}{n^2}\cos nt\approx 1.415$$



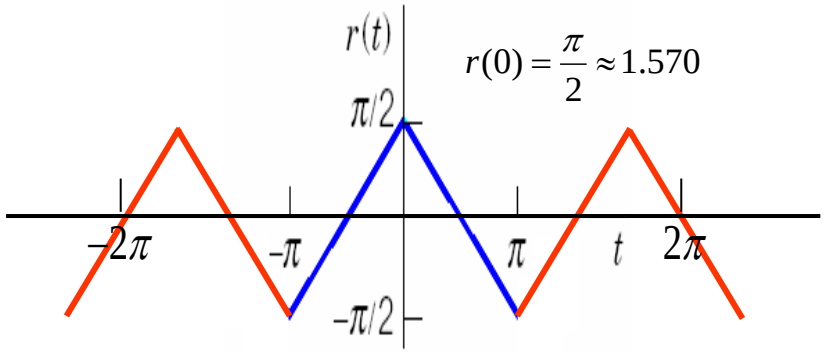
for $n=5$

$$r(0)=\frac{4}{\pi}\sum_{n=1}^{3}\frac{1}{n^2}\cos nt\approx 1.466$$



for $n=7$

$$r(0)=\frac{4}{\pi}\sum_{n=1}^{3}\frac{1}{n^2}\cos nt\approx 1.491$$



$$r(0)=\frac{\pi}{2}\approx 1.570$$

$$r(t)=\begin{cases} t+\dfrac{\pi}{2} & \text{if } -\pi<t<0 \\ -t+\dfrac{\pi}{2} & \text{if } 0<t<\pi \end{cases}$$

$$r(t+2\pi)=r(t)$$

$$r(t)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nt,\ n=1,3,\ldots$$

Forced Oscillations

$$m\ddot{z} + c\dot{z} + kz = F_0 \cos \omega t$$

- **Mass-Spring system**

$$m\ddot{y} + c\dot{y} + ky = r(t)$$

- **If $r(t)$ is a sine or cosine function and if there is damping ($c > 0$), then the steady-state solution is a harmonic oscillation with frequency equal to that of $r(t)$.**

- **If $r(t)$ is not a pure sine or cosine function but is any other periodic function, then the steady-state solution will be a superposition of harmonic oscillations with frequencies equal to that of $r(t)$ and integer multiples of the latter.**

1) Dennis G. zill, Advanced Engineering Mathematics 3rd, Jones and Bartlett, p 113

Particular Solution – Another Superposition

Theorem 3.7

Superposition Principle– Nonhomogeneous Equations

Let $y_{p1}, y_{p2}, \ldots, y_{pk}$ be k particular solution of the nonhomogeneous linear nth-order differential equation on an interval I corresponding, in turn, to k distinct functions $g_1, g_2, \ldots, g_k$. That is, y_{pi} denotes a particular solution of the corresponding differential equation $a_n(x)y^n + a_{n-1}(x)y^{n-1} + \ldots + a_1(x)y' + a_0(x)y = g_i(x)$ where $i = 1,2,\ldots,k$.

Then $y_p = y_{p1}(x) + y_{p2}(x) + \ldots + y_{pk}(x)$ is a particular solution of

$$a_n(x)y^n + a_{n-1}(x)y^{n-1} + \ldots + a_1(x)y' + a_0(x)y = g_1(x) + g_2(x) + \ldots + g_k(x)$$

Let y_n is one particular solution of following ODE corresponding to distinct functions.

$$\ddot{y} + 0.05\dot{y} + 25y = r(t),\ r(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nt$$

$$\ddot{y}_n + 0.05\dot{y}_n + 25y_n = \frac{4}{n^2\pi}\cos nt,\ (n = 1,\ 3,\ 5,\ldots)$$

then, the particular solution of the given ODE is

$$y_p = y_1 + y_3 + y_5 + \ldots$$

Solving Particular Solution

$$\ddot{y}_n + 0.05\dot{y}_n + 25y_n = \frac{4}{n^2\pi}\cos nt \ , \ (n = 1,3,5,\cdots)$$

· solution of above ODE y_n

Let $y_n = A_n \cos nt + B_n \sin nt$

Then $\dot{y}_n = -nA_n \sin nt + nB_n \cos nt$

$$\ddot{y}_n = -n^2 A_n \cos nt - n^2 B_n \sin nt$$

Substituting y_n, $\dot{y}_n$, $\ddot{y}_n$

$$\ddot{y}_n + 0.05\dot{y}_n + 25y_n = -n^2 A_n \cos nt - n^2 B_n \sin nt$$
$$+0.05(nB_n \cos nt - nA_n \sin nt)$$
$$+25(A_n \cos nt + B_n \sin nt)$$
$$= \frac{4}{n^2\pi}\cos nt$$

$$-n^2 A_n \cos nt - n^2 B_n \sin nt + 0.05(nB_n \cos nt - nA_n \sin nt) + 25(A_n \cos nt + B_n \sin nt) = \frac{4}{n^2\pi}\cos nt$$

$$(-n^2 A_n + 25A_n + 0.05nB_n)\cos nt - (0.05nA_n + n^2 B_n - 25B_n)\sin nt = \frac{4}{n^2\pi}\cos nt + 0 \cdot \sin nt$$

$$(n^2 - 25)A_n - 0.05nB_n = -\frac{4}{n^2\pi}, \ 0.05nA_n + (n^2 - 25)B_n = 0$$

the method of undetermined coefficient (Sec. 2.7)

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n (n = 0,1,\cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k\cos\omega x$ $k\sin\omega x$	$K\cos\omega x + M\sin\omega x$
$ke^{\alpha x}\cos\omega x$ $ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(K\cos\omega x + M\sin\omega x)$

Solving Particular Solution

$(A_1 \approx 0.05305,\ B_1 \approx 0.00011),\ (A_7 \approx -0.0011,\ B_7 \approx 0.00002)$
$(A_3 \approx 0.0088,\ B_3 \approx 0.00008),\ (A_9 \approx -0.0003,\ B_9 \approx 0.000002)$
$(A_5 \approx 0.0088,\ B_5 \approx 0.20372),\ (A_{11} \approx -0.0001,\ B_{11} \approx 0.000001)$

$$(n^2-25)A_n - 0.05nB_n = -\frac{4}{n^2\pi}$$
$$0.05nA_n + (n^2-25)B_n = 0$$

$$\Rightarrow \begin{pmatrix} (n^2-25) & -0.05n \\ 0.05n & (n^2-25) \end{pmatrix}\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} -\frac{4}{n^2\pi} \\ 0 \end{pmatrix}$$

·Cramer's Rule for linear system

$$\therefore A_n = \frac{\frac{4}{n^2\pi}(25-n^2)}{D_n},\ B_n = \frac{\frac{4}{n^2\pi}(0.05n)}{D_n},\ D_n = \begin{vmatrix} (n^2-25) & -0.05n \\ 0.05n & (n^2-25) \end{vmatrix} = (n^2-25)^2 + (0.05n)^2$$

$$y_n = A_n \cos nt + B_n \sin nt$$

·Superposition Principle

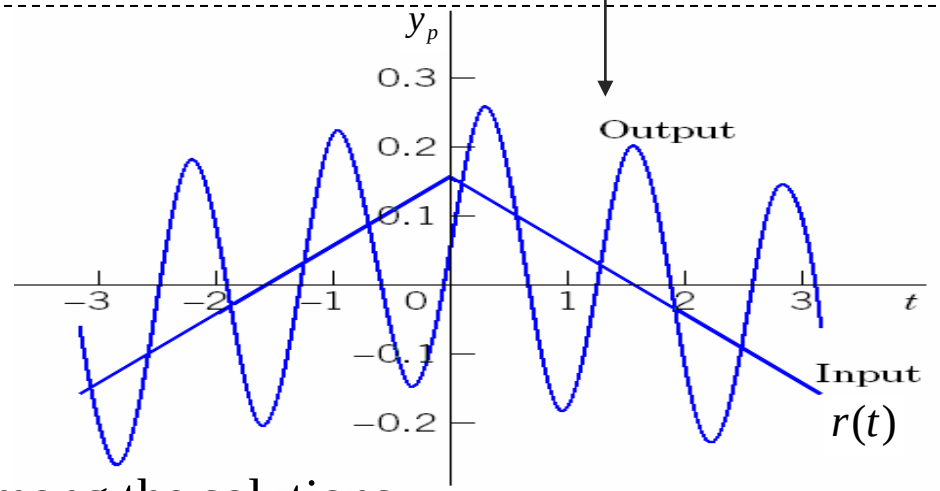
$$A_n = \frac{4(25-n^2)}{n^2\pi D_n},\ B_n = \frac{0.2n}{n^2\pi D_n},\ n = 1,3,5,\ldots$$

$$y_p = y_1 + y_3 + y_5 + \ldots$$

·Amplitude of solution y_n

$$C_n = \sqrt{A_n^{\,2} + B_n^{\,2}}$$

$C_1 = 0.05305,\ C_7 = 0.01083$
$C_3 = 0.00884,\ C_9 = 0.00028$
$C_5 = 0.20372,\ C_{11} = 0.00011$



·C_5 is so large that y_5 is dominating term among the solutions.

BACKGROUND OF FOURIER SERIES

Background of Fourier Series

① **Sturm-Liouville Boundary Value Problem**

Solve (Equation): $\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$

Subject to (Boundary Condition): $A_1 y(a) + B_1 y'(a) = 0$

$A_2 y(b) + B_2 y'(b) = 0$

Theorem 12.3 Properties of the Regular Sturm-Liouville Problem

(a) There exist an infinite number of real eigenvalues that can be arranged in increasing order $\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots \lambda_n \to \infty$ such that $n \to \infty$ as

(b) For each eigenvalues there is only one eigenfunction (except for nonzero constant multiples)

(c) Eigenfunctions corresponding to different eigenvalues are linearly independent

(d) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ on interval $[a, b]$

② Solutions of Sturm-Liouville equation are Eigenfunctions (All of the solutions are linearly independent and Orthogonal)

Definition 12.4 Orthogonal Set/Weight Function

A set of real-valued functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \cdots\}$ is said to be orthogonal with respect to a weight function $w(x)$ on an interval $[a, b]$ if

$$\int_a^b w(x)\phi_m(x)\phi_n(x)dx = 0, \quad m \neq n$$

③ Solutions are Basis functions.

ex) **Sturm-Liouville Boundary Value Problem**

$$y'' + \lambda y = 0 \quad (\lambda = k^2)$$

boundary condition: $y(\pi) = y(-\pi)$, $y'(\pi) = y'(-\pi)$

solution

$$y(x) = A\cos kx + B\sin kx \quad (k = 0, 1, 2, \cdots)$$

'Basis functions' and 'Orthogonal Set'

④ Series solution using Orthogonal Set

Fourier Sine/Cosine Series

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Fourier-Bessel Series

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x)$$

Fourier-Legendre Series

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$$

$f(x)$:a given function

Complex Fourier Series

For some engineering problem, it is actually more convenient to represent a function in an infinite series of *complex-valued* function of a real variables x such as $e^{inx}, (n = 1, 2, 3..)$

Definition 12.7

Complex Fourier Series

The complex Fourier series of functions f defined on an interval $(-p, p)$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p}$$

where,

$$c_n = \frac{1}{2p}\int_{-p}^{p} f(x)e^{-in\pi x/p}dx, \quad n = 0, \pm 1, \pm 2, \ldots$$

Complex Fourier Series

$$e^{ix} = \cos x + i \sin x \qquad e^{-ix} = \cos x - i \sin x$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos \frac{n\pi}{p} x = \frac{e^{in\pi x/p} + e^{-in\pi x/p}}{2}$$

$$\sin \frac{n\pi}{p} x = \frac{e^{in\pi x/p} - e^{-in\pi x/p}}{2i}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \frac{e^{in\pi x/p} + e^{-in\pi x/p}}{2} + b_n \frac{e^{in\pi x/p} - e^{-in\pi x/p}}{2i} \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{1}{2}[a_n - ib_n] e^{in\pi x/p} + \frac{1}{2}[a_n + ib_n] e^{-in\pi x/p} \right)$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/p} + \sum_{n=1}^{\infty} c_{-n} e^{-in\pi x/p}$$

where,

$$c_0 = \frac{1}{2} \cdot a_0, \quad c_n = \frac{1}{2}[a_n - ib_n], \quad c_{-n} = \frac{1}{2}[a_n + ib_n]$$

Complex Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x \right)$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/p} + \sum_{n=1}^{\infty} c_{-n} e^{-in\pi x/p}$$

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}xdx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}xdx$$

$$c_0 = \frac{1}{2}a_0 = \frac{1}{2}\cdot\frac{1}{p}\int_{-p}^{p} f(x)dx$$

$$c_n = \frac{1}{2}\left[\frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}xdx - i\frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}xdx \right]$$

$$= \frac{1}{2}\left[\frac{1}{p}\int_{-p}^{p} f(x)\left(\cos\frac{n\pi}{p}x - i\sin\frac{n\pi}{p}x \right)dx \right]$$

$$= \frac{1}{2}\left[\frac{1}{p}\int_{-p}^{p} f(x)e^{-\frac{n\pi}{p}x}dx \right]$$

$$e^{ix} = \cos x + i\sin x$$

$$e^{-ix} = \cos x - i\sin x$$

Complex Fourier Series

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x \right)$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}xdx$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/p} + \sum_{n=1}^{\infty} c_{-n} e^{-in\pi x/p}$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}xdx$$

$$c_0 = \frac{1}{2}a_0 = \frac{1}{2}\cdot\frac{1}{p}\int_{-p}^{p} f(x)dx$$

$$c_0 = \frac{1}{2}\cdot a_0,\ c_n = \frac{1}{2}[a_n - ib_n],\ c_{-n} = \frac{1}{2}[a_n + ib_n]$$

$$c_{-n} = \frac{1}{2}\left[\frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}xdx + i\frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}xdx\right]$$

$$= \frac{1}{2}\left[\frac{1}{p}\int_{-p}^{p} f(x)\left(\cos\frac{n\pi}{p}x + i\sin\frac{n\pi}{p}x\right)dx\right]$$

$$e^{ix} = \cos x + i\sin x$$

$$e^{-ix} = \cos x - i\sin x$$

$$= \frac{1}{2p}\int_{-p}^{p} f(x)e^{in\pi x/p}dx$$

Complex Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x \right)$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/p} + \sum_{n=1}^{\infty} c_{-n} e^{-in\pi x/p}$$

← **Complex Fourier Series**

$$c_{-n} = \frac{1}{2p}\int_{-p}^{p} f(x)e^{in\pi x/p}dx$$

$$c_n = \frac{1}{2p}\int_{-p}^{p} f(x)e^{-in\pi x/p}dx$$

$$c_0 = \frac{1}{2p}\int_{-p}^{p} f(x)dx$$

where,

$$c_0 = \frac{1}{2}\cdot a_0$$

$$c_n = \frac{1}{2}[a_n - ib_n],\ c_{-n} = \frac{1}{2}[a_n + ib_n]$$

Proof: f(x) is a real-valued function-> Home work

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x \right)$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/p} + \sum_{n=1}^{\infty} c_{-n} e^{-in\pi x/p}$$

← **Complex Fourier Series**

$$c_0 = \frac{1}{2}\cdot a_0, \quad c_n = \frac{1}{2}[a_n - ib_n], \quad c_{-n} = \frac{1}{2}[a_n + ib_n]$$

Complex Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x \right)$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/p} + \sum_{n=1}^{\infty} c_{-n} e^{-in\pi x/p}$$ ← Complex Fourier Series

$$= \left(\frac{1}{2p}\int_{-p}^{p} f(x)dx \right) + \sum_{n=1}^{\infty}\left(\frac{1}{2p}\int_{-p}^{p} f(x)e^{-in\pi x/p}dx \right) e^{in\pi x/p} + \sum_{n=1}^{\infty}\left(\frac{1}{2p}\int_{-p}^{p} f(x)e^{in\pi x/p}dx \right) e^{-in\pi x/p}$$

$$\text{when, } n=0 \quad \sum_{n=0}^{0}\left(\frac{1}{2p}\int_{-p}^{p} f(x)\underbrace{e^{-in\pi x/p}}_{=1}dx \right)\underbrace{e^{in\pi x/p}}_{=1}$$

$$n \to -n$$

$$\sum_{n=-\infty}^{-1}\left(\frac{1}{2p}\int_{-p}^{p} f(x)e^{-in\pi x/p}dx \right) e^{in\pi x/p}$$

$$= \sum_{n=-\infty}^{\infty}\left(\frac{1}{2p}\int_{-p}^{p} f(x)e^{-in\pi x/p}dx \right) e^{in\pi x/p}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p} \quad , \quad c_n = \frac{1}{2p}\int_{-p}^{p} f(x)e^{-in\pi x/p}dx \quad , \; n = 0, \pm 1, \pm 2, \ldots$$

← in a more compact manner

$$f(x)=\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p},\quad c_n=\frac{1}{2p}\int_{-p}^{p} f(x)e^{-in\pi x/p}dx,\quad n=0,\pm1,\pm2,\ldots$$

☑ Example 1

Complex Fourier Series

Expand

in a complex Fourier series

$$f(x)=e^{-x},\quad -\pi<x<\pi$$

$$f(x)=\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p},\quad c_n=\frac{1}{2p}\int_{-p}^{p} f(x)e^{-in\pi x/p}dx$$

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-x}e^{-inx}dx=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-(in+1)x}dx$$

$$=-\frac{1}{2\pi(in+1)}[e^{-(in+1)\pi}-e^{(in+1)\pi}]$$

$$e^{ix}=\cos x+i\sin x$$

$$e^{-ix}=\cos x-i\sin x$$

$$e^{-(in+1)\pi}=e^{-\pi}(\cos n\pi-i\sin n\pi)=(-1)^n e^{-\pi}$$

$$e^{(in+1)\pi}=e^{\pi}(\cos n\pi+i\sin n\pi)=(-1)^n e^{\pi}$$

since $\cos n\pi=(-1)^n$ *and* $\sin n\pi=0$

Hence

$$c_n=(-1)^n\frac{(e^{\pi}-e^{-\pi})}{2(in+1)\pi}=(-1)^n\frac{(e^{\pi}-e^{-\pi})}{2}\frac{1}{\pi}\frac{1}{(in+1)}$$

$$=(-1)^n\frac{\sinh\pi}{\pi}\frac{(1-in)}{(n^2+1)}$$

then

$$f(x)=\frac{\sinh\pi}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\frac{(1-in)}{(n^2+1)}e^{inx}$$

Proof: f(x) is a real-valued function-> Home work

$$f(x) = e^{-x},\ -\pi < x < \pi \quad \Longrightarrow \quad f(x) = \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty} c_n e^{inx}$$:complex

$$c_n = (-1)^n \frac{1-in}{n^2+1}$$:complex

Is $f(x)$ complex too?

Complex Fourier Series

▪Fundamental Frequency

Fundamental period of the function is $T = 2p$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x \right) \quad \textbf{and} \quad f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p}$$

$$p = T/2$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi}{T}x + b_n \sin\frac{2n\pi}{T}x \right) \quad \textbf{and} \quad f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2in\pi x/T}$$

$$\boxed{\omega = \frac{2\pi}{T}}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\omega x + b_n \sin n\omega x \right) \quad \textbf{and} \quad f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega x}$$

Since T is **fundamental period**, ω is **fundamental angular frequency.**

Frequency Spectrum

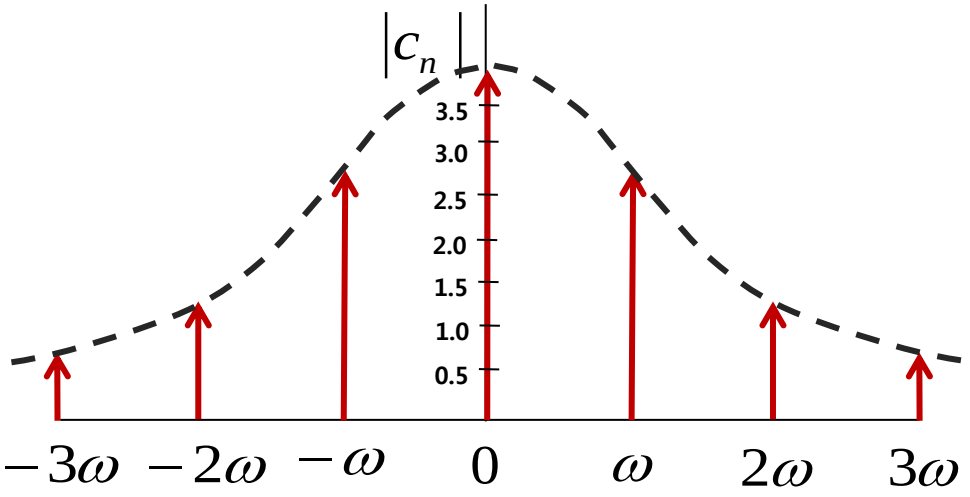
$$f(x)=\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p},\quad c_n=\frac{1}{2p}\int_{-p}^{p} f(x)e^{-in\pi x/p}dx,\quad n=0,\pm1,\pm2,\ldots$$

▪If f is periodic and has fundamental period T, the plot of the points $(n\omega, |c_n|)$, where ω is the fundamental angular frequency and c_n are the coefficient, is called frequency spectrum of f.

$$\omega=\frac{2\pi}{T}=\frac{2\pi}{2p}=\frac{\pi}{p}$$

▪Frequency Spectrum

$$f(x)=\sum_{n=-\infty}^{\infty} c_n e^{in\omega x}$$



$$f(x)=\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p},\quad c_n=\frac{1}{2p}\int_{-p}^{p} f(x)e^{-in\pi x/p}dx,\quad n=0,\pm1,\pm2,\ldots$$

$$\omega=\frac{2\pi}{T}=\frac{2\pi}{2p}=\frac{\pi}{p}$$

☑ Example 2 : Frequency Spectrum

$$f(x)=e^{-x},\quad -\pi<x<\pi \qquad p=\pi$$

$$\Rightarrow f(x)=\frac{\sinh\pi}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\frac{1-in}{n^2+1}e^{inx}$$

Since T is fundamental period,

ω is fundamental angular frequency. $\quad \omega=\dfrac{2\pi}{T}$

In Example 1, $\omega = 1$ so that $n\omega$ takes on the values $\pm 1, \pm 2, \ldots$

Then

$$f(x)=\sum_{n=-\infty}^{\infty}(-1)^n\frac{\sinh\pi}{\pi}\left(\frac{1-in}{n^2+1}\right)e^{inx}$$

$$f(x)=\sum_{n=-\infty}^{\infty}c_n e^{inx}$$

$$c_n=\frac{\sinh\pi}{\pi}(-1)^n\frac{1-in}{n^2+1} \quad \text{:complex} \quad \therefore c_n=\alpha+i\beta$$

$$c_n=\frac{\sinh\pi}{\pi}(-1)^n\frac{1}{n^2+1}+\left(-\frac{\sinh\pi}{\pi}(-1)^n\frac{in}{n^2+1}\right)$$

$$\alpha=\frac{\sinh\pi}{\pi}(-1)^n\frac{1}{n^2+1}$$

$$\beta=\frac{\sinh\pi}{\pi}(-1)^{n+1}\frac{n}{n^2+1}$$

Using $|\alpha+i\beta|=\sqrt{\alpha^2+\beta^2}$

$$|c_n|=\sqrt{\alpha^2+\beta^2}$$

$$|c_n|=\frac{\sinh\pi}{\pi}\frac{1}{\sqrt{n^2+1}}$$

$$f(x)=\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p},\quad c_n=\frac{1}{2p}\int_{-p}^{p} f(x)e^{-in\pi x/p}dx,\quad n=0,\pm1,\pm2,\ldots$$

$$\omega=\frac{2\pi}{T}=\frac{2\pi}{2p}=\frac{\pi}{p}$$

$$|c_n|=\frac{\sinh\pi}{\pi}\frac{1}{\sqrt{n^2+1}}$$

$$|c_0|=\frac{\sinh\pi}{\pi}\frac{1}{\sqrt{1}}\approx 3.676$$

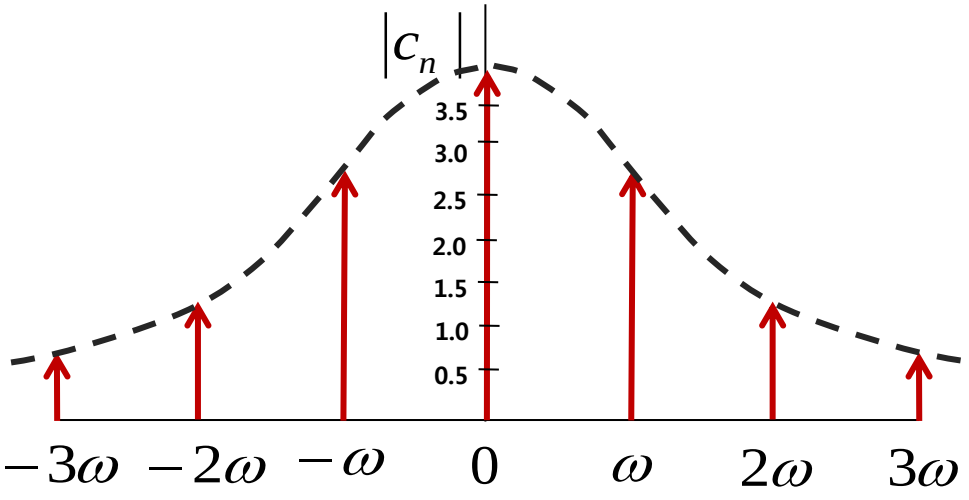
$$|c_1|=\frac{\sinh\pi}{\pi}\frac{1}{\sqrt{1^2+1}}\approx 2.599$$

$$|c_2|=\frac{\sinh\pi}{\pi}\frac{1}{\sqrt{2^2+1}}\approx 1.644$$

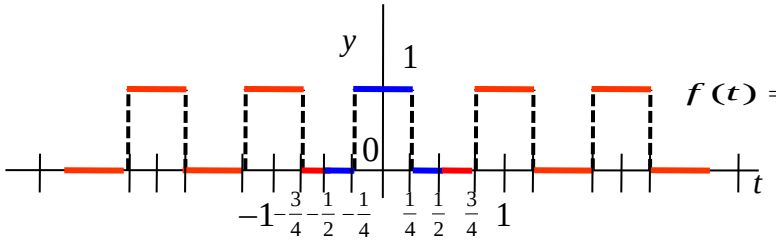
$$|c_3|=\frac{\sinh\pi}{\pi}\frac{1}{\sqrt{3^2+1}}=1.162$$

▪If f is periodic and has fundamental period T, the plot of the points $(n\omega,|c_n|)$, where ω is the fundamental angular frequency and c_n are the coefficient, is called frequency spectrum of f .

n	-3	-2	-1	0	1	2	3
$\lvert c_n\rvert$	1.162	1.644	2.599	3.676	2.599	1.644	1.162



Example 3: Find the amplitude as a function of angular frequency of the periodic square wave or periodic pulse shown in Fig. The wave is periodic extension of function $f(t)$.



$$f(t)=\begin{cases}0, & -\frac{1}{2}<t<-\frac{1}{4}\\ 1, & -\frac{1}{4}<t<\frac{1}{4}\\ 0, & \frac{1}{4}<t<\frac{1}{2}\end{cases}$$

$$\omega=\frac{\pi}{p},\ p=1/2$$

$$f(t)=\sum_{n=-\infty}^{\infty}c_n e^{in\pi t/p},\ c_n=\frac{1}{2p}\int_{-p}^{p}f(t)e^{-in\pi t/p}dt$$

$$c_n=\frac{1}{2p}\int_{-p}^{p}f(t)e^{-in\pi t/p}dt\ ,n=0,\pm1,\pm2,\ldots$$

$$=\int_{-1/4}^{1/4}e^{-2in\pi t}dt$$

$$=\frac{e^{-2in\pi x}}{-2in\pi}\Big|_{-1/4}^{1/4}=\frac{1}{n\pi}\frac{-e^{-in\pi/2}+e^{in\pi/2}}{2i}=\frac{1}{n\pi}\sin\frac{n\pi}{2}$$

$$c_{-n}=\frac{1}{-n\pi}\sin\frac{-n\pi}{2}=\frac{1}{n\pi}\sin\frac{n\pi}{2}$$

Since last term is not valid at n=0, we compute that term separately:

$$c_0=\int_{-1/4}^{1/4}1\cdot dx=\frac{1}{2}$$

$$\therefore f(t)=\frac{1}{2}+\sum_{-\infty}^{\infty}\left(\frac{1}{n\pi}\sin\frac{n\pi}{2}\right)e^{2in\pi t},\ n\neq0$$

Recast the complex Fourier series into the Fourier series

$$f(t)=\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{1}{n\pi}\sin n\pi\right)e^{2in\pi t}+\sum_{n=-\infty}^{-1}\left(\frac{1}{n\pi}\sin n\pi\right)e^{2in\pi t}$$

$$=\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{1}{n\pi}\sin n\pi\right)e^{2in\pi t}+\sum_{n=1}^{\infty}\left(\frac{1}{n\pi}\sin n\pi\right)e^{2i(-n)\pi t}$$

$$=\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{1}{n\pi}\sin n\pi\right)(\cos2n\pi t+i\sin2n\pi t)$$

$$+\sum_{n=1}^{\infty}\left(\frac{1}{n\pi}\sin n\pi\right)(\cos2(-n)\pi t+i\sin2(-n)\pi t)$$

$$=\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{1}{n\pi}\sin n\pi\right)\cos2n\pi t+\sum_{n=1}^{\infty}i\left(\frac{1}{n\pi}\sin n\pi\right)\sin2n\pi t$$

$$+\sum_{n=1}^{\infty}\left(\frac{1}{n\pi}\sin n\pi\right)\cos2n\pi t-\sum_{n=1}^{\infty}i\left(\frac{1}{n\pi}\sin n\pi\right)\sin2n\pi t$$

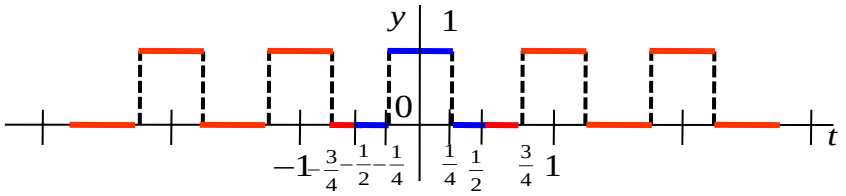
= amplitude

=frequency

$$\therefore f(t)=\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\sin n\pi\right)\cos2n\pi t$$

1) Dennis G. zill, Advanced Engineering Mathematics 3rd, Jones and Bartlett, p 670

Function values for different n



For $n = 0$

$$f(t) = c_0 = \frac{1}{2}$$

For $n = 1$

$$f(t) = \frac{1}{2} + \sum_{n=1}^{1}\left(\frac{2}{n\pi}\sin\frac{n\pi}{2}\right)\cos 2n\pi t = \frac{1}{2} + \frac{2}{\pi}\cos 2\pi t$$

For $n = 3$

$$f(t) = \frac{1}{2} + \sum_{n=1}^{3}\left(\frac{2}{n\pi}\sin\frac{n\pi}{2}\right)\cos 2n\pi t = \frac{1}{2} + \frac{2}{\pi}\cos 2\pi t - \frac{2}{3\pi}\cos 6\pi t$$

For $n = 5$

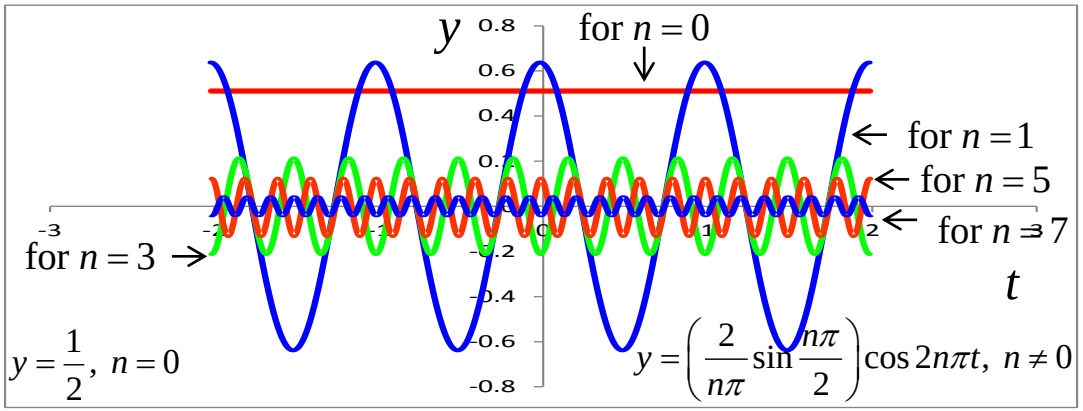
$$f(t) = \frac{1}{2} + \sum_{n=1}^{5}\left(\frac{2}{n\pi}\sin\frac{n\pi}{2}\right)\cos 2n\pi t = \frac{1}{2} + \frac{2}{\pi}\cos 2\pi t - \frac{2}{3\pi}\cos 6\pi t + \frac{2}{5\pi}\cos 10\pi t$$

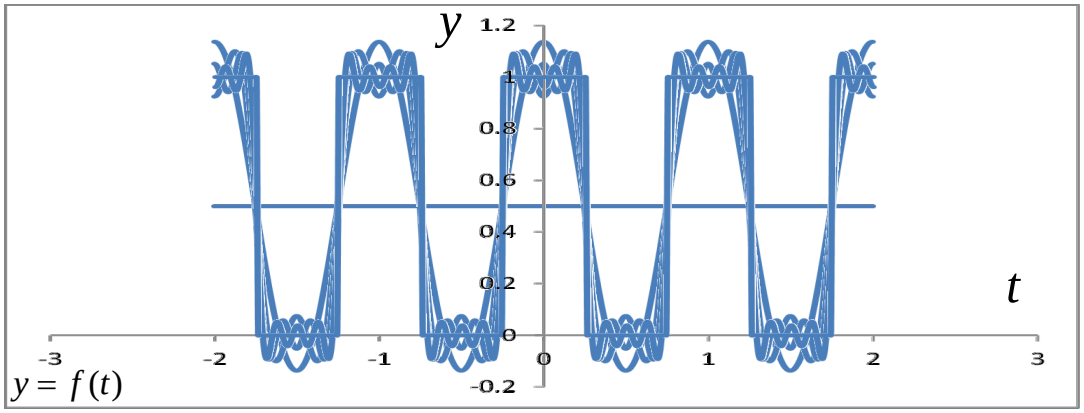
For $n = 7$

$$f(t) = \frac{1}{2} + \sum_{n=1}^{7}\left(\frac{2}{n\pi}\sin\frac{n\pi}{2}\right)\cos 2n\pi t = \frac{1}{2} + \frac{2}{\pi}\cos 2\pi t - \frac{2}{3\pi}\cos 6\pi t + \frac{2}{5\pi}\cos 10\pi t - \frac{2}{7\pi}\cos 14\pi t$$

For $n = \infty$

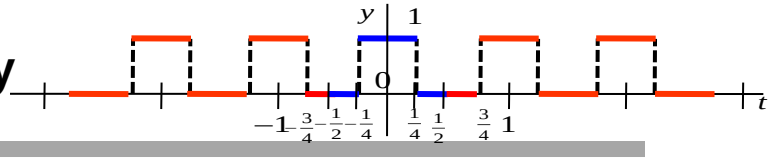
$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\sin\frac{n\pi}{2}\right)\cos 2n\pi t = \frac{1}{2} + \frac{2}{\pi}\cos 2\pi t - \frac{2}{3\pi}\cos 6\pi t + \frac{2}{5\pi}\cos 10\pi t - \frac{2}{7\pi}\cos 14\pi t + \frac{2}{9\pi}\cos 18\pi t + \ldots$$

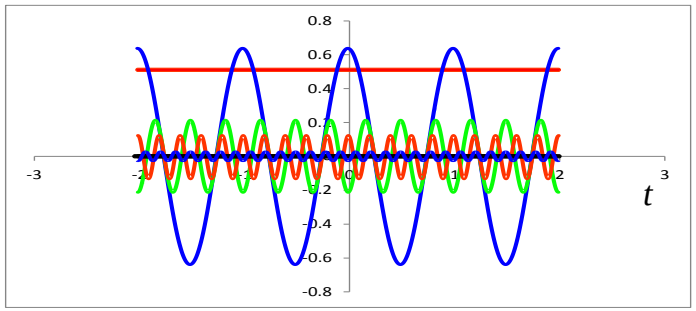




$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\sin\frac{n\pi}{2}\right)\cos 2n\pi t$$

The amplitude as a function of angular frequency

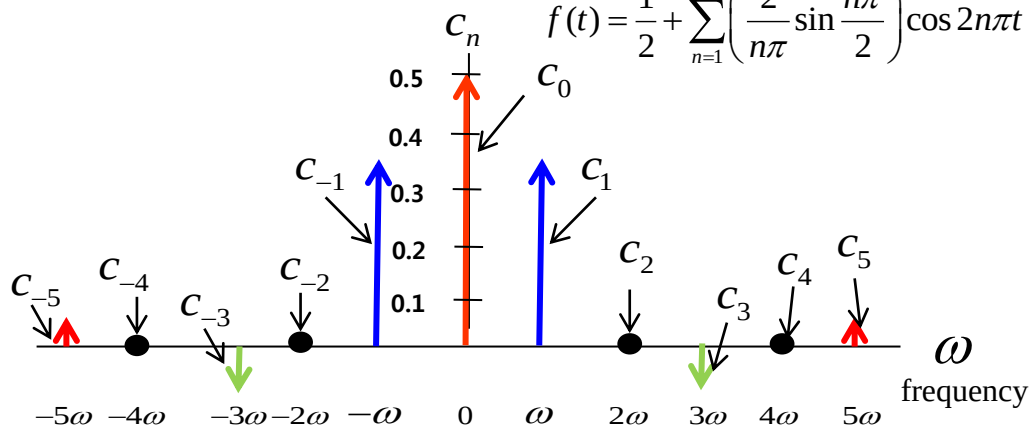




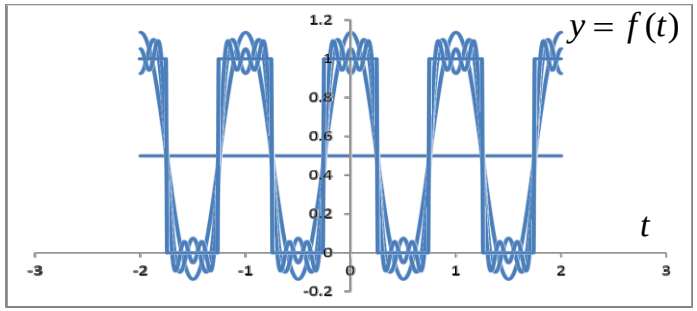
$$y=\frac{1}{2},\ n=0$$

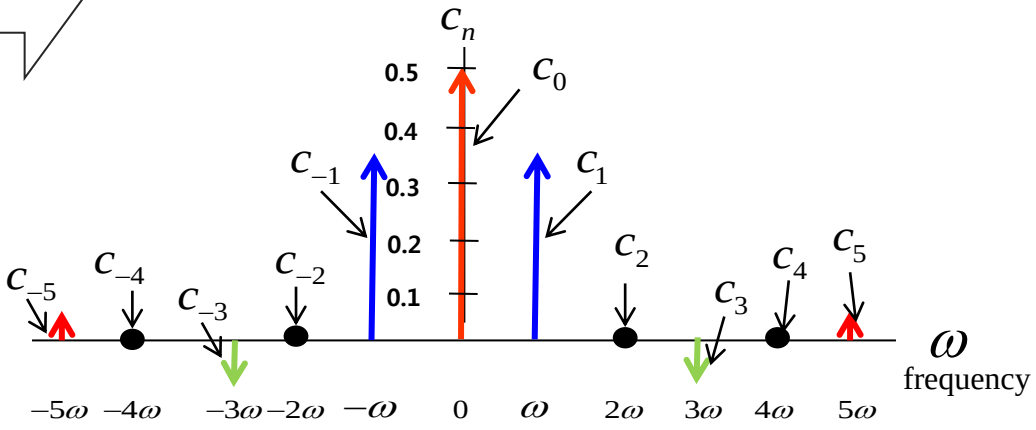
$$y=\left(\frac{2}{n\pi}\sin\frac{n\pi}{2}\right)\cos 2n\pi t,\ n\neq 0$$

$$f(t)=\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\sin\frac{n\pi}{2}\right)\cos 2n\pi t$$



the amplitude as a function of angular frequency





$$f(x)=\frac{1}{2}+\frac{2}{\pi}\cos 2\pi x+\frac{2}{2\pi}\sin\frac{2\pi}{2}\cos 4\pi x-\frac{2}{3\pi}\cos 6\pi x+\frac{2}{4\pi}\sin\frac{4\pi}{2}\cos 8\pi x+\frac{2}{5\pi}\cos 10\pi x+\frac{2}{6\pi}\sin\frac{6\pi}{2}\cos 12\pi x-\frac{2}{7\pi}\cos 14\pi x+\frac{2}{8\pi}\sin\frac{8\pi}{2}\cos 16\pi x+\frac{2}{9\pi}\cos 18\pi x+\ldots$$

Frequency Spectrum

$$\therefore f(t) = \frac{1}{2} + \sum_{-\infty}^{\infty} \left(\frac{1}{n\pi} \sin \frac{n\pi}{2} \right) e^{2in\pi t}, \; n \neq 0$$

$$c_0 = \frac{1}{2}$$

$$c_1 = c_{-1} = \frac{1}{\pi} \sin \frac{\pi}{2} = \frac{1}{\pi}$$

$$c_2 = c_{-2} = \frac{1}{2\pi} \sin \frac{2\pi}{2} = 0$$

$$c_3 = c_{-3} = \frac{1}{3\pi} \sin \frac{3\pi}{2} = -\frac{1}{3\pi}$$

$$c_4 = c_{-4} = \frac{1}{4\pi} \sin \frac{4\pi}{2} = 0$$

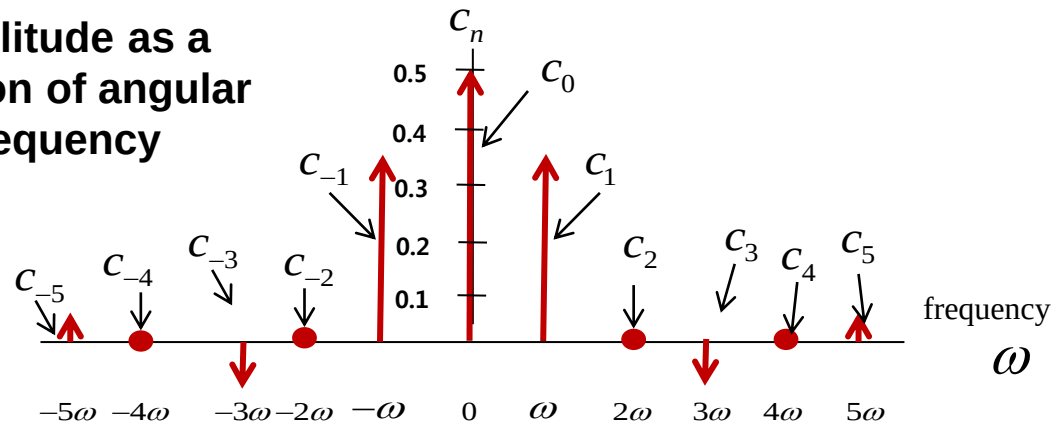
$$c_5 = c_{-5} = \frac{1}{5\pi} \sin \frac{5\pi}{2} = \frac{1}{5\pi}$$

$$\vdots$$

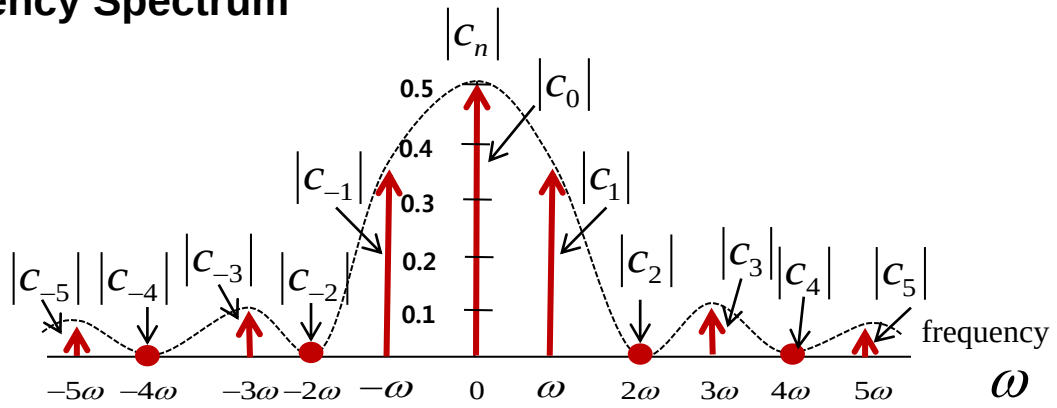
$$c_n = c_{-n} = \frac{1}{n\pi} \sin \frac{n\pi}{2}$$

n	-5	-4	-3	-2	-1	0	1	2	3	4	5
c_n	$\frac{1}{5\pi}$	0	$-\frac{1}{3\pi}$	0	$\frac{1}{\pi}$	$\frac{1}{2}$	$\frac{1}{\pi}$	0	$-\frac{1}{3\pi}$	0	$\frac{1}{5\pi}$

Amplitude as a function of angular frequency



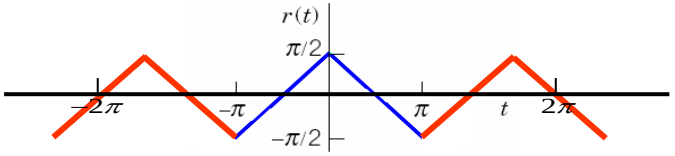
Frequency Spectrum



Application of Fourier Series (2)

$$\ddot{y}+0.05\dot{y}+25y=r(t),\quad r(t)=\begin{cases} t+\dfrac{\pi}{2} & \text{if } -\pi<t<0 \\ -t+\dfrac{\pi}{2} & \text{if } 0<t<\pi \end{cases}$$
$$r(t+2\pi)=r(t)$$

✓ Application 2) Transform between time domain and frequency domain.



ex) Interpretation of Fourier series

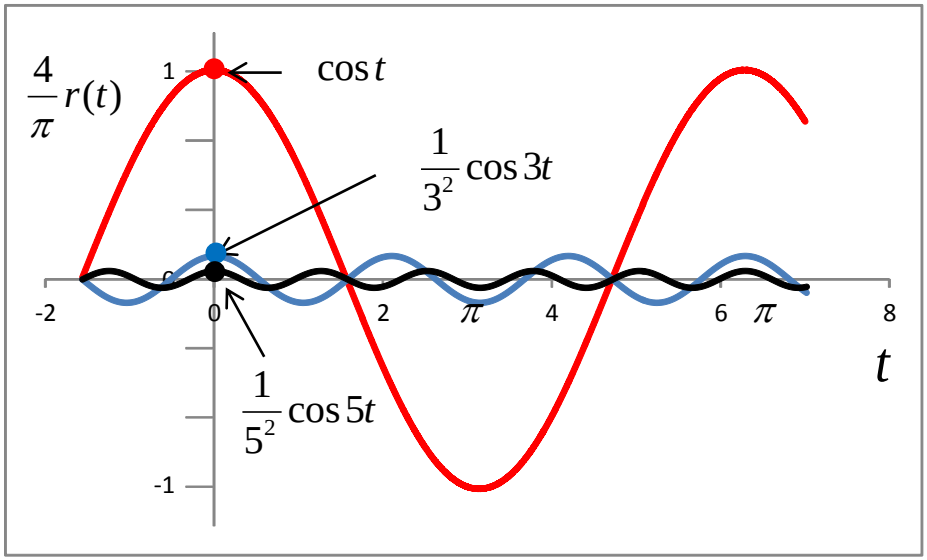
Time Domain ⟵ Inverse ⟵ Frequency Domain

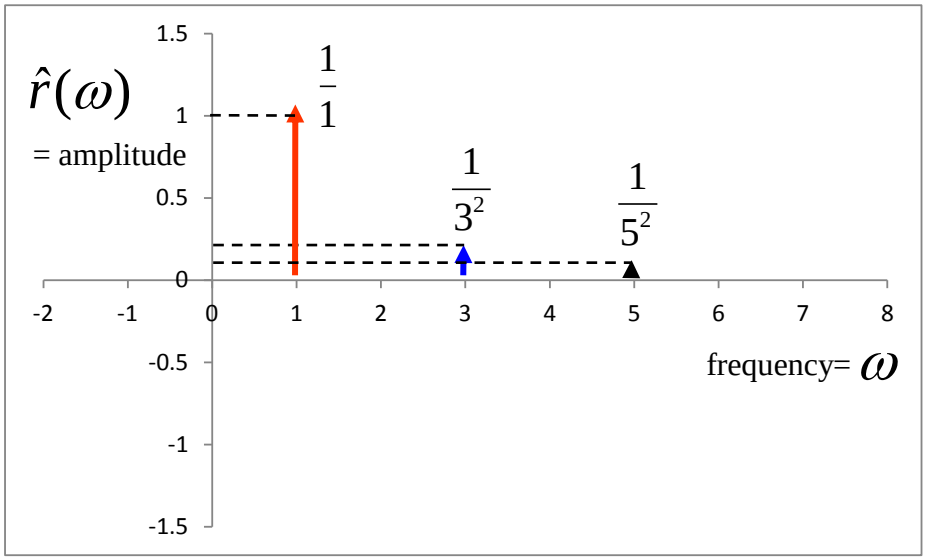
Frequency and amplitude ⟶

By definition

$$\omega=\frac{2\pi}{T}=\frac{2\pi}{2p}=\frac{\pi}{p}$$
$$T=1$$
$$p=1/2,\quad \omega=2\pi$$

$$r(t)=\frac{4}{\pi}\left(\cos t+\frac{1}{3^2}\cos 3t+\frac{1}{5^2}\cos 5t+\cdots\right)$$





If we represent frequency spectrum using frequency and amplitude,
we can reconstruct the function of time f(t).

HOMEWORK(120408)-SOLUTION

Proof: f(x) is a real-valued function-> Homework 1

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{p}x+b_n\sin\frac{n\pi}{p}x\right)$$

$$=c_0+\sum_{n=1}^{\infty}c_n e^{in\pi x/p}+\sum_{n=1}^{\infty}c_{-n}e^{-in\pi x/p}$$

← **Complex Fourier Series**

$$c_0=\frac{1}{2}\cdot a_0,\quad c_n=\frac{1}{2}[a_n-ib_n],\quad c_{-n}=\frac{1}{2}[a_n+ib_n]$$

Proof: f(x) is a real-valued function

$$f(x)=c_0+\sum_{n=1}^{\infty}c_n e^{in\pi x/p}+\sum_{n=1}^{\infty}c_{-n}e^{-in\pi x/p}=\frac{a_0}{2}+\sum_{n=1}^{\infty}\frac{1}{2}[a_n-ib_n]e^{in\pi x/p}+\sum_{n=1}^{\infty}\frac{1}{2}[a_n+ib_n]e^{-in\pi x/p}$$

$$=\frac{a_0}{2}+\frac{1}{2}\sum_{n=1}^{\infty}\left\{[a_n-ib_n](\cos\frac{n\pi}{p}x+i\sin\frac{n\pi}{p}x)+[a_n+ib_n](\cos\frac{n\pi}{p}x-i\sin\frac{n\pi}{p}x)\right\}$$

$$=\frac{a_0}{2}+\frac{1}{2}\sum_{n=1}^{\infty}\left\{\left[a_n\cos\frac{n\pi}{p}x+b_n\sin\frac{n\pi}{p}x+i\left(a_n\sin\frac{n\pi}{p}x-b_n\cos\frac{n\pi}{p}x\right)\right]+\left[a_n\cos\frac{n\pi}{p}x+b_n\sin\frac{n\pi}{p}x+i\left(-a_n\sin\frac{n\pi}{p}x+b_n\cos\frac{n\pi}{p}x\right)\right]\right\}$$

$$\therefore f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{p}x+b_n\sin\frac{n\pi}{p}x\right)$$ ← **real-valued function**

Proof: f(x) is a real-valued function-> Homework 2

$$f(x)=e^{-x},\ -\pi<x<\pi \quad \Rightarrow \quad f(x)=\frac{\sinh\pi}{\pi}\sum_{n=-\infty}^{\infty}c_n e^{inx}$$:complex

$f(x)$ complex too?

$$c_n=(-1)^n\frac{1-in}{n^2+1}$$:complex

Proof:

from $f(x)=\frac{\sinh\pi}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\frac{1-in}{n^2+1}e^{inx}$

$$(1-in)(\cos nx+i\sin nx)=\cos nx+n\sin nx+i(\sin nx-n\cos nx)$$

For positive integer : n=m (m>0): $\cos mx+m\sin mx+i(\sin mx-m\cos mx)$

For negative integer : n=-m (m>0): $\cos mx+m\sin mx+i(-\sin mx+m\cos mx)$

:opposite in sign

→**real part of** $(-1)^n\frac{1-in}{n^2+1}e^{inx}$ **will be remained in the summation**

$\therefore f(x):real\ \ function$

A real function $f(x)$ is represented by a complex series ; a series in which the coefficients c_n are complex numbers*

Half Range Expansions of Fourier Series -> Homework

$$f(x) = x^2, \quad 0 < x < L$$

(a) In a cosine series,
(b) In a sine series,
(c) In a Fourier series

Fourier Series $(0, 2p) = (0, L)$

$$a = 0, \quad 2p = L$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x \right)$$

where,

$$a_0 = \frac{1}{p}\int_0^{2p} f(x)dx, \; a_n = \frac{1}{p}\int_0^{2p} f(x)\cos\frac{n\pi}{p}xdx$$

$$b_n = \frac{1}{p}\int_0^{2p} f(x)\sin\frac{n\pi}{p}xdx$$

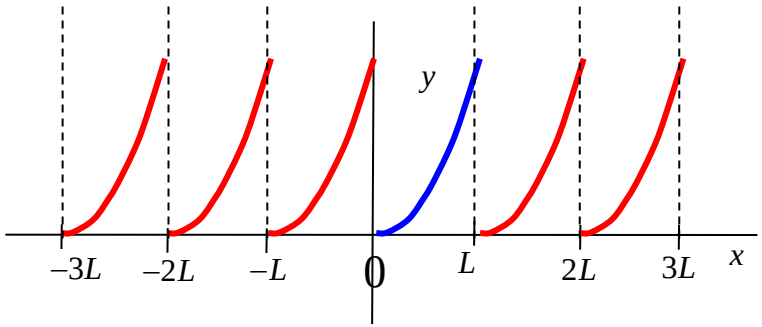
(c)

$$a_0 = \frac{1}{p}\int_0^{2p} x^2 dx = \frac{2}{L}\int_0^{L} x^2 dx = \frac{2}{3}L^2$$

$$a_n = \frac{1}{p}\int_0^{2p} x^2 \cos\frac{2n\pi}{2p}xdx = \frac{2}{L}\int_0^{L} x^2 \cos\frac{2n\pi}{L}xdx = \frac{L^2}{n^2\pi^2}$$

$$b_n = \frac{1}{p}\int_0^{2p} x^2 \sin\frac{2n\pi}{2p}xdx = \frac{2}{L}\int_0^{L} x^2 \sin\frac{2n\pi}{L}xdx = \frac{L^2}{n^2\pi^2}$$

$$f(x) = \frac{L^2}{3} + \frac{L^2}{\pi}\sum_{n=1}^{\infty}\{\frac{1}{n^2\pi}\cos\frac{2n\pi}{L}x - \frac{1}{n}\sin\frac{2n\pi}{L}x\}$$



참 고 자 료

Complex Fourier Series

▪**Fundamental Frequency**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x \right) \qquad f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{in\pi x/p}$$

Fundamental period **of the function** $T = 2p \longrightarrow$ $\omega = \frac{2\pi}{T} = 2\pi f = \frac{2\pi}{2p} = \frac{\pi}{p}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\omega x + b_n \sin n\omega x\right) \qquad f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{in\omega x}$$

$$\text{where, } \omega = \frac{2\pi}{T}$$

Fundamental angular frequency

Fourier Transform & Fourier Integral

Seoul National Univ.
@SDAL Advanced Ship Design Automation Lab.
http://asdal.snu.ac.kr

Fourier Series and Fourier Integral*

Function expansion

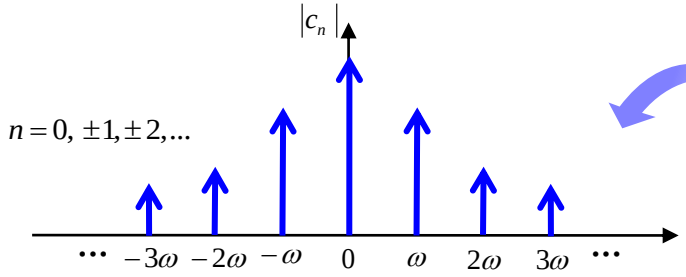
Fourier Series

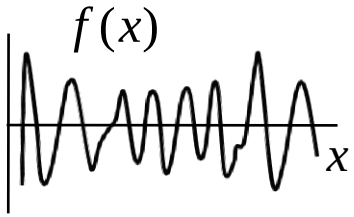
- *series of sine, cosine, and complex***
- expand a periodic function
- infinite but discrete set of frequencies

$$\omega = 2\pi f = \frac{2\pi}{T} = \frac{2\pi}{2p} = \frac{\pi}{p}$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p}$$

$$c_n = \frac{1}{2p}\int_{-p}^{p} f(x) e^{-in\pi x/p} dx$$





Q1. is it possible to represent a function which is not periodic by something analoguos to a Fourier series?

Q2. can we somehow extend or modify Fourier series to cover the case of a continuous set of frequencies?

recall, an 'integral' is a 'limit of a sum'

$$\sum_{n=-\infty}^{\infty} \rightarrow \int_{-\infty}^{\infty}$$

Fourier Series $\rightarrow$ Fourier Integral

A real function $f(x)$ is represented by a complex series ; a series in which the coefficients C_n are complex numbers**

*Mary L. Boas, Mathematical Methods in the Physical Science., Second Edition, John Wiley & Sons, 1966, p647-648

**Zill D.G., & Cullen M.R., Advanced Engineering Mathematics, Third Edition, Jones and Bartlett, 2006, p670

Fourier Series and Fourier Integral*

Function expansion

Fourier Series

-in *series of sine, cosine, and complex***

-expand a periodic function

-infinite but discrete set of frequencies

$$\omega = 2\pi f = \frac{2\pi}{T} = \frac{2\pi}{2p} = \frac{\pi}{p}$$

extend

recall, an 'integral' is a 'limit of a sum'

$$\sum_{n=-\infty}^{\infty} \rightarrow \int_{-\infty}^{\infty}$$

Fourier Series → Fourier Integral

Fourier Integral

- *integral of sine, cosine, and complex***
- represent nonperiodic functions
- continuous set of frequencies

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p}$$

$$c_n = \frac{1}{2p}\int_{-p}^{p} f(x)e^{-in\pi x/p}dx$$

corresponding

interval-valued variable n

⇩

continuous variable ω

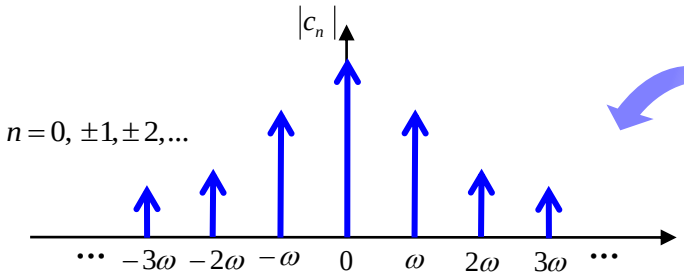
the set of coefficients c_n

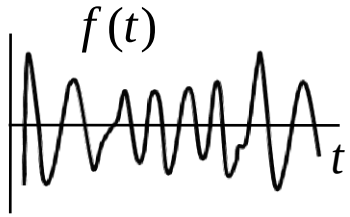
⇩

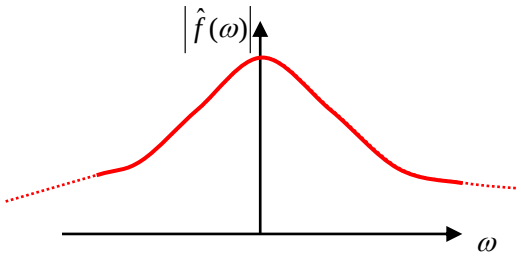
becomes a function $\hat{f}(\omega)$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$







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Fourier Series and Fourier Transform*

Function expansion

Fourier Series

-in *series of sine, cosine, and complex***

-expand a periodic function

-infinite but discrete set of frequencies

$$\omega = 2\pi f = \frac{2\pi}{T} = \frac{2\pi}{2p} = \frac{\pi}{p}$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p}$$

$$c_n = \frac{1}{2p}\int_{-p}^{p} f(x)e^{-in\pi x/p}dx$$

extend

recall, an 'integral' is a 'limit of a sum'

$$\sum_{n=-\infty}^{\infty} \rightarrow \int_{-\infty}^{\infty}$$

Fourier Series → Fourier Integral

corresponding

interval-valued variable n

⇩

continuous variable ω

the set of coefficients c_n

⇩

become a function $\hat{f}(\omega)$

Fourier Transform

-*integral of sine, cosine, and complex***

-represent nonperiodic functions

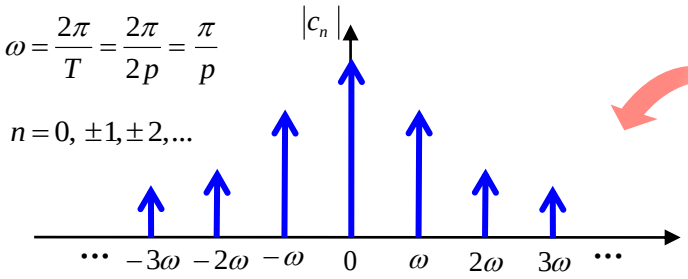
-continuous set of frequencies

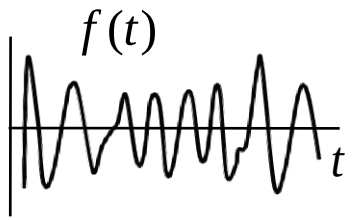
Inverse Fourier Transform of $\hat{f}(\omega)$

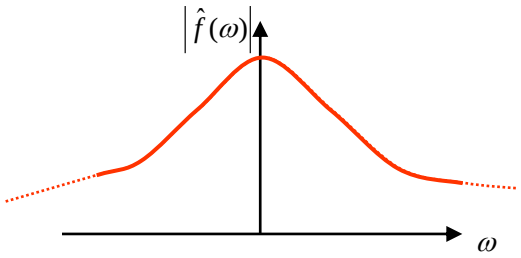
$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

Fourier Transform of $f(x)$







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$$\omega = 2\pi f = \frac{2\pi}{T} = \frac{2\pi}{2p} = \frac{\pi}{p}$$

Function expansion

Fourier Series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p}$$

$$c_n = \frac{1}{2p}\int_{-p}^{p} f(x)e^{-in\pi x/p}dx$$

Fourier Transform

Inverse Fourier Transform of $\hat{f}(\omega)$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

Fourier Transform of $f(x)$

Try to represent a function which is not periodic by letting the period (-p,p) increase to (-∞, ∞)

If we call $\frac{n\pi}{p} = \omega_n$ **and** $\Delta\omega = \omega_{n+1} - \omega_n = \frac{\pi}{p}$ **then,** $\frac{1}{2p} = \frac{\Delta\omega}{2\pi}$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p} \Rightarrow f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n x}$$

$$c_n = \frac{1}{2p}\int_{-p}^{p} f(x)e^{-in\pi x/p}dx \Rightarrow c_n = \frac{\Delta\omega}{2\pi}\int_{-p}^{p} f(u)e^{-i\omega_n u}du$$

Integration value changed *x→u* to avoid confusion

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Fourier Series and Fourier Transform*

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Inverse Fourier Transform of $\hat{f}(\omega)$

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Try to represent a function which is not periodic by letting the period (-p,p) increase to (-∞, ∞)

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n x}$$

$$c_n = \frac{\Delta\omega}{2\pi}\int_{-p}^{p} f(u)e^{-i\omega_n u}du$$

$$\Rightarrow \quad f(x) = \sum_{n=-\infty}^{\infty}\left[\frac{\Delta\omega}{2\pi}\int_{-p}^{p} f(u)e^{-i\omega_n u}du\right]e^{i\omega_n x}$$

$$= \sum_{n=-\infty}^{\infty}\frac{\Delta\omega}{2\pi}\int_{-p}^{p} f(u)e^{i\omega_n (x-u)}du$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(\int_{-p}^{p} f(u)e^{i\omega_n (x-u)}du\right)\Delta\omega$$

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Fourier Series and Fourier Transform*

$$\omega = 2\pi f = \frac{2\pi}{T} = \frac{2\pi}{2p} = \frac{\pi}{p}$$

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Fourier Transform of $f(x)$

Try to represent a function which is not periodic by letting the period (-p,p) increase to (-∞, ∞)

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(\int_{-p}^{p} f(u)e^{i\omega_n(x-u)}du\right)\Delta\omega$$

$$\Delta\omega = \frac{\pi}{p} \to 0 \text{ , as } p \to \infty$$

$$\Rightarrow f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} F(\omega_n)\Delta\omega$$

$$\sum_{n=-\infty}^{\infty} F(\omega_n)\Delta\omega \to \int_{-\infty}^{\infty} F(\omega)d\omega$$

$\omega_n \to \omega$ **continuous variable**

also,

$$\text{where, } F(\omega_n) = \int_{-p}^{p} f(u)e^{i\omega_n(x-u)}du \to F(\omega) = \int_{-\infty}^{\infty} f(u)e^{i\omega(x-u)}du$$

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Fourier Transform

Inverse Fourier Transform of $\hat{f}(\omega)$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

Fourier Transform of $f(x)$

Try to represent a function which is not periodic by letting the period (-p,p) increase to (-∞, ∞)

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} F(\omega_n)\Delta\omega$$

$$where,\ F(\omega_n) = \int_{-p}^{p} f(u)e^{i\omega_n(x-u)}du$$

$$\Delta\omega = \frac{\pi}{p} \to 0\ ,as\ \ p \to \infty$$

$$\sum_{n=-\infty}^{\infty} F(\omega_n)\Delta\omega \to \int_{-\infty}^{\infty} F(\omega)d\omega$$

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$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)d\omega$$

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Fourier Transform of $f(x)$

Try to represent a function which is not periodic by letting the period (-p,p) increase to (-∞, ∞)

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)d\omega$$

$$where,\ F(\omega) = \int_{-\infty}^{\infty} f(u)e^{i\omega(x-u)}du$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)e^{i\omega(x-u)}dud\omega$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-i\omega u}du\right)e^{i\omega x}d\omega$$

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$$\omega = 2\pi f = \frac{2\pi}{T} = \frac{2\pi}{2p} = \frac{\pi}{p}$$

Fourier Series and Fourier Transform*

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Fourier Transform

Inverse Fourier Transform of $\hat{f}(\omega)$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

Fourier Transform of $f(x)$

Try to represent a function which is not periodic by letting the period (-p,p) increase to (-∞, ∞)

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-i\omega u}du\right)e^{i\omega x}d\omega$$

If we define

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-i\omega u}du$$

then,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega$$

Fourier Sine Transform*

$f_{odd} \cdot f_{even}$: odd

$f_{odd} \cdot f_{odd}$: even

$f_{even} \cdot f_{even}$: even

f is even, if $f(-x) = f(x)$

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

f is odd, if $f(-x) = -f(x)$

$$\int_{-a}^{a} f(x)dx = 0$$

Fourier Transform

Fourier Transform of $f(x)$**:**

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

Inverse Fourier Transform of $\hat{f}(\omega)$**:**

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega$$

By the Euler formula $e^{-i\omega x} = \cos\omega x - i\sin\omega x$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\left[\cos\omega x - i\sin\omega x\right]dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\underbrace{\cos\omega x}_{\text{even}}dx - i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\underbrace{\sin\omega x}_{\text{odd}}dx$$

if, $f(x)$: odd

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underbrace{f(x)}_{\text{odd}}\underbrace{\cos\omega x}_{\text{even}}dx - i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underbrace{f(x)}_{\text{odd}}\underbrace{\sin\omega x}_{\text{odd}}dx = 0 - i\frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} f(x)\sin\omega x dx$$

$$\therefore \hat{f}(\omega) = -i\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\sin\omega x dx$$

Fourier Sine Transform*

f is even, if $f(-x)=f(x)$

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

f is odd, if $f(-x)=-f(x)$

$$\int_{-a}^{a} f(x)dx = 0$$

Fourier Transform

Fourier Transform of $f(x)$:

$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

Inverse Fourier Transform of $\hat{f}(\omega)$:

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega$$

---

cf, $\hat{f}(\omega)$: **odd function?**

if, $f(x)$: odd

$$\hat{f}(\omega)=-i\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\sin\omega x dx$$

$$\hat{f}(-\omega)=-i\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\sin(-\omega x)dx$$

$$=i\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\sin(\omega x)dx=-\hat{f}(\omega)$$

odd function!

$$\therefore f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\underbrace{\hat{f}(\omega)}_{\text{odd}}\underbrace{\cos\omega x}_{\text{even}}d\omega+i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\underbrace{\hat{f}(\omega)}_{\text{odd}}\underbrace{\sin\omega x}_{\text{odd}}d\omega=0+i\frac{2}{\sqrt{2\pi}}\int_{0}^{\infty}\hat{f}(\omega)\sin\omega x d\omega$$

$$=i\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\hat{f}(\omega)\sin\omega x d\omega$$

Fourier Sine Transform*

f is even, if $f(-x) = f(x)$

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

f is odd, if $f(-x) = -f(x)$

$$\int_{-a}^{a} f(x)dx = 0$$

Fourier Transform

Fourier Transform of $f(x)$:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

Inverse Fourier Transform of $\hat{f}(\omega)$:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega$$

if, $f(x)$: odd

$$\hat{f}(\omega) = -i\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\sin\omega x dx$$

$$f(x) = i\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} \hat{f}(\omega)\sin\omega x d\omega$$ ➡ **The numerical factor** $$f(x) = \left(i\sqrt{\frac{2}{\pi}}\right)\left(-i\sqrt{\frac{2}{\pi}}\right) = \frac{2}{\pi}$$

Thus the imaginary factors are not needed, the factor $\frac{2}{\pi}$ may multiply either of the two integrals, or each integral may be multiplied by $\sqrt{\frac{2}{\pi}}$.

Let us make the latter choice in giving the following definition

$$f(x) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} \hat{f}(\omega)\cdot(\sin\omega x)d\omega \qquad \hat{f}(\omega) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\sin\omega x\, dx$$

*Mary L. Boas, Mathematical Methods in the Physical Science., Second Edition, John Wiley & Sons, 1966, p647-648

Fourier Sine Transform

f is even, if $f(-x) = f(x)$

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

f is odd, if $f(-x) = -f(x)$

$$\int_{-a}^{a} f(x)dx = 0$$

Fourier Transform

Fourier Transform of $f(x)$:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

Inverse Fourier Transform of $\hat{f}(\omega)$:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(x) e^{i\omega x} d\omega$$

Fourier Sine Transform

The Fourier transform of an odd function on the interval $(-\infty, \infty)$ is the sine transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \hat{f}(\omega) \sin \omega x d\omega$$

Where,

$$\hat{f}(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin \omega x \, dx$$

Fourier Cosine Transform

f is even, if $f(-x) = f(x)$

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

f is odd, if $f(-x) = -f(x)$

$$\int_{-a}^{a} f(x)dx = 0$$

Fourier Transform

Fourier Transform of $f(x)$:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

Inverse Fourier Transform of $\hat{f}(\omega)$:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(x) e^{i\omega x} d\omega$$

In a similar way

Fourier Cosine Transform

The Fourier transform of an even function on the interval $(-\infty, \infty)$ is the cosine transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \hat{f}(\omega) \cos \omega x d\omega$$

Where,

$$\hat{f}(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos \omega x \, dx$$

Fourier Transform

☑ Convergence of a Fourier Transform

Theorem 15.1

Conditions for Convergence

Let f and f' be piecewise continuous on every finite interval, and let f be absolutely integrable on $(-\infty, \infty)$*. Then the Fourier transform of f on the interval converges to $f(x)$ at a point of continuity. At a point of discontinuity, the Fourier integral will converge to the average

$$\frac{f(x+) + f(x-)}{2}$$

Where $f(x+)$ and $f(x-)$ denote t he limit of f at x from the right and from the left, respectively.

* This means that the integral $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges.

Fourier Integral of Impulse function

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega$$
$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx$$

When f : odd

$$f(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\hat{f}(\omega)\sin\omega x d\omega$$
$$\hat{f}(\omega)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}f(x)\sin\omega x\,dx$$

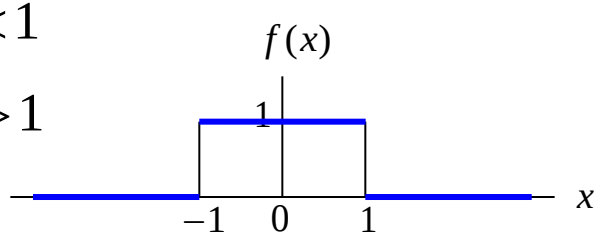
When f : even

$$f(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\hat{f}(\omega)\cos\omega x d\omega$$
$$\hat{f}(\omega)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}f(x)\cos\omega x\,dx$$

☑ Example 1: Impulse function

Find the Fourier integral representation of an impulse function

$$f(x)=\begin{cases}1, & \text{if } |x|<1\\ 0, & \text{if } |x|>1\end{cases}$$



Approach 1: By using Fourier Transform

$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}1\cdot e^{-i\omega x}dx$$

$$=\frac{1}{\sqrt{2\pi}}\frac{1}{-i\omega}\left[e^{-i\omega x}\right]_{-1}^{1}=\frac{1}{\sqrt{2\pi}}\frac{1}{-i\omega}(e^{-i\omega}-e^{i\omega})$$

$$=\frac{1}{\sqrt{2\pi}}\frac{1}{-i\omega}(-2i\sin\omega)=\sqrt{\frac{2}{\pi}}\frac{\sin\omega}{\omega}$$

By the Euler formula

$$e^{-i\omega}=\cos\omega-i\sin\omega$$
$$e^{i\omega}=\cos\omega+i\sin\omega$$

Fourier Integral of Impulse function

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega$$
$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx$$

When f : odd

$$f(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\hat{f}(\omega)\sin\omega x d\omega$$
$$\hat{f}(\omega)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}f(x)\sin\omega x\,dx$$

When f : even

$$f(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\hat{f}(\omega)\cos\omega x d\omega$$
$$\hat{f}(\omega)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}f(x)\cos\omega x\,dx$$

$\frac{\sin\omega}{\omega}$: odd / odd → even

$$\hat{f}(\omega)=\sqrt{\frac{2}{\pi}}\frac{\sin\omega}{\omega}$$

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\sqrt{\frac{2}{\pi}}\frac{\sin\omega}{\omega}\right)e^{i\omega x}d\omega=\frac{1}{\pi}\int_{-\infty}^{\infty}\underbrace{\frac{\sin\omega}{\omega}}_{\text{even}}\Big[\underbrace{\cos\omega x}_{\text{even}}+i\underbrace{\sin\omega x}_{\text{odd}}\Big]d\omega$$

$$\therefore\ f(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\omega}{\omega}\cos\omega x d\omega=\frac{2}{\pi}\int_{0}^{\infty}\frac{\sin\omega\cos\omega x}{\omega}d\omega$$

Fourier Cosine Integral of Impulse function

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega$$
$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx$$

When f : odd

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\hat{f}(\omega)\sin\omega x d\omega$$
$$\hat{f}(\omega)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\sin\omega x\,dx$$

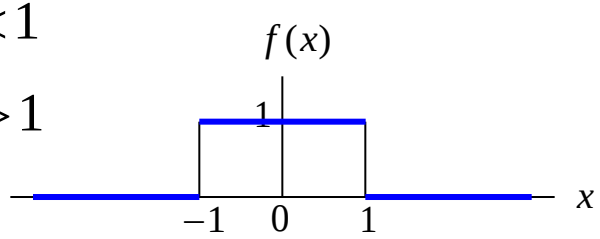
When f : even

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\hat{f}(\omega)\cos\omega x d\omega$$
$$\hat{f}(\omega)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\cos\omega x\,dx$$

☑ Impulse function

Find the Fourier cosine integral representation of an impulse function

$$f(x)=\begin{cases}1, & \text{if} \quad |x|<1\\ 0, & \text{if} \quad |x|>1\end{cases}$$



Approach 2: By using Fourier Cosine Transform of even function f(x)

$$\hat{f}(\omega)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\cos\omega x\,dx=\sqrt{\frac{2}{\pi}}\int_0^{1}1\cdot\cos\omega x\,dx=\sqrt{\frac{2}{\pi}}\left(\frac{\sin\omega x}{\omega}\Big|_0^1\right)=\sqrt{\frac{2}{\pi}}\frac{\sin\omega}{\omega}$$

$$\therefore\ f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\sqrt{\frac{2}{\pi}}\frac{\sin\omega}{\omega}\cos\omega x d\omega=\frac{2}{\pi}\int_0^{\infty}\frac{\sin\omega\cos\omega x}{\omega}d\omega$$

Because f(x) is even function, the results of Fourier transform and Fourier Cosine transform are identical

Fourier Sine Integral of Impulse function

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega$$
$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx$$

When f : odd

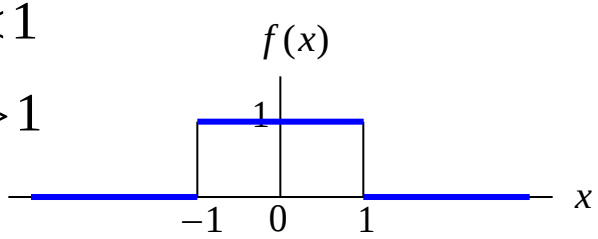
$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\hat{f}(\omega)\sin\omega x d\omega$$
$$\hat{f}(\omega)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\sin\omega x\,dx$$

When f : even

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\hat{f}(\omega)\cos\omega x d\omega$$
$$\hat{f}(\omega)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\cos\omega x\,dx$$

Find the Fourier sine integral representation of an impulse function

$$f(x)=\begin{cases}1, & \text{if} \quad |x|<1\\ 0, & \text{if} \quad |x|>1\end{cases}$$



Approach 3: Fourier Sine Transform

If f(x) is even function, we cannot apply Fourier Sine Transform formula

$$\hat{f}(\omega)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\sin\omega x\,dx$$

But if try to integrate f with sine over x

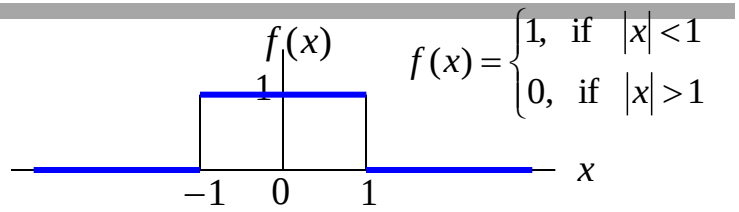
$$\hat{f}(\omega)=\int_{-\infty}^{\infty}f(x)\sin\omega x\,dx=\int_{-1}^{1}1\cdot\sin\omega x\,dx$$

$$=\left(-\frac{\cos\omega x}{\omega}\Big|_{-1}^{1}\right)=-\left(\frac{\cos\omega}{\omega}-\frac{\cos(-\omega)}{\omega}\right)=0 \Rightarrow$$

It remains only cosine term, that means approach 3 is identical with the cosine integral approach.

Dirichlet's discontinuous factor

$$f(x)=\frac{2}{\pi}\int_0^{\infty}\frac{\cos\omega x\sin\omega}{\omega}d\omega \quad \Leftarrow$$



$$f(x)=\begin{cases}1, & \text{if } |x|<1\\ 0, & \text{if } |x|>1\end{cases}$$

The average of the left- and right-hand limits of *f* (*x*) at *x* = 1 is equal to (1+0)/2, that is, 1/2. (Theorem 15.1)

Furthermore, multipling by $\pi/2$ we obtain by

$$\frac{\pi}{2}f(x)=\int_0^{\infty}\frac{\cos\omega x\sin\omega}{\omega}d\omega=\begin{cases}1\cdot\dfrac{\pi}{2}=\dfrac{\pi}{2}, & |x|<1\\ \dfrac{1}{2}\cdot\dfrac{\pi}{2}=\dfrac{\pi}{4}, & x=1\\ 0\cdot\dfrac{\pi}{2}=0, & x>1\end{cases}$$

This integral is called Dirichlet's discontinuous factor.

Fourier Cosine Integral -> Homework

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega$$
$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx$$

When f : odd

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\hat{f}(\omega)\sin\omega x d\omega$$
$$\hat{f}(\omega)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\sin\omega x\,dx$$

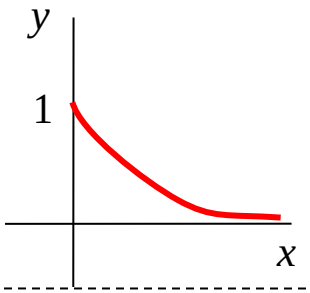
When f : even

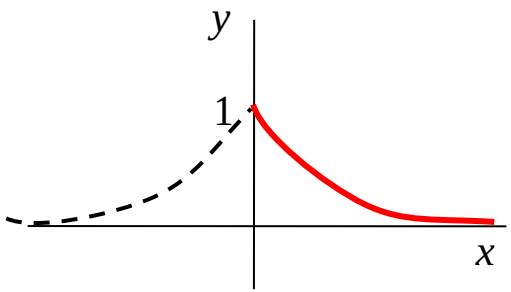
$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\hat{f}(\omega)\cos\omega x d\omega$$
$$\hat{f}(\omega)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\cos\omega x\,dx$$

☑ Example 2: Cosine Integral Representation

Represent $f(x)=e^{-x},\ x>0$

by a cosine integral and
by a sine integral





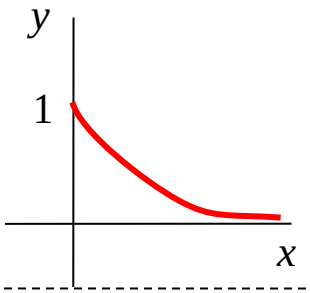
Cosine integral

Fourier Sine Integral -> Homework

Sine Integral Representation

Represent $f(x) = e^{-x},\ x > 0$

by a sine integral



Definition 15.2 **Fourier Cosine and Sine Integrals**

(*i*) The Fourier Integral of an even function on the interval $(-\infty, \infty)$ is the cosine integral

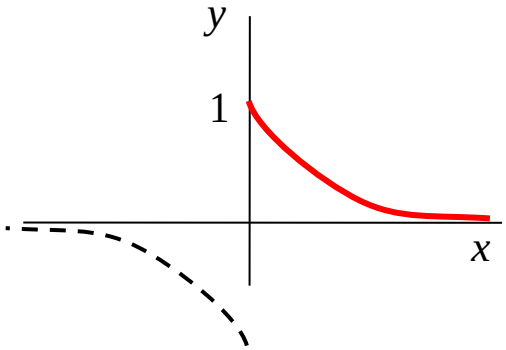
$$f(x) = \int_0^\infty A(\omega) \cdot (\cos \omega x) d\omega$$

Where, $$A(\omega) = \frac{2}{\pi} \int_0^\infty f(x) \cos \omega x \, dx,$$

(*ii*) The Fourier Integral of an odd function on the interval $(-\infty, \infty)$ is the sine integral

$$f(x) = \int_0^\infty B(\omega) \cdot (\sin \omega x) d\omega$$

Where, $$B(\omega) = \frac{2}{\pi} \int_0^\infty f(x) \sin \omega x \, dx$$



sine integral

FOURIER TRANSFORM

HOMEWORK -SOLUTION

Fourier Cosine Integral -> Homework solution

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega$$
$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx$$

When f : odd

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\hat{f}(\omega)\sin\omega x d\omega$$
$$\hat{f}(\omega)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\sin\omega x\,dx$$

When f : even

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\hat{f}(\omega)\cos\omega x d\omega$$
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by a cosine integral
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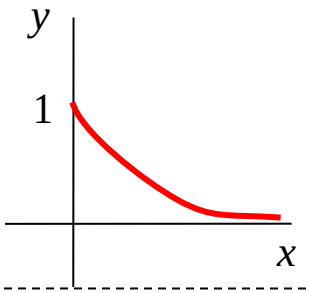
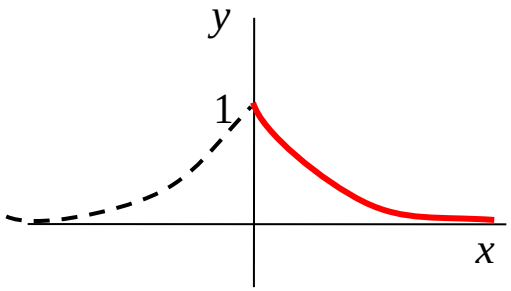


(a) By a cosine integral

$$\hat{f}(\omega)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}e^{-x}\cos\omega x\ dx=\sqrt{\frac{2}{\pi}}\frac{1}{1+\omega^2}$$

$$\therefore f(x)=\frac{2}{\pi}\int_0^{\infty}\frac{\cos\omega x}{1+\omega^2}d\omega$$



Cosine integral

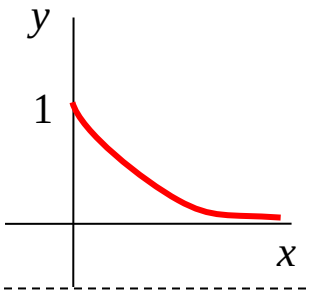
Using Integration by parts

$$\int_0^{\infty}e^{-x}\cos\omega x\ dx=\left[e^{-x}\frac{\sin\omega x}{\omega}\right]_0^{\infty}-\int_0^{\infty}(-e^{-x})\frac{\sin\omega x}{\omega}dx$$
$$=0+\int_0^{\infty}e^{-x}\frac{\sin\omega x}{\omega}dx$$
$$=\left[e^{-x}\frac{(-\cos\omega x)}{\omega^2}\right]_0^{\infty}-\int_0^{\infty}(-e^{-x})\frac{(-\cos\omega x)}{\omega^2}dx$$
$$=\frac{1}{\omega^2}-\frac{1}{\omega^2}\int_0^{\infty}e^{-x}\cos\omega x dx$$
$$\therefore\int_0^{\infty}e^{-x}\cos\omega x dx=\frac{1}{1+\omega^2}$$

Fourier Sine Integral -> Homework solution

Sine Integral Representation

Represent $f(x)=e^{-x},\ x>0$ **by a sine integral**



Definition 15.2 **Fourier Cosine and Sine Integrals**

(*i*) The Fourier Integral of an even function on the interval $(-\infty,\infty)$ is the cosine integral

$$f(x)=\int_0^\infty A(\omega)\cdot(\cos\omega x)d\omega$$

Where, $$A(\omega)=\frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\,dx,$$

(*ii*) The Fourier Integral of an odd function on the interval $(-\infty,\infty)$ is the sine integral

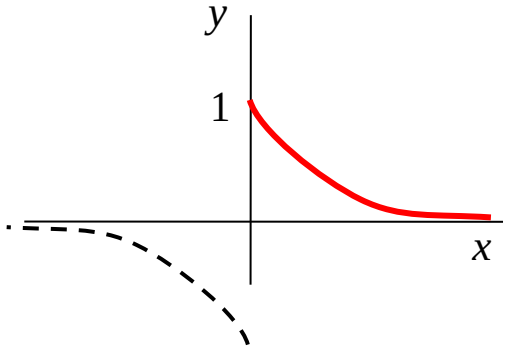
$$f(x)=\int_0^\infty B(\omega)\cdot(\sin\omega x)d\omega$$

Where, $$B(\omega)=\frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\,dx$$

(b) By a sine integral

$$\hat{f}(\omega)=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin\omega x\,dx=\sqrt{\frac{2}{\pi}}\frac{\omega}{1+\omega^2}$$

$$\therefore\ f(x)=\frac{2}{\pi}\int_0^\infty\frac{\omega\sin\omega x}{1+\omega^2}d\omega$$



sine integral

$$\int_0^\infty e^{-x}\sin\omega x\ dx$$

Using Integration by parts

$$=\left[e^{-x}\frac{(-\cos\omega x)}{\omega}\right]_0^\infty-\int_0^\infty(-e^{-x})\frac{(-\cos\omega x)}{\omega}dx$$

$$=\frac{1}{\omega}-\left\{\left[e^{-x}\frac{\sin\omega x}{\omega^2}\right]_0^\infty-\int_0^\infty(-e^{-x})\frac{(\sin\omega x)}{\omega^2}dx\right\}$$

$$=\frac{1}{\omega}-\left\{0+\frac{1}{\omega^2}\int_0^\infty e^{-x}\sin\omega x dx\right\}$$

$$\therefore\int_0^\infty e^{-x}\sin\omega x dx=\frac{\omega}{1+\omega^2}$$

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Naval Architecture & Ocean Engineering

Engineering Mathematics

03-2
Fourier Transform Analysis

Spring 2012
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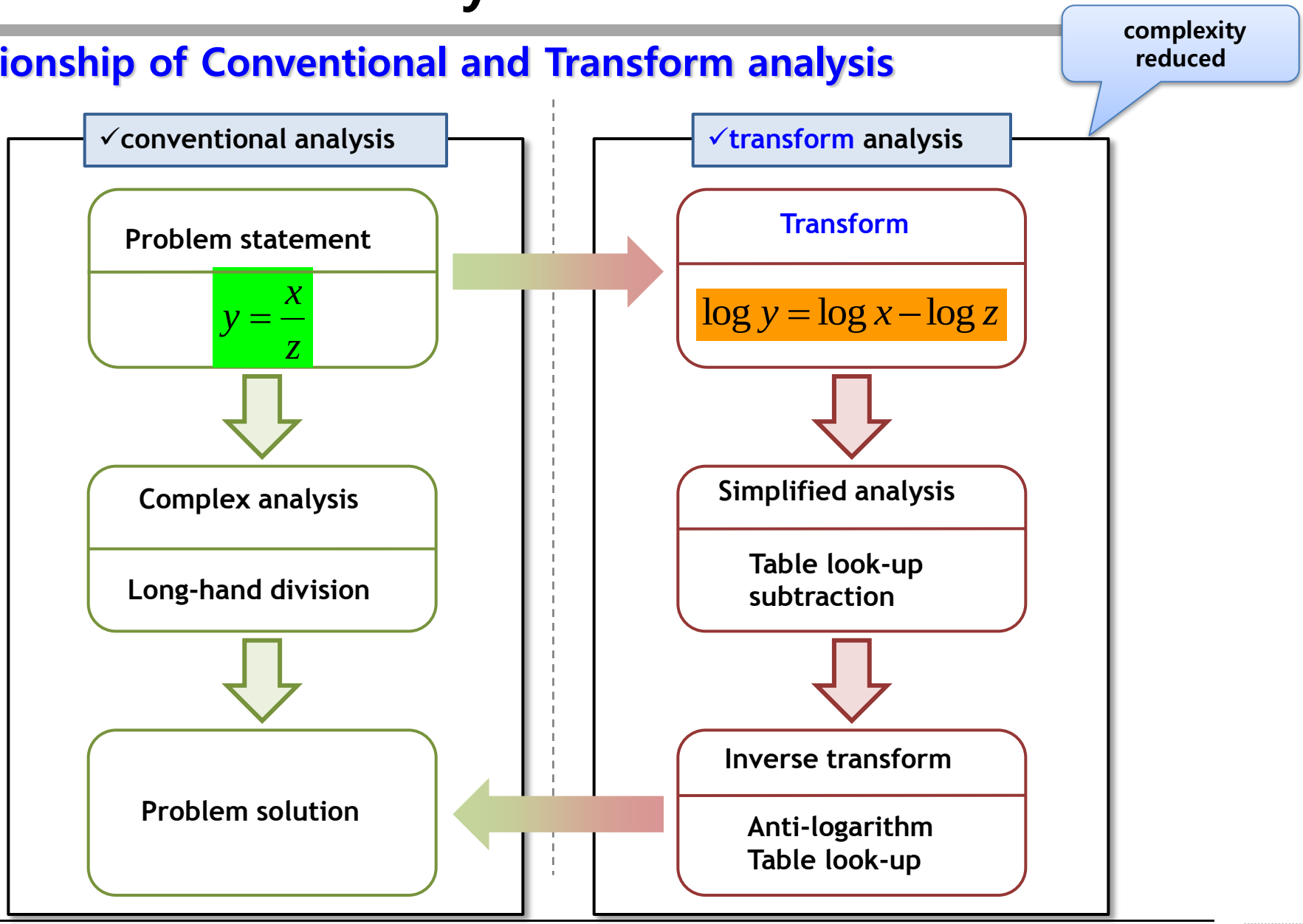
BASIC FOURIER TRANSFORM ANALYSIS

Basic Transform Analysis

Relationship of Conventional and Transform analysis



Basic Transform Analysis

What kind of Transform ?

$$y = \frac{x}{z} \xrightarrow{\log} \log y = \log x - \log z$$

$$y = x^2 \xrightarrow{\frac{d}{dx}} \frac{d}{dx} y = 2x$$

$$y = x^2 \xrightarrow{\int dx} \int y dx = \frac{1}{3} x^3 + c$$

Laplace Transform

$$f(t) \xrightarrow{\int_0^\infty e^{-st}\, dt} \int_0^\infty e^{-st} f(t) dt = \mathcal{L}\{f(t)\}$$

Fourier Transform

$$s(t) \xrightarrow{\int_{-\infty}^\infty e^{-i\omega t}\, dt} \int_0^\infty e^{-i\omega t} f(t) dt = \hat{f}(\omega)$$

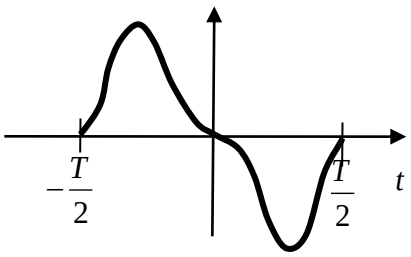
Basic Fourier Transform Analysis

$$\omega = 2\pi f = \frac{2\pi}{T} = \frac{2\pi}{2p} = \frac{\pi}{p}$$

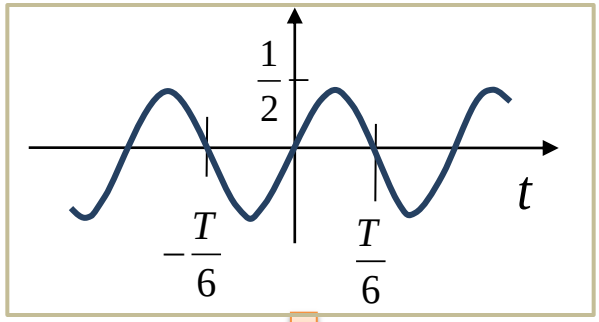
Interpretation of the Fourier Series*

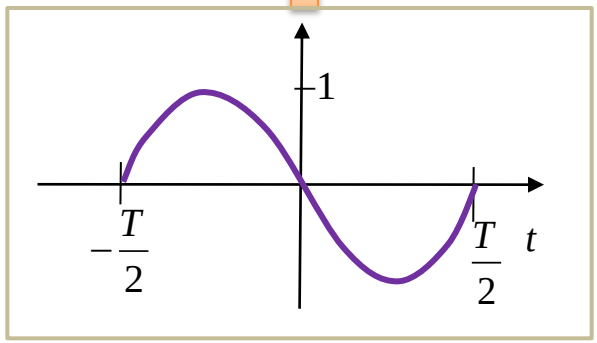
$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi t/p} \qquad c_n = \frac{1}{2p}\int_{-p}^{p} f(t)e^{-in\pi t/p}dt$$

Waveform defined From -∞ to+ ∞



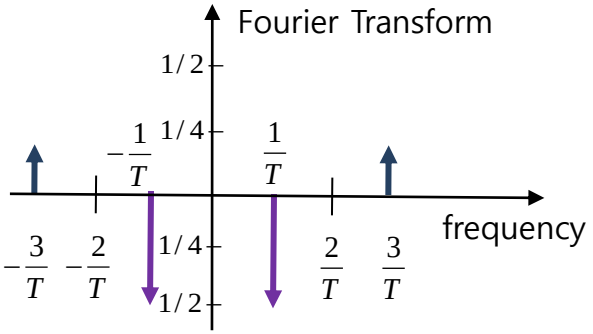
Fourier Transform
Synthesize a summation of sinusoids which add to give the waveform





Fourier Series

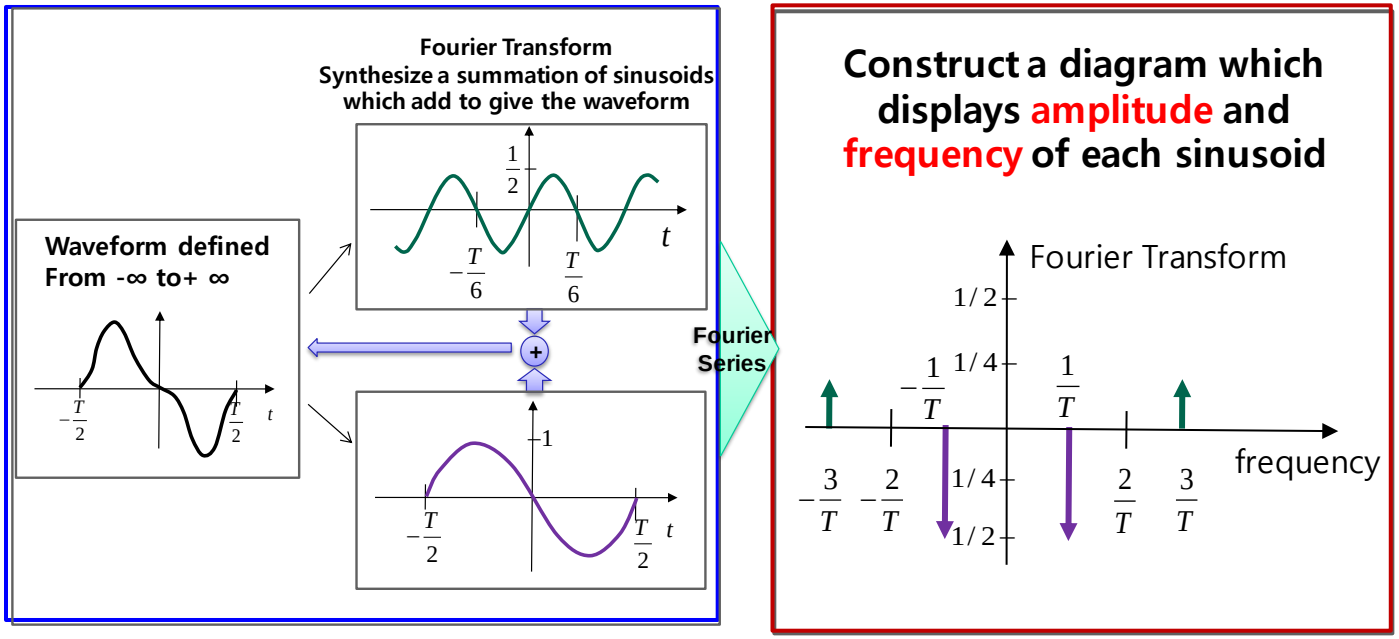
Construct a diagram which displays amplitude and frequency of each sinusoid



Basic Fourier Transform Analysis

$$\omega = 2\pi f = \frac{2\pi}{T} = \frac{2\pi}{2p} = \frac{\pi}{p}$$

Interpretation of the Fourier Transform* $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi t/p}$ $c_n = \frac{1}{2p}\int_{-p}^{p} f(t)e^{-in\pi t/p}dt$



Usual convention : displaying both positive and negative frequency sinusoids for each frequencies; the amplitude has been halved accordingly

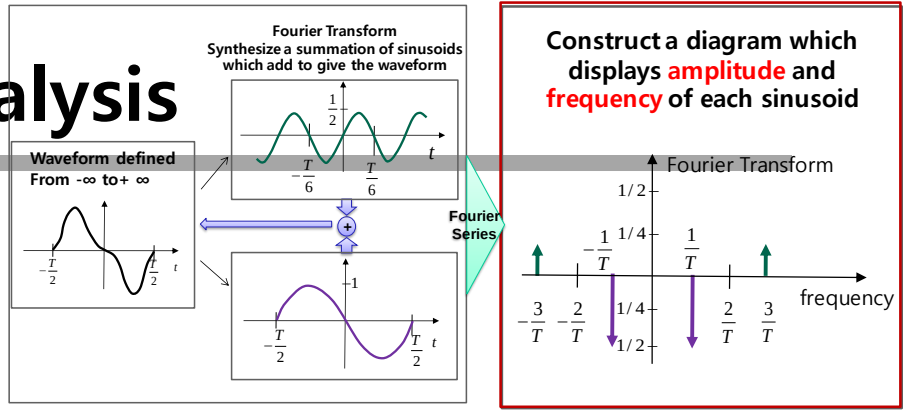
The essence of the Fourier transform of a waveform is

to decompose or separate the waveform into a sum of sinusoids of different frequencies

The pictorial representation of the Fourier transform is a diagram which displays

the amplitude and frequencies of each of the determined sinusoids

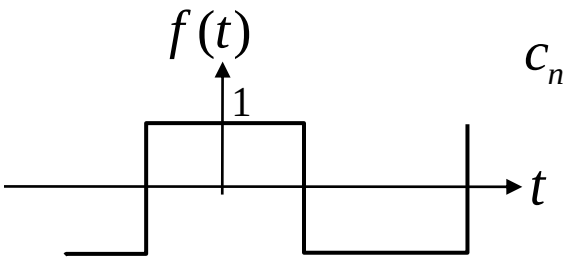
Basic Fourier Transform Analysis



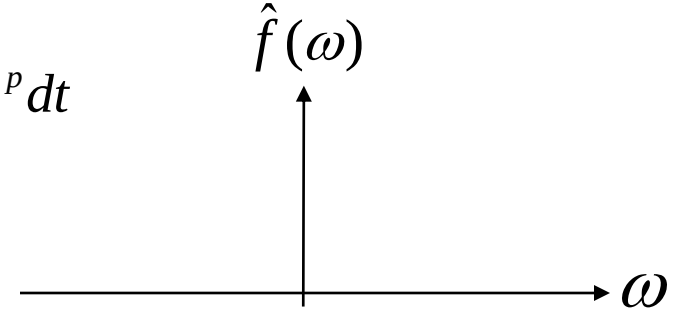
Ex.) Fourier Transform of square wave function

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi t/p}$$

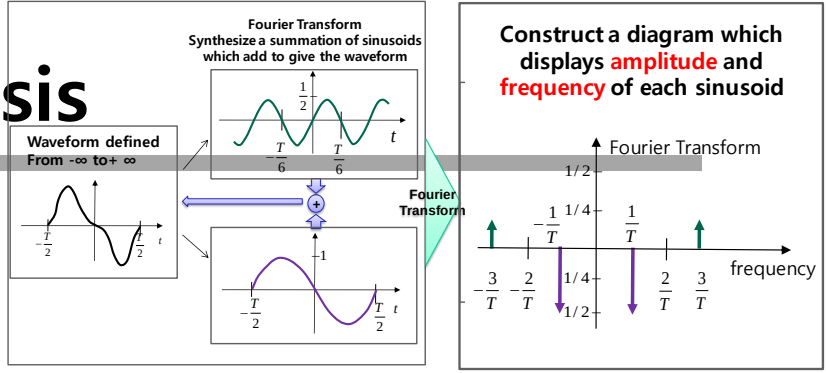
$$c_n = \frac{1}{2p}\int_{-p}^{p} f(t)e^{-in\pi t/p}dt$$



Fourier Series



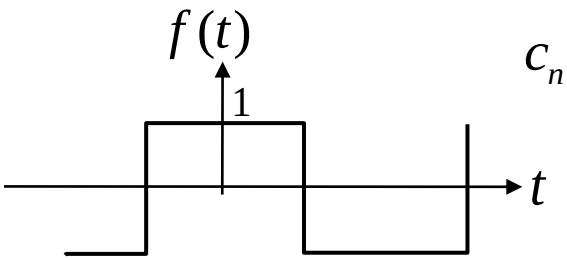
Basic Fourier Transform Analysis



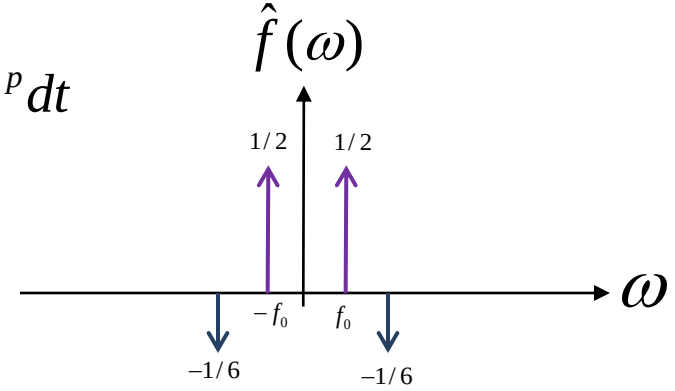
Ex.) Fourier Transform of square wave function

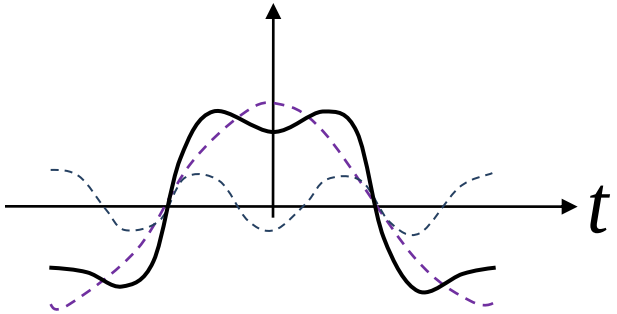
$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi t/p}$$

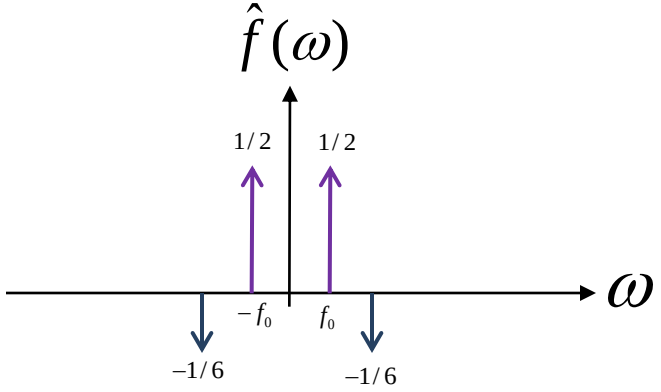
$$c_n = \frac{1}{2p}\int_{-p}^{p} f(t)e^{-in\pi t/p}dt$$



Fourier Series

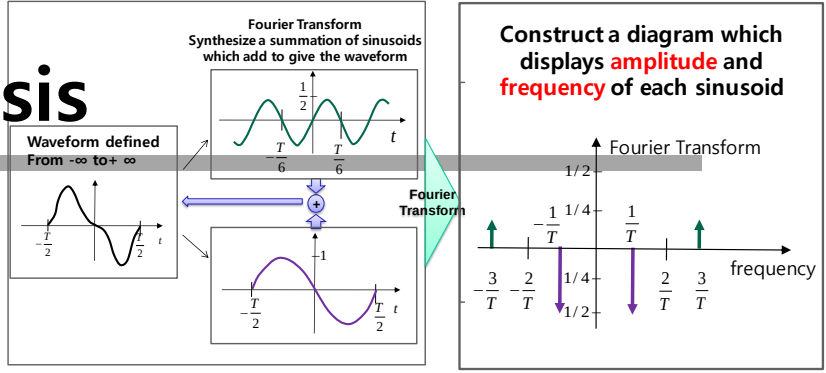






$$s_1(t) = \cos(2\pi f_0 t) - \frac{1}{3}\cos(6\pi f_0 t)$$

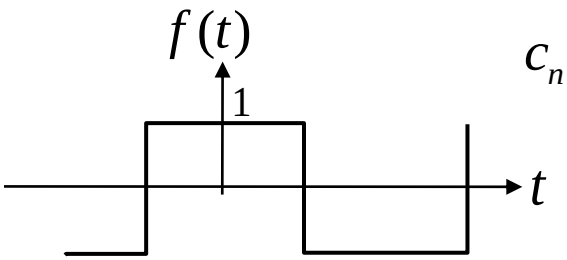
Basic Fourier Transform Analysis



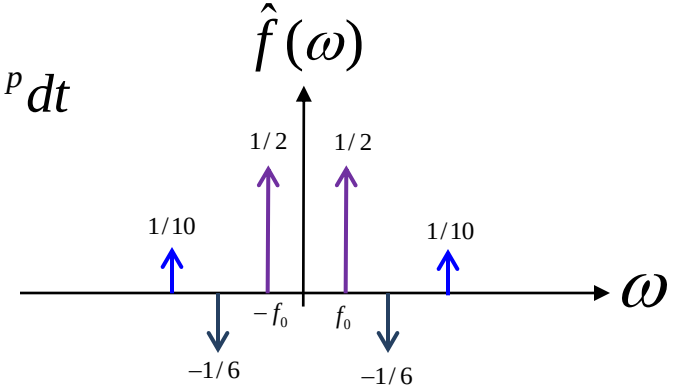
Ex.) Fourier Transform of square wave function

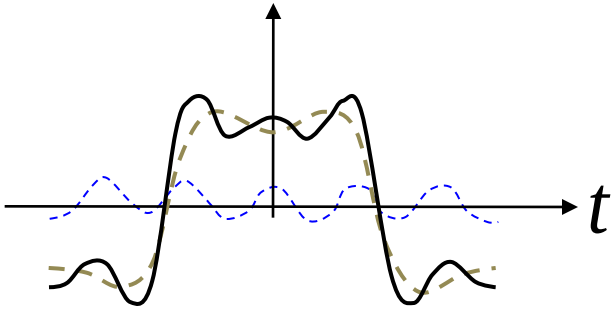
$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi t/p}$$

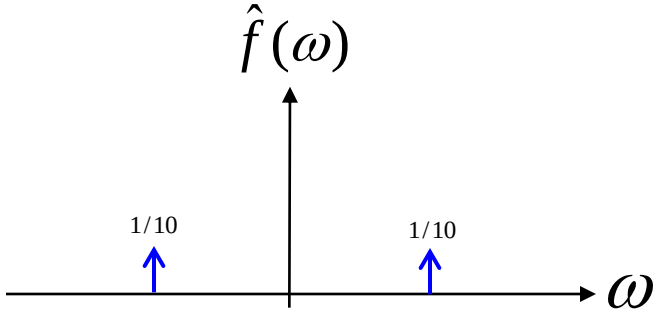
$$c_n = \frac{1}{2p}\int_{-p}^{p} f(t)e^{-in\pi t/p}dt$$



Fourier Series

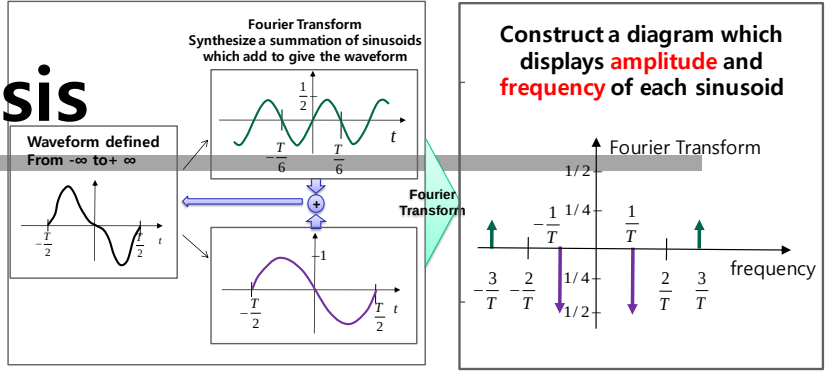






$$s_2(t) = \cos(2\pi f_0 t) - \frac{1}{3}\cos(6\pi f_0 t) + \frac{1}{5}\cos(10\pi f_0 t)$$

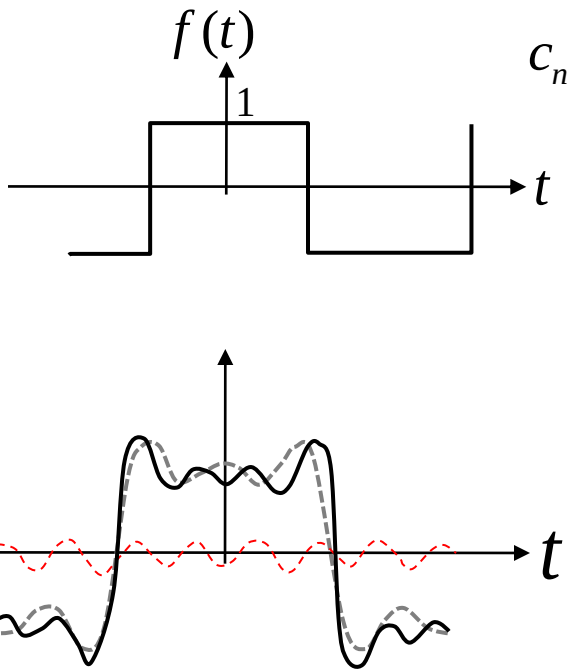
Basic Fourier Transform Analysis



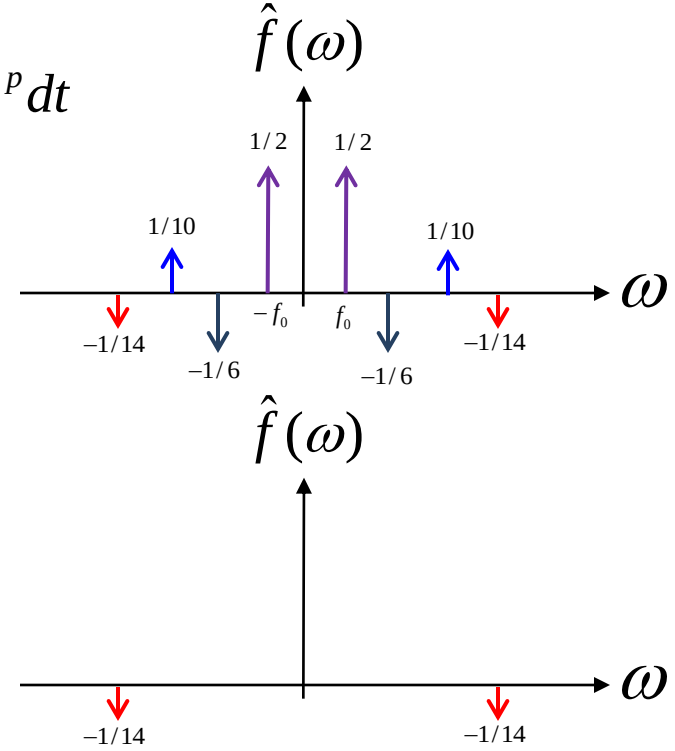
Ex.) Fourier Transform of square wave function

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi t/p}$$

$$c_n = \frac{1}{2p}\int_{-p}^{p} f(t)e^{-in\pi t/p}dt$$

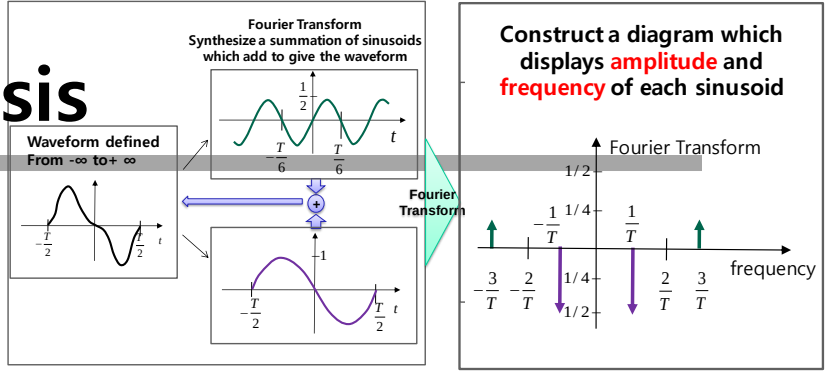


Fourier Series



$$s_3(t) = \cos(2\pi f_0 t) - \frac{1}{3}\cos(6\pi f_0 t) + \frac{1}{5}\cos(10\pi f_0 t) - \frac{1}{7}\cos(14\pi f_0 t)$$

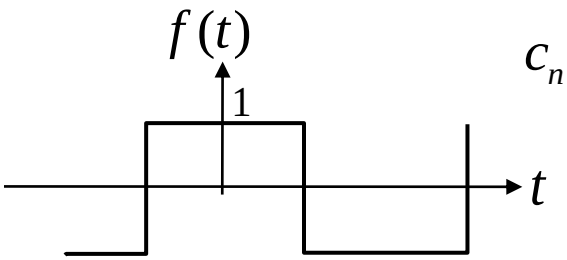
Basic Fourier Transform Analysis



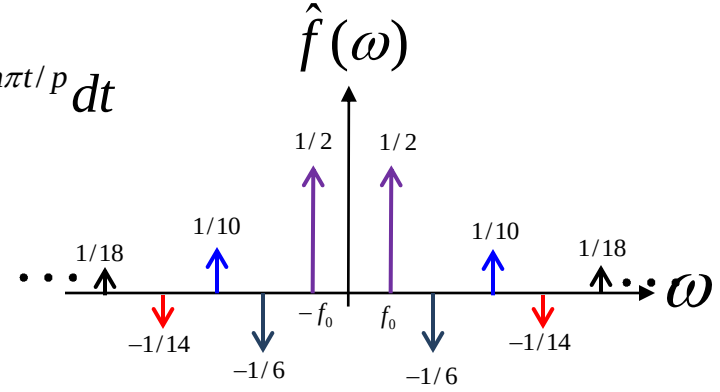
Ex.) Fourier Transform of square wave function

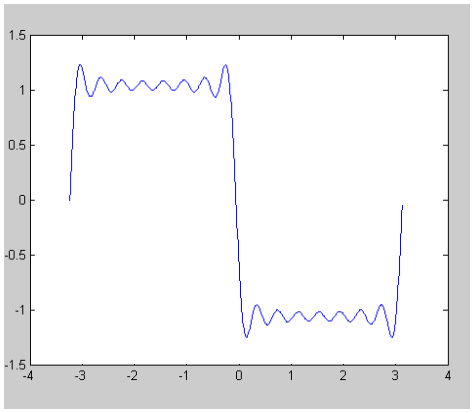
$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi t/p}$$

$$c_n = \frac{1}{2p}\int_{-p}^{p} f(t)e^{-in\pi t/p}dt$$



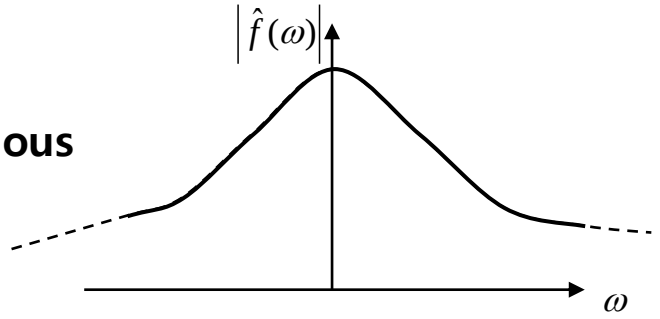
Fourier Series





$$\sum_{n=-\infty}^{\infty} \rightarrow \int_{-\infty}^{\infty}$$

Discrete **Continuous**

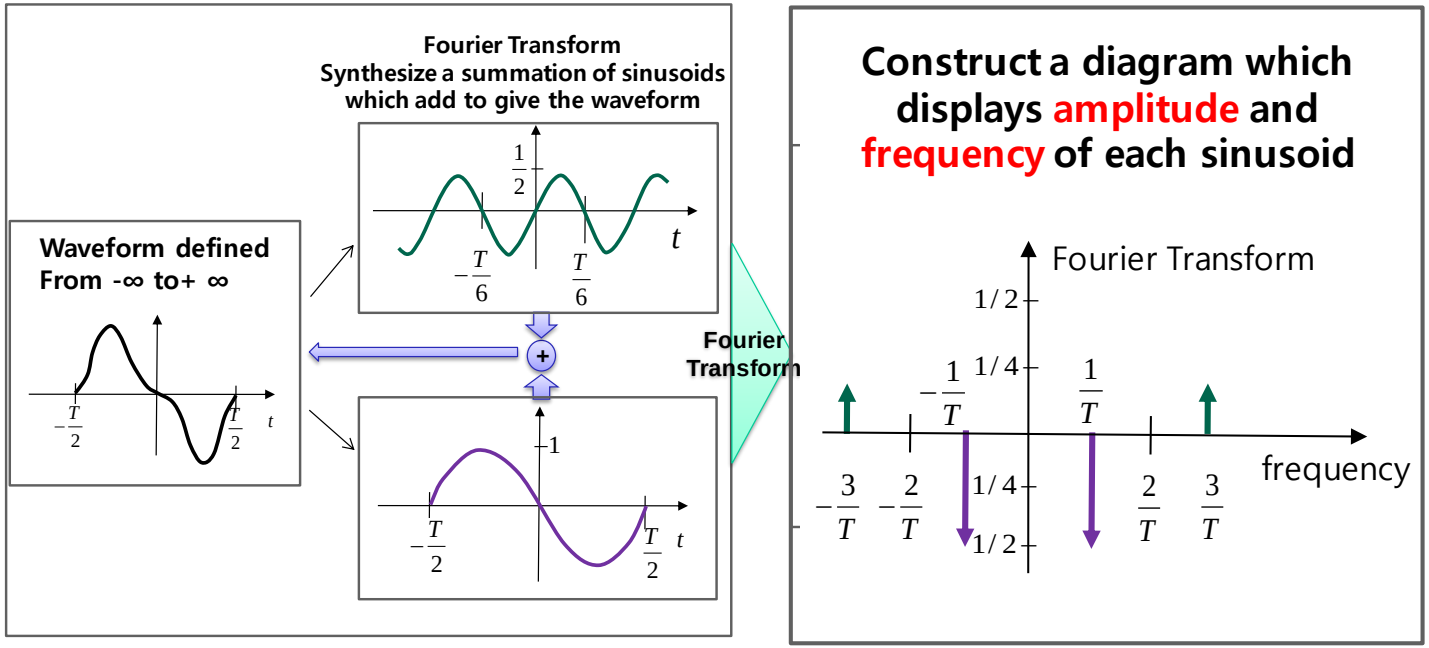


$$s(t) = s_\infty(t) = \cos(2\pi f_0 t) - \frac{1}{3}\cos(6\pi f_0 t) + \frac{1}{5}\cos(10\pi f_0 t) - \frac{1}{7}\cos(14\pi f_0 t) + \ldots$$

Basic Fourier Transform Analysis

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2p} = \frac{\pi}{p}$$

Interpretation of the Fourier Transform* $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi t/p}$ $\quad c_n = \frac{1}{2p}\int_{-p}^{p} f(t)e^{-in\pi t/p}dt$



The Fourier transform is , then, a frequency domain representation of a function

Fourier transform frequency domain contains exactly the same information as that of the original function ; they differ only in the manner of presentation of the information

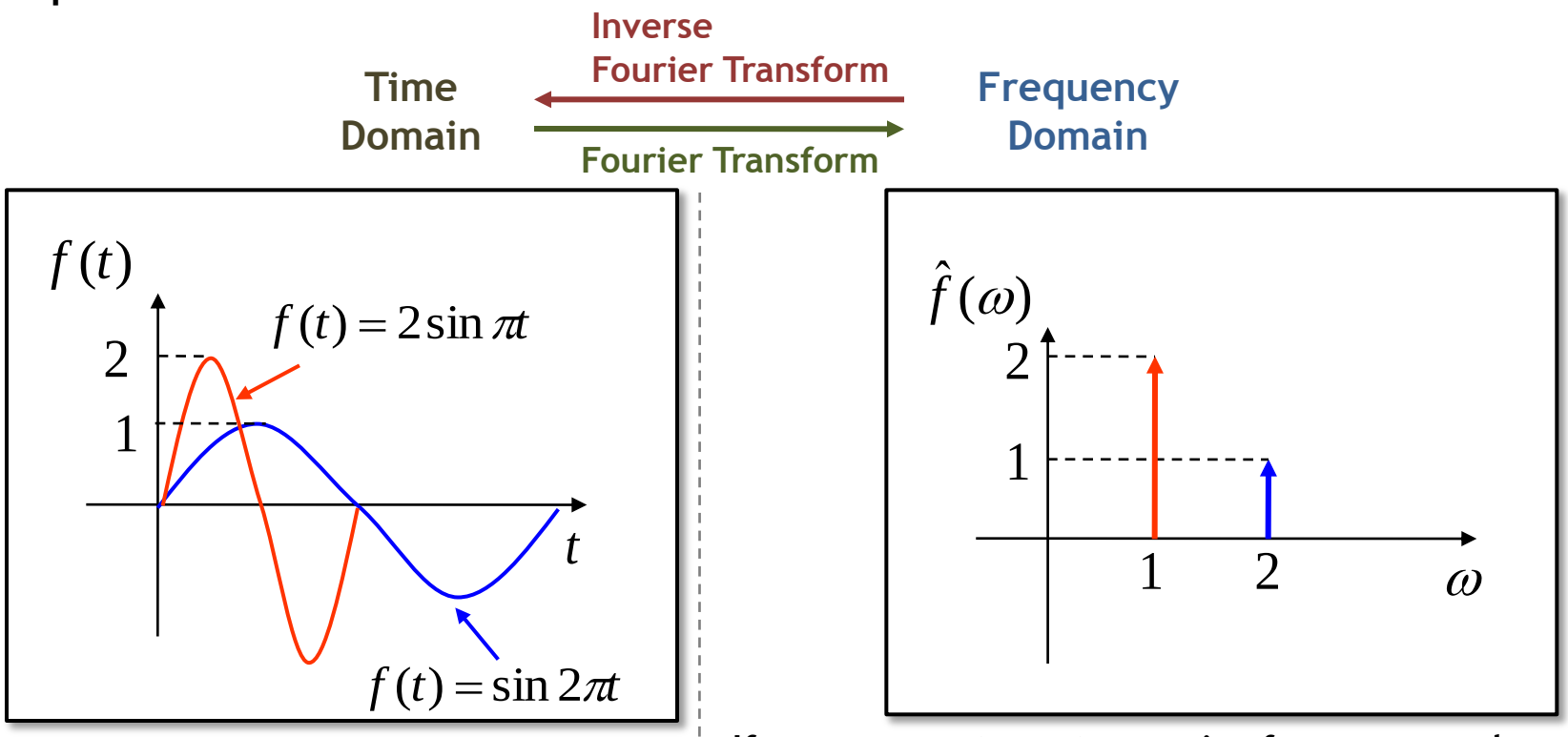
WAVE SPECTRUM : APPLICATION OF FOURIER SERIES AND FOURIER TRANSFORM

Application of Fourier Series and Fourier Transform (2)

✓ Application 2) Fourier transform

: Transform between time domain and frequency domain.

ex) Interpretation of the Fourier transform



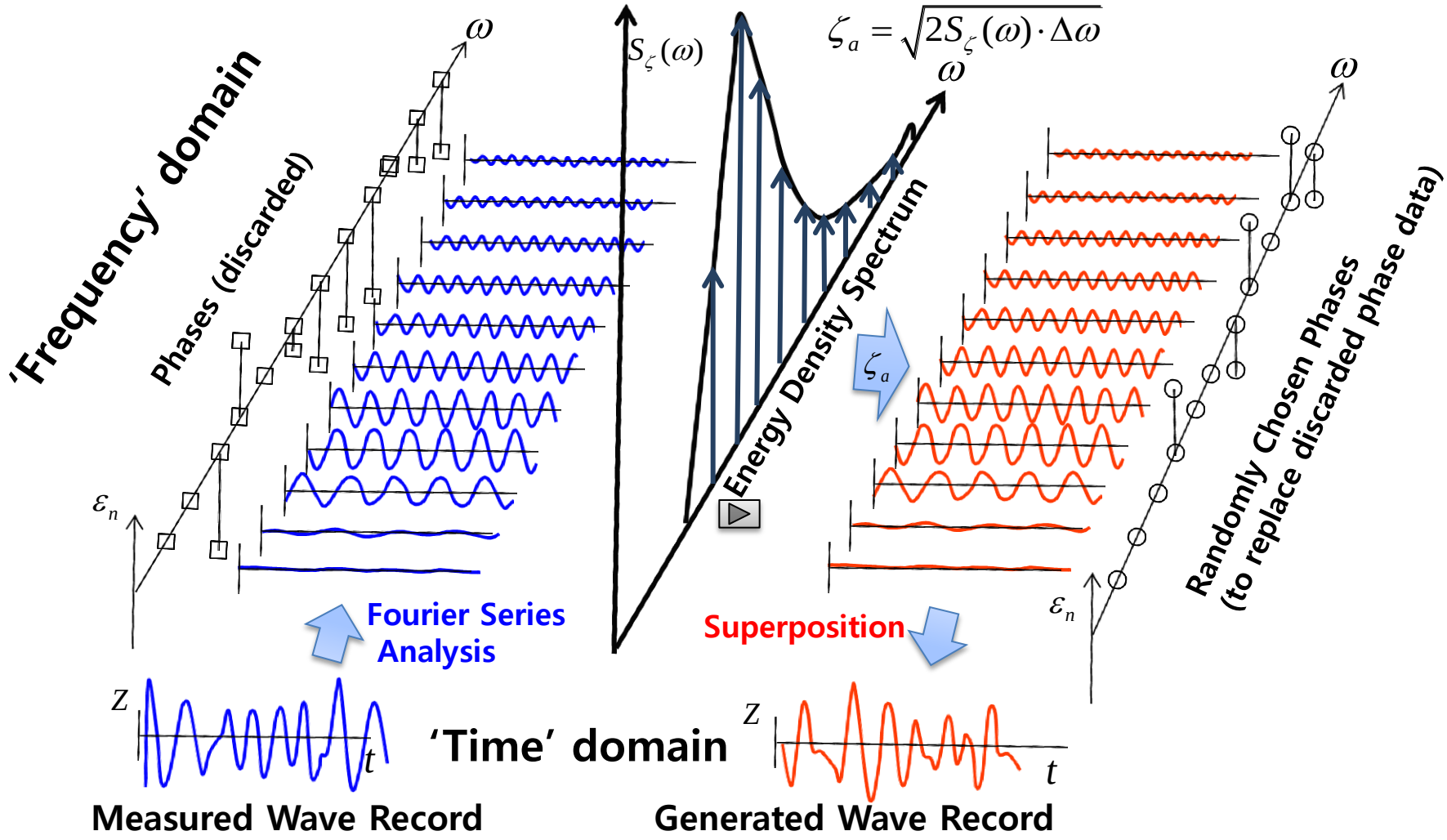
If we represent spectrum using frequency and amplitude,

we can reconstruct the function of time f(t).

Application of Fourier Series

✓ Application 2) Fourier transform

: Transform between time domain and frequency domain. ▷ wave spectrum



ζ_a:wave amplitude

Wave Spectrum

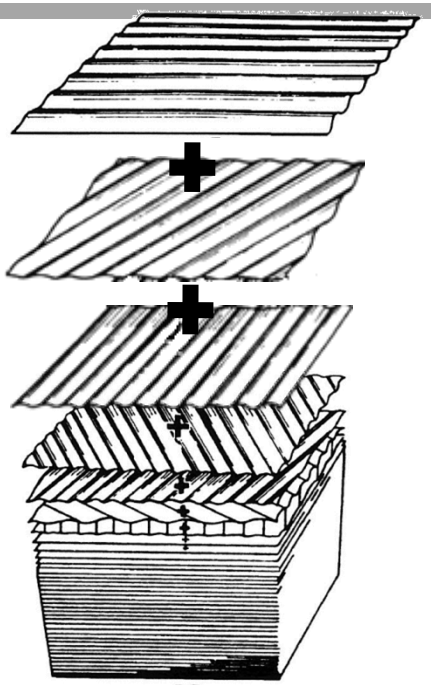
→ 3학년 해양파 역학

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2p} = \frac{\pi}{p}$$

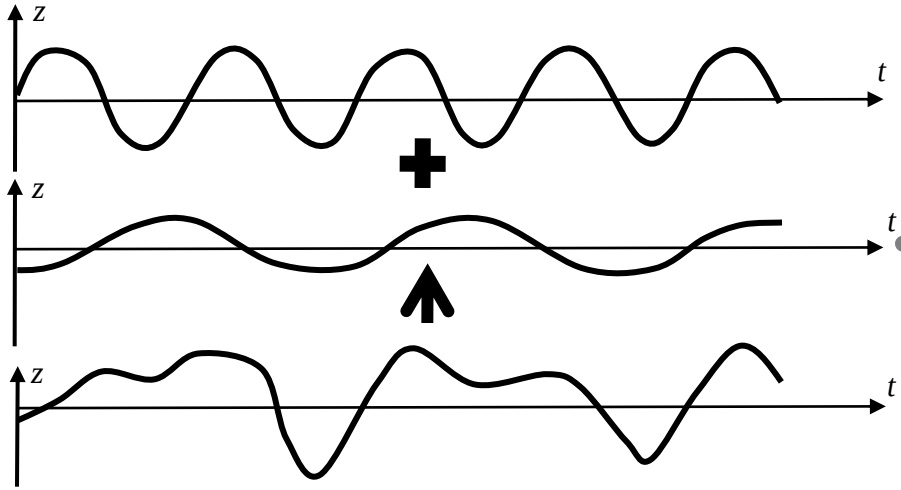
•Linear wave

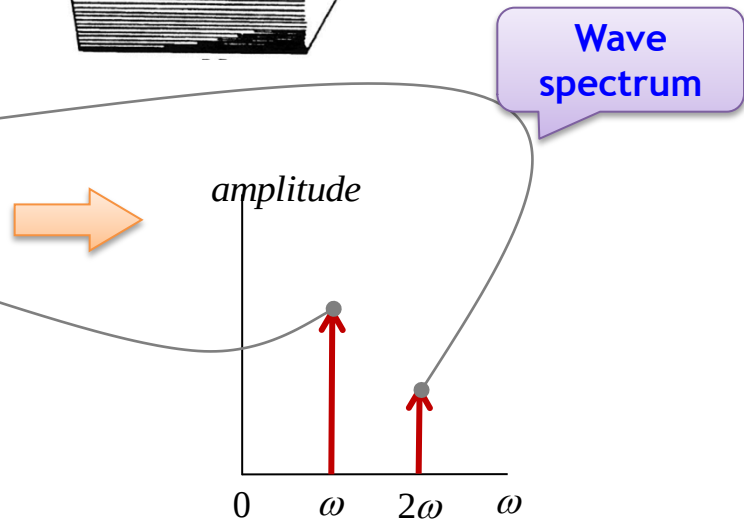
-Sum of many simple sine waves makes an irregular sea*

* *



-Superposition of two uni-directional harmonic waves***





frequency domain contains exactly the same information as that of the time domain

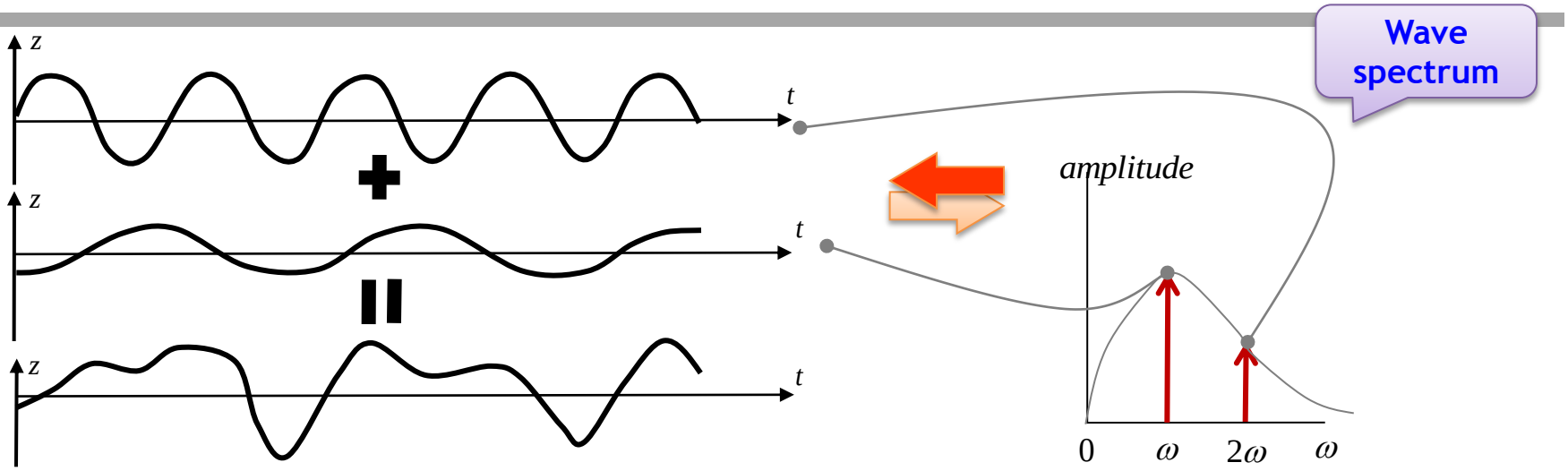
* Journee J.M.J. and Massie W.W., Offshore Hydrodynamics, first edition, Delft University of Technology, 2001, 5-3, Figure 5.1

** http://i.pbase.com/o6/47/624647/1/71472987.Kc5MyTSE.seasurface.jpg

***Journee J.M.J. and Massie W.W., Offshore Hydrodynamics, first edition, Delft University of Technology, 2001, 5-29, Figure 5.22

Wave Spectrum

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2p} = \frac{\pi}{p}$$



If you know wave spectrum, can 're-construct' the original wave

Standard Wave Spectrum

•Bretschneider (Modified Two-Parameter Pierson-Moskowitz/ ITTC/ ISSC) Wave Spectrum*

$$S(\omega) = \frac{173 \cdot H_{1/3}^2}{T_1^4} \cdot \omega^{-5} \cdot \exp\left\{\frac{-692}{T_1^4} \cdot \omega^{-4}\right\}$$

$H_{1/3}$: **Significant Wave Height**

T_1 : **Mean Centroid Wave Period**

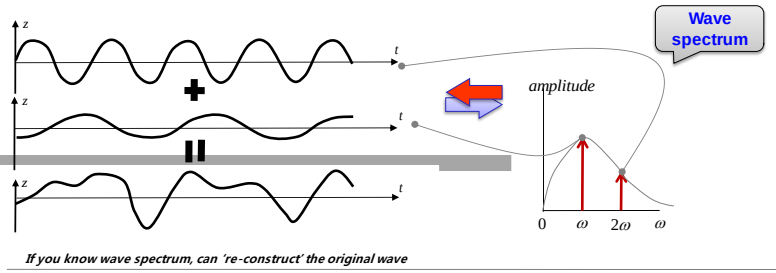
•JONSWAP Wave Spectrum

$$S(\omega) = 155 \frac{H_{1/3}^2}{T_1^4 \omega^5} \exp\left\{\frac{-944}{T_1^4 \omega^4}\right\} (3.3)^Y \quad , Y = \exp\left[-\left(\frac{0.191\omega T_1 - 1}{2^{1/2}\sigma}\right)^2\right] \quad , \sigma = \begin{cases} 0.07 & (\omega \le 5.24/T_1) \\ 0.09 & (\omega > 5.24/T_1) \end{cases}$$

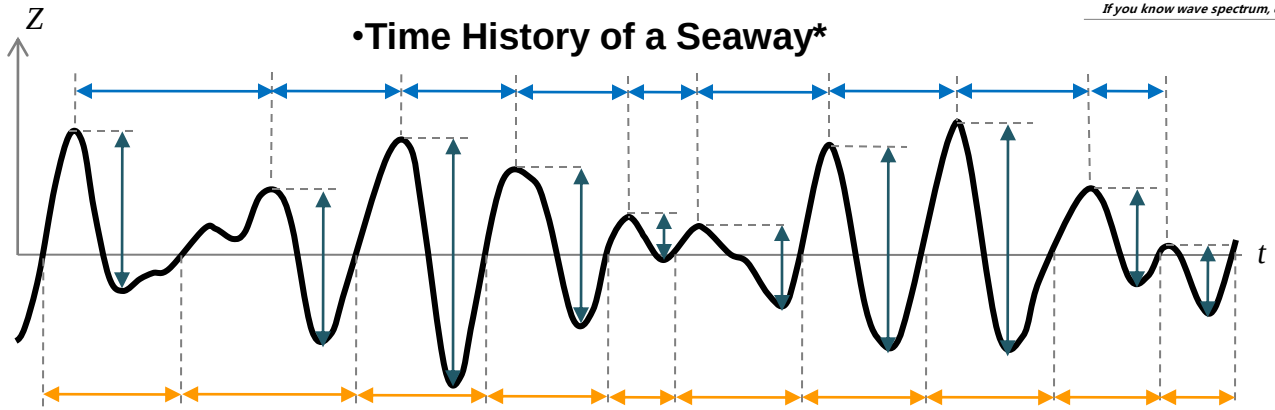
ITTC : International Towing Tank Conference/ISSC : International Ship Structures Congress
JONSWAP : the Joint North Sea Wave Project
Engineering Mat* Journee J.M.J. and Massie W.W., Offshore Hydrodynamics, first edition, Delft University of Technology, 2001, 5-43~5-44

Wave Spectrum

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2p} = \frac{\pi}{p}$$



How to use Standard Wave Spectrum



•Time History of a Seaway*

Measuring

- Wave Height
- Peak-to-Peak Wave Period
- Zero-Crossing Wave Period

Parameters

$H_{1/3}$: Significant Have Height
the average of the highest 1/3 the waves in record

T_1 ← T_2 : Mean Zero Crossing Wave Period

T_1 ← T_0 : Mean Peak-to-Peak Wave Period (Modal Period)

•Bretschneider (Modified Two-Parameter Pierson-Moskowitz/ ITTC/ ISSC) Wave Spectrum*

$$S(\omega) = \frac{173 \cdot H_{1/3}^2}{T_1^4} \cdot \omega^{-5} \cdot \exp\left\{\frac{-692}{T_1^4} \cdot \omega^{-4}\right\}$$

•JONSWAP Wave Spectrum

$$S(\omega) = 155\frac{H_{1/3}^2}{T_1^4 \omega^5}\exp\left\{\frac{-944}{T_1^4\omega^4}\right\}(3.3)^Y$$

$$,Y = \exp\left[-\left(\frac{0.191\omega T_1 - 1}{2^{1/2}\sigma}\right)^2\right] \quad ,\sigma = \begin{cases} 0.07 & (\omega \le 5.24/T_1) \\ 0.09 & (\omega > 5.24/T_1) \end{cases}$$

Parameter Input

•Bretschneider (Modified Two-Parameter Pierson-Moskowitz/ ITTC/ ISSC) Wave Spectrum

$$T_1 = 1.086T_2, \quad T_0 = 1.408T_2$$

•JONSWAP Wave Spectrum

$$T_1 = 0.834T_0 = 1.073T_2$$

* Journee J.M.J. and Massie W.W., Offshore Hydrodynamics, first edition, Delft University of Technology, 2001, 5-32. Figure 5.26

Wave Spectrum

ζ_a:wave amplitude

$S_\zeta(\omega)$:energy density spectrum

$T_1,\ H_{1/3}$

•**Bretschneider (Modified Two-Parameter Pierson-Moskowitz/ ITTC/ ISSC) Wave Spectrum***

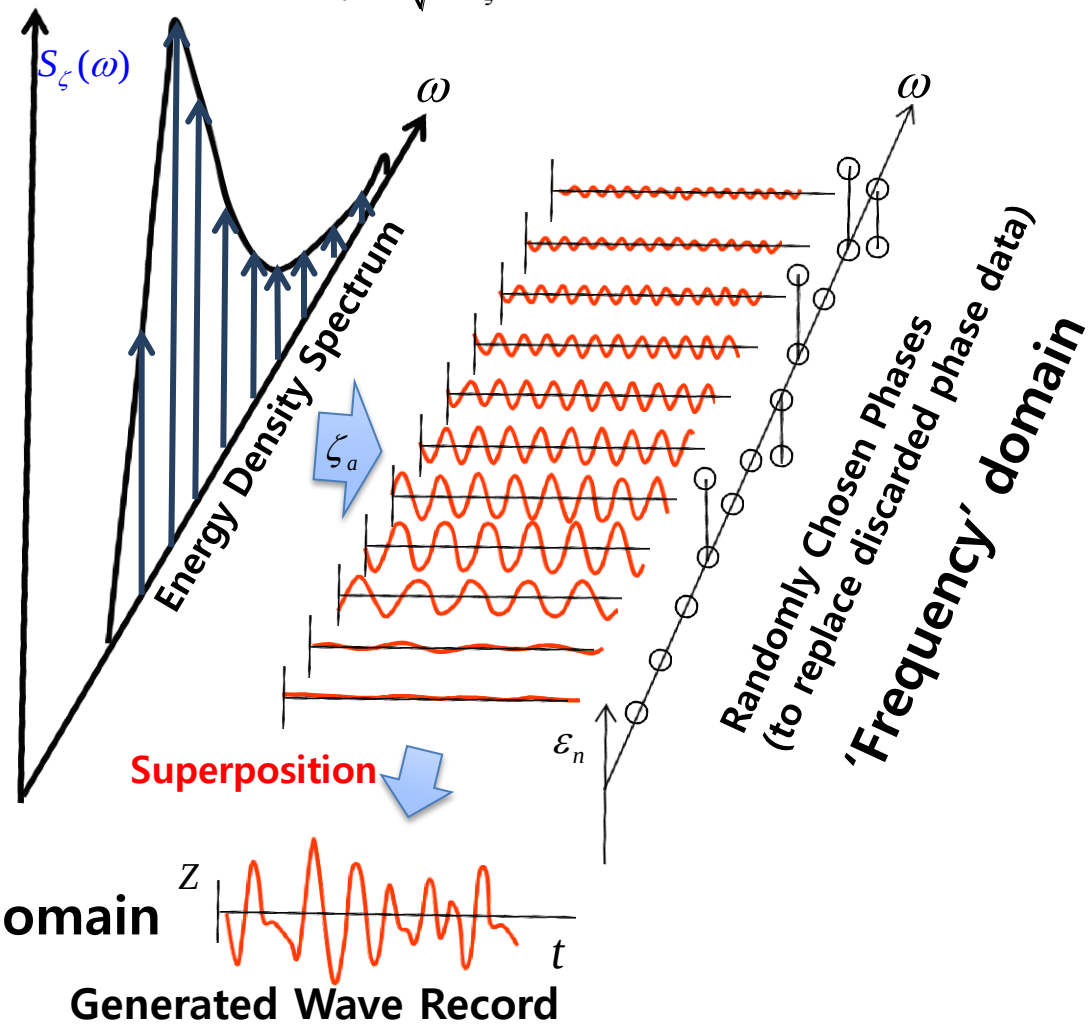
$$S(\omega) = \frac{173 \cdot H_{1/3}^2}{T_1^4} \cdot \omega^{-5} \cdot \exp\left\{\frac{-692}{T_1^4} \cdot \omega^{-4}\right\}$$

•**JONSWAP Wave Spectrum**

$$S(\omega) = 155 \frac{H_{1/3}^2}{T_1^4 \omega^5} \exp\left\{\frac{-944}{T_1^4 \omega^4}\right\} (3.3)^Y$$

$$,Y = \exp\left[-\left(\frac{0.191\omega T_1 - 1}{2^{1/2}\sigma}\right)^2\right] \quad ,\sigma = \begin{cases} 0.07 & (\omega \le 5.24/T_1) \\ 0.09 & (\omega > 5.24/T_1) \end{cases}$$

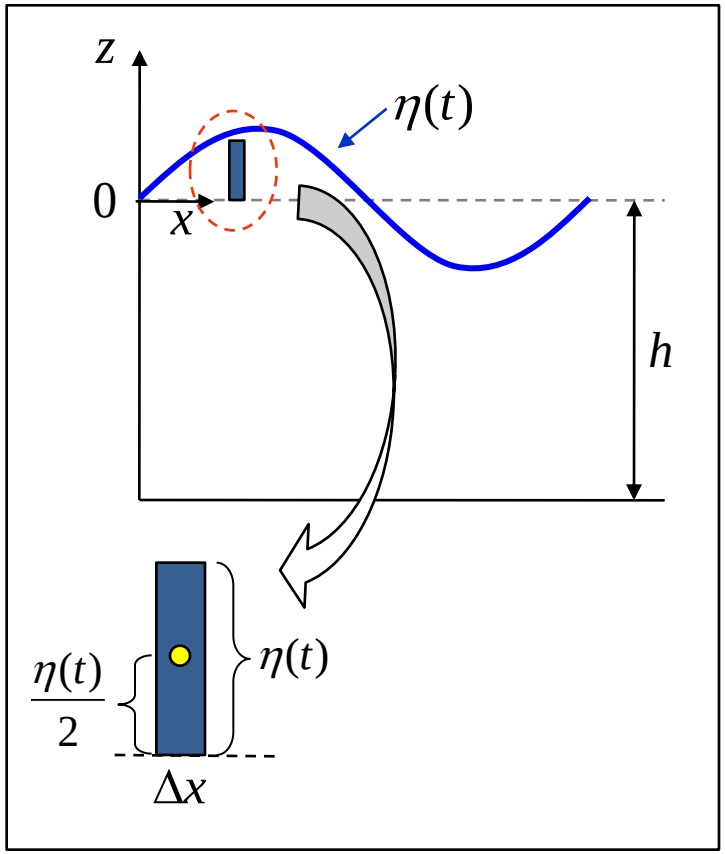
$$\zeta_a = \sqrt{2 S_\zeta(\omega) \cdot \Delta\omega}$$



*Journee J. M. J., Massie W. W., Offshore Hydrodynamics, First Edition, Delft University of Technology, 2001, Figure 5.38

WAVE ENERGY

Wave Energy



(Q) 한 주기 파의 Potential Energy 합은?

(A) 0 이 아니다.

sin 파의 앞쪽 반주기 동안은 물입자를 들어올리는데 일을 하였고, 뒤의 반주기 동안은 내리는데 일을 하였음.

- 미소 부피에 대한 Potential Energy

$$dE = \rho g \eta(t) dx \times \frac{\eta(t)}{2} = \frac{\rho g \{\eta(t)\}^2}{2} dx$$

- 한 주기 파의 평균 Potential Energy

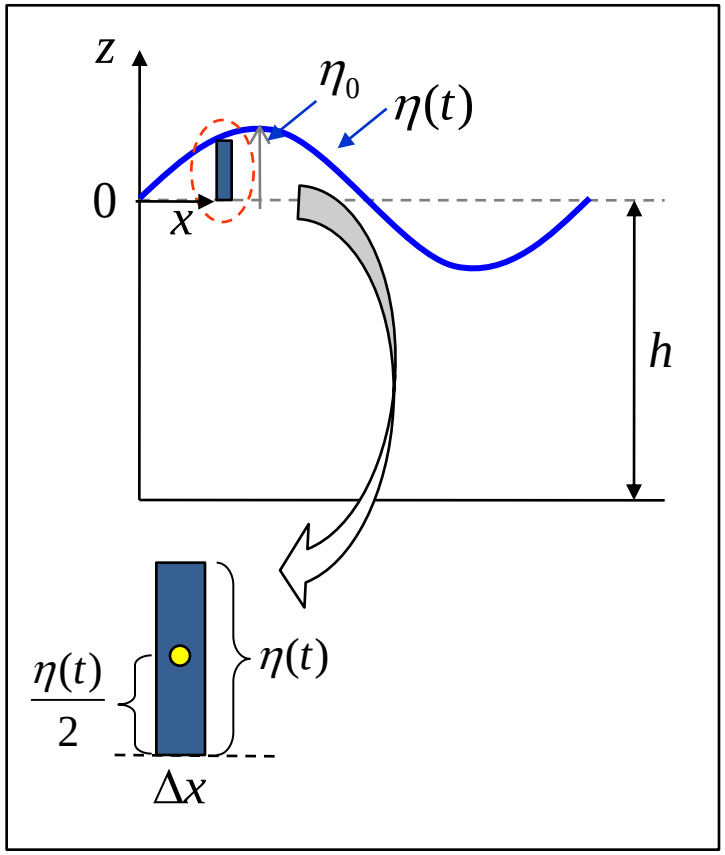
$$PE = \frac{1}{L}\int dE = \frac{1}{L}\int_0^L \frac{\rho g \{\eta(t)\}^2}{2} dx$$

$$= \frac{\rho g}{2L}\int_0^L \eta_0^2 \sin^2(kx - \omega t) dx = \left[\frac{1}{2}x - \frac{2}{k}\sin(kx - wt)\right]_0^L$$

$$= \frac{\rho g}{2L} \cdot \frac{\eta_0^2 L}{2} = \frac{1}{4}\rho g \eta_0^2$$

Wave Energy

Wave velocity potential* $\Phi = \frac{g\eta_0}{\omega} \cdot \frac{\cosh k(z+h)}{\cosh kh} \cdot \sin(kx - \omega t)$



- 미소 부피에 대한 Kinetic Energy

$$dE = \frac{dm}{2} \cdot |\nabla\Phi|^2 = \frac{\rho}{2}(u^2 + w^2)dzdx$$

$$u = \frac{\partial\Phi}{\partial x} = \frac{g\eta_0}{\omega} \cdot \frac{\cosh k(z+h)}{\cosh kh} \cdot k\cos(kx - \omega t)$$

$$w = \frac{\partial\Phi}{\partial z} = \frac{g\eta_0}{\omega} \cdot k\frac{\sinh k(z+h)}{\cosh kh} \cdot \sin(kx - \omega t)$$

$$dE = \frac{\rho}{2} \cdot \left(\frac{g\eta_0 k}{\omega\cosh kh}\right)^2 \left(\begin{array}{l}\cosh^2 k(z+h)\cos^2(kx-\omega t) \\ +\sinh^2 k(z+h)\sin^2(kx-\omega t)\end{array}\right) dzdx$$

*권순홍 외 공역, 해양환경하중, 동명사, 1990, p27 표2.1, (원저: Faltisen O.M., Sea Load on Ships and Offshore Structures)

Wave Energy

$$dE = \frac{\rho}{2} \cdot \left(\frac{g\eta_0 k}{\omega \cosh kh} \right)^2 \underline{\left(\cosh^2 k(z+h)\cos^2(kx-\omega t) + \sinh^2 k(z+h)\sin^2(kx-\omega t) \right)} dzdx$$

$$\cosh^2 k(z+h)\cos^2(kx-\omega t) + \sinh^2 k(z+h)\sin^2(kx-\omega t)$$
$$= \cosh^2 k(z+h)\cos^2(kx-\omega t) + \sinh^2 k(z+h)\left\{1-\cos^2(kx-\omega t)\right\}$$
$$= \cosh^2 k(z+h)\cos^2(kx-\omega t) - \sinh^2 k(z+h)\cos^2(kx-\omega t) + \sinh^2 k(z+h)$$
$$= \left\{\cosh^2 k(z+h) - \sinh^2 k(z+h)\right\}\cos^2(kx-\omega t) + \sinh^2 k(z+h)$$
$$= \cos^2(kx-\omega t) + \sinh^2 k(z+h)$$
$$= \frac{1+\cos 2(kx-\omega t)}{2} + \frac{\cosh 2k(z+h) - 1}{2}$$
$$= \frac{\cos 2(kx-\omega t) + \cosh 2k(z+h)}{2}$$

$$dE = \frac{\rho}{2} \cdot \left(\frac{g\eta_0 k}{\omega \cosh kh} \right)^2 \left(\frac{\cos 2(kx-\omega t) + \cosh 2k(z+h)}{2} \right) dzdx$$

Wave Energy

- 미소 부피에 대한 Kinetic Energy

$$dE = \frac{\rho}{2} \cdot \left(\frac{g\eta_0 k}{\omega \cosh kh} \right)^2 \left(\frac{\cos 2(kx-\omega t) + \cosh 2k(z+h)}{2} \right) dzdx$$

- 한 주기 파의 평균 Kinetic Energy

$$KE = \frac{1}{L}\int dE = \frac{1}{L}\int_0^L \int_{-h}^0 \frac{\rho}{2}\left(\frac{g\eta_0 k}{\omega \cosh kh} \right)^2 \left(\frac{\cos 2(kx-\omega t) + \cosh 2k(z+h)}{2} \right) dzdx$$

$$= \frac{\rho}{4L}\left(\frac{g\eta_0 k}{\omega \cosh kh} \right)^2 \int_0^L \int_{-h}^0 \left[\cos 2(kx-\omega t) + \cosh 2k(z+h)\right] dzdx$$

$$= \frac{\rho}{4L} \cdot \frac{g^2\eta_0^2 k^2}{\omega^2 \cosh^2 kh} \int_0^L \left[z\cos 2(kx-\omega t) + \frac{\sinh 2k(z+h)}{2k} \right]_{-h}^0 dx$$

$$= \frac{\rho}{4L} \cdot \frac{g^2\eta_0^2 k^2}{\omega^2 \cosh^2 kh} \int_0^L \left[h\cos 2(kx-\omega t) + \frac{\sinh 2kh}{2k} \right] dx$$

$$= \frac{\rho}{4L} \cdot \frac{g^2\eta_0^2 k^2}{\omega^2 \cosh^2 kh} \left[\frac{h\cos 2(kx-\omega t)}{2k} + \frac{x\sinh 2kh}{2k} \right]_0^L$$

$$= \frac{\rho}{4L} \cdot \frac{g^2\eta_0^2 k^2}{\omega^2 \cosh^2 kh} \left[\frac{L\sinh 2kh}{2k} \right]$$

Wave Energy

(Continue)

$$KE = \frac{\rho}{4L} \cdot \frac{g^2 \eta_0^2 k^2}{\omega^2 \cosh^2 kh} \left[\frac{L \sinh 2kh}{2k} \right]$$

$$= \frac{\rho}{8} \cdot \frac{g^2 \eta_0^2 k}{\omega^2} \cdot \frac{\sinh 2kh}{\cosh^2 kh}$$

$$= \frac{\rho}{8} \cdot \frac{g^2 \eta_0^2 k}{\omega^2} \cdot \frac{2(\cosh kh)(\sinh kh)}{\cosh^2 kh}$$

$$= \frac{\rho}{4} \cdot \frac{g^2 \eta_0^2 k}{\omega^2} \cdot \frac{\sinh kh}{\cosh kh}$$

$$= \frac{\rho}{4} \cdot g \eta_0^2 \cdot \frac{gk \tanh kh}{\omega^2}$$

$$= \boxed{\frac{\rho g \eta_0^2}{4}}$$

$$\sinh 2kh = \frac{e^{2kh} - e^{-2kh}}{2} = \frac{\left(e^{kh} + e^{-kh}\right)\left(e^{kh} - e^{-kh}\right)}{2}$$

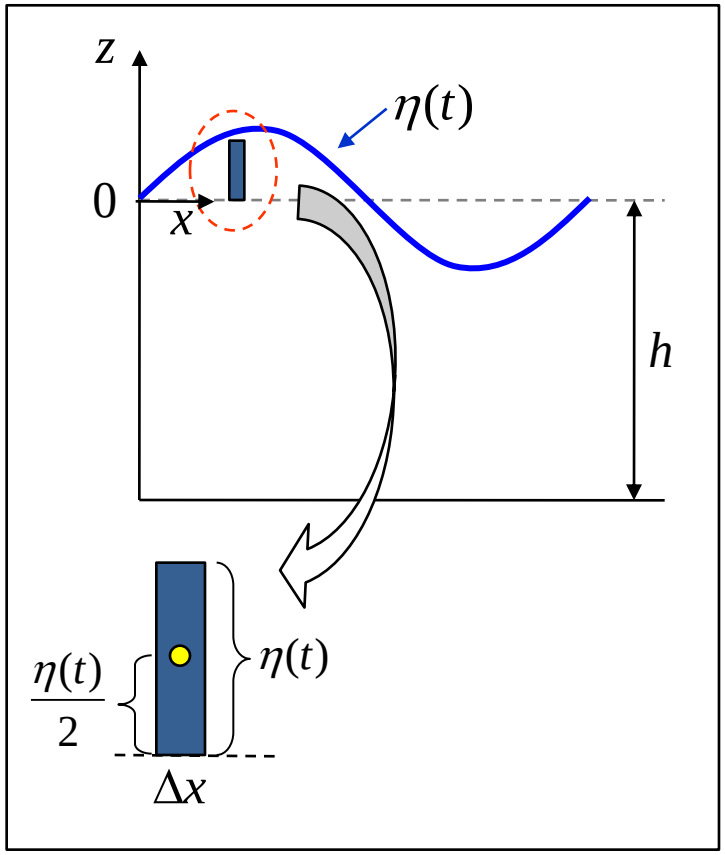
$$= 2\frac{\left(e^{kh} - e^{-kh}\right)}{2}\frac{\left(e^{kh} - e^{-kh}\right)}{2} = 2(\cosh kh)(\sinh kh)$$

Dispersion Relation

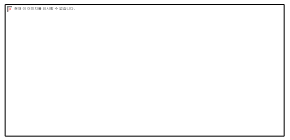
$$\omega^2 = gk \tanh kh$$

=> 한 주기 파의 평균 Kinetic Energy

Wave Energy



- 한 주기 파의 평균 Potential Energy



- 한 주기 파의 평균 Kinetic Energy

$$KE = \frac{1}{4}\rho g \eta_0^2$$

- 한 주기 파의 평균 Energy

$$E = PE + KE = \frac{1}{2}\rho g \eta_0^2$$

=> Energy는 (파고)^{2}에 비례함

Physical Interpretation : Spectrum

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \cdots (7)$$

The nature of the representation (7) of $f(x)$ becomes clear if we think of it as a superposition of sinusoidal oscillations of all possible frequencies, called a spectral representation.

This name is suggested by optics, where light is such a superposition of colors (frequencies).

In (7), the "spectral density" $\hat{f}(\omega)$ measures the intensity of $f(x)$ in the frequency interval between w and $w + \Delta w$(Δw small, fixed). We claim that in connection with vibrations, the integral

$$\int_{-\infty}^{\infty} \left| \hat{f}(\omega) \right|^2 d\omega$$

can be interpreted as the total energy of the physical system. Hence an integral of $\left| \hat{f}(\omega) \right|^2$ from a to b gives the contribution of the frequency w between a and b to the total energy.

Physical Interpretation : Spectrum

$\int_{-\infty}^{\infty}\left|\hat{f}(\omega)\right|^2 d\omega$: **total energy** of the physical system

To make this plausible, we begin with a mechanical system giving a single frequency, namely, the harmonic oscillator (mass on a spring)

$$my''+ky=0.$$ **(Here we denote time *t* by *x*)**

Multiplication by y' **gives** $my''y'+kyy'=0.$

Integrating with respect to *x*,

L.H.S: $\int m\frac{d}{dx}(v)\cdot v dx+\int ky\frac{dy}{dx}dx$ $\qquad y''=\frac{dv}{dx},\ v=y'=\frac{dy}{dx},$

$$=\frac{1}{2}mv^2+\frac{1}{2}ky^2+c_1=c_2$$ **: R.H.S**

Physical Interpretation : Spectrum

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega\cdots(7)$$

$$\int_{-\infty}^{\infty}\left|\hat{f}(\omega)\right|^2 d\omega$$

$$my''+ky=0,\quad \frac{1}{2}mv^2+\frac{1}{2}ky^2+c_1=c_2$$

$$\frac{1}{2}mv^2+\frac{1}{2}ky^2=E_0=\text{const}$$

kinetic energy + potential energy = total energy of the system

general solution of the above ODE

$$y(x)=a_1\cos\omega_0 x+b_1\sin\omega_0 x=c_1e^{i\omega_0 x}+c_{-1}e^{-i\omega_0 x},\ \left(\omega_0^2=\frac{k}{m}\right)$$

$$c_1=\frac{a_1-ib_1}{2},\ c_{-1}=\bar{c}_1=\frac{a_1+ib_1}{2}$$

Physical Interpretation : Spectrum

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega\cdots(7)$$

$$\int_{-\infty}^{\infty}\left|\hat{f}(\omega)\right|^2 d\omega$$

$$my''+ky=0,\quad \frac{1}{2}mv^2+\frac{1}{2}ky^2=E_0=\text{const}$$

$$y(x)=a_1\cos\omega_0 x+b_1\sin\omega_0 x=c_1e^{i\omega_0 x}+c_{-1}e^{-i\omega_0 x},\quad\left(\omega_0^2=\frac{k}{m}\right)$$

$$c_1=\frac{a_1-ib_1}{2},\ c_{-1}=\bar{c}_1=\frac{a_1+ib_1}{2}$$

Writng simply $A=c_1e^{i\omega_0 x}, B=c_{-1}e^{-i\omega_0 x}$

$$y(x)=A+B,\quad y'(x)=v=A'+B'=i\omega_0c_1e^{i\omega_0 x}-i\omega_0c_{-1}e^{-i\omega_0 x}$$

$$=i\omega_0(A-B)$$

Substitution of *v* and *y* on the left side of the equation for E_0 gives

$$E_0=\frac{1}{2}m(iw_0)^2(A-B)^2+\frac{1}{2}k(A+B)^2$$

Physical Interpretation : Spectrum

$$A = c_1 e^{i\omega_0 x},\ B = c_{-1} e^{-i\omega_0 x},\quad \left(\omega_0^2 = \frac{k}{m}\right)$$

$$E_0 = \frac{1}{2} m (i\omega_0)^2 (A-B)^2 + \frac{1}{2} k (A+B)^2$$

$$= \frac{1}{2} m\omega_0^2 \left[-(A-B)^2\right] + \frac{1}{2} k (A+B)^2$$

$$= \frac{1}{2} k \left[-(A-B)^2\right] + \frac{1}{2} k (A+B)^2$$

$$= \frac{1}{2} k \left[(A+B)^2 - (A-B)^2\right]$$

$$= 2kAB \ = 2kc_1 e^{i\omega_0 x} \cdot c_{-1} e^{-i\omega_0 x}$$

$$= 2kc_1 \cdot c_{-1}$$

$$= 2k|c_1|^2 \qquad \because c_{-1} = \bar{c}_1$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega$$

$$\int_{-\infty}^{\infty} \left|\hat{f}(\omega)\right|^2 d\omega$$

the energy is proportional to the square of the amplitude $|c_1|$.

Physical Interpretation : Spectrum

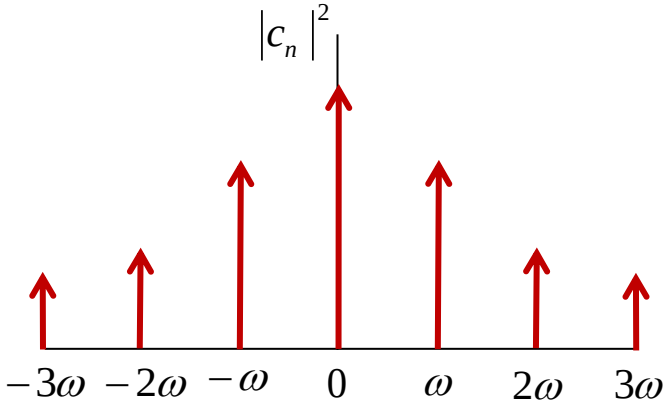
$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p}$$

$$E_0 = 2k|c_1|^2$$

$$c_n = \frac{1}{2p}\int_{-p}^{p} f(x)e^{-in\pi x/p}dx$$

Hence the energy is proportional to the square of the amplitude $|c_1|$.

As the next step, if a more complicated system leads to a periodic solution $y = f(x)$ that can be represented by a Fourier series, then instead of the single energy term $|c_1|^2$ we get a series of squares $|c_n|^2$ of Fourier coefficients c_n given by (6). In this case we have a "discrete spectrum" (or "point spectrum") consisting of countably many isolated frequencies (infinitely many, in general), the corresponding $|c_n|^2$ being the contributions to the total energy.



FOURIER TRANSFORM

Fourier Cosine Transform

For an even function *f* (*x*), the Fourier integral is the Fourier cosine integral

$$f(x) = \int_0^\infty A(\omega)\cos\omega x\, d\omega \cdots (1a)$$

$$\text{where } A(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\cos\omega v\, dv \cdots (1b)$$

$$\text{Now we set } A(\omega) = \sqrt{2/\pi}\,\hat{f}_c(\omega), \quad \text{where } c \text{ suggests "cosine"}$$

Then from (1b), writing *v* = *x*

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\omega x\, dx \cdots (2)$$

: Fourier cosine transform of *f* (*x*)

and from (1a),

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(\omega)\cos\omega x\, d\omega \cdots (3)$$

: inverse Fourier cosine transform of $\hat{f}_c(\omega)$

Fourier Sine Transform

For an odd function *f* (*x*), the Fourier integral is the Fourier sine integral

$$f(x) = \int_0^\infty B(\omega)\sin\omega x\, d\omega \cdots (4a)$$

$$\text{where } B(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\sin\omega v\, dv \cdots (4b)$$

$$\text{Now we set } B(\omega) = \sqrt{2/\pi}\,\hat{f}_s(\omega), \quad \text{where } s \text{ suggests "sine"}$$

Then from (4b), writing *v* = *x*

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin\omega x\, dx \cdots (5)$$

: Fourier sine transform of *f* (*x*)

and from (4a),

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(\omega)\sin\omega x\, d\omega \cdots (6)$$

: inverse Fourier sine transform of $\hat{f}_s(\omega)$

Fourier Cosine and Sine Transform

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx \cdots (2)$$

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx \cdots (5)$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(\omega) \cos \omega x \, d\omega \cdots (3)$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(\omega) \sin \omega x \, d\omega \cdots (6)$$

Other notations

$$\mathcal{F}_c(f) = \hat{f}_c(\omega), \quad \mathcal{F}_s(f) = \hat{f}_s(\omega)$$

$\mathcal{F}_c^{-1}(f)$, $\mathcal{F}_s^{-1}(f)$: inverses of $\mathcal{F}_c$ and $\mathcal{F}_s$, respectively.

Fourier Cosine and Sine Transform

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx \cdots (5)$$

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx \cdots (2)$$

Ex.)Find the Fourier cosine and Fourier sine transforms of the function

$$f(x) = \begin{cases} k & if \;\; 0 < x < a \\ 0 & if \;\; x > a \end{cases}$$

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx = \sqrt{\frac{2}{\pi}} k \int_0^a \cos \omega x \, dx + \sqrt{\frac{2}{\pi}} \int_a^\infty 0 \cdot \cos \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} k \left(\frac{\sin a\omega}{\omega} \right)$$

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} k \int_0^a \sin \omega x \, dx = \sqrt{\frac{2}{\pi}} k \left(\frac{1 - \cos a\omega}{\omega} \right)$$

Fourier Cosine and Sine Transform

Ex.)Find the Fourier cosine and Fourier sine transforms of the function

$$f(x) = e^{-x}$$

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx \cdots (5)$$

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx \cdots (2)$$

$$\mathcal{F}_c(e^{-x}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos \omega x \, dx = \sqrt{\frac{2}{\pi}} \frac{1}{1+\omega^2}$$

$$\mathcal{F}_s(e^{-x}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin \omega x \, dx = \sqrt{\frac{2}{\pi}} \frac{\omega^2}{1+\omega^2}$$

$$\int_0^\infty e^{-x} \cos \omega x \, dx = \left[e^{-x} \frac{\sin \omega x}{\omega} \right]_0^\infty - \int_0^\infty (-e^{-x}) \frac{\sin \omega x}{\omega} dx$$

$$= \left[e^{-x} \frac{(-\cos \omega x)}{\omega^2} \right]_0^\infty - \int_0^\infty (-e^{-x}) \frac{(-\cos \omega x)}{\omega^2} dx$$

$$= \frac{1}{\omega^2} - \frac{1}{\omega^2} \int_0^\infty e^{-x} \cos \omega x dx$$

$$\therefore \int_0^\infty e^{-x} \cos \omega x dx = \frac{1}{1+\omega^2}$$

$$\int_0^\infty e^{-x} \sin \omega x \, dx = \left[e^{-x} \frac{(-\cos \omega x)}{\omega} \right]_0^\infty - \int_0^\infty (-e^{-x}) \frac{(-\cos \omega x)}{\omega} dx$$

$$= 1 - \left\{ \left[e^{-x} \frac{\sin \omega x}{\omega^2} \right]_0^\infty - \int_0^\infty (-e^{-x}) \frac{(\sin \omega x)}{\omega^2} dx \right\}$$

$$= 1 - \left\{ 0 + \frac{1}{\omega^2} \int_0^\infty e^{-x} \sin \omega x dx \right\}$$

$$\therefore \int_0^\infty e^{-x} \sin \omega x dx = \frac{\omega^2}{1+\omega^2}$$

Fourier Cosine and Sine Transform

Linearity, Transforms of Derivatives

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx \cdots (5)$$

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx \cdots (2)$$

If $f(x)$ is absolutely integrable on the positive x-axis and piecewise continuous on every finite interval, then the Fourier cosine and sine transforms of f exist.

If f and g have Fourier cosine and sine transforms, so does $af+bg$ for any constants a and b, and by (2)

$$\mathcal{F}_c(af + bg) = \sqrt{\frac{2}{\pi}} \int_0^\infty [af(x) + bg(x)] \cos \omega x \, dx$$

$$= a\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx + b\sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \cos \omega x \, dx$$

Fourier Cosine and Sine Transform

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx \cdots (5)$$

Linearity, Transforms of Derivatives

$$\mathcal{F}_c(af + bg) = a\mathcal{F}_c(f) + b\mathcal{F}_c(g)$$

Similarly for F_s, by (5)

$$\mathcal{F}_s(af + bg) = \sqrt{\frac{2}{\pi}} \int_0^\infty [af(x) + bg(x)] \sin \omega x \, dx$$

$$= a\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx + b\sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \sin \omega x \, dx$$

$$= a\mathcal{F}_s(f) + b\mathcal{F}_s(g)$$

This shows that the Fourier cosine and sine transforms are linear operations,

$$F_c(af + bg) = aF_c(f) + bF_c(g), \cdots (7a)$$

$$\mathcal{F}_s(af + bg) = a\mathcal{F}_s(f) + b\mathcal{F}_s(g) \cdots (7b)$$

Fourier Cosine and Sine Transform

Linearity, Transforms of Derivatives

Theorem1) Cosine and Sine Transforms of Derivatives

Let $f(x)$ be continuous and absolutely integrable on the x-axis (Fourier cosine and sine transforms exist), let $f'(x)$ be piecewise continuous on every finite interval, and let $f(x) \to 0$ as $x \to \infty$. Then

$$\mathcal{F}_c\{f'(x)\} = w\mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0), \cdots (8a)$$

$$\mathcal{F}_s\{f'(x)\} = -w\mathcal{F}_c\{f(x)\} \cdots (8b)$$

Fourier Cosine and Sine Transform

Linearity, Transforms of Derivatives

$$\mathcal{F}_c\{f'(x)\} = \omega\mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}}f(0), \cdots (8a)$$

$$\mathcal{F}_s\{f'(x)\} = -\omega\mathcal{F}_c\{f(x)\} \cdots (8b)$$

proof) $\mathcal{F}_c\{f'(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos\omega x\,dx$

$$= \sqrt{\frac{2}{\pi}}\left[f(x)\cos\omega x\Big|_0^\infty + \omega\int_0^\infty f(x)\sin\omega x\,dx \right]$$

$$= -\sqrt{\frac{2}{\pi}}f(0) + \omega F_s\{f(x)\};$$

$$\mathcal{F}_s\{f'(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\sin\omega x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[f(x)\sin\omega x\Big|_0^\infty - \omega\int_0^\infty f(x)\cos\omega x\,dx \right]$$

$$= 0 - \omega F_c\{f(x)\}$$

Fourier Cosine and Sine Transform

Linearity, Transforms of Derivatives

Formula (8a) with f' instead of f gives (when f', f'' satisfy the respective assumptions for f', f'' in Theorem 1)

$$\mathcal{F}_c\{f''(x)\} = \omega\mathcal{F}_s\{f'(x)\} - \sqrt{\frac{2}{\pi}}f'(0);$$

hence, by (8b)

$$\mathcal{F}_c\{f''(x)\} = \omega^2\mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}}f'(0)\cdots(9a)$$

Similarly,

$$\mathcal{F}_s\{f''(x)\} = -\omega^2\mathcal{F}_s\{f(x)\} + \sqrt{\frac{2}{\pi}}\omega f(0).\cdots(9b)$$

Fourier Cosine and Sine Transform

Linearity, Transforms of Derivatives

$$\mathcal{F}_c\{f''(x)\}=\omega^2\mathcal{F}_c\{f(x)\}-\sqrt{\frac{2}{\pi}}f'(0)\cdots(9a)$$

$$\mathcal{F}_s\{f''(x)\}=-\omega^2\mathcal{F}_s\{f(x)\}+\sqrt{\frac{2}{\pi}}\omega f(0).\cdots(9b)$$

Ex.)Find the Fourier cosine transform of $\mathcal{F}_c(f),\ f(x)=e^{-ax},\ a>0$

$$f'(x)=(e^{-ax})'=ae^{-ax}$$

$$f''(x)=(e^{-ax})''=a^2e^{-ax}$$

$$\therefore f''(x)=a^2f(x)$$

From this and the linearity (7a)

$$\mathcal{F}_c(f'')=\mathcal{F}_c(a^2f)=a^2\mathcal{F}_c(f)$$

From (9a),

$$\mathcal{F}_c(f'')=-\omega^2\mathcal{F}_c(f)-\sqrt{\frac{2}{\pi}}f'(0)$$

$$\therefore \mathcal{F}_c(f'')=-\omega^2\mathcal{F}_c(f)+a\sqrt{\frac{2}{\pi}}$$

Hence,

$$a^2\mathcal{F}_c(f)=\mathcal{F}_c(f'')=-\omega^2\mathcal{F}_c(f)+a\sqrt{\frac{2}{\pi}}$$

$$(a^2+\omega^2)\mathcal{F}_c(f)=a\sqrt{\frac{2}{\pi}}$$

$$\therefore \mathcal{F}_c(e^{-ax})=\sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+\omega^2}\right)\quad(a>0)$$

Fourier Transform

Fourier transform of $f(x)$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \cdots (6)$$

inverse Fourier transform of $\hat{f}(w)$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \cdots (7)$$

Another notation for the Fourier transform is

$$\hat{f} = \mathcal{F}(f)$$: Fourier transform of $f(x)$ or Fourier transform method.

so that

$$f = \mathcal{F}^{-1}(\hat{f})$$

Fourier Transform

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \cdots (6)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \cdots (7)$$

Theorem1) Existence of the Fourier Transform

Let *f* (*x*) be continuous and absolutely integrable on the *x*-axis and piecewise continuous on every finite interval, then the Fourier transform $\hat{f}(\omega)$ of *f* (*x*) given by (6) exists.

Fourier Transform

$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx\cdots(6)$$

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega\cdots(7)$$

Ex.)Find the Fourier transform of $f(x)$

$$f(x)=\begin{cases}1, & \text{if } |x|<1\\ 0, & \text{otherwise}\end{cases}$$

$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}e^{-i\omega x}dx$$

$$=\frac{1}{\sqrt{2\pi}}\cdot\frac{e^{-i\omega x}}{-i\omega}\Bigg|_{-1}^{1}$$

$$=\frac{1}{-i\omega\sqrt{2\pi}}\left(e^{-i\omega}-e^{i\omega}\right)$$

By Euler formula $(e^{\pm i\omega}=\cos\omega\pm i\sin\omega)$

$$\hat{f}(\omega)=\frac{1}{-i\omega\sqrt{2\pi}}\left[(\cos\omega-i\sin\omega)-(\cos\omega+i\sin\omega)\right]$$

$$=\frac{1}{-i\omega\sqrt{2\pi}}(2i\sin\omega)$$

$$\therefore\ \hat{f}(\omega)=\sqrt{\frac{\pi}{2}}\frac{\sin\omega}{\omega}$$

Fourier Transform

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \cdots (6)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \cdots (7)$$

Find the Fourier transform of

$$f(x) = \begin{cases} e^{-ax}, & \text{if } x > 0 \\ 0, & \text{if } x < 0 \end{cases},$$

$$(a > 0)$$

$$\mathcal{F}\left(e^{-ax}\right) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-ax} e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \frac{e^{-(a+i\omega)x}}{-(a+i\omega)}\Bigg|_{x=0}^{\infty} = \frac{1}{\sqrt{2\pi}(a+i\omega)}$$

Fourier Transform : Linearity

$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx\cdots(6)$$

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega\cdots(7)$$

Theorem2) Linearity of the Fourier Transform

The Fourier transform is a linear operation; that is, for any functions $f(x)$ and $g(x)$ whose Fourier transforms exist and any constants a and , the Fourier transform of $af+bg$ exists, and

$$\mathcal{F}(af+bg)=a\mathcal{F}(f)+b\mathcal{F}(g)\cdots(8)$$

proof)

$$\mathcal{F}(af+bg)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[af(x)+bg(x)\right]e^{-i\omega x}dx$$

$$=a\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx+b\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(x)e^{-i\omega x}dx$$

$$=a\mathcal{F}(f)+b\mathcal{F}(g)$$

Fourier Transform : Derivative

$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx\cdots(6)$$

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega\cdots(7)$$

Theorem3) Fourier Transform of the Derivative of $f(x)$

Let $f(x)$ be continuous on the x-axis and $f(x)\to 0$ as $|x|\to\infty$.

Furthermore, let $f'(x)$ be absolutely integrable on the x-axis. Then

$$\mathcal{F}\{f'(x)\}=i\omega\mathcal{F}\{f(x)\}\cdots(9)$$

proof)

$$\mathcal{F}\{f'(x)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f'(x)e^{-i\omega x}dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[f(x)e^{-i\omega x}\Big|_{-\infty}^{\infty}-(-i\omega)\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx\right]$$

$$=0+i\omega\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx \quad =i\omega F\{f(x)\}$$

Fourier Transform : Derivative

$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx\cdots(6)$$

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega\cdots(7)$$

$$\mathcal{F}\{f'(x)\}=i\omega\mathcal{F}\{f(x)\}\cdots(9)$$

Two successive application of (9) give

$$\mathcal{F}\{f''(x)\}=i\omega\mathcal{F}\{f'(x)\}$$

$$=i\omega i\omega\mathcal{F}\{f(x)\}$$

$$\therefore\mathcal{F}\{f''(x)\}=-\omega^2\mathcal{F}\{f(x)\}\cdots(10)$$

Fourier Transform : Derivative

$$\mathcal{F}\{f'(x)\} = i\omega\mathcal{F}\{f(x)\}\cdots(9)$$

Find the Fourier transform of xe^{-x^2}

$$\mathcal{F}\left(xe^{-x^2}\right) = \mathcal{F}\left\{-\frac{1}{2}\left(e^{-x^2}\right)'\right\}$$

$$= -\frac{1}{2}\mathcal{F}\left\{\left(e^{-x^2}\right)'\right\}$$

$$= -\frac{1}{2}i\omega\mathcal{F}\left(e^{-x^2}\right)$$

$$= -\frac{1}{2}i\omega\frac{1}{\sqrt{2}}e^{-\omega^2/4}$$

$$= -\frac{i\omega}{2\sqrt{2}}e^{-\omega^2/4}$$

Fourier Transforms : P.D.E

$$\mathcal{F}\{f'(x)\} = i\omega\mathcal{F}(\omega)$$
$$\mathcal{F}\{f''(x)\} = -\omega^2\mathcal{F}(\omega)$$

☑ Example 1 Using the Fourier Transform

Solve the heat equation $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t},\ -\infty < x < \infty,\ t > 0$

Subject to $u(x,0) = f(x)$, **where** $f(x) = \begin{cases} u_0, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

define $\mathcal{F}\{u(x,t)\} = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}dx = U(\omega,t)$

Transforming the equation

$$\mathcal{F}\left\{k\frac{\partial^2 u}{\partial x^2}\right\} = \mathcal{F}\left\{\frac{\partial u}{\partial t}\right\}$$

$$-k\omega^2 U = \frac{dU}{dt} \quad or \quad \frac{dU}{dt} + k\omega^2 U = 0 \xrightarrow{\text{solution}} U = ce^{-k\omega^2 t}$$

Fourier Transforms : P.D.E

$$\mathcal{F}\{f'(x)\} = i\omega\mathcal{F}(\omega)$$
$$\mathcal{F}\{f''(x)\} = -\omega^2\mathcal{F}(\omega)$$

☑ Example 1 Using the Fourier Transform

Solve the heat equation $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t},\ -\infty < x < \infty,\ t > 0$

Subject to $u(x,0) = f(x)$, **where** $f(x) = \begin{cases} u_0, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

$$\frac{dU}{dt} + k\omega^2 U(\omega,t) = 0 \quad \Rightarrow \quad U(\omega,t) = ce^{-k\omega^2 t}$$

Initial condition transformation

$$\mathcal{F}\{u(x,0)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} u_0 e^{-i\omega x}dx = u_0\frac{e^{i\omega} - e^{-i\omega}}{\sqrt{2\pi}i\omega}$$

$$U(\omega,0) = \mathcal{F}\{u(x,0)\} = \frac{2}{\sqrt{2\pi}}u_0\frac{\sin\omega}{\omega}$$

$$\therefore U(\omega,t) = \frac{2}{\sqrt{2\pi}}u_0\frac{\sin\omega}{\omega}e^{-k\omega^2 t}$$

Fourier Transforms : P.D.E

$$\mathcal{F}\{f'(x)\} = i\omega\mathcal{F}(\omega)$$
$$\mathcal{F}\{f''(x)\} = -\omega^2\mathcal{F}(\omega)$$

☑ Example 1 Using the Fourier Transform

Solve the heat equation $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t},\ -\infty < x < \infty,\ t > 0$

Subject to $u(x,0) = f(x)$, **where** $f(x) = \begin{cases} u_0, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

$$U(\omega,t) = \frac{2}{\sqrt{2\pi}} u_0 \frac{\sin\omega}{\omega} e^{-k\omega^2 t}$$

$$\because \frac{u_0}{\pi} i\int_{-\infty}^{\infty} \left(\underbrace{\frac{1}{\omega}}_{\text{odd}} \underbrace{\sin\omega}_{\text{odd}} \underbrace{e^{-k\omega^2 t}}_{\text{even}} \underbrace{\sin\omega x}_{\text{odd}} \right) d\omega = 0$$

odd function of w

Inverse transformation

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{2}{\sqrt{2\pi}} u_0 \frac{\sin\omega}{\omega} e^{-k\omega^2 t} \right) e^{i\omega x} d\omega$$

$$= \frac{u_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin\omega}{\omega} e^{-k\omega^2 t} e^{i\omega x} d\omega = \frac{u_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin\omega}{\omega} e^{-k\omega^2 t} (\cos\omega x + i\sin\omega x) d\omega$$

$$= \frac{u_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin\omega\cos\omega x}{\omega} e^{-k\omega^2 t} d\omega$$

Fourier Transforms

$$\mathcal{F}_c\{f''(x)\} = \omega^2 \mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0) \cdots (9a)$$

$$\mathcal{F}_s\{f''(x)\} = -\omega^2 \mathcal{F}_s\{f(x)\} + \sqrt{\frac{2}{\pi}} \omega f(0). \cdots (9b)$$

☑ Example 2 Using the Cosine Transform

The steady-state temperature in a semi-infinite plate is determined from

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < \pi,\ y > 0$$

$$u(0, y) = 0,\ u(\pi, u) = e^{-y},\ y > 0$$

Subject to

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \quad 0 < x < \pi.$$

Cosine transform is suitable

Define:

$$\mathcal{F}_c\{u(x, y)\} = \int_0^\infty u(x, y) \cos \omega y \, dy = U(x, \omega)$$

$$u(x, y) = \frac{2}{\pi} \int_0^\infty U(x, \omega) \cos \omega y \, d\omega$$

$$\mathcal{F}_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} + \mathcal{F}_c\left\{\frac{\partial^2 u}{\partial y^2}\right\} = \mathcal{F}_c\{0\}$$

Fourier Transforms

$$\mathcal{F}_c\{f''(x)\} = \omega^2 \mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0) \cdots (9a)$$

$$\mathcal{F}_s\{f''(x)\} = -\omega^2 \mathcal{F}_s\{f(x)\} + \sqrt{\frac{2}{\pi}} \omega f(0). \cdots (9b)$$

☑ Example 2 Using the Cosine Transform

The steady-state temperature in a semi-infinite plate is determined from

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < \pi,\ y > 0$$

Subject to

$$u(0, y) = 0,\ u(\pi, u) = e^{-y},\ y > 0$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \quad 0 < x < \pi.$$

Cosine transform is suitable

$$\mathcal{F}_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} + \mathcal{F}_c\left\{\frac{\partial^2 u}{\partial y^2}\right\} = \mathcal{F}_c\{0\}$$

$$\frac{d^2 U}{dx^2} - \omega^2 U(x, \omega) - u_y(x, 0) = 0 \ \Rightarrow\ \frac{d^2 U}{dx^2} - \omega^2 U = 0\ (\because \text{boundary condition})$$

$$\therefore U(x, \omega) = c_1 \cosh \omega x + c_2 \sinh \omega x$$

solution

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx \cdots (2)$$

☑ Example 2 Using the Cosine Transform

The steady-state temperature in a semi-infinite plate is determined from

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \; 0 < x < \pi, \; y > 0$$

Subject to

$$u(0, y) = 0, \; u(\pi, u) = e^{-y}, \; y > 0$$

$$\left. \frac{\partial u}{\partial y} \right|_{y=0} = 0, \quad 0 < x < \pi.$$

$$U(x, \omega) = c_1 \cosh \omega x + c_2 \sinh \omega x$$

Boundary condition

$$\mathcal{F}_c \{u(0, y)\} = U(0, \omega) = \mathcal{F}_c \{0\}$$

$$\therefore U(0, \omega) = 0$$

$$\mathcal{F}_c \{u(\pi, y)\} = U(\pi, \omega) = \mathcal{F}_c \{e^{-y}\}$$

$$\therefore U(\pi, \omega) = \frac{1}{1 + \omega^2}$$

$$\int_0^\infty e^{-x} \cos \omega x \, dx = \left[e^{-x} \frac{\sin \omega x}{\omega} \right]_0^\infty - \int_0^\infty (-e^{-x}) \frac{\sin \omega x}{\omega} dx$$

$$= \left[e^{-x} \frac{(-\cos \omega x)}{\omega^2} \right]_0^\infty - \int_0^\infty (-e^{-x}) \frac{(-\cos \omega x)}{\omega^2} dx$$

$$= \frac{1}{\omega^2} - \frac{1}{\omega^2} \int_0^\infty e^{-x} \cos \omega x dx$$

$$\therefore \int_0^\infty e^{-x} \cos \omega x dx = \frac{1}{1 + \omega^2}$$

Fourier Transforms

$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx\cdots(6)$$

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega\cdots(7)$$

☑ Example 2 Using the Cosine Transform

The steady-state temperature in a semi-infinite plate is determined from

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\ 0<x<\pi,\ y>0$$

Subject to

$$u(0,y)=0,\ u(\pi,u)=e^{-y},\ y>0$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0}=0,\quad 0<x<\pi.$$

$$U(x,\omega)=c_1\cosh\omega x+c_2\sinh\omega x$$

Boundary condition

$$U(0,\omega)=0\ ,U(\pi,\omega)=\frac{1}{1+\omega^2}$$

$$U(0,\omega)=c_1=0$$

$$U(\pi,\omega)=c_2\sinh\omega\pi=\frac{1}{1+\omega^2}\ \therefore c_2=\frac{1}{(1+\omega^2)\sin\omega\pi}$$

$$\therefore U(x,\omega)=\frac{\sinh\omega x}{(1+\omega^2)\sinh\omega\pi}$$

$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx\cdots(6)$$

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega\cdots(7)$$

☑ Example 2 Using the Cosine Transform

The steady-state temperature in a semi-infinite plate is determined from

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\ 0<x<\pi,\ y>0$$

Subject to

$$u(0,y)=0,\ u(\pi,u)=e^{-y},\ y>0$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0}=0,\quad 0<x<\pi.$$

$$U(x,\omega)=\frac{\sinh\omega x}{(1+\omega^2)\sinh\omega\pi}$$

Recall, definition

$$\mathcal{F}_c\{u(x,y)\}=\int_0^{\infty}u(x,y)\cos\omega y\,dy=U(x,\omega)$$

$$u(x,y)=\frac{2}{\pi}\int_0^{\infty}U(x,\omega)\cos\omega y\,d\omega$$

$$\therefore u(x,y)=\frac{2}{\pi}\int_0^{\infty}\frac{\sinh\omega x}{(1+\omega^2)\sinh\omega\pi}\cos\omega y\,d\omega$$

CHAPTER 4. SERIES SOLUTION

Review

Political, Economical, Social, and Cultural Phenomena

Physical Phenomena and Engineering Problems

Observation, Insight

Analysis, Prediction

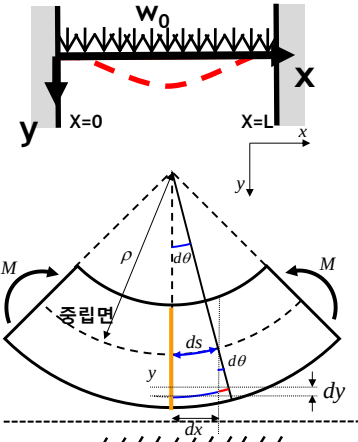
Solution

Mathematical Model

Nonlinear Mathematical Model → Numerical Method

Linearization

Linear Mathematical Model → Analytic Solution



$$\rho \cdot d\theta = ds \Rightarrow \frac{d\theta}{ds} = \frac{1}{\rho}$$

$$dF = \sigma dA = E \cdot \frac{y}{\rho} dA$$

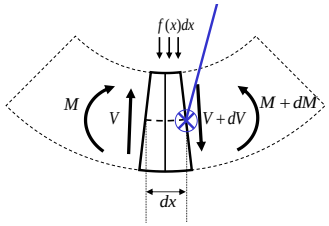
$$dM = -y\sigma dA \Rightarrow \frac{M}{EI} = -\frac{1}{\rho}$$

$$\therefore \frac{d\theta}{ds} = -\frac{M}{EI}$$

⑦ Assume that

$$ds \approx dx,\ \theta \approx \tan(\theta) = \frac{dy}{dx}$$

$$\therefore \frac{d\theta}{ds} = \frac{d^2y}{dx^2} \Rightarrow \boxed{\frac{d^2y}{dx^2} = -\frac{M}{EI}}$$



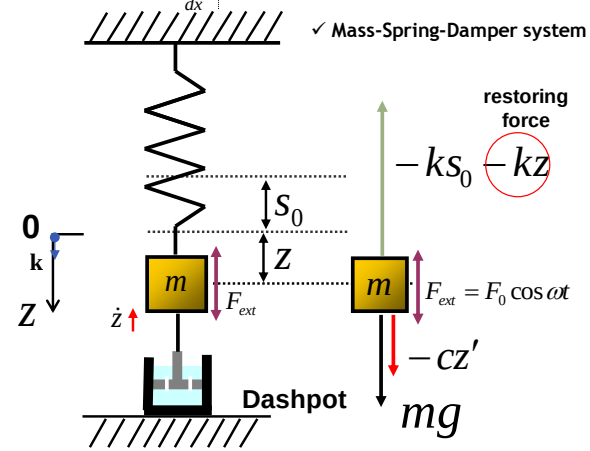
$f(x)$: 분포하중

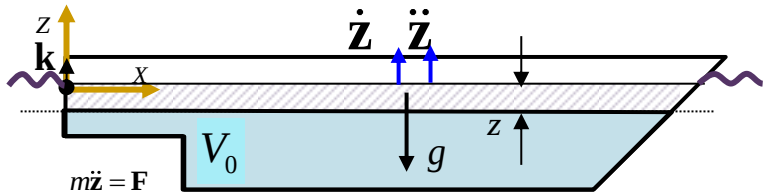
$$\frac{dV}{dx} = -f(x) \quad \frac{dM}{dx} = V(x) \quad \therefore \boxed{EI\frac{d^4y}{dx^4} = f(x)}$$

Differential Equation

$$EI\frac{d^4y(x)}{dx^4} = w_0$$

$$y(x) = c_1 + c_2x + c_3x^2 + c_4x^3 + \frac{w_0}{24EI}x^4$$





$$m\ddot{\mathbf{z}} = \mathbf{F}$$

$$= \mathbf{F}_{gravity} + \mathbf{F}_{static} + \mathbf{F}_{radiation} + \mathbf{F}_{exciting}$$

$$= -mg\mathbf{k} + \rho g V_0\mathbf{k} - \rho g A_{wp}\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}} + \mathbf{F}_{exciting}$$

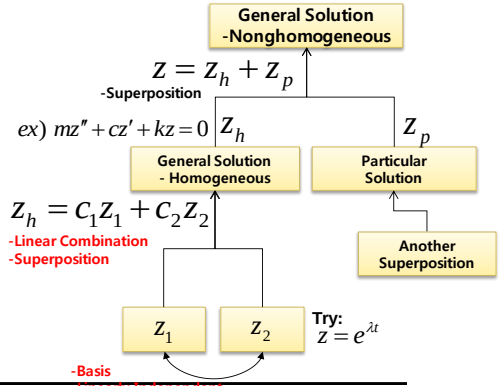
$$= -\rho g A_{wp}\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}} + \mathbf{F}_{exciting}$$

$$= -k\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}} + \mathbf{F}_{exciting}$$

k : Linearized Restoring Force
c : damping coefficient
m_a : added mass

Differential Equation

$$(m + m_a)\ddot{\mathbf{z}} + c\dot{\mathbf{z}} + k\mathbf{z} = \mathbf{F}_{exciting}$$



4-1. Power Series

@ Introduction to Series Solutions*

$$\frac{dy(x)}{dx} = 2xy(x)$$

$$\int \frac{dy(x)}{dx}dx = \int 2xy(x)dx \qquad \int \frac{dy}{y} = \int 2xdx$$

Integration $\quad y = \int 2xy(x)dx + c$ $\quad$ Then? $\quad \ln|y| = x^2 + C$

$$y = e^{x^2+C} = \tilde{C}e^{x^2}$$

Another general approach: Assume a solution in the form of a power series,

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots = \sum_{n=0}^{\infty} a_nx^n$$

$$\frac{dy}{dx} = 2xy$$ Substitute $a's$ are to be found. Differentiating term by term

$$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} + \cdots = \sum_{n=0}^{\infty} na_nx^{n-1}$$

$$a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} + \cdots = 2x(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots)$$

$$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + na_nx^{n-1} + \cdots = 2a_0x + 2a_1x^2 + 2a_2x^3 + \cdots + 2a_{n-2}x^{n-1} + \cdots$$

$$a_1 = 0 \qquad a_3 = \frac{2}{3}a_1 = 0$$

$$a_2 = a_0 \qquad a_4 = \frac{1}{2}a_2 = \frac{1}{2}a_0$$

In general, $\quad na_n = 2a_{n-2} \quad \Rightarrow \quad a_n = \begin{cases} 0 & ,\text{n : odd} \\ \frac{2}{n}a_{n-2} & ,\text{n: even} \end{cases}$

*Boas M.L., Mathematical Methods in the Physical Sciences, John Wiley & Sons, 1966, p538~539

Assume a solution in the form of a power series,

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

$$a_n = \begin{cases} 0 & ,\text{n : odd} \\ \dfrac{2}{n} a_{n-2} & ,\text{n: even} \end{cases}$$

Putting *n=2m* (since only even terms appear in this series)

$$a_{2m} = \frac{2}{2m} a_{2m-2} = \frac{1}{m}\frac{1}{m-1} a_{2m-4} = \cdots = \frac{1}{m} a_0$$

$$\therefore y = a_0 + a_0 x^2 + \frac{1}{2!} a_0 x^4 + \cdots + \frac{1}{m!} a_0 x^{2m} + = a_0 \sum_{m=0}^{\infty} \frac{x^{2m}}{m!}$$

Compare this with the solution by separation of variables

$$\int \frac{dy}{y} = \int 2x dx$$

$$\ln|y| = x^2 + C$$

$$y = e^{x^2 + C} = \tilde{C} e^{x^2}$$

Expanding this in a series of powers of x^2

$$y = \tilde{C} e^{x^2} = \tilde{C}(1 + x^2 + \frac{x^4}{2!} + \cdots)$$

$$= \tilde{C} \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

Maclaurin Series

$$e^x = (1 + x + \frac{x^2}{2!} + \cdots)$$

$$e^{x^2} = (1 + x^2 + \frac{x^4}{2!} + \cdots)$$

@ Linear ODE with variable coefficients

Linear second-order ODE with variable coefficients

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

$$y = B_0 y_1 + B_1 y_2$$

-Linear Combination
-Superposition

-Fundamental set of solutions (Basis)

$[y_1]$ $[y_2]$

Linearly Independent [series]

Try: $y = \sum_{n=0}^{\infty} c_n x^n$

Cf: Linear ODE with constant coefficients

$ex)\ my'' + cy' + ky = f(x)$

$$y'' + \alpha y = 0$$

Ref. Initial Value Problem Vs. Boundary Value Problem

Try: $y = e^{\lambda x}$

Linear ODE with variable coefficients

$$x^2 y'' + xy' - \alpha y = 0$$

$$y'' + xy = 0$$

$$x^2 y'' + y' + \alpha^2 x^2 y = 0$$

$$(1 - x^2)y'' - 2xy + n(n+1)y = 0$$

$where,\ y = y(x),$

$n = 0,1,2\ldots$

@ Review: I.V.P Vs. B.V.P[1)]

I.V.P : Initial Value Problem
B.V.P :Boundary Value Problem

$$y'' + \nu^2 y = 0 \quad , \nu > 0$$

The solution of this is known as

$$y = A\cos\nu x + B\sin\nu x$$

Initial Conditions : $y(0) = 0,\ y'(0) = 1$

$$y(0) = A\cos\nu\cdot 0 + B\sin\nu\cdot 0 = A = 0$$

$$y'(0) = -\nu A\sin\nu\cdot 0 + \nu B\cos\nu\cdot 0 = \nu B = 1$$

B= $\frac{1}{\nu}$

$$\therefore y = \frac{1}{\nu}\sin\nu x$$

Boundary Conditions : $y(0) = 0,\ y(\pi) = 0$

$$y(0) = A\cos\nu\cdot 0 + B\sin\nu\cdot 0 = A = 0$$

$y(\pi) = B\sin\nu\cdot\pi = 0$ **satisfied for** $\nu = n\ , n = 1, 2, 3\cdots$

$\therefore y = B\sin\nu x$ **: Eigen function**

ν **: Eigen value**

1) 김찬중, 길잡이 공업수학, 문운당, 1998, p293

@ Properties of Power Series

Power Series centered at a

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

Important fact about power series

1) Convergence

$\sum_{n=0}^{\infty} c_n (x-a)^n$ is **convergent** at a specified value of x

if its sequence of partial sums $\{S_n(x)\}$ converges;

that is, if $\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \sum_{n=0}^{N} c_n (x-a)^n$ exists.

Otherwise, the series is said to be divergent

nth partial sum

$$S_n(x) = \sum_{k=0}^{n} c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$$

remainder

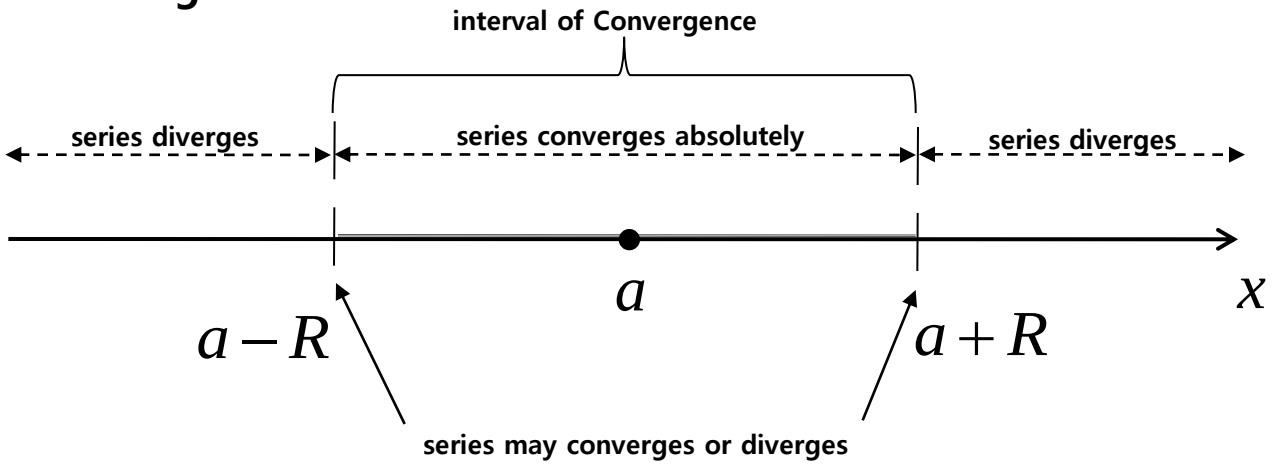
$$R_n(x) = \sum_{k=n+1}^{\infty} c_k (x-a)^k = c_{n+1}(x-a)^{n+1} + c_{n+2}(x-a)^{n+2} + \cdots$$

$$\begin{aligned}
S_1(x) &= c_0 \\
S_2(x) &= c_0 + c_1(x-a) \\
S_3(x) &= c_0 + c_1(x-a) + c_2(x-a)^2 \\
&\vdots \\
S_n(x) &= c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n
\end{aligned}$$

2) Interval of Convergence

Every power series has an interval of convergence.

The interval of convergence is the set of all real x numbers for which the series converges.



R radius of convergence

3) Radius of Convergence

Every power series has a radius of convergence R .

If $R>0$, $\sum_{n=0}^{\infty} c_n (x-a)^n$

- converges for $|x-a|<R$
- diverges for $|x-a|>R$

4) Absolute Convergence

If x is a real number in the interval of convergence and is not an end-point of the interval, then the series of absolute values $\sum_{n=0}^{\infty} \left| c_n (x-a)^n \right|$ converges.

If $R>0$, $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges for $|x-a|<R$; diverges for $|x-a|>R$

Power Series centered at a

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

5) Convergence test by the Ratio Test

Convergence of power series can be often be determined by the ratio test.

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = |x-a|\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L$$

- $L<1$: converges
- $L>1$: diverges
- $L=1$: inconclusive

Radius of convergence R :

$$|x-a|\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| < 1$$

$$\therefore |x-a| < \frac{1}{\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|} \to R$$

@ Example) Ratio Test

$$\sum_{n=1}^{\infty}\frac{(x-3)^n}{2^n n} \Rightarrow \lim_{n\to\infty}\left|\frac{\frac{(x-3)^{n+1}}{2^{n+1}(n+1)}}{\frac{(x-3)^n}{2^n n}}\right| = |x-3|\lim_{n\to\infty}\frac{n}{2(n+1)} = \frac{1}{2}|x-3| = L$$

- $L<1$: converges
- $L>1$: diverges
- $L=1$: inconclusive

converges $\quad \frac{1}{2}|x-3|<1 \Rightarrow |x-3|<\underbrace{2}_{\to R} \Rightarrow 1<x<5$

diverges $\quad \frac{1}{2}|x-3|>1 \Rightarrow |x-3|>2 \Rightarrow x<1, 5<x$

inconclusive $\quad \frac{1}{2}|x-3|=1 \Rightarrow |x-3|=2 \Rightarrow x=1,\ x=5$

$x=1$ $\sum_{n=1}^{\infty}\frac{(x-3)^n}{2^n n}$ $x=5$

$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$: converges $\qquad \sum_{n=1}^{\infty}\frac{1}{n}$: diverges

Ref. Proof

interval of converges [1,5)
convergence radius R=2

Leibniz's Test

Theorem 2 : Leibniz's test* (alternating series test)

A (real) alternating series of which the absolute values of the terms form a monotonic null sequences, is invariably convergent

If $\sum_{\nu=1}^{\infty}(-1)^{\nu}b_{\nu}$ ***, with*** $b_{\nu}\to 0$ ***, is such a series, then its value lies between*** b_0

and b_0-b_1 ***, more generally, between any two successive partial sums.***

Proof.

For arbitrary natural ν **and** ρ**,**

$$\left|(-1)^{\nu+1}b_{\nu+1}+\cdots+(-1)^{\nu+\rho}b_{\nu+\rho}\right|=\left|b_{\nu+1}-b_{\nu+2}+\cdots+(-1)^{\rho-1}b_{\nu+\rho}\right|$$

The sum between the absolute-value signs can be written in the form

$$\left(b_{\nu+1}-b_{\nu+2}\right)+\left(b_{\nu+3}-b_{\nu+4}\right)+\cdots+\begin{cases}\left(b_{\nu+\rho-1}-b_{\nu+\rho}\right) & \text{,if } \rho \text{ is even,}\\ b_{\nu+\rho} & \text{,if } \rho \text{ is odd,}\end{cases}$$

*Knopp K., "Infinite Sequences and Series", Dover, 1956, p 68 -Theorem 2

Proof.

$$(b_{\nu+1}-b_{\nu+2})+(b_{\nu+3}-b_{\nu+4})+\cdots+\begin{cases}(b_{\nu+\rho-1}-b_{\nu+\rho}) & \text{,if } \rho \text{ is even,}\\ b_{\nu+\rho} & \text{,if } \rho \text{ is odd,}\end{cases}$$

Since $\{b_\nu\}$ is decreasing, this shows that this sum is ≥ 0 and therefore the absolute-value signs on the right may be removed. If this sum is then written in the form

$$b_{\nu+1}-(b_{\nu+2}-b_{\nu+3})-\cdots-\begin{cases}b_{\nu+\rho} & \text{,if } \rho \text{ is even,}\\ (b_{\nu+\rho-1}-b_{\nu+\rho}) & \text{,if } \rho \text{ is odd,}\end{cases}$$

*Knopp K., "Infinite Sequences and Series", Dover, 1956, p 68 -Theorem 2

Then this shows further that the sum is $\leq b_{\nu+1}$

Since $b_\nu \to 0$ this is $\leq \varepsilon$ for all $\nu > \mu$, if we choose so that $b_\mu < \varepsilon$.

Proof for Convergence of $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$

Using convergence test : Test for alternating series*

An alternating series converges if the absolute value of the terms decrease steadily to zero, that is, if $|a_{n+1}| \le |a_n|$ **and** $\lim_{n\to\infty} a_n = 0$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-+\cdots$$

$$\left|\frac{1}{n+1}\right| < \left|\frac{1}{n}\right| \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{n} = 0 \quad \text{: converges}$$

*Boas M.L., Mathematical Methods in the Physical Sciences, Third Edition, John Wiley & Sons, 2006, p17

Proof for Divergence of $\sum_{n=1}^{\infty}\frac{1}{n}$

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots$$

$$=1+\left[\frac{1}{2}\right]+\left[\frac{1}{3}+\frac{1}{4}\right]+\left[\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right]+\left[\frac{1}{9}+\cdots\right]+\cdots$$

$$>1+\left[\frac{1}{2}\right]+\left[\frac{1}{4}+\frac{1}{4}\right]+\left[\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right]+\left[\frac{1}{16}+\cdots\right]+\cdots$$

$$=1+\left[\frac{1}{2}\right]+\left[\frac{1}{2}\right]+\left[\frac{1}{2}\right]+\left[\frac{1}{2}\right]+\cdots$$ **: diverges***

*Boas M.L., Mathematical Methods in the Physical Sciences, Third Edition, John Wiley & Sons, 2006, p11 problem 2

Proof for Divergence of $\sum_{n=1}^{\infty}\frac{1}{n}$

Using convergence test : The Integral Test*

If $0 < a_{n+1} < a_n$ for $n > N$, then $\sum_{n=1}^{\infty} a_n$ converge if $\int^{\infty} a_n dn$ is finite and diverge if the integral is infinite (The integral is to be evaluated only at the upper limit; no lower limit is needed)

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots$$

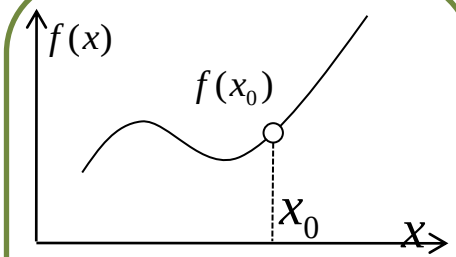
$$\int^{\infty}\frac{1}{n}dn = \ln n\Big|^{\infty} = \infty \quad \text{: diverges}$$

*Boas M.L., Mathematical Methods in the Physical Sciences, Third Edition, John Wiley & Sons, 2006, p11-12

@ A Power Series defines a function.

6) A Power Series defines a function.

$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n$ domain is the interval of convergence R.



$f(x_0)$ can not be defined at $x=x_0$.
not continuous, not differentiable, not integrable : not analytic.

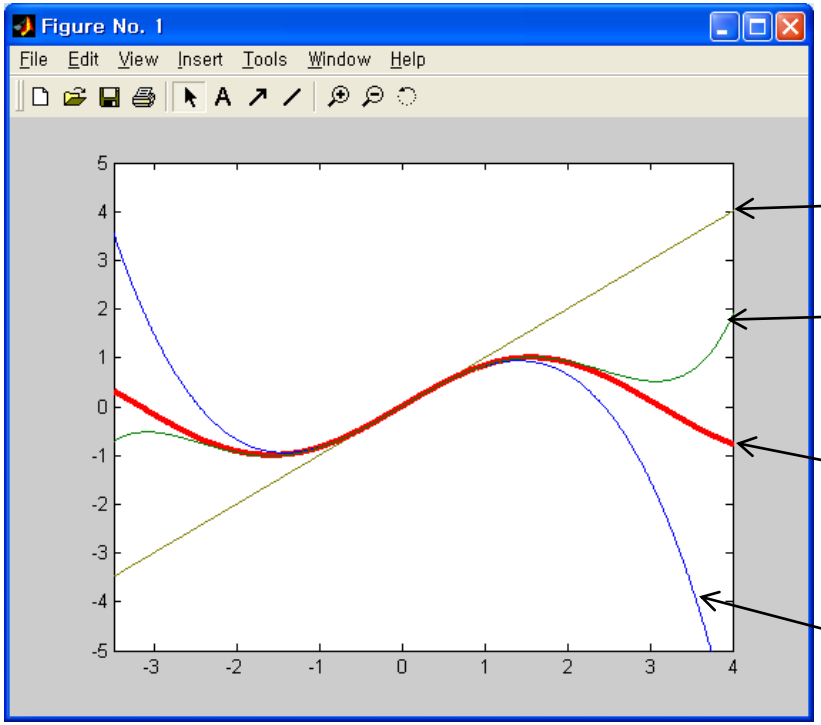
If $R>0$ then, $f(x)$ is continuous, differentiable, and integrable on $(a-R, a+R)$

Moreover, $f'(x), \int f(x)dx$ can be founded by term-by-term differentiation and integration.

8) Analytic at a point

A function $f(x)$ is analytic at a point a, if it can be represented by a power series in $x-a$ with a positive radius of convergence.

@ Ex) A Power Series defines a function



$$ex^{*})\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$

x

$$x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$\sin x$

$$x - \frac{x^3}{3!}$$

*see also Larson R., Hostetler R.P., Edwards B.H., Calculus with Analytic Geometry, English Edition, Houghton Mifflin Company, 2006, p679 example 2

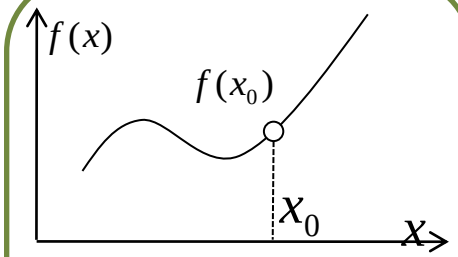
7) Identity Property

If $\sum_{n=0}^{\infty} c_n (x-a)^n = 0, R > 0$, for all numbers x in the interval of convergence,

then $c_n = 0$ for all n.

8) Analytic at a Point

A function $f(x)$ is analytic at a point a, if it can be represented by a power series in $x-a$ with a positive radius of convergence.



$f(x_0)$ can not be defined at $x = x_0$.
not continuous, not differentiable, not integrable : not analytic.

Taylor series centered at 0, called Maclaurin series, show that

$e^x, \sin x,$ and $\cos x$ are analytic at $x = 0$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots,\ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,\ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

6) A Power Series Defines a Function

$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ domain is the interval of convergence R.

if $R > 0$ then $f(x)$ is continuous, differentiable, and integrable on $(a-R, a+R)$

Moreover, $f'(x), \int f(x)dx$ can be founded by term-by-term differentiation and integration.

@ Summary

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = |x-a|\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L \qquad L<1 \text{ : converges}$$

$$\therefore |x-a| < \frac{1}{\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|} \to R$$

	Power Series	Ratio Test		Radius of convergence
$\sin x$	$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$	$\lim_{n\to\infty}\left\|\frac{(-1)^{n+1}\frac{x^{2n+3}}{(2n+3)!}}{(-1)^n\frac{x^{2n+1}}{(2n+1)!}}\right\| = \|x^2\|\lim_{n\to\infty}\left\|\frac{-1}{2n+3}\right\| = \|x^2\|\cdot 0 = L$	$L<1$ **Converges for all x**	$\|x\|\cdot 0<1$ $\|x\|<\frac{1}{0}$ $\|x\|<\infty \to R$
$\sum_{n=1}^{\infty}\frac{(x-3)^n}{2^n n}$	$\lim_{n\to\infty}\left\|\frac{\frac{(x-3)^{n+1}}{2^{n+1}(n+1)}}{\frac{(x-3)^n}{2^n n}}\right\| = \|x-3\|\lim_{n\to\infty}\frac{n}{2(n+1)} = \frac{1}{2}\|x-3\| = L$	$L<1$ **Converges for** $1<x<5$	$\frac{1}{2}\|x-3\|<1$ $\|x-3\|<2 \to R$	
	$x=1$ $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ **: converges**	$x=5$ $\sum_{n=1}^{\infty}\frac{1}{n}$ **: diverges**	**interval of converges [1,5) convergence radius R=2**	

9) Arithmetic of Power Series

Power series can be combined through the operations of addition, multiplication and division.

$$e^x \sin x = (1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots)(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots)$$

$$= (1)x + (1)x^2 + (-\frac{1}{6} + \frac{1}{2})x^3 + (-\frac{1}{6} + \frac{1}{6})x^4 + \cdots$$

$$= x + x^2 + \frac{x^3}{3} + \cdots$$

@ Ordinary & Singular Points

Linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \cdots (1)$$

Divided by $a_2(x)$ ↓

$$y'' + P(x)y' + Q(x)y = 0 \cdots (2)$$ **Standard form**

Definition 5.1 **Ordinary and Singular Points**

A point x_0 is said to be an **ordinary point** of the differential equation (1),

if both $P(x)$ and $Q(x)$ in the standard form (2) are **analytic** at x_0 .

A point that is not an ordinary point is said to be a singular point on the equation

analytic at x_0 means

continuous, differentiable and **integrable** at x_0

@ Ex) Ordinary & Singular Points

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \cdots (1)$$

$$y'' + P(x)y' + Q(x)y = 0 \cdots (2)$$ **Standard form**

Ex)

$y'' + e^x y' + (\sin x)y = 0$ **Every x is an ordinary point including** $x = 0$

$y'' + e^x y' + (\ln x)y = 0$ **Singular points** $x = 0$

$(x^2 - 1)y'' + 2xy' + 6y = 0$ **Singular points** $x = \pm 1$

$(x^2 + 1)y'' + 2xy' + 6y = 0$ **Singular points** $x = \pm i$

@@ Power Series Solutions

Theorem 5.1 — **Existence of Power Series Solutions**

If $x = x_0$ is an ordinary point of the second-order differential equation, we can always find two linearly independent solutions in the form of a power series centered at x_0, that is $y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$.

A series solution converges at least on some interval defined by $|x - x_0| < R$, where R is the distance from x_0 to the closest singular point.

A solution of the form $y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$ is said to be a solution about the ordinary point x_0

@ Power Series Solutions about the ordinary Point $x = 0$

In the examples that follows, we shall, for the sake of simplicity, only find power series solutions about the ordinary point $x = 0$.
If it is necessary to find a power series solution of an ODE about an ordinary point $x_0 \neq 0$, we can simply make the change of variable $t = x - x_0$ in the equation(this transforms $x = x_0$ to $t = 0$), find solutions of the new equation of the form $y = \sum_{n=0}^{\infty} c_n t^n$, and then resubstitute $t = x - x_0$

@ Linear ODE with variable /constant coefficients

Power Series Solution for Linear ODE with variable coefficients

$$y'' + P(x)y' + Q(x)y = 0 \longleftarrow y = \sum_{n=0}^{\infty} c_n x^n$$

$$y = c_0[y_1] + c_1[y_2]$$

A power series — Another power series

Linearly independent → **Linear combination** → General solution $y = B_0 y_1 + B_1 y_2$

Vs.

Linear DE with constant coefficients

$$mz'' + cz' + kz = 0 \longleftarrow z = e^{\lambda t}$$

$$z = c_1 z_1 + c_2 z_2 \quad (= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t})$$

$$where,\ \lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

z_1, z_2 are linearly independent? → three cases depending on $c^2 - 4mk$

cases	$c^2 - 4mk$	Linearly Independent	Linear combination →	General Solution
Case I	$c^2 - 4mk > 0$	$e^{\lambda_1 t}, e^{\lambda_2 t}$	→	$z = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$
Case II	$c^2 - 4mk = 0$	$e^{\lambda_1 t}, te^{\lambda_1 t}$	→	$z = c_1 e^{\lambda_1 t} + c_2 te^{\lambda_1 t}$
Case III	$c^2 - 4mk < 0$	$e^{\alpha t}\cos\beta t, e^{\alpha t}\cos\beta t)$	→	$z = e^{\alpha t}(c_1 \cos\beta t + c_2 \cos\beta t)$

Example 1) Power Series Solutions

☑ Power Series Solution

solve $y'' + xy = 0$ **x=0 is an ordinary point**

$$y = \sum_{n=0}^{\infty} c_n x^n \quad , \quad y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$$

$$y'' + xy = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + x\sum_{n=0}^{\infty} c_n x^n = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$$

$$= 2c_2 + \sum_{k=1}^{\infty} [(k+1)(k+2)c_{k+2} + c_{k-1}]x^k = 0$$

$$2c_2 = 0,$$

$$(k+1)(k+2)c_{k+2} + c_{k-1} = 0, \quad k = 1,2,3,\cdots$$

$$c_{k+2} = -\frac{c_{k-1}}{(k+1)(k+2)}, \quad k = 1,2,3,\cdots$$

$$k=1, c_3 = -\frac{c_0}{2\cdot 3}$$

$$k=2, c_4 = -\frac{c_1}{3\cdot 4}$$

$$k=3, c_5 = -\frac{c_2}{4\cdot 5} = 0$$

$$k=4, c_6 = -\frac{c_3}{5\cdot 6} = \frac{1}{2\cdot 3\cdot 5\cdot 6} c_0$$

$$k=5, c_7 = -\frac{c_4}{6\cdot 7} = \frac{1}{3\cdot 4\cdot 6\cdot 7} c_1$$

$$k=6, c_8 = -\frac{c_5}{7\cdot 8} = 0$$

$$k=7, c_9 = -\frac{c_6}{8\cdot 9} = -\frac{1}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} c_0$$

$$k=8, c_{10} = -\frac{c_7}{9\cdot 10} = -\frac{1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10} c_1$$

$$k=9, c_{11} = -\frac{c_8}{10\cdot 11} = 0$$

$$\vdots$$

$$y'' + xy = 0$$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5$$

$$+ c_6 x^6 + c_7 x^7 c_8 x^8 + c_9 x^9 + c_{10} x^{10} + c_{11} x^{11} + \cdots$$

$$= c_0 + c_1 x + 0 - \frac{c_0}{2 \cdot 3} x^3 - \frac{c_1}{3 \cdot 4} x^4 + 0 + \frac{c_0}{2 \cdot 3 \cdot 5 \cdot 6} x^6$$

$$+ \frac{c_1}{3 \cdot 4 \cdot 6 \cdot 7} x^7 + 0 - \frac{c_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} x^9 - \frac{c_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} x^{10} + 0 + \cdots$$

$$k = 1, c_3 = -\frac{c_0}{2 \cdot 3} \qquad k = 4, c_6 = -\frac{c_3}{5 \cdot 6} = \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} c_0$$

$$k = 2, c_4 = -\frac{c_1}{3 \cdot 4} \qquad k = 5, c_7 = -\frac{c_4}{6 \cdot 7} = \frac{1}{3 \cdot 4 \cdot 6 \cdot 7} c_1$$

$$k = 3, c_5 = -\frac{c_2}{4 \cdot 5} = 0 \qquad 2c_2 = 0,$$

$$k = 6, c_8 = -\frac{c_5}{7 \cdot 8} = 0$$

$$k = 7, c_9 = -\frac{c_6}{8 \cdot 9} = -\frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} c_0$$

$$k = 8, c_{10} = -\frac{c_7}{9 \cdot 10} = -\frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} c_1$$

$$k = 9, c_{11} = -\frac{c_8}{10 \cdot 11} = 0$$

$$\vdots$$

$$\square : c_0 \left[1 - \frac{1}{2 \cdot 3} x^3 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} x^6 - \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} x^9 + \cdots \right] = c_0 y_1(x)$$

$$\bigcirc : c_1 \left[x - \frac{1}{3 \cdot 4} x^4 + \frac{1}{3 \cdot 4 \cdot 6 \cdot 7} x^7 - \frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} x^{10} + \cdots \right] = c_1 y_2(x)$$

Linearly independent

General solution

$$y = B_1 y_1(x) + B_2 y_2(x)$$

Linear combination

Example 2) Power Series Solutions

solve $(x^2+1)y'' + xy' - y = 0$ **x=0 is an ordinary point**

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^n - \sum_{n=0}^{\infty} c_n x^n$$

$$= 2c_2 x^0 - c_0 x^0 + 6c_3 x + c_1 x - c_1 x$$

$$+ \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=4}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=2}^{\infty} nc_n x^n - \sum_{n=2}^{\infty} c_n x^n$$

$$= 2c_2 - c_0 + 6c_3 x + \sum_{k=2}^{\infty} [k(k-1)c_k + (k+2)(k+1)c_{k+2} + kc_k - c_k]x^k = 0$$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$, \quad y' = \sum_{n=0}^{\infty} c_n n x^{n-1}$$

$$, \quad y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}$$

From this last identity we conclude that

$$2c_2 - c_0 = 0 \quad , \quad 6c_3 = 0 \quad , \quad (k+1)(k-1)c_k + (k+2)(k+1)c_{k+2} = 0$$

$$c_2 = \frac{1}{2}c_0 \quad , \quad c_3 = 0 \quad , \quad c_{k+2} = \frac{1-k}{k+2}c_k, \quad k = 2, 3, 4, \cdots$$

$$c_2 = \frac{1}{2}c_0 \quad , \quad c_3 = 0 \quad , \quad c_{k+2} = \frac{1-k}{k+2}c_k, \quad k = 2,3,4,\cdots$$

$$c_4 = -\frac{1}{4}c_2 = -\frac{1}{2\cdot 4}c_0 = -\frac{1}{2^2\cdot 2!}c_0$$

$$c_6 = -\frac{3}{6}c_4 = \frac{3}{2\cdot 4\cdot 6}c_0 = \frac{1\cdot 3}{2^3\cdot 3!}c_0$$

$$c_8 = -\frac{5}{8}c_6 = -\frac{1}{2\cdot 4\cdot 6\cdot 8}c_0 = -\frac{1\cdot 3\cdot 5}{2^4\cdot 4!}c_0$$

$$c_{10} = -\frac{7}{10}c_8 = \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}c_0 = \frac{1\cdot 3\cdot 5\cdot 7}{2^5\cdot 5!}c_0$$

$$c_5 = -\frac{2}{5}c_3 = 0$$

$$c_7 = -\frac{4}{7}c_5 = 0$$

$$c_9 = -\frac{6}{9}c_7 = 0$$

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + c_7x^7 + c_8x^8 + c_9x^9 + c_{10}x^{10} + \cdots$$

$$= c_0\left[1 + \frac{1}{2}x^2 - \frac{1}{2^2 2!}x^4 + \frac{1\cdot 3}{2^3 3!}x^6 - \frac{1\cdot 3\cdot 5}{2^4 4!}x^8 + \frac{1\cdot 3\cdot 5\cdot 7}{2^5 5!}x^{10} - \cdots\right] + c_1x$$

$$= c_0y_1(x) + c_1y_2(x)$$

$$, y_1(x) = 1 + \frac{1}{2}x^2 + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n n!}x^{2n} \qquad , y_2(x) = x$$

Example 3) Power Series Solutions

☑ Three-Term Recurrence Relation

$$y'' - (1+x)y = 0$$

x=0 is an ordinary point

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$$

We obtain $c_2 = \frac{c_0}{2}$ and

(three-term)recurrence relation

$$c_{k+2} = \frac{c_k + c_{k-1}}{(k+1)(k+2)}, \quad k = 1,2,3,\cdots$$

Examination of the formula shows that the coefficients $c_3, c_4, c_5 \cdots$ are expressed in terms of both c_0 and c_1

To simplify the calculus, we can first choose

$$c_0 \neq 0, c_1 = 0$$

$$c_2 = \frac{1}{2} c_0$$

$$c_3 = \frac{c_1 + c_0}{2 \cdot 3} = \frac{c_0}{2 \cdot 3} = \frac{1}{6} c_0$$

$$c_4 = \frac{c_2 + c_1}{3 \cdot 4} = \frac{c_0}{2 \cdot 3 \cdot 4} = \frac{1}{24} c_0$$

$$c_5 = \frac{c_3 + c_2}{4 \cdot 5} = \frac{c_0}{4 \cdot 5}\left[\frac{1}{6} + \frac{1}{2}\right] = \frac{1}{30} c_0$$

$$\vdots$$

$$y_1(x) = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{30}x^5 + \cdots$$

$$y_1(x) = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{30}x^5 + \cdots$$

$$c_2 = \frac{c_0}{2}$$

$$c_{k+2} = \frac{c_k + c_{k-1}}{(k+1)(k+2)}, \quad k = 1,2,3,\cdots$$

Examination of the formula shows that the coefficients $c_3, c_4, c_5 \cdots$ are expressed in terms of both c_0 and c_1

Next, by choosing

$$c_0 = 0, c_1 \neq 0$$

$$c_2 = \frac{1}{2}c_0 = 0$$

$$c_3 = \frac{c_1 + c_0}{2 \cdot 3} = \frac{c_1}{2 \cdot 3} = \frac{1}{6}c_1$$

$$c_4 = \frac{c_2 + c_1}{3 \cdot 4} = \frac{c_1}{2 \cdot 3 \cdot 4} = \frac{1}{12}c_1$$

$$c_5 = \frac{c_3 + c_2}{4 \cdot 5} = \frac{c_1}{4 \cdot 5 \cdot 6} = \frac{1}{120}c_1$$

$$\vdots$$

$$y_2(x) = x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5 + \cdots$$

$$\therefore y = C_1 y_1(x) + C_2 y_2(x)$$

$$y_1(x) = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{30}x^5 + \cdots$$

$$y_2(x) = x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5 + \cdots$$

Example 4) Power Series Solutions

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

☑ ODE with Nonpolynomial Coefficients

solve $y'' + (\cos x)y = 0$ **x=0 is an ordinary point**

$$y'' + (\cos x)y = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots)\sum_{n=0}^{\infty} c_n x^n$$

$$= 2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \cdots + (1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots)\cdot(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots)$$

$$= 2c_2 + c_0 + (6c_3 + c_1)x + (12c_4 + c_2 - \frac{1}{2}c_0)x^2 + (20c_5 + c_3 - \frac{1}{2}c_1)x_3 + \cdots = 0$$

$$2c_2 + c_0 = 0 \qquad c_2 = -\frac{1}{2}c_0$$

$$6c_3 + c_1 = 0 \qquad c_3 = -\frac{1}{6}c_1$$

$$12c_4 + c_2 - \frac{1}{2}c_0 = 0 \qquad c_4 = \frac{1}{12}c_0$$

$$20c_5 + c_3 - \frac{1}{2}c_1 = 0 \qquad c_5 = \frac{1}{30}c_1$$

$$y_1(x) = 1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 - \cdots$$

$$y_2(x) = x - \frac{1}{6}x^3 + \frac{1}{30}x^5 - \cdots$$

4-2. Solutions about Singular Points

@ Solutions about Singular Points

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \cdots (1)$$

$$y'' + P(x)y' + Q(x)y = 0 \cdots (2) \quad \text{standard form}$$

analytic at x_0 means

continuous, differentiable and integrable at x_0

Definition 5.2 Regular/Irregular Singular Points

A singular point x_0 is said to be a regular singular point of the differential equation (1) if the functions $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2 Q(x)$ are both analytic at x_0 .

A singular point that is not regular is said to be an irregular point of the equation

In case $P(x), Q(x)$ are not analytic at x_0 and $x = x_0$ is a regular singular point then we multiply (2) by $(x - x_0)^2$, then

$$(x - x_0)^2 y'' + (x - x_0)p(x)y' + q(x)y = 0$$

solution → Method of Frobenius

where $p(x), q(x)$ are analytic at $x = x_0$

@ Check of Regular Singular Points

We can now determine whether x0 is regular singular point by a quick visual check of denominators: If (x-x0) appears at most to the first power in the denominator of P(x) and at most to the second power in the denominator of Q(x), then x= x0 is a regular singular point. Moreover, observe that if x=x0 is a regular singular point and we multiply (2) $(x-x_0)^2$, then the original DE can be put into the form

$$(x-x_0)^2 y'' + (x-x_0)^2 P(x) y' + (x-x_0)^2 Q(x) y = 0$$

$$(x-x_0)^2 y'' + (x-x_0)[(x-x_0)P(x)] y' + [(x-x_0)^2 Q(x)] y = 0$$

$$(x-x_0)^2 y'' + (x-x_0)\underline{p(x)} y' + \underline{q(x)} y = 0$$

analytic at x_0

standard form

$$x^3 y'' - 2xy' + 8y = 0 \Rightarrow y'' + (-\frac{2}{x^2}) y' + \frac{8}{x^3} y = 0$$

$Q(x)$

not analytic at $x = 0$

multiply x^2

$$\Rightarrow x^2 y'' + x(\frac{-2}{x}) y' + \frac{8}{x} y =$$

$p(x)$ $q(x)$

not analytic at $x = 0$

$$(x^2+9)y''-3xy'+(1-x)y=0 \quad \Rightarrow \quad y''+\left(\frac{-3x}{x^2+9}\right)y'+\left(\frac{1-x}{x^2+9}\right)y=0$$

$P(x)$ $Q(x)$

multiply $(x^2+9)^2$

$$(x^2+9)^2 y''+(x^2+9)(-3x)y'+(x^2+9)(1-x)y=0$$

$p(x)$ $q(x)$

not analytic at $x=\pm 3i$

analytic at $x=\pm 3i$

Example 1) Regular Singular Points

☑ It should be clear that x=2 and x=-2 are singular points of

$$(x^2-4)^2 y'' + 3(x-2)y' + 5y = 0$$

Dividing the equation by

$$(x^2-4)^2 = (x-2)^2(x+2)^2$$

and reducing the coefficients to lowest terms, we find that

$$P(x) = \frac{3}{(x-2)(x+2)^2}, \quad Q(x) = \frac{5}{(x-2)^2(x+2)^2}$$

$$p(x) = (x-2)P(x) = \frac{3}{(x+2)^2}$$

$$q(x) = (x-2)^2 Q(x) = \frac{5}{(x+2)^2}$$

In order for x=2 to be a regular singular point, the factor x-2 can appear at most to the first power in the denominator of P(x) and at most to the second power in the denominator of Q(x). These conditions are satisfied, so x=2 is a regular singular point.

$$(x-2)^2 y'' + (x-2)\frac{3}{(x+2)^2} y' + \frac{5}{(x+2)^2} y = 0$$

$p(x)$ analytic at $x=2$

$q(x)$ analytic at $x=2$

@ Method of Frobenius

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \cdots (1)$$

Theorem 5.2 — **Frobenius's Theorem**

If $x = x_0$ is a regular singular point of the differential equation (1), then there exists at least one solution of the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

where the number r is a constant to be determined .

The series will Converge at least on some interval $0 < x - x_0 < R$

Example 2) Method of Frobenius

☑ Two Series Solutions

Because x=0 is a regular singular point of the DE

$$3xy'' + y' - y = 0$$

we try to find a solution of the form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ **. Now**

$$y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

$$\begin{aligned} 3xy'' + y' - y &= 3\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} \\ &= \sum_{n=0}^{\infty} (n+r)(3n+3r-2) c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} \\ &= x^r [r(3r-2)c_0 x^{-1} + \underbrace{\sum_{n=1}^{\infty} (n+r)(3n+3r-2) c_n x^{n-1}}_{k=n-1} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n}] \\ &= x^r [r(3r-2)c_0 x^{-1} + \sum_{k=0}^{\infty} [(k+r+1)(3k+3r+1)c_{k+1} - c_k] x^k] = 0 \end{aligned}$$

$$= x^r [r(3r-2)c_0 x^{-1} + \sum_{k=0}^{\infty} [(k+r+1)(3k+3r+1)c_{k+1} - c_k]x^k] = 0$$

which implies $r(3r-2)c_0 = 0$

$$(k+r+1)(3k+3r+1)c_{k+1} - c_k = 0,$$
$$k = 0,1,2,\cdots$$

We must then have

$r(3r-2) = 0$ **"Indicial equation"**

$$c_{k+1} = \frac{c_k}{(k+r+1)(3k+3r+1)},$$
$$k = 0,1,2,\cdots$$

$$r_1 = \frac{2}{3}, \quad c_{k+1} = \frac{c_k}{(3k+5)(k+1)}, \quad k = 0,1,2,\cdots$$

$$c_1 = \frac{c_0}{5 \cdot 1}$$

$$c_2 = \frac{c_1}{8 \cdot 2} = \frac{c_0}{2!5 \cdot 8}$$

$$c_3 = \frac{c_2}{11 \cdot 3} = \frac{c_0}{3!5 \cdot 8 \cdot 11}$$

$$c_4 = \frac{c_3}{14 \cdot 4} = \frac{c_0}{4!5 \cdot 8 \cdot 11 \cdot 14}$$

$$\vdots$$

$$c_n = \frac{c_0}{n!5 \cdot 8 \cdot 11 \cdots (3n+2)}$$

$$r_2 = 0, \quad c_{k+1} = \frac{c_k}{(k+1)(3k+1)}, \quad k = 0,1,2,\cdots$$

$$c_1 = \frac{c_0}{1 \cdot 1}$$

$$c_2 = \frac{c_1}{2 \cdot 4} = \frac{c_0}{2!1 \cdot 4}$$

$$c_3 = \frac{c_2}{3 \cdot 7} = \frac{c_0}{3!1 \cdot 4 \cdot 7}$$

$$c_4 = \frac{c_3}{4 \cdot 10} = \frac{c_0}{4!1 \cdot 4 \cdot 7 \cdot 10}$$

$$\vdots$$

$$c_n = \frac{(-1)^n c_0}{n!1 \cdot 4 \cdot 7 \cdots (3n-2)}$$

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

Omitting c0, the series solutions are

$$y_1(x) = x^{2/3}[1 + \sum_{n=1}^{\infty} \frac{1}{n!5 \cdot 8 \cdot 11 \cdots (3n+2)} x^n]$$

$$y_2(x) = x^{0}[1 + \sum_{n=1}^{\infty} \frac{1}{n!1 \cdot 4 \cdot 7 \cdots (3n-2)} x^n]$$

By the ratio test it can be demonstrated that above equations converge for all x. The two solutions, y1(x) and y2(x) are linearly independent. Hence by the superposition principle y= C1 y1(x) + C2 y2(x) is also solution.

Indicial Equation of Frobenius Method

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \cdots (1)$$

$$x^2y'' + x[xP(x)]y' + [x^2Q(x)]y = 0 \longleftarrow \qquad y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$\downarrow$$

$$r(r-1) + a_0 r + b_0 = 0$$

General indicial equation

$$p(x) = xP(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$q(x) = x^2Q(x) = b_0 + b_1 x + b_2 x^2 + \cdots$$

After substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the given DE and simplifying, the indicial equation is a quadratic equation in r that results from equating the total coefficient of the lowest power of x to zero.

Solutions about Singular Points

☑ Example 3

Solve $2xy'' + (1+x)y' + y = 0$

Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ **gives**

$$2xy'' + (1+x)y' + y = 2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty}c_n x^{n+r}$$

$$= \sum_{n=0}^{\infty}(n+r)(2n+2r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r+1)c_n x^{n+r}$$

$$= x^r[r(2r-1)c_0 x^{-1} + \underbrace{\sum_{n=1}^{\infty}(n+r)(2n+2r-1)c_n x^{n-1}}_{k=n-1} + \underbrace{\sum_{n=0}^{\infty}(n+r+1)c_n x^n}_{k=n}]$$

$$= x^r[r(2r-1)c_0 x^{-1} + \sum_{k=0}^{\infty}[(k+r+1)(2k+2r+1)c_{k+1} + (k+r+1)c_k]x^k]$$

Solutions about Singular Points

which implies $r(2r-1)=0$

$$(k+r+1)(2k+2r+1)c_{k+1}+(k+r+1)c_k=0,\ k=0,1,2,\cdots$$

$$r_1=\frac{1}{2},\quad c_{k+1}=\frac{-c_k}{2(k+1)},\quad k=0,1,2,\cdots$$

$$c_1=\frac{-c_0}{2\cdot 1}$$

$$c_2=\frac{-c_1}{2\cdot 2}=\frac{c_0}{2^2\cdot 2!}$$

$$c_3=\frac{-c_2}{2\cdot 3}=\frac{-c_0}{2^3\cdot 3!}$$

$$c_4=\frac{-c_3}{2\cdot 4}=\frac{c_0}{c^4\cdot 4!}$$

$$c_n=\frac{(-1)^n c_0}{2^n n!}$$

$$r_2=0,\quad c_{k+1}=\frac{-c_k}{2k+1},\quad k=0,1,2,\cdots$$

$$c_1=\frac{-c_0}{1}$$

$$c_2=\frac{-c_1}{3}=\frac{c_0}{1\cdot 3}$$

$$c_3=\frac{-c_2}{5}=\frac{-c_0}{1\cdot 3\cdot 5}$$

$$c_4=\frac{-c_3}{7}=\frac{c_0}{1\cdot 3\cdot 5\cdot 7}$$

$$c_n=\frac{(-1)^n c_0}{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}$$

Solutions about Singular Points

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y_1(x) = x^{1/2}[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n!} x^n] = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{n+1/2}, \; x \geq 0$$

$$y_2(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)} x^n, \; |x| < \infty$$

$$\therefore y = C_1 y_1(x) + C_2 y_2(x) \quad , (0, \infty)$$

@ Only One Series Solution

Solve $xy'' + y = 0$

From xP(x)=0 and x^2 Q(x)=x and the fact that 0 and x are their own power series centered at 0, we conclude a0=0 and b0=0 and so the indicial equation is r(r-1)= 0.
You should verify that the two recurrence relations corresponding to the indicial roots r1=1 and r2=0 yields exactly the same set of coefficients. In other words, in this case the method of Frobenius produces only a single series solution

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} x^{n+1}$$

$$= x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots$$

@ Determination of second series solution, when only one series solution is known

☑ Method of Frobenius

$$x^2y'' + x[xP(x)]y' + [x^2Q(x)]y = 0 \longleftarrow y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

↓

find the unknown exponent r (indicial equation)

indicial equation

$$r(r-1) + a_0 r + b_0 = 0 \longrightarrow r_1, r_2$$

Case I	distinct and do not differ by an integer	$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \; y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2}$
Case II	if $r_1 - r_2 = N$ N : positive integer $r_1 > r_2$	$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, c_0 \neq 0,$ $\quad y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx$ Frobenius fail to give a second series solution $y_2(x) = Cy_1(x)\ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}, b_0 \neq 0, C$ could be zero
Case III	if $r_1 = r_2$ Analogous to the solution of a Cauchy-Euler equation	$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \; c_0 \neq 0 \longrightarrow y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx$ Frobenius fail to give a second series solution $y_2(x) = y_1(x)\ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}$

@ Example 4) Determination of second series solution, when only one series solution is known

Find the general solution of

$$xy'' + y = 0$$

From the known solution given in Example 4,

$$y_1(x) = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots$$

We can construct a second solution y2(x) using formula of case 2:

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx$$

$$y_2(x) = y_1(x)\int \frac{e^{-\int 0\,dx}}{[y_1(x)]^2}dx$$

$$= y_1(x)\int \frac{dx}{[x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots]^2}$$

$$= y_1(x)\int \frac{dx}{[x^2 - x^3 + \frac{5}{12}x^4 - \frac{7}{72}x^5 + \cdots]}$$

$$= y_1(x)\int [\frac{1}{x^2} + \frac{1}{x} + \frac{7}{12} + \frac{19}{72}x + \cdots]dx$$

$$= y_1(x)[-\frac{1}{x} + \ln x + \frac{7}{12}x + \frac{19}{144}x^2 + \cdots]$$

or $y_2(x) = y_1(x)\ln x$

$$+ y_1(x)[-\frac{1}{x} + \frac{7}{12}x + \frac{9}{144}x^2 + \cdots]$$

4-3. Two Special Equations:

1) Bessel's Equation

2) Legendre's Equation

Two Special Equations

The two differential equations occur frequently in applied mathematics, physics, and engineering

	Bessel's equation	Legendre's equation
Equation	$x^2y'' + xy' + (x^2 - \nu^2)y = 0$	$(1-x^2)y'' - 2xy' + n(n+1)y = 0$
Order	$\nu \geq 0$	n : **nonnegative integer**
Solution	**Bessel Function**	**Legendre Function**
$x = 0$	**Regular singular point**	**Ordinary point**

Solution of Bessel's Equation

Equation	Bessel's equation
	$x^2y''+xy'+(x^2-\nu^2)y=0$
Order	ν , $\nu \geq 0$
Solution	Bessel Function
$x=0$	Regular singular point

$$x^2y''+xy'+(x^2-\nu^2)y=0$$

Standard form

$$\downarrow$$

$$y''+\frac{1}{x}y'+(1-\frac{\nu}{x^2}^2)y=0$$

$$x^2y''+xy'+(x^2-\nu^2)y$$ ⇐ **Method of Frobenius** $$y=\sum_{n=0}^{\infty}c_nx^{n+r}$$

$$=\sum_{n=0}^{\infty}c_n(n+r)(n+r-1)x^{n+r}+\sum_{n=0}^{\infty}c_n(n+r)x^{n+r}+\sum_{n=0}^{\infty}c_nx^{n+r+2}-\nu^2\sum_{n=0}^{\infty}c_nx^{n+r}$$

$$=c_0(r^2-r+r-\nu^2)x^r+x^r\sum_{n=1}^{\infty}c_n[(n+r)(n+r-1)+(n+r)-\nu^2]x^n+x^r\sum_{n=0}^{\infty}c_nx^{n+2}$$

$$=c_0(r^2-\nu^2)x^r+x^r\sum_{n=1}^{\infty}c_n[(n+r)^2-\nu^2]x^n+x^r\sum_{n=0}^{\infty}c_nx^{n+2}=0\cdots(1)$$

Indicial roots $r_1=\nu$, $r_2=-\nu$

Bessel Functions

$$c_0(r^2-\nu^2)x^r + x^r\sum_{n=1}^{\infty} c_n[(n+r)^2-\nu^2]x^n + x^r\sum_{n=0}^{\infty} c_n x^{n+2} = 0 \cdots (1)$$

Indicial roots $r_1 = \nu\,, r_2 = -\nu$

When $r_1 = \nu$, (1) becomes

$$x^\nu\sum_{n=1}^{\infty} c_n n(n+2\nu)x^n + x^\nu\sum_{n=0}^{\infty} c_n x^{n+2}$$

$$= x^\nu\left[(1+2\nu)c_1 x + \sum_{n=2}^{\infty} c_n n(n+2\nu)x^n + \sum_{n=0}^{\infty} c_n x^{n+2}\right]$$

$$= x^\nu\left[(1+2\nu)c_1 x + \sum_{k=0}^{\infty}[(k+2)(k+2+2\nu)c_{k+2} + c_k]x^{k+2}\right] = 0$$

($k = n-2$ for the first sum, $k = n$ for the second sum)

$$(1+2\nu)c_1 = 0$$

$$c_{k+2} = \frac{-c_k}{(k+2)(k+2+2\nu)},\quad k = 0,1,2,....$$

$$(1+2\nu)c_1 = 0$$

The choice $c_1 = 0$ **implies** $c_3 = c_5 = c_7 = \cdots = 0$

so, for $k = 0, 2, 4, \ldots$ **we find, after letting** $k+2 = 2n, n = 1, 2, 3, \ldots.,$ **that**

$$c_{2n} = -\frac{c_{2n-2}}{2^2 n(n+\nu)},$$

Thus

$$c_2 = -\frac{c_0}{2^2 \cdot 1 \cdot (1+\nu)}$$

$$c_4 = -\frac{c_2}{2^2 \cdot 2(2+\nu)} = \frac{c_0}{2^4 \cdot 1 \cdot 2(1+\nu)(2+\nu)}$$

$$c_6 = -\frac{c_4}{2^2 \cdot 3(3+\nu)} = -\frac{c_0}{2^6 \cdot 1 \cdot 2 \cdot 3(1+\nu)(2+\nu)(3+\nu)}$$

$$\vdots$$

$$c_{2n} = \frac{(-1)^n c_0}{2^{2n} n!(1+\nu)(2+\nu)\cdots(n+\nu)}, \quad n = 1, 2, 3, \ldots$$

Gamma Function

Gamma function

Definition

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

Recurrence relation

$$\Gamma(x+1) = x\Gamma(x)$$

It is standard practice to choose c0 to be a specific value –namely,

$$c_0 = \frac{1}{2^\nu \Gamma(1+\nu)}$$

where $\Gamma(1+\nu)$ **is the gamma function.**

Since this function possesses the convenient property $\Gamma(1+x) = x\Gamma(x)$ **, we can reduce the product in the denominator of c2n to one term. For example,**

$$\Gamma(1+\nu) = \nu\Gamma(\nu)$$

$$\Gamma(1+\nu+2) = (\nu+2)\Gamma(\nu+2) = (\nu+2)(\nu+1)\Gamma(\nu+1)$$

$$\therefore c_{2n} = \frac{(-1)^n}{2^{2n} n!(1+\nu)(2+\nu)\cdots(n+\nu)2^\nu \Gamma(1+\nu)}$$

$$= \frac{(-1)^n}{2^{2n+\nu} n!\Gamma(1+\nu+n)}, n = 1,2,3,\ldots$$

Bessel Functions of the First Kind of order ν

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

solution

$$\therefore y = \sum_{n=0}^{\infty} c_n x^{n+r} = \sum_{n=0}^{\infty} c_{2n} x^{2n+\nu} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

denoted by $J_\nu(x)$

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

What happen at x=0?

Also, for the second exponent $r_2 = -\nu$ **we obtain, in exactly the same manner,**

$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1-\nu+n)} \left(\frac{x}{2}\right)^{2n-\nu}$$

The functions $J_\nu(x) and J_{-\nu}(x)$ **are called Bessel functions of the first kind of order** ν

Determination of Series Solution depending on the value of ν

$$x^2y'' + xy' + (x^2 - \nu^2)y$$

Indicial roots

$r_1 = \nu, r_2 = -\nu$

When $r_1 = \nu$, $J_\nu(x) =$

When $r_2 = -\nu$, $J_{-\nu}(x) =$

Case I	distinct and do not differ by an integer	Two linearly independent series solution $y_1(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1}, y_2(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2}$
Case II	if $r_1 - r_2 = N$ N : positive integer $r_1 > r_2$	Two linearly independent series solution $y_1(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1}, c_0 \neq 0,$ $y_2(x) = C y_1(x) \ln x + \sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2}, b_0 \neq 0$ Then, this indicates what the second solution looks like Frobenius fail to give a second series solution
		$y_1(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1}, c_0 \neq 0,$ $y_2(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2}, b_0 \neq 0$
Case III	if $r_1 = r_2$	Two linearly independent series solution Frobenius fail to give a second series solution $y_1(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1}, c_0 \neq 0 \longrightarrow y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$ $y_2(x) = y_1(x) \ln x + \sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2}$

$r_1 = r_2 \longrightarrow$ Case III

$\nu = 0 : J_\nu(x) = J_{-\nu}(x)$

$\nu > 0 : r_1 - r_2 = \nu - (-\nu) = 2\nu$

Not a positive integer → Case I

$J_\nu(x), J_{-\nu}(x)$: **linearly independent solutions**

so general solution is $y = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$

a positive integer ⟶ Case II

Two possibilities

- $\nu = m =$ **Positive integer**

  $J_{-m}(x) = (-1)^m J_m(x),$

  not linearly independent solutions

- $2\nu =$ **Positive integer**

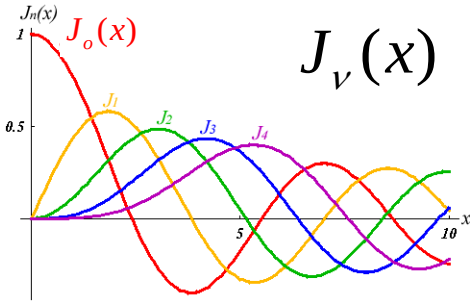
  $(\nu =$ **Positive half an odd integer)**

  $J_\nu(x), J_{-\nu}(x)$: **linearly independent solutions**

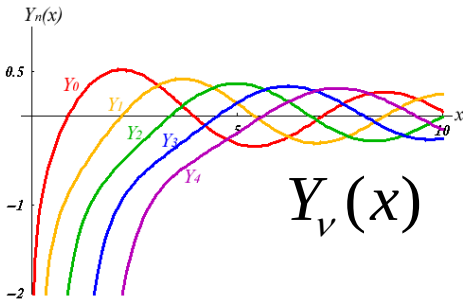
  so general solution is $y = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$

Then how?

@@ Bessel functions of the second kind of order ν



$J_\nu(x)$



$Y_\nu(x)$

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \cdots (1)$$

If $\nu \neq$ integer, the function defined by $Y_\nu(x) = \dfrac{\cos \nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}$ and the function $J_\nu(x)$ are linearly independent solutions of (1) . Thus another form of the general solution of (1) is $y = c_1 J_\nu(x) + c_2 Y_\nu(x)$ provided $\nu \neq$ integer. As $\nu \to m$, m an integer, $Y_\nu(x) = \dfrac{\cos \nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}$ has the indeterminate form 0/0. However, it can be shown by L'Hopital's rule that $\lim_{\nu \to m} Y_\nu(x)$ exists. Moreover, the function

$$Y_m(x) = \lim_{\nu \to m} Y_\nu(x)$$

and $J_m(x)$ are linearly independent solutions of $x^2y'' + xy' + (x^2 - m^2)y = 0$

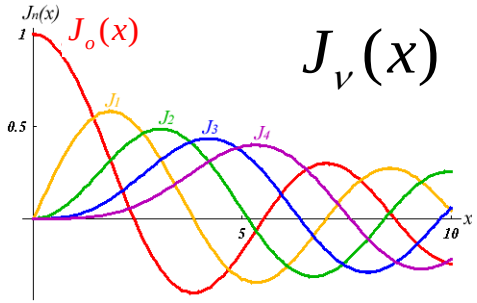
Hence for any value of ν the general solution of (1) on $(0, \infty)$ can be written as

$$y = c_1 J_\nu(x) + c_2 Y_\nu(x)$$

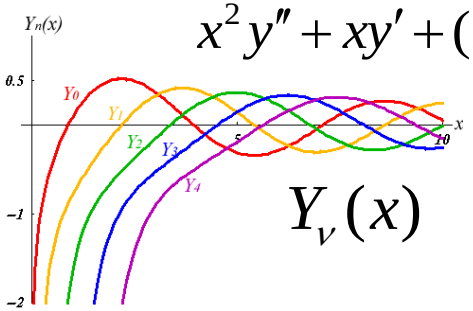
$Y_\nu(x)$ is called the Bessel functions of the second kind of order ν [1)]

1) Zill D.G., Cullen M.R., Advanced Engineering Mathematics, The Third Edition, Jones and Bartlett, 2006, p262

General solution of the Bessel's equation of order zero($\nu = 0$) for $x > 0$



$J_\nu(x)$



$Y_\nu(x)$

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \cdots (1)$$

the function $Y_\nu(x)$ and $J_\nu(x)$ are linearly independent solutions!!

$$Y_\nu(x) = \frac{\cos \nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}$$

(defined)

$$Y_0(x) = \left[\frac{2}{\pi}\left(\gamma + \ln\frac{x}{2} \right) J_0(x) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2} x^{2m} \right], \quad x > 0$$

The general solution of the Bessel equation of order zero for $x > 0$

$$y = c_1 J_\nu(x) + c_2 Y_\nu(x)$$

Note that $J_0(x) \to 1 \; as \; x \to 0$ **and that** $Y_0(x)$ **has a logarithm singularity at** $x = 0$;

that is $Y_0(x)$ **behaves as** $(2/\pi)\ln x$ **when** $x \to 0$ **through positive values.**

Thus, if we are interested in solutions of Bessel's equation of order zero that are finite at the origin, which is often the case, we must discard $Y_0(x)$ [1)]

1) Boyce W.E., DiPrima R.C., Elementary Differential Equations and Boundary Value Problems, Eight edition, 2005, John Wiley & Sons, Inc., p297

Series Solution of Bessel's Equation Depending on ν

☑ Example 1 General Solution: ν Not an Integer

By identifying $\nu^2 = \frac{1}{4}$ and $\nu = \frac{1}{2}$ we can see that the general solution of the equation

$$x^2y'' + xy' + (x^2 - \left(\frac{1}{2}\right)^2)y = 0 \quad on\ (0,\infty)$$

is $y = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$

☑ Example 2) General Solution: ν an Integer

By identifying $\nu^2 = 9$ and $\nu = 3$ we can see that the general solution of the equation

$$x^2y'' + xy' + (x^2 - 9)y = 0 \quad on\ (0,\infty)$$

is $y = c_1 J_3(x) + c_2 Y_3(x)$

Parametric Bessel equation of order ν

☑ DEs solvable in terms of Bessel Functions

$$x^2y'' + xy' + (\alpha^2x^2 - \nu^2)y = 0$$

Equation	Bessel's equation
	$x^2y'' + xy' + (x^2 - \nu^2)y = 0$
Order	ν , $\nu \geq 0$
Solution	Bessel Function
$x = 0$	Regular singular point

To transform into the Bessel equation, let $t = \alpha x$

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - \nu^2)y = 0$$

$x^2y'' + xy' + (\alpha^2x^2 - \nu^2)y = 0$ **"Parametric Bessel equation of order** ν **"**

The general solution : $y = c_1J_\nu(\alpha x) + c_2Y_\nu(\alpha x)$

Very important in the study of certain boundary-value problems involving partial differential equation that are expressed in cylindrical coordinates.

☑ DEs solvable in terms of Bessel Functions

$$x^2y'' + xy' - (x^2 + \nu^2)y = 0$$

Equation	Bessel's equation
	$x^2y'' + xy' + (x^2 - \nu^2)y = 0$
Order	ν , $\nu \geq 0$
Solution	Bessel Function
$x = 0$	Regular singular point

To transform into the Bessel equation, let $t = ix$

$$t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - \nu^2)y = 0$$

$$x^2y'' + xy' - (x^2 + \nu^2)y = 0$$ "Modified Bessel equation of order ν"

The general solution : $y = c_1 I_\nu(x) + c_2 K_\nu(x)$

Complex-valued solution

$$J_\nu(ix)\ Y_\nu(ix)$$

Real-valued solution

$$I_\nu(x) = i^{-\nu} J_\nu(ix)$$

"Modified Bessel function of the first kind of order ν"

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin\nu\pi}$$

"Modified Bessel function of the second kind of order ν"

when $\nu =$ integer

Transformation of Bessel's Equation

☑ DEs solvable in terms of Bessel Functions

$$y'' + \frac{1-2a}{x} y' + (b^2 c^2 x^{2c-2} + \frac{a^2 - p^2 c^2}{x^2}) y = 0,\ p \geq 0$$

Equation	Bessel's equation
	$x^2 y'' + xy' + (x^2 - \nu^2) y = 0$
Order	ν , $\nu \geq 0$
Solution	Bessel Function
$x = 0$	Regular singular point

To transform into the Bessel equation, let $z = bx^c,\ y(x) = \left(\frac{z}{b}\right)^{a/c} w(z)$

$$z^2 \frac{d^2 w(z)}{dz^2} + z \frac{dw(z)}{dz} + (z^2 - p^2) w = 0$$

The general solution : $y = x^a \left[c_1 J_p(bx^c) + c_2 Y_p(bx^c) \right]$

Special Functions

☑ Example 3

Find the general solution of

$$xy'' + 3y' + 9y = 0 \quad on\ (0, \infty)$$

By writing the given DE as

$$y'' + \frac{3}{x}y' + \frac{9}{x}y = 0$$

We can make the following identifications

$$1 - 2a = 3,\ b^2c^2 = 9,$$
$$2c - 2 = -1,\ a^2 - p^2c^2 = 0$$

$$a = -1,\ c = \frac{1}{2},\ b = 6,\ p = 2$$

$$y = x^{-1}[c_1 J_2(6x^{1/2}) + c_2 Y_2(6x^{1/2})]$$

☑ Example 4
The Aging Spring Revisited

$$mx'' + ke^{-\alpha t}x = 0, \quad \alpha > 0$$

$$s^2\frac{d^2x}{ds^2} + s\frac{dx}{ds} + s^2x = 0$$

$$x = c_1 J_0(s) + c_2 Y_0(s)$$

$$x(t) = c_1 J_0\left(\frac{2}{\alpha}\sqrt{\frac{k}{m}e^{-\alpha t/2}}\right)$$

$$+ c_2 Y_0\left(\frac{2}{\alpha}\sqrt{\frac{k}{m}}e^{-\alpha t/2}\right)$$

Derivation Using the Series Definition

Derive the formula

$$xJ_\nu'(x) = \nu J_\nu(x) - xJ_{\nu+1}(x)$$

$$xJ_\nu'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\nu)}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

$$= \nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

$$+2\sum_{n=0}^{\infty} \frac{(-1)^n n}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

$$= \nu J_\nu(x) + x\sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu-1}$$

$$= \nu J_\nu(x) - x\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(2+\nu+k)} \left(\frac{x}{2}\right)^{2k+\nu+1}$$

$$= \nu J_\nu(x) - xJ_{\nu+1}(x)$$

Spherical Bessel Function

☑ DEs solvable in terms of Bessel Functions

when $\nu =$ half and odd integer $\pm\frac{1}{2},\pm\frac{3}{2},\pm\frac{5}{2}\cdots$

Equation	Bessel's equation
	$x^2y''+xy'+(x^2-\nu^2)y=0$
Order	ν , $\nu \geq 0$
Solution	Bessel Function
$x=0$	Regular singular point

$$J_\nu(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)x^{2n+\nu}$$

Bessel functions of the first kind of order $J_\nu(x)$ can be expressed in terms of the elementary functions $\sin x, \cos x,$ and powers of x →"spherical Bessel function"

In case of $\nu=\frac{1}{2}$

$$J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x$$

$$J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x$$

Bessel Functions at x=0

$$J_\nu(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu}$$

What happen at x=0?

When x=0 and v = 0

$$J_0(0)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+0+n)}\left(\frac{0}{2}\right)^{2n+0}=\underbrace{\frac{(-1)^0}{0!\Gamma(1+0+0)}\left(\frac{0}{2}\right)^{2\cdot0+0}}_{1}+\underbrace{\frac{(-1)^n}{1!\Gamma(1+0+1)}\left(\frac{0}{2}\right)^{2\cdot1+0}+\cdots}_{0}$$

$0!=1^{1)}$

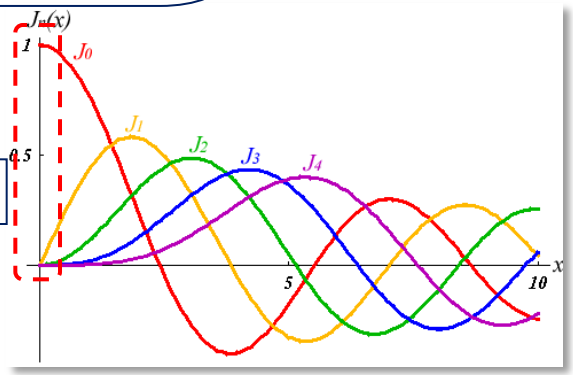
$0^0=1^{2)}$

$\Gamma(1)=1^{3)}$

When x=0 and v ≠ 0

$$J_\nu(0)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+0+n)}\left(\frac{0}{2}\right)^{2n+\nu}=\underbrace{\frac{(-1)^0}{0!\Gamma(1+0+0)}\left(\frac{0}{2}\right)^{2\cdot0+\nu}+\frac{(-1)^n}{1!\Gamma(1+0+1)}\left(\frac{0}{2}\right)^{2\cdot1+\nu}+\cdots}_{0}$$

Bessel functions $J_\nu(x)$ are defined at x=0 (regular singular point) as well



1) Larson R., Hostetler R.P., Edwards B.H., Calculus with Analytic Geometry, English Edition, Houghton Mifflin Company, 2006, p597
2) Donald E. Knuth, Two notes on notation, Amer. Math. Monthly 99 no. 5 (May 1992), 403–422, see also http://en.wikipedia.org/wiki/Exponentiation
3) Zill D.G., Cullen M.R., Advanced Engineering Mathematics, The Third Edition, Jones and Bartlett, 2006, App-4

Bessel Functions at x=0

$$x^2y'' + xy' + (x^2 - \nu^2)y$$

$$= \boxed{c_0(r^2 - \nu^2)}x^r + x^r\sum_{n=1}^{\infty} c_n[(n+r)^2 - \nu^2]x^n + x^r\sum_{n=0}^{\infty} c_n x^{n+2} \cdots (1)$$

Indicial roots $r_1 = \nu, r_2 = -\nu$

When $r_1 = \nu$,

$$c_{2n} = \frac{(-1)^n c_0}{2^{2n} n!(1+\nu)(2+\nu)\cdots(n+\nu)}, \quad n = 1,2,3,\ldots$$

Solution denoted by

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n c_0}{2^{2n} n!(1+\nu)(2+\nu)\cdots(n+\nu)} x^{2n+\nu}$$

To make $J_0(0) = 1$

$$J_0(0) = \frac{(-1)^0 c_0}{2^0 \cdot 0! \cdot 1} 0^0 + \frac{(-1)^1 c_0}{2^2 n!(1\cdot 2)}(0)^{2\cdot 1} + \frac{(-1)^n}{1!\Gamma(1+0+1)}\left(\frac{0}{2}\right)^{2\cdot 1+0} + \cdots$$

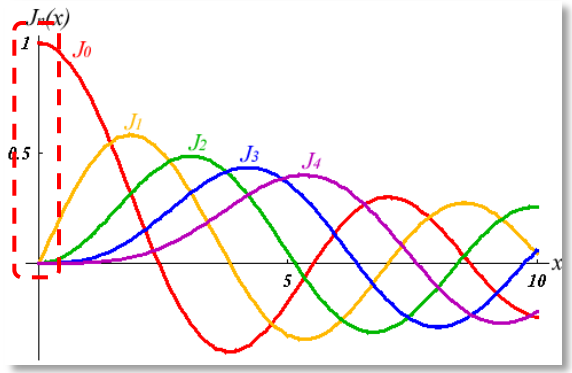
at x=0

The equation: $x^2y'' + xy' + (x^2 - \nu^2)y$

$$\Rightarrow -\nu^2 y = 0$$

When $\nu = 0$, equation is satisfied for any

When $\nu \neq 0$, $y = 0$ ↓ choose 1



4-3. Two Special Equations:

1) Bessel's Equation

2) Legendre's Equation

Legendre's Equation

Legendre's equation
$(1-x^2)y''-2xy'+n(n+1)y=0$
n , n : nonnegative integer
Legendre Function
Ordinary point

$$y=\sum_{k=0}^{\infty}c_k x^k$$

$$(1-x^2)y''-2xy'+n(n+1)y$$

$$=[n(n+1)c_0+2c_2]+[(n-1)(n+2)c_1+6c_3]x$$

$$+\sum_{j=2}^{\infty}[(j+2)(j+1)c_{j+2}+(n-j)(n+j+1)c_j]x^j=0$$

Which implies

$$n(n+1)c_0+2c_2=0$$

$$(n-1)(n+2)c_1+6c_3=0$$

$$(j+2)(j+1)c_{j+2}+(n-j)(n+j+1)c_j=0$$

$$c_2=-\frac{n(n+1)c_0}{2!}$$

$$c_3=-\frac{(n-1)(n+2)c_1}{3!}$$

$$c_{j+2}=-\frac{(n-j)(n+j+1)}{(j+2)(j+1)}c_j,$$

$$j=2,3,4\ldots$$

Legendre's Equation

Legendre's equation
$(1-x^2)y''-2xy'+n(n+1)y=0$
n , n : nonnegative integer
Legendre Function
Ordinary point

$$y=\sum_{k=0}^{\infty}c_k x^k$$

$$(1-x^2)y''-2xy'+n(n+1)y$$

$$=[n(n+1)c_0+2c_2]+[(n-1)(n+2)c_1+6c_3]x$$

$$+\sum_{j=2}^{\infty}[(j+2)(j+1)c_{j+2}+(n-j)(n+j+1)c_j]x^j=0$$

Which implies

$$n(n+1)c_0+2c_2=0$$

$$(n-1)(n+2)c_1+6c_3=0$$

$$(j+2)(j+1)c_{j+2}+(n-j)(n+j+1)c_j=0$$

$$\Rightarrow\quad c_{j+2}=-\frac{(n-j)(n+j+1)}{(j+2)(j+1)}c_j$$

Ratio test for convergence

$$\lim_{j\to\infty}\left|\frac{c_{j+2}x^{j+2}}{c_j x^j}\right|=\lim_{j\to\infty}\left|-\frac{(n-j)(n+j+1)x^2}{(j+2)(j+1)}\right|=\left|x^2\right|<1$$

$\therefore$ **converge for** $-1<x<1$

$\therefore$ converge for $-1 < x < 1$

Legendre's equation

$(1-x^2)y'' - 2xy' + n(n+1)y = 0$

n , n : nonnegative integer

Legendre Function

Ordinary point

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad y = \sum_{k=0}^{\infty} c_k x^k$$

$$c_2 = -\frac{n(n+1)c_0}{2!}$$

$$c_3 = -\frac{(n-1)(n+2)c_1}{3!}$$

$$c_4 = -\frac{(n-2)(n+3)}{(4)(3)}c_2 = \frac{(n-2)n(n+1)(n+3)}{4!}c_0$$

$$c_5 = -\frac{(n-3)(n+4)}{(5)(4)}c_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}c_1$$

$$c_6 = -\frac{(n-4)(n+5)}{(6)(5)}c_4 = \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}c_0$$

$$c_7 = -\frac{(n-5)(n+6)}{(7)(6)}c_5 = -\frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}c_1$$

$$c_{j+2} = -\frac{(n-j)(n+j+1)}{(j+2)(j+1)}c_j,\ j = 2,3,4\ldots$$

$\therefore$ converge for

$-1 < x < 1$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad y = \sum_{k=0}^{\infty} c_k x^k$$

thus

$$y_1(x) = c_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}x^6 + \cdots\right]$$

$$y_2(x) = c_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}x^7 + \cdots\right]$$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad y = \sum_{k=0}^{\infty} c_k x^k$$

If n =	even integer	odd integer
$y_1(x) = c_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}x^6 + \cdots\right]$	terminate with x^n	infinite series
$y_2(x) = c_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}x^7 + \cdots\right]$	infinite series	terminate with x^n

If n = nonnegative obtain an nth-degree polynomial solution of Legendre's equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$y = \sum_{k=0}^{\infty} c_k x^k$$

If n =	even integer	odd integer
$y_1(x) = c_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}x^6 + \cdots\right]$	terminate with x^n	infinite series
$y_2(x) = c_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}x^7 + \cdots\right]$	infinite series	terminate with x^n

If n = nonnegative integer, obtain an nth-degree polynomial solution of Legendre's equation

A constant multiple of a solution of Legendre's equation is also a solution.

It is *traditional to choose* specific values for c_0, c_1

n	c_0	c_1
0	1	
1		1
even	$c_0 = (-1)^{n/2}\frac{1\cdot 3\cdots(n-1)}{2\cdot 4\cdots n}$	
odd		$c_0 = (-1)^{(n-1)/2}\frac{1\cdot 3\cdots n}{2\cdot 4\cdots(n-1)}$

$$(1-x^2)y''-2xy'+n(n+1)y=0 \qquad y=\sum_{k=0}^{\infty} c_k x^k$$

If n =	even integer	odd integer
$y_1(x)=c_0\left[1-\frac{n(n+1)}{2!}x^2+\frac{(n-2)n(n+1)(n+3)}{4!}x^4-\frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}x^6+\cdots\right]$	terminate with x^n	infinite series
$y_2(x)=c_1\left[x-\frac{(n-1)(n+2)}{3!}x^3+\frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5-\frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}x^7+\cdots\right]$	infinite series	terminate with x^n

ex.) $n=4$

$$y_1(x)=(-1)^{4/2}\frac{1\cdot 3}{2\cdot 4}\left(1-\frac{4(4-1)}{2!}x^2+\frac{(4-2)4(4+1)(4+3)}{4!}x^4\right)$$

$$=\frac{1}{8}(35x^2-30x^2+3)$$

$$\rightarrow P_4(x)$$

n	c_0	c_1
0	1	
1		1
even	$c_0=(-1)^{n/2}\frac{1\cdot 3\cdots(n-1)}{2\cdot 4\cdots n}$	
odd		$c_0=(-1)^{(n-1)/2}\frac{1\cdot 3\cdots n}{2\cdot 4\cdots(n-1)}$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$y = \sum_{k=0}^{\infty} c_k x^k$$

in the same manner

$$P_{n=0}(x) = 1$$

$$P_{n=1}(x) = x$$

$$P_{n=2}(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_{n=3}(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_{n=4}(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_{n=5}(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

→ **Legendre Polynomials**

If n =	even integer	odd integer
$y_1(x) = c_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}x^6 + \cdots\right]$	terminate with x^n	infinite series
$y_2(x) = c_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}x^7 + \cdots\right]$	infinite series	terminate with x^n

n	c_0	c_1
0	1	
1		1
even	$c_0 = (-1)^{n/2}\frac{1\cdot 3\cdots(n-1)}{2\cdot 4\cdots n}$	
odd		$c_0 = (-1)^{(n-1)/2}\frac{1\cdot 3\cdots n}{2\cdot 4\cdots(n-1)}$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad y = \sum_{k=0}^{\infty} c_k x^k$$

n	Legendre Equation	$y = y_1(x) + y_2(x)$
0	$(1-x^2)y'' - 2xy' = 0$	$y = [P_0(x)] + [Q_0(x)]$
1	$(1-x^2)y'' - 2xy' + 2y = 0$	$y = [Q_1(x)] + [P_1(x)]$
2	$(1-x^2)y'' - 2xy' + 6y = 0$	$y = [P_2(x)] + [Q_2(x)]$
3	$(1-x^2)y'' - 2xy' + 12y = 0$	$y = [Q_3(x)] + [P_3(x)]$
4	$(1-x^2)y'' - 2xy' + 20y = 0$	$y = [P_4(x)] + [Q_4(x)]$
5	$(1-x^2)y'' - 2xy' + 30y = 0$	$y = [Q_5(x)] + [P_5(x)]$

$Q_n(x)$ **:an infinite series**

$$y = \sum_{k=0}^{\infty} c_k x^k$$

$Q_n(x)$:an infinite series

$(1-x^2)y'' - 2xy' + n(n+1)y = 0$ $\therefore$ converge for $-1 < x < 1$ $\therefore$ converge for $-1 \le x \le 1$

n	Legendre Equation	$y = y_1(x) + y_2(x)$	Legendre Polynomials
0	$(1-x^2)y'' - 2xy' = 0$	$y = [P_0(x)] + [Q_0(x)]$	$P_0(x) = 1$
1	$(1-x^2)y'' - 2xy' + 2y = 0$	$y = [Q_1(x)] + [P_1(x)]$	$P_1(x) = x$
2	$(1-x^2)y'' - 2xy' + 6y = 0$	$y = [P_2(x)] + [Q_2(x)]$	$P_2(x) = \frac{1}{2}(3x^2 - 1)$
3	$(1-x^2)y'' - 2xy' + 12y = 0$	$y = [Q_3(x)] + [P_3(x)]$	$P_3(x) = \frac{1}{2}(5x^3 - 3x)$
4	$(1-x^2)y'' - 2xy' + 20y = 0$	$y = [P_4(x)] + [Q_4(x)]$	$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$
5	$(1-x^2)y'' - 2xy' + 30y = 0$	$y = [Q_5(x)] + [P_5(x)]$	$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$

You should be aware of the fact that Legendre's equation possesses solutions that are bounded on the closed interval $-1 \le x \le 1$ only in the case when $n = 0, 1, 2, ..$
More to the point, the only Legendre functions that are bounded on the closed interval $-1 \le x \le 1$ are the Legendre polynomials $P_n(x)$ or constant multiples of these polynomials.[1)]

1) Zill D.G., Cullen M.R., Advanced Engineering Mathematics, The Third Edition, Jones and Bartlett, 2006, p270 'remark'

Legendre Functions[1)]

$$y=\sum_{k=0}^{\infty}c_k x^k$$

$Q_n(x)$:an infinite series

$(1-x^2)y''-2xy'+n(n+1)y=0$ ∴converge for $-1<x<1$ ∴converge for $-1\le x\le 1$

n	Legendre Equation	$y=y_1(x)+y_2(x)$	Legendre Polynomials
0	$(1-x^2)y''-2xy'=0$	$y=[P_0(x)]+[Q_0(x)]$	$P_0(x)=1$
1	$(1-x^2)y''-2xy'+2y=0$	$y=[Q_1(x)]+[P_1(x)]$	$P_1(x)=x$
2	$(1-x^2)y''-2xy'+6y=0$	$y=[P_2(x)]+[Q_2(x)]$	$P_2(x)=\frac{1}{2}(3x^2-1)$
3	$(1-x^2)y''-2xy'+12y=0$	$y=[Q_3(x)]+[P_3(x)]$	$P_3(x)=\frac{1}{2}(5x^3-3x)$
4	$(1-x^2)y''-2xy'+20y=0$	$y=[P_4(x)]+[Q_4(x)]$	$P_4(x)=\frac{1}{8}(35x^4-30x^2+3)$
5	$(1-x^2)y''-2xy'+30y=0$	$y=[Q_5(x)]+[P_5(x)]$	$P_5(x)=\frac{1}{8}(63x^5-70x^3+15x)$

[boundary condition]

$$(1-x^2)y''-2xy'+n(n+1)y=0$$
$$-1<x<1$$

Because that Legendre's equation has singularities at both endpoints $x=\pm1$. Thus in order to be assured of generating a set of orthogonal eigenfunctions, we must impose the special boundary conditions

$$y(x),y'(x):\ finite\ as\ x\to-1^+,x\to1^-$$

The remaining infinite series in each case defines a second linearly independent solution, multiples of which are called *Legendre functions of the second kind*. In general,

$$Q_n(x)=P_n(x)Q_0(x)-\sum_{k=0}^{(n-1)/2}\frac{(2n-4k-1)}{(2k+1)(n-k)}P_{n-2k-1}(x)\quad ,Q_0(x)=\frac{1}{2}\log\frac{1+x}{1-x}$$

Note that all $Q_n(x)$, $n=0,1,2..$ have singularities at $x=\pm1$ due to the presence of the logarithm term in $Q_0(x)$ Hence, because the Legendre functions of the second kind do not satisfy the boundary condition (should be bounded at $x=\pm1$), they cannot be eigenfunctions of Legendre's equation[1)]

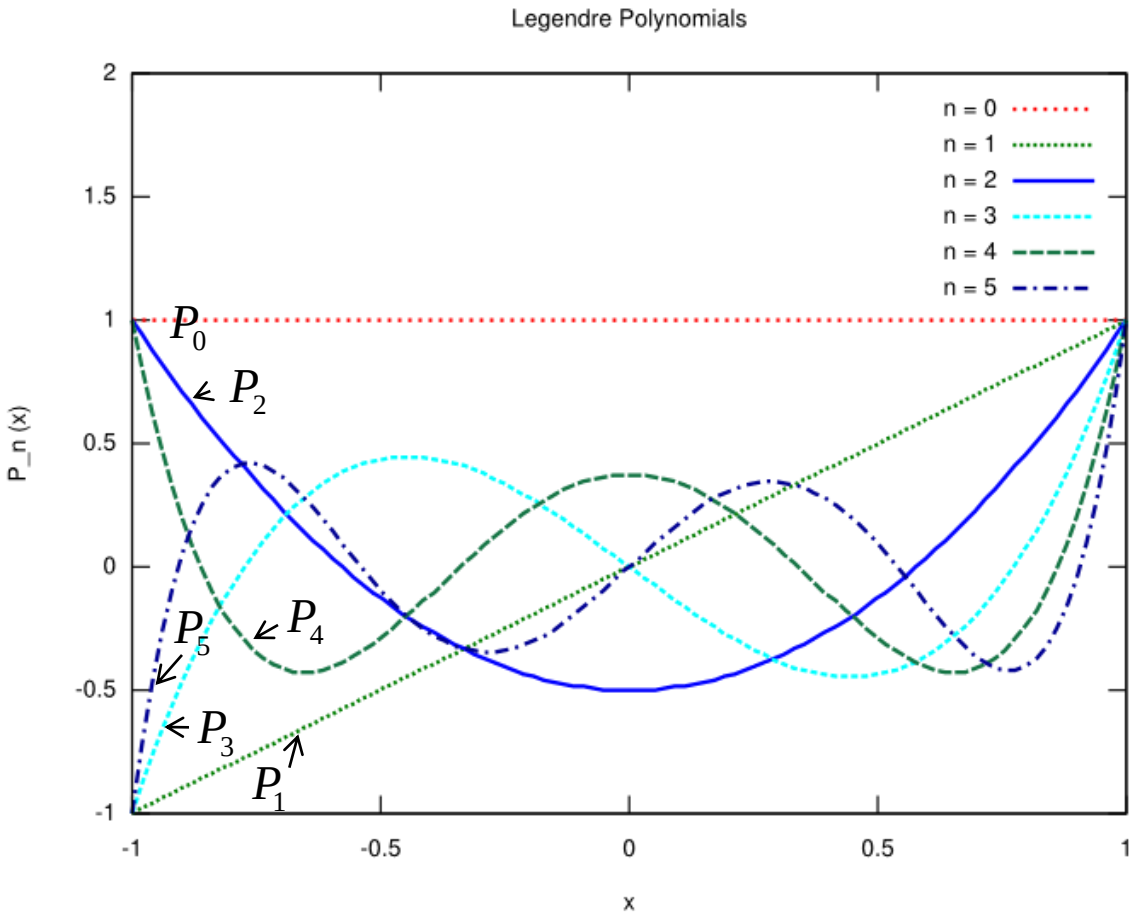
1) Andrew L.C., Elementary Partial Differential Equations with Boundary Value Problems, Academic Press, 1986, p386-389

Legendre Functions

$Q_n(x)$:an infinite series

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$y = \sum_{k=0}^{\infty} c_k x^k$$



Legendre Polynomials

$$P_0(x) = 1$$

$$P_1(x) = x$$

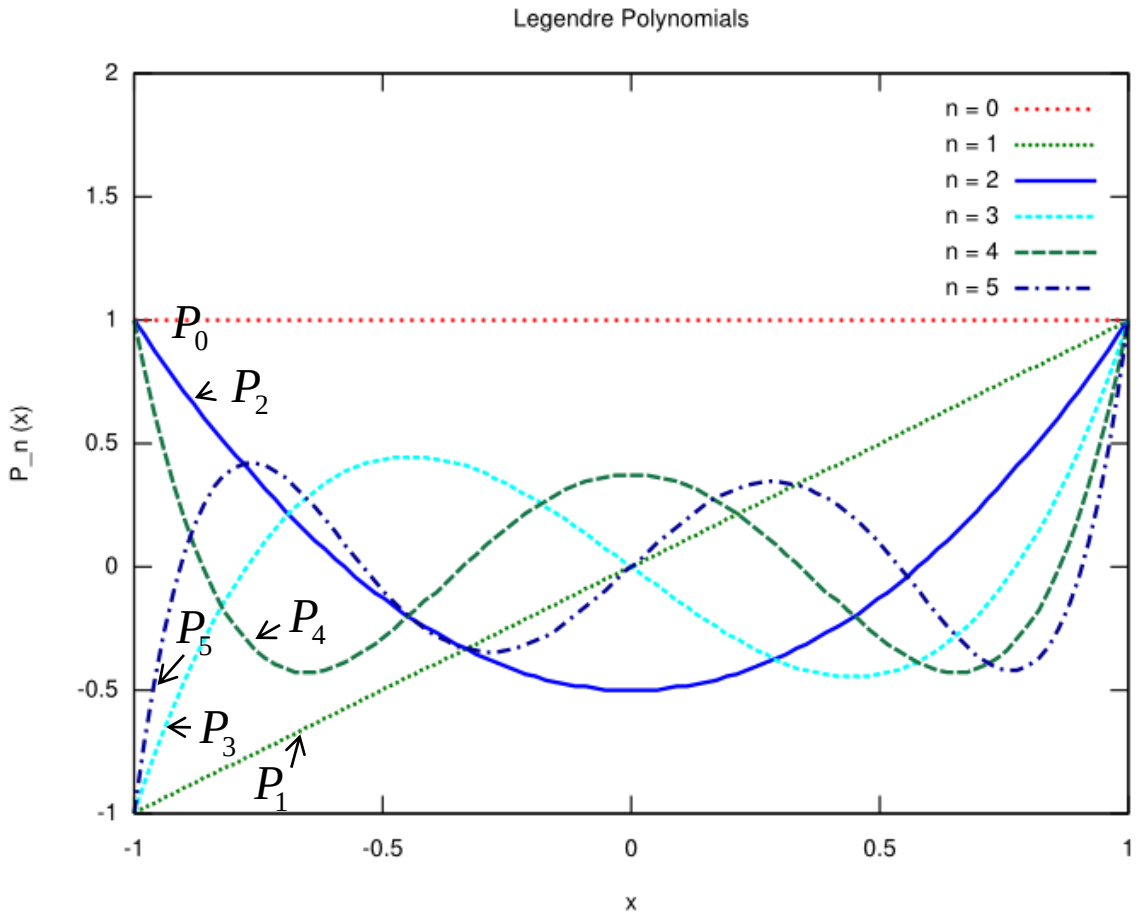
$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad y = \sum_{k=0}^{\infty} c_k x^k$$



Legendre Polynomials Properties

$$P_n(-x) = (-1)^n P_n(x)$$

$$P_n(1) = 1$$

$$P_n(-1) = (-1)^n$$

$$P_n(0) = 0, \quad n : \text{odd}$$

$$P_n'(0) = 0, \quad n : \text{even}$$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad y = \sum_{k=0}^{\infty} c_k x^k$$

- **Recurrence Relation**

$$(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x) = 0$$

Ex) if wish to find $P_6(x)$, use $k = 5$

Then $P_6(x)$ is expresses in the known $P_5(x)$ and $P_4(x)$

- **Rodrigues' Formula**

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \quad n = 0,1,2\cdots$$

Legendre Polynomials

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

[Reference]

Series solution : example

example

다음의 D.E.이 주어져 있을 때 다음 물음에 단계별로 답하시오.*

$$8x^2y'' + 10xy' + (x-1)y = 0$$

1) x=0 이 ordinary point 인지 regular singular point 인지 다음의 정의를 참고하여 판별하시오.

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \cdots (1)$$

standard form $y'' + P(x)y' + Q(x)y = 0 \cdots (2)$

Definition 5.2 **Regular/Irregular Singular Points**

A singular point x_0 is said to be a regular singular point of the differential equation (1) if the functions $p(x) = (x-x_0)P(x)$ and $q(x) = (x-x_0)^2 Q(x)$ are both analytic at x_0.

A singular point that is not regular is said to be an irregular point of the equation

*Bronson R., Costa G., Schaum's Outlines, Differential Equations Third Edition, Mc Graw Hill, 2006, p278, 28.6문제 응용

Problem

다음의 D.E.이 주어져 있을 때 다음 물음에 단계별로 답하시오.*

$$8x^2y'' + 10xy' + (x-1)y = 0$$

2) 해의 형태를 가정하고 그렇게 가정한 이유를 1)의 결과와 연관지어 설명하시오.

3) 가정한 해를 주어진 문제에 대입하여 indicial equation과 recurrence formula를 각각 구하시오.

4) equation에서 linearly independent 한 solution을 몇 개 찾을 수 있는지 3)의 indicial equation의 형태와 연관하여 설명하시오.

5) x=0 부근에서 General Solution을 구하시오. 단 series를 나타낼 때는 처음 세 번째 항까지 표시하시오.

*Bronson R., Costa G., Schaum's Outlines, Differential Equations Third Edition, Mc Graw Hill, 2006, p278, 28.6문제 응용

Solution

다음의 D.E.이 주어져 있을 때 다음 물음에 단계별로 답하시오.

$$8x^2y'' + 10xy' + (x-1)y = 0$$

1) x=0 이 ordinary point 인지 regular singular point 인지 판별하시오.

Standard form을 바꾸면 $y'' + \frac{5}{4x}y' + \frac{(x-1)}{8x^2}y = 0$

$P(x) = \frac{5}{4x}$, $Q(x) = \frac{(x-1)}{8x^2}$ 는 x=0에서 singular point 이다.

$(x-0)^2$ 을 곱하면 $x^2y'' + x(\frac{10}{8})y' + \frac{(x-1)}{8}y = 0$

$p(x) = \frac{5}{4}$, $q(x) = \frac{(x-1)}{8}$ 는 x=0에서 analytic 하다.

∴x=0은 regular singular point이다

Solution

다음의 D.E.이 주어져 있을 때 다음 물음에 단계별로 답하시오.

$$8x^2y'' + 10xy' + (x-1)y = 0$$

2) 해의 형태를 가정하고 그렇게 가정한 이유를 1)의 결과와 연관지어 설명하시오.

x=0에서 regular singular point이기 때문에 Frobenius method를 적용하면 적어도 하나의 solution을 구할 수 있다. 따라서 해를 다음과 같이 가정한다.

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

Solution

다음의 D.E.이 주어져 있을 때 다음 물음에 단계별로 답하시오.

$$8x^2y'' + 10xy' + (x-1)y = 0$$

3) 가정한 해를 주어진 문제에 대입하여 indicial equation과 recurrence formula를 각각 구하시오.

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

Solution

다음의 D.E.이 주어져 있을 때 다음 물음에 단계별로 답하시오.

$$8x^2y'' + 10xy' + (x-1)y = 0$$

3) 가정한 해를 주어진 문제에 대입하여 indicial equation과 recurrence formula를 각각 구하시오.

식에 대입하면

$$8x^2y'' + 10xy' + (x-1)y$$

$$= 8x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + 10x\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + (x-1)\sum_{n=0}^{\infty}c_n x^{n+r}$$

$$= \sum_{n=0}^{\infty}8(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty}10(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty}c_n x^{n+r+1} - \sum_{n=0}^{\infty}c_n x^{n+r}$$

$$= x^r\left\{\sum_{n=0}^{\infty}8(n+r)(n+r-1)c_n x^{n} + \sum_{n=0}^{\infty}10(n+r)c_n x^{n} + \sum_{n=0}^{\infty}c_n x^{n+1} - \sum_{n=0}^{\infty}c_n x^{n}\right\}$$

Solution

다음의 D.E.이 주어져 있을 때 다음 물음에 단계별로 답하시오.

$$8x^2y'' + 10xy' + (x-1)y = 0$$

3) 가정한 해를 주어진 문제에 대입하여 indicial equation과 recurrence formula를 각각 구하시오.

Continue …

$$= x^r \left\{ \sum_{n=0}^{\infty} 8(n+r)(n+r-1)c_n x^n + \sum_{n=0}^{\infty} 10(n+r)c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+1} - \sum_{n=0}^{\infty} c_n x^n \right\}$$

$$= x^r \left\{ \sum_{n=0}^{\infty} 8(n+r)(n+r-1)c_n x^n + \sum_{n=0}^{\infty} 10(n+r)c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+1} - \sum_{n=0}^{\infty} c_n x^n \right\}$$

$$= x^r \left\{ \left[8(r)(r-1)c_0 x^0 + 10(r)c_0 x^0 - c_0 x^0 \right] + \right.$$

$$\left. \left[\sum_{n=1}^{\infty} 8(n+r)(n+r-1)c_n x^n + \sum_{n=1}^{\infty} 10(n+r)c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+1} - \sum_{n=1}^{\infty} c_n x^n \right] \right\}$$

Solution

다음의 D.E.이 주어져 있을 때 다음 물음에 단계별로 답하시오.

$$8x^2y'' + 10xy' + (x-1)y = 0$$

3) 가정한 해를 주어진 문제에 대입하여 indicial equation과 recurrence formula를 각각 구하시오.

Continue …

$$= x^r \left\{ \left[8r(r-1) + 10r - 1 \right] c_0 x^0 + \right.$$

$$\left. \left[\sum_{k=1}^{\infty} 8(k+r)(k+r-1)c_k x^k + \sum_{k=1}^{\infty} 10(k+r)c_k x^k + \sum_{k=1}^{\infty} c_{k-1} x^k - \sum_{k=1}^{\infty} c_k x^k \right] \right\}$$

$$= x^r \left\{ \left[8r^2 + 2r - 1 \right] c_0 x^0 + \right.$$

$$\left. \left[x^k \sum_{k=1}^{\infty} \left(8(k+r)(k+r-1) + 10(k+r) - 1 \right) c_k + c_{k-1} \right] \right\}$$

Solution

다음의 D.E.이 주어져 있을 때 다음 물음에 단계별로 답하시오.

$$8x^2y'' + 10xy' + (x-1)y = 0$$

3) 가정한 해를 주어진 문제에 대입하여 indicial equation과 recurrence formula를 각각 구하시오.

Continue …

$$= x^r \left\{ \left[(4r-1)(2r+1) \right] c_0 x^0 + \left[x^k \sum_{k=1}^{\infty} \left(8(k+r)(k+r-1) + 10(k+r) - 1 \right) c_k + c_{k-1} \right] \right\}$$

$$when,\ r = \frac{1}{4},$$

$$2k(4k+3)c_k + c_{k-1} = 0$$

$$\therefore c_k = -\frac{1}{2k(4k+3)} c_{k-1}\ ,(k = 1,2,3,..)$$

$$when,\ r = -\frac{1}{2},$$

$$2k(4k-3)c_k + c_{k-1} = 0$$

$$\therefore c_k = -\frac{1}{2k(4k-3)} c_{k-1}\ ,(k = 1,2,3,..)$$

Solution

다음의 D.E.이 주어져 있을 때 다음 물음에 단계별로 답하시오.

$$8x^2y'' + 10xy' + (x-1)y = 0$$

4) equation에서 linearly independent 한 solution을 몇 개 찾을 수 있는지 3)의 indicial equation의 형태와 연관하여 설명하시오.

indicial equation $(4r-1)(2r-1)$ 에서 $r = \frac{1}{4}, r = -\frac{1}{2}$ 은

서로 다르고 차이가 정수가 아니므로 linearly independent 한 두 개의

Solution이 존재한다.

Solution

다음의 D.E.이 주어져 있을 때 다음 물음에 단계별로 답하시오.

$$8x^2y'' + 10xy' + (x-1)y = 0$$

5) x=0 부근에서 General Solution을 구하시오. 단 series를 나타낼 때는 처음 세 번째 항까지 표시하시오.

$$r = \frac{1}{4},$$

$$c_k = -\frac{1}{2k(4k+3)} c_{k-1}, (k = 1,2,3,..)$$

$$c_1 = -\frac{1}{2\cdot 7} c_0 = -\frac{1}{14} c_0$$

$$c_2 = -\frac{1}{4\cdot 11} c_1 = \frac{1}{4\cdot 11}\cdot\frac{1}{2\cdot 7} c_0 = \frac{1}{616} c_0$$

$$y = x^r \sum_{n=0}^{\infty} c_n x^n$$

$$y_1 = x^{1/4}\left(c_0 x^0 + c_1 x^1 + c_2 x^2 + \cdots\right)$$

$$= c_0 x^{1/4}\left(x^0 - \frac{1}{14} x^1 + \frac{1}{616} x^2 + \cdots\right)$$

Solution

다음의 D.E.이 주어져 있을 때 다음 물음에 단계별로 답하시오.

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$$r = -\frac{1}{2},$$

$$c_k = -\frac{1}{2k(4k-3)} c_{k-1}, (k = 1,2,3,..)$$

$$c_1 = -\frac{1}{2\cdot 1} c_0 = -\frac{1}{2} c_0$$

$$c_2 = -\frac{1}{4\cdot 5} c_1 = \frac{1}{4\cdot 5}\cdot\frac{1}{2\cdot 1} c_0 = \frac{1}{40} c_0$$

$$y = x^r \sum_{n=0}^{\infty} c_n x^n$$

$$y_2 = x^{-1/2}\left(c_0 x^0 + c_1 x^1 + c_2 x^2 + \cdots\right)$$

$$= c_0 x^{-1/2}\left(x^0 - \frac{1}{2}x^1 + \frac{1}{40}x^2 + \cdots\right)$$

Solution

다음의 D.E.이 주어져 있을 때 다음 물음에 단계별로 답하시오.

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$$\therefore y = C_0 y_1 + C_1 y_2$$

$$y_1 = x^{1/4}\left(x^0 - \frac{1}{14}x^1 + \frac{1}{616}x^2 + \cdots\right)$$

$$y_2 = x^{-1/2}\left(x^0 - \frac{1}{2}x^1 + \frac{1}{40}x^2 + \cdots\right)$$

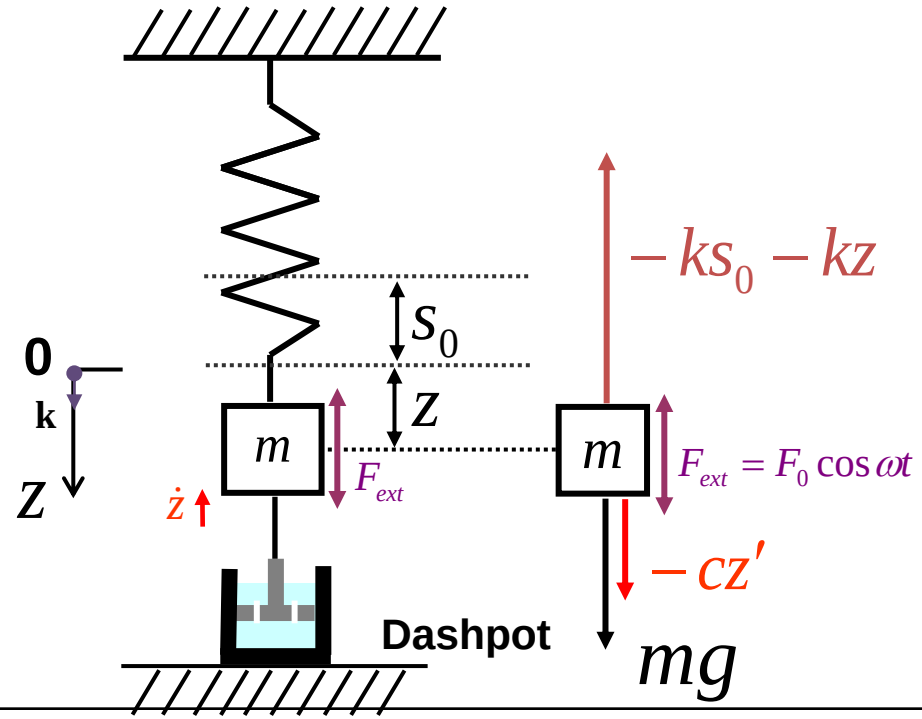
CHAPTER 5. LAPLACE TRANSFORMS

1) COMPARISON

1) Homogeneous & Nonhomogeneous Solutions.
2) Transient & Steady-state Solutions
3) Zero-Input & Zero_initial Solutions

Linear Model
Spring/Mass Systems: Driven Motion

$$z = z(t)\ ,\ z'' = \frac{d^2 z}{dt^2}$$



$$mz'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F}_0 \cos \omega t$$
$$= -kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F}_0 \cos \omega t$$

$$mz'' + cz' + kz = \mathbf{F}_0 \cos \omega t$$

$$mz'' + cz' + kz = F_0 \cos \omega t$$ (z component)

- Solution of 2nd O.E.D.

Linear Model (Linear Equation)

$$mz'' + cz' + kz = F_0 \cos \omega t$$

General Solution -Nonghomogeneous

$$z = z_h + z_p$$

-Superposition

$mz'' + cz' + kz = 0$ → z_h : **General Solution - Homogeneous**

z_p : **Particular Solution**

$$z_h = c_1 z_1 + c_2 z_2$$

-Linear Combination
-Superposition

Another Superposition

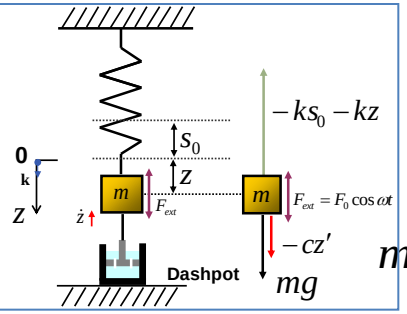
-Fundamental set of solutions (Basis): z_1, z_2

Linearly Independent

Try: $z = e^{\lambda t}$

Linear Model
Spring/Mass Systems: Driven Motion

$$z = z(t)\ ,\ z'' = \frac{d^2 z}{dt^2}$$



Linear Model (Linear Equation)

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-Linear Combination
-Superposition

Another Superposition

-Fundamental set of solutions (Basis)

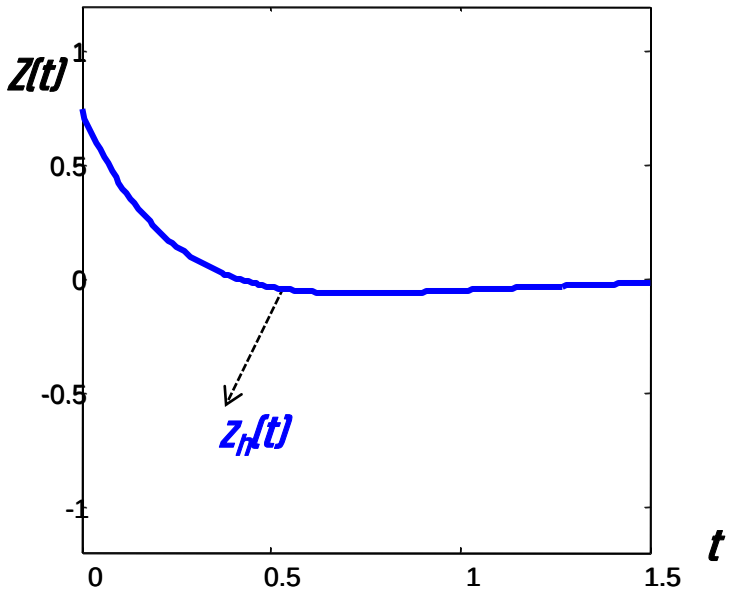
z_1, z_2

Try: $z = e^{\lambda t}$

Linearly Independent

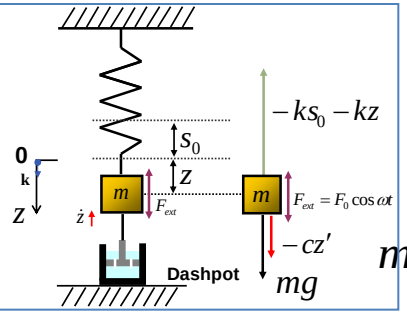
$$0.2z'' + 1.2z' + 2z = 5\cos 4t$$

$$z(0) = \frac{1}{2},\ z'(0) = 0$$



Linear Model
Spring/Mass Systems: Driven Motion

$$z = z(t)\ ,\ z'' = \frac{d^2 z}{dt^2}$$



Linear Model (Linear Equation)

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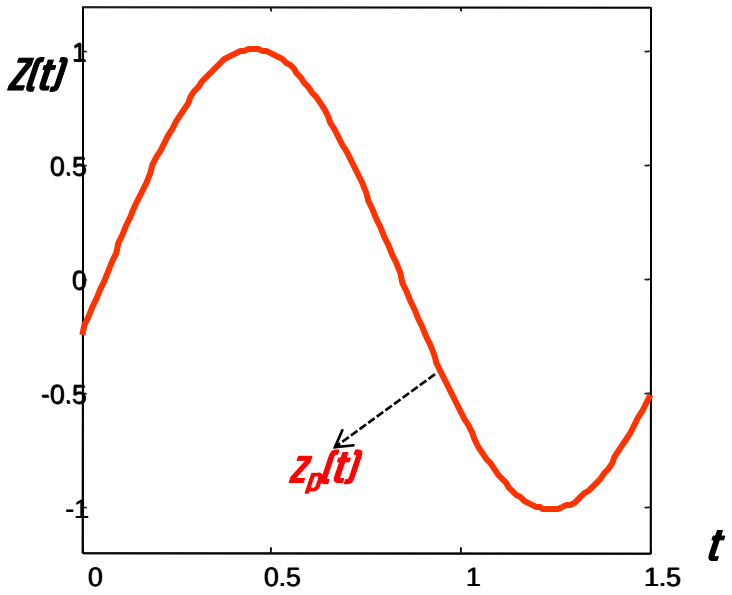

Linear Model
Spring/Mass Systems: Driven Motion

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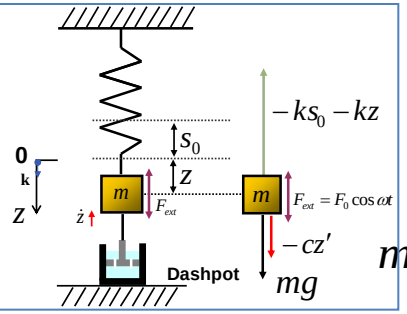

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-Linear Combination
-Superposition

Another Superposition

-Fundamental set of solutions (Basis)

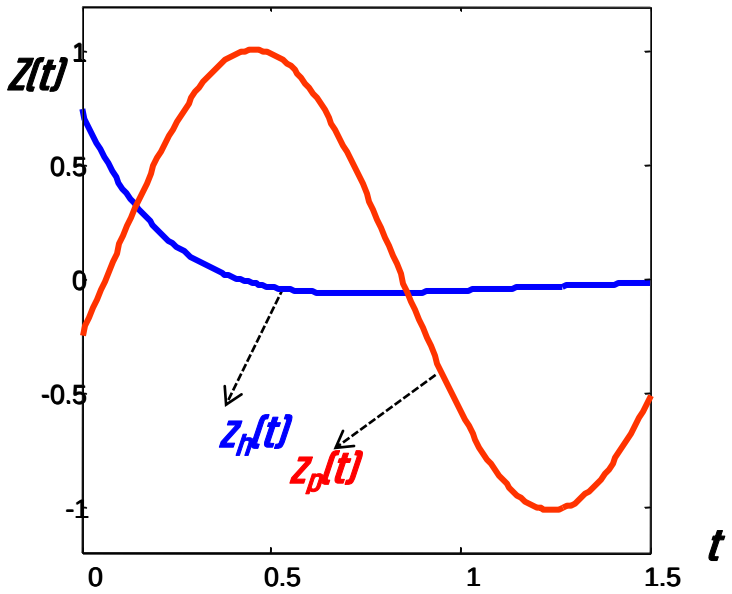
z_1, z_2

Try: $z = e^{\lambda t}$

Linearly Independent

$$0.2z'' + 1.2z' + 2z = 5\cos 4t$$

$$z(0) = \frac{1}{2},\ z'(0) = 0$$



Linear Model
Spring/Mass Systems: Driven Motion

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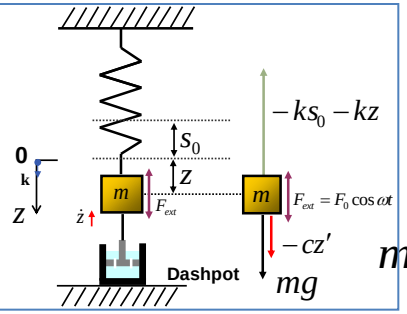

Linear Model (Linear Equation)

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-Linear Combination
-Superposition

Another Superposition

-Fundamental set of solutions (Basis)

z_1, z_2 : Linearly Independent

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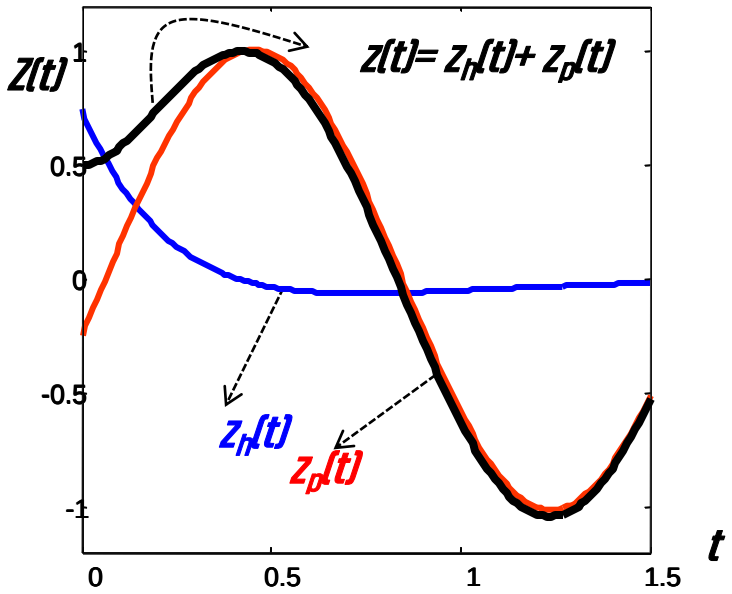
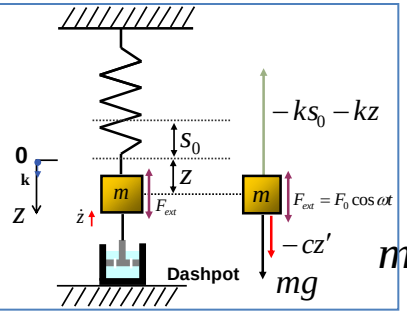


Linear Model
Spring/Mass Systems: Driven Motion

$$z = z(t)\ ,\ z'' = \frac{d^2 z}{dt^2}$$



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Linear Model (Linear Equation)

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$$z = z_h + z_p$$

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$$z_h = c_1 z_1 + c_2 z_2$$

-Linear Combination
-Superposition

Another Superposition

-Fundamental set of solutions (Basis)

z_1, z_2 — Linearly Independent

Try: $z = e^{\lambda t}$

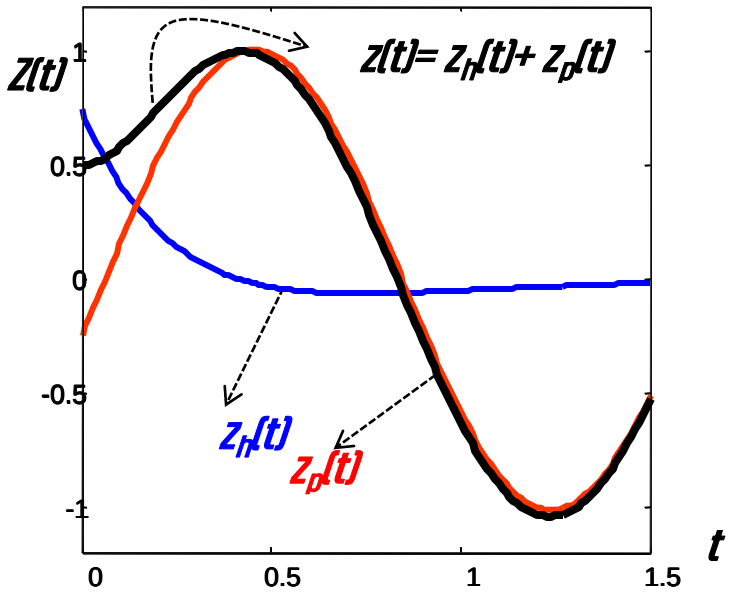
$$mz'' + cz' + kz = F_0 \cos \omega t$$

Transient and Steady-State Terms

When *F(t)* is periodic function, general solution have

$z_h(t)$ nonperiodic function

$z_p(t)$ and periodic function



$z_h(t)$: transient solution

$z_p(t)$: steady-state solution

Comparison

Method to find the solution of the 2nd-order O.D.E

$mz'' + cz' + kz = u(t)$

$z(t) = e^{\lambda t}$ → **Solution** → $z_{\text{homogeneous}}, z_{\text{particular}}$

Laplace Transform $\mathcal{L}\{z(t)\}$ → **Solution** → $z_{\text{zero-input}}, z_{\text{zero-state}}$

Comparison

Method to find the solution of the 2nd-order O.D.E

$mz'' + cz' + kz = u(t)$

$z(t) = e^{\lambda t}$ → **Solution** → $z_{\text{transient}}, z_{\text{steady}}$

Laplace Transform → $\mathcal{L}\{z(t)\}$ → **Solution** → $z_{\text{zero-input}}, z_{\text{zero-state}}$

$$z_{\text{general}} = z_{\text{transient}} + z_{\text{steady}}$$

$$\| \qquad \| \qquad \|$$

$$z_{\text{zero-input}} = z_{\text{transient of zero-input}} + z_{\text{steady of zero-input}}$$

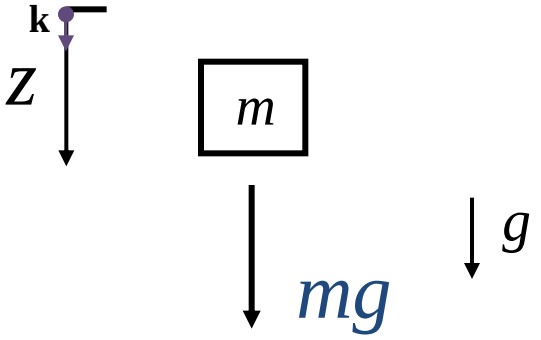
$$+ \qquad + \qquad +$$

$$z_{\text{zero-state}} = z_{\text{transient of zero-state}} + z_{\text{steady of zero-state}}$$

Example

Action/Reaction, Equilibrium

①



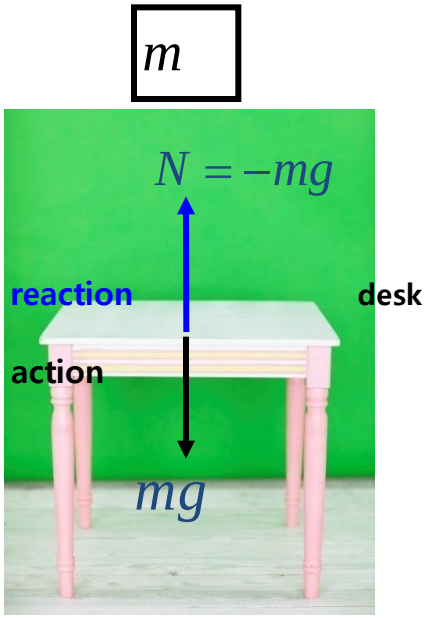
:Gravity force

By Newton's second law,

$$m\mathbf{z}'' = \mathbf{F}$$

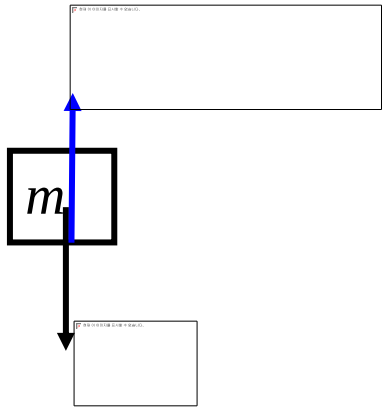
$$= mg\mathbf{k}$$

②



③

Free body diagram



By Newton's second law,

$$m\mathbf{z}'' = \mathbf{F}$$

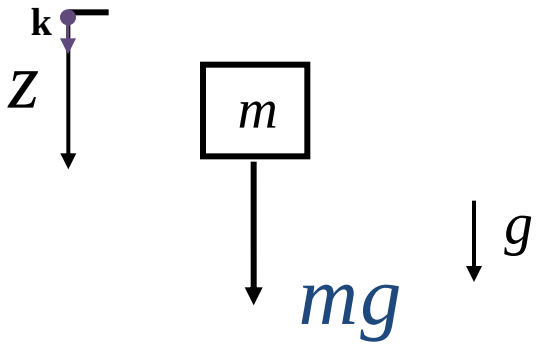
$$= 0$$

: Static equilibrium

Spring/Mass Systems: Driven Motion

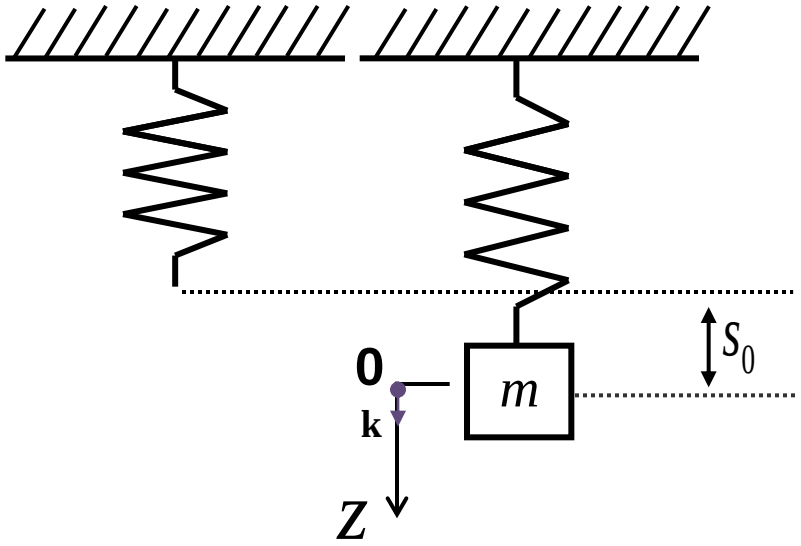
$z = z(t)\,, \quad z'' = \dfrac{d^2 z}{dt^2}$

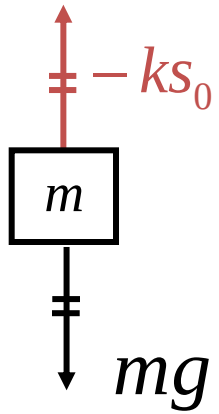
①



:Gravity force

②





By Newton's 2nd law,

$$m\mathbf{z}'' = \mathbf{F}$$
$$= mg\mathbf{k}$$

Nonlinearity of spring

$$\mathbf{F}(z) = -k\mathbf{z} - k_1\mathbf{z}^3$$

linearize

Hooke's law

$$F \propto z$$
$$\mathbf{F}_{spring} = -k\mathbf{z}$$

k : spring constant

$$m\mathbf{z}'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k}$$
$$= 0 \quad (\because z'' = 0)$$

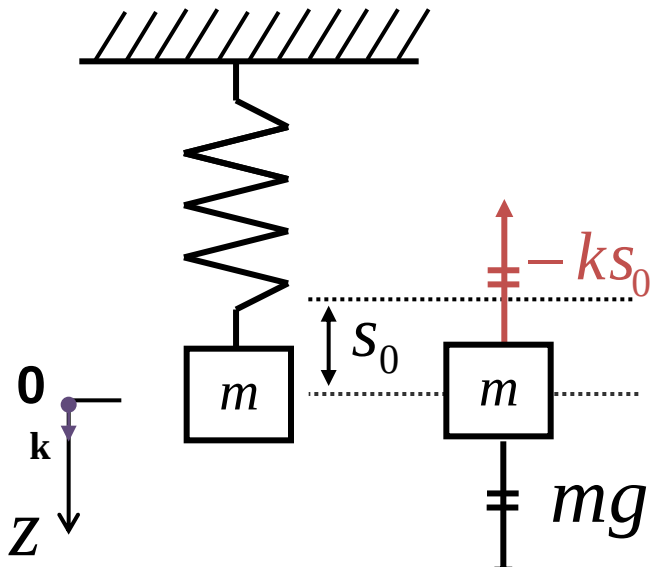
: static equilibrium

○ **opposite to the direction of displacement**

Spring/Mass Systems: Driven Motion

$$z = z(t), \quad z'' = \frac{d^2 z}{dt^2}$$

②



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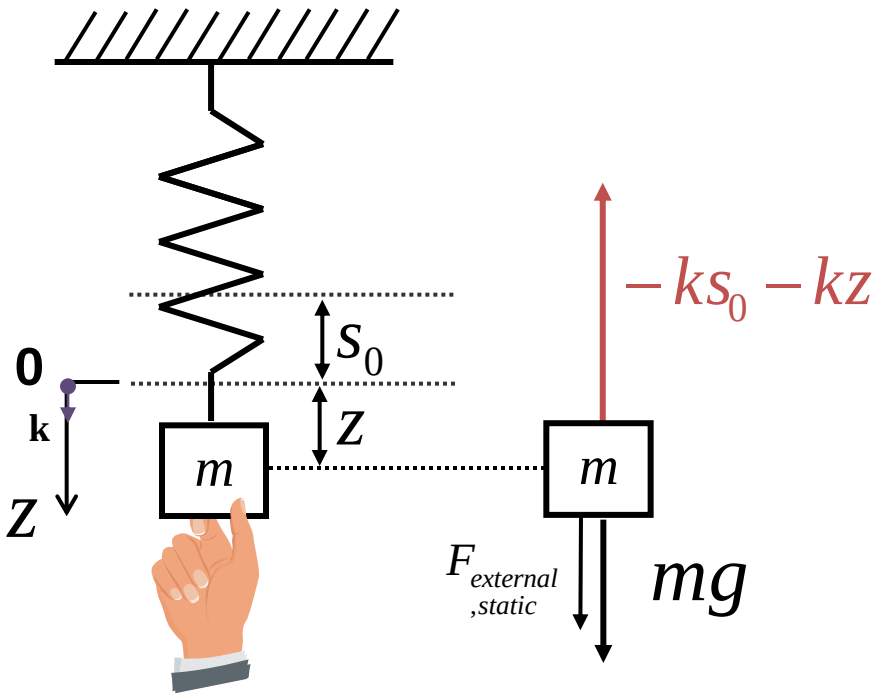

$$m\mathbf{z}'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} + \mathbf{F}_{external,static}$$
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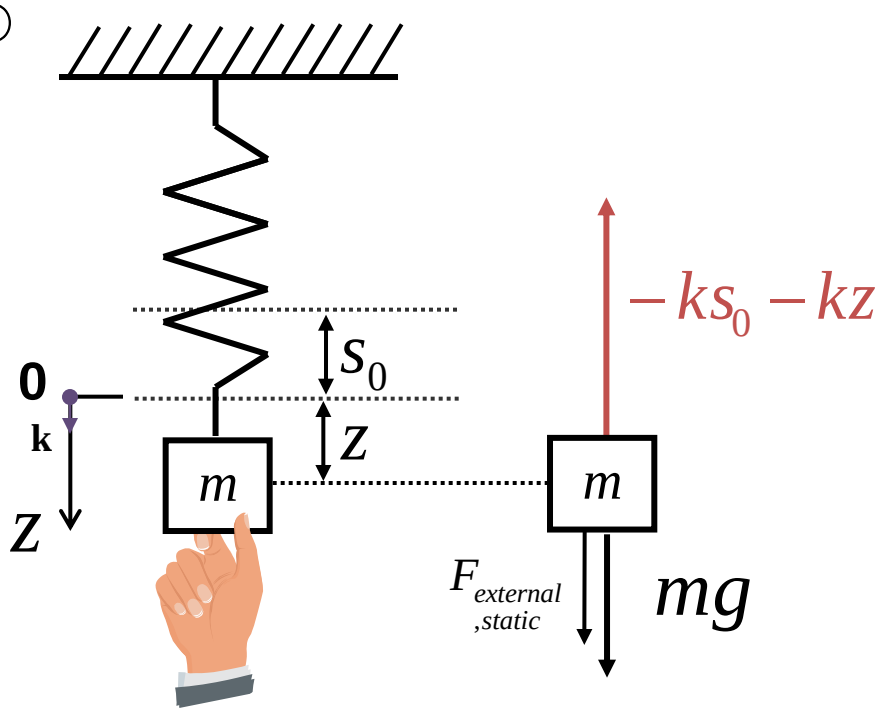
$$= 0 \quad (\because z'' = 0)$$

Spring/Mass Systems: Driven Motion

$$z = z(t), \quad z'' = \frac{d^2 z}{dt^2}$$

③



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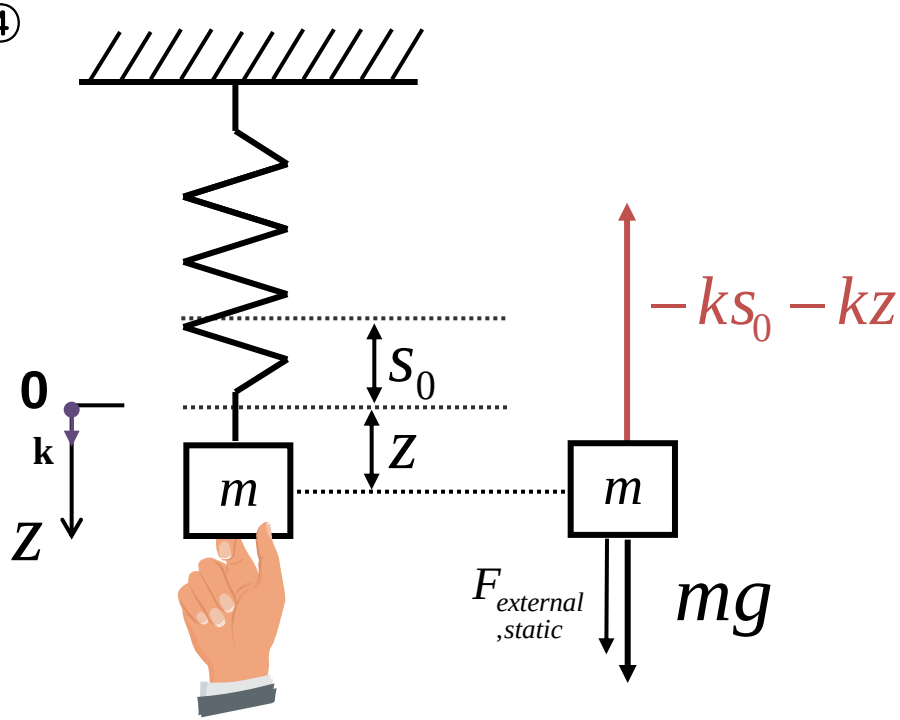
④



$$m\mathbf{z}'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} + \mathbf{F}_{external,static}$$
$$= -kz\mathbf{k} + \mathbf{F}_{external,static}$$

restoring force

Physical Phenomenon

Mathematical Equation

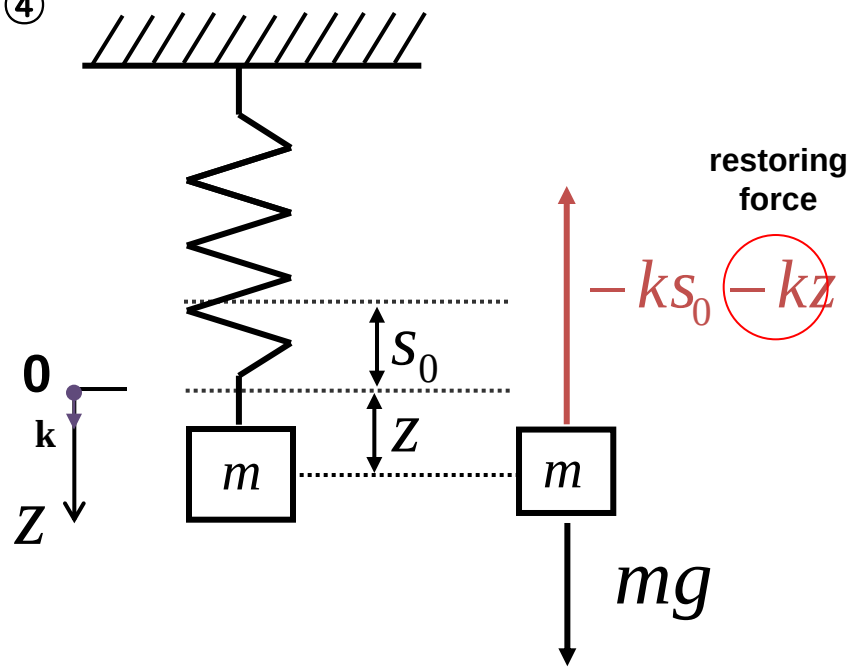
$$m\mathbf{z}'' + k\mathbf{z} = 0$$

Oscillation by the restoring force

Spring/Mass Systems: Driven Motion

$$z = z(t), \quad z'' = \frac{d^2 z}{dt^2}$$

④



$$\begin{aligned} m\mathbf{z}'' &= \mathbf{F} \\ &= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} \\ &= -kz\mathbf{k} \end{aligned}$$

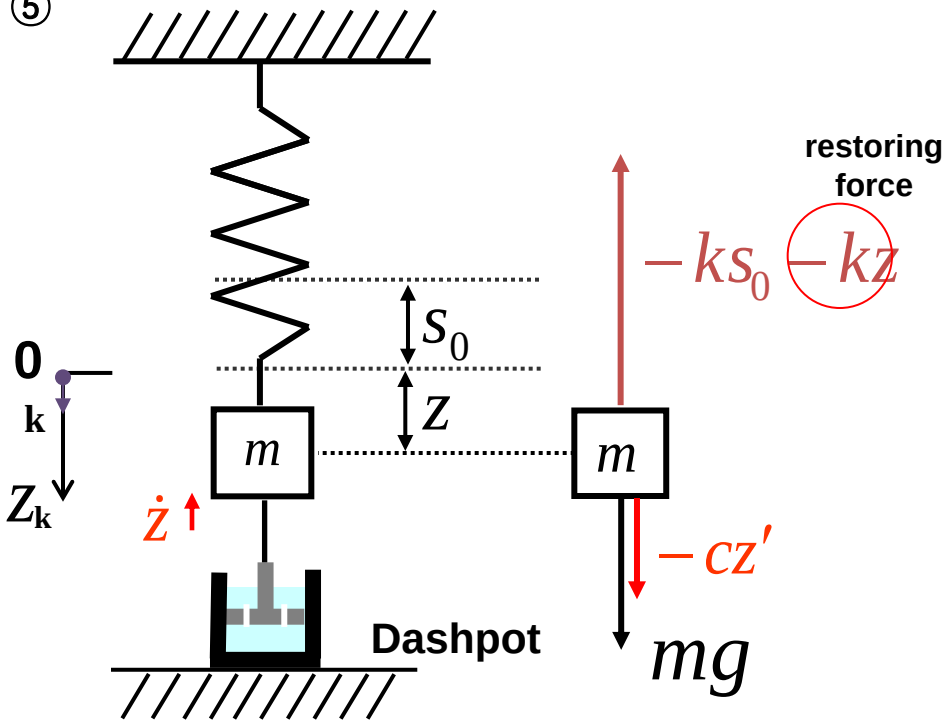
$$mz'' + kz = 0$$

oscillation by the restoring force

Physical Phenomenon

Mathematical Equation

⑤



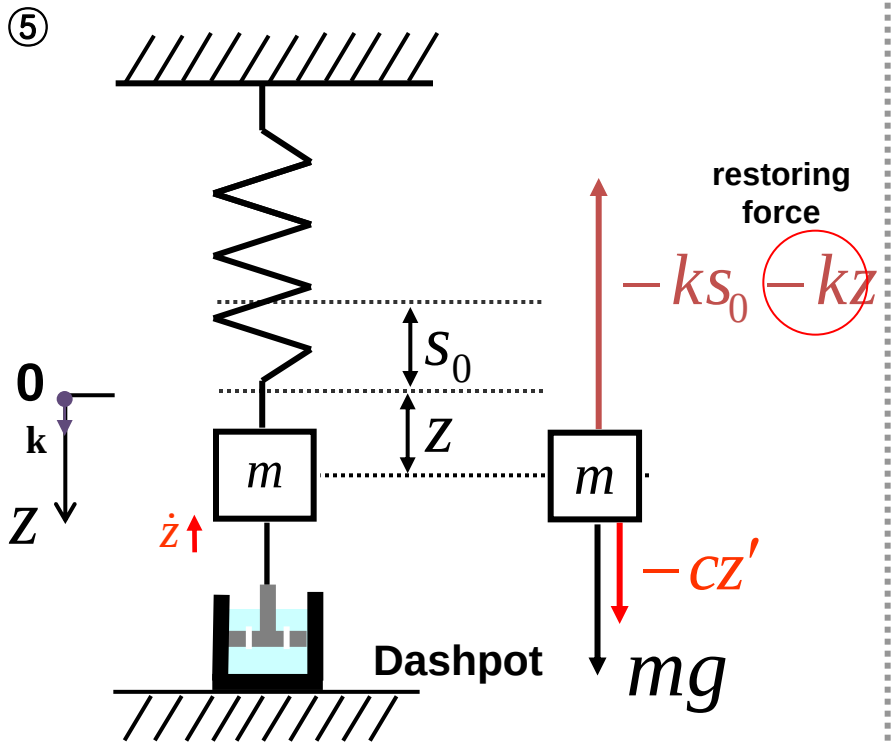
$$\begin{aligned} m\mathbf{z}'' &= \mathbf{F} \\ &= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - cz'\mathbf{k} \\ &= -kz\mathbf{k} - cz'\mathbf{k} \end{aligned}$$

$$mz'' + cz' + kz = 0$$

Spring/Mass Systems: Driven Motion

$$z = z(t), \quad z'' = \frac{d^2 z}{dt^2}$$

⑤

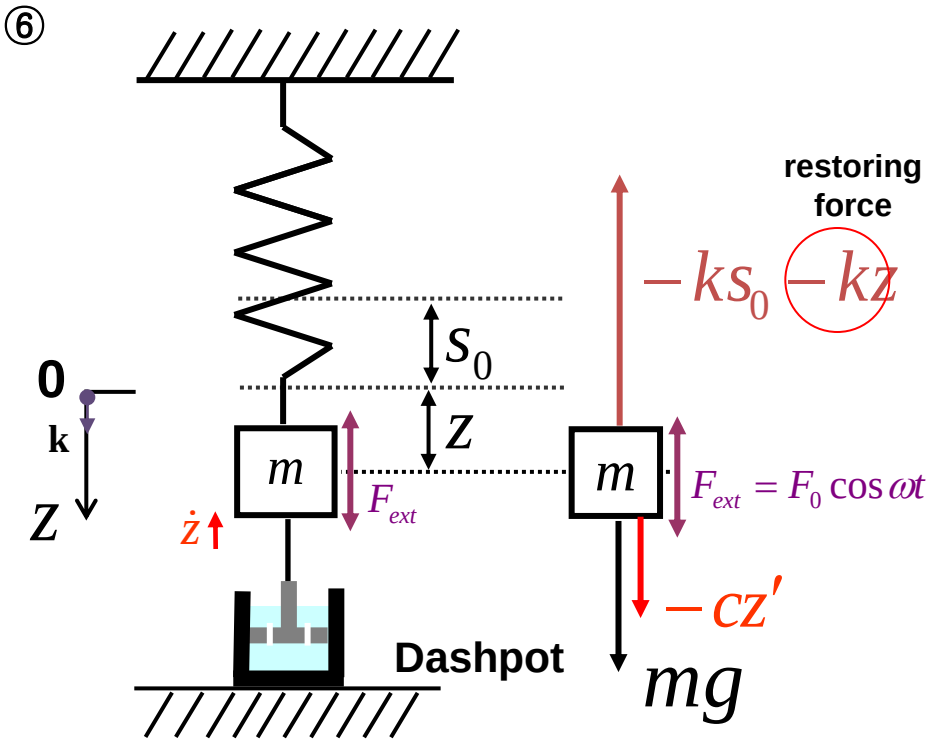


$$mz'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} \quad -cz'\mathbf{k}$$
$$= -kz\mathbf{k} \quad -cz'\mathbf{k}$$

Physical Phenomenon
Mathematical Equation

$$mz'' + cz' + kz = 0$$

⑥



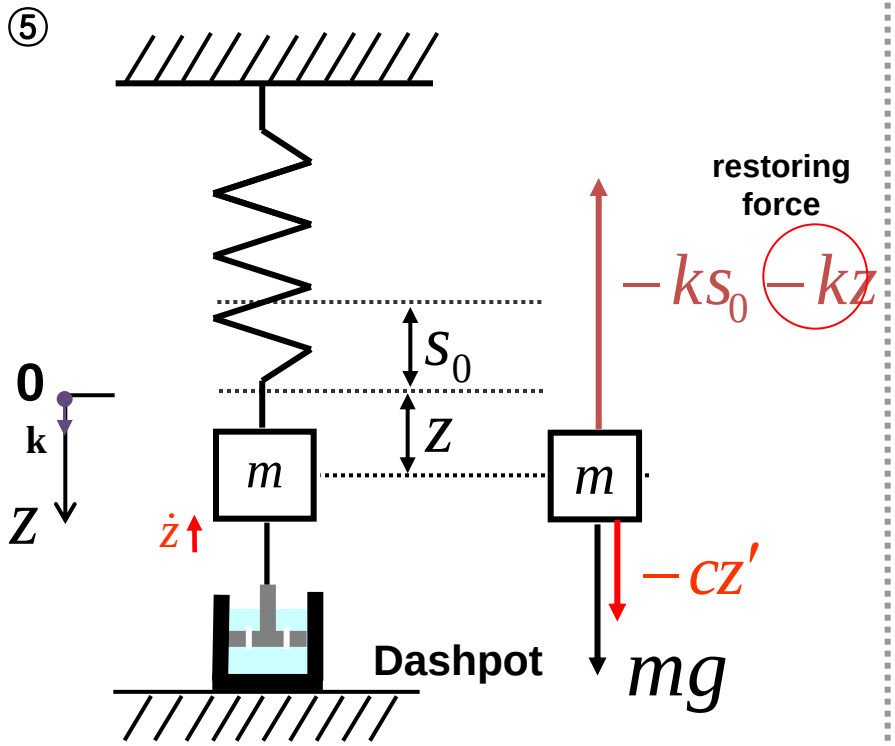
$$mz'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} \quad -cz'\mathbf{k} + \mathbf{F}_0 \cos \omega t$$
$$= -kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F}_0 \cos \omega t$$

$$mz'' + cz' + kz = \mathbf{F}_0 \cos \omega t$$

Spring/Mass Systems: Driven Motion

$$z = z(t), \quad z'' = \frac{d^2 z}{dt^2}$$

⑤

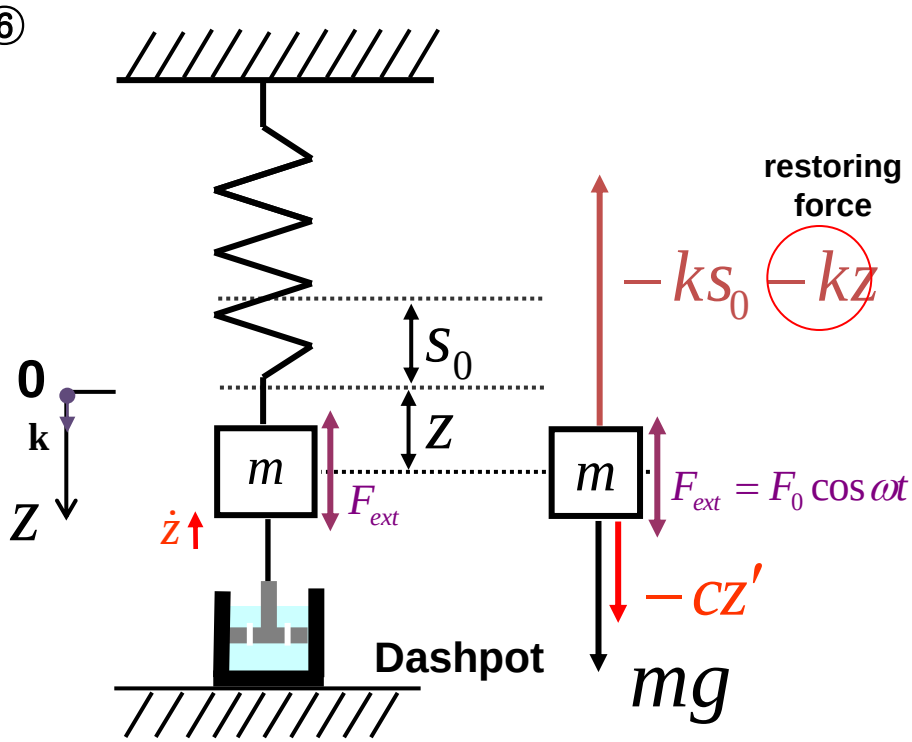


$$\begin{aligned} m\mathbf{z}'' &= \mathbf{F} \\ &= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - cz'\mathbf{k} \\ &= -kz\mathbf{k} - cz'\mathbf{k} \end{aligned}$$

Physical Phenomenon
Mathematical Equation

$$mz'' + cz' + kz = 0$$

⑥



$$\begin{aligned} m\mathbf{z}'' &= \mathbf{F} \\ &= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F}_0 \cos \omega t \\ &= -kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F}_0 \cos \omega t \end{aligned}$$

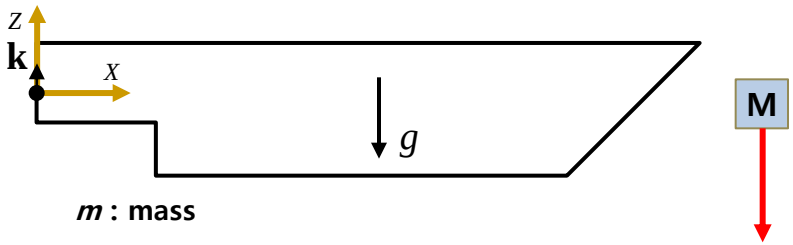
$$mz'' + cz' + kz = \mathbf{F}_0 \cos \omega t$$

Nonlinearity

Nonlinearity of the nature → Nonlinear Mathematical Model → Numerical Method

Nonlinear Mathematical Model → (Linearization) → Linear Mathematical Model → Analytic Solution

Ex) Heave Motion of a Ship – step 1

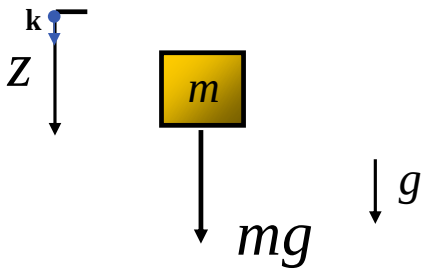


m **: mass**

$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= \mathbf{F}_{gravity}$$
$$= -mg\mathbf{k}$$

✓ Mass-Spring-Damper system

①



By Newton's 2nd law,

$$m\mathbf{z}'' = \mathbf{F}$$
$$= mg\mathbf{k}$$

Nonlinearity

Nonlinearity of the nature → Nonlinear Mathematical Model → Numerical Method

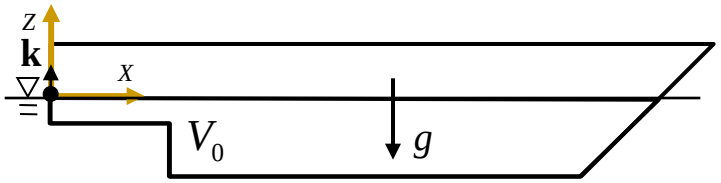
Linearization

Linear Mathematical Model → Analytic Solution

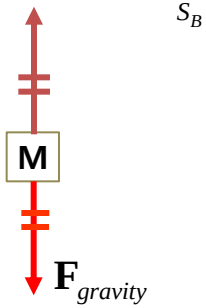
Ex) Heave Motion of a Ship – step 2

$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS = \rho g V_0 \mathbf{k}$$

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0 \mathbf{k}$

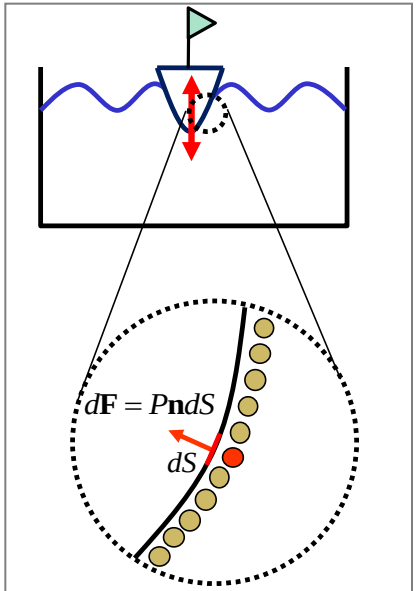


m : mass
ρ : density of sea water
V_0 : submerged volume
S_B : submerged surface area



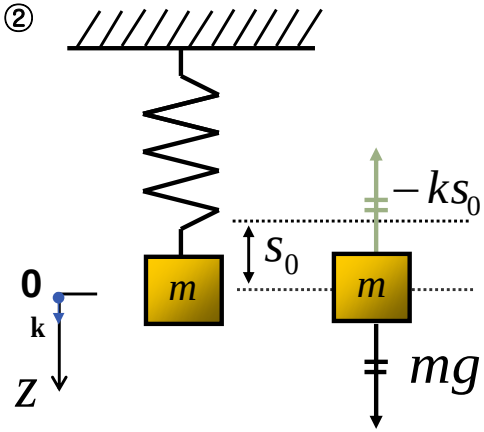
$$= -mg\mathbf{k}$$

$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= \mathbf{F}_{gravity} + \mathbf{F}_{static}$$
$$= -mg\mathbf{k} + \rho g V_0 \mathbf{k}$$
$$= 0 \ (\because \ddot{z} = 0) \quad \text{: static equilibrium}$$



dS : infinitesimal submerged surface area
$d\mathbf{F}$: force exerted by the infinitesimal fluid element on dS
$\mathbf{n}$: normal vector of dS

✓ Mass-Spring-Damper system



$$m\mathbf{z}'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k}$$
$$= 0 \ (\because z'' = 0)$$

: static equilibrium

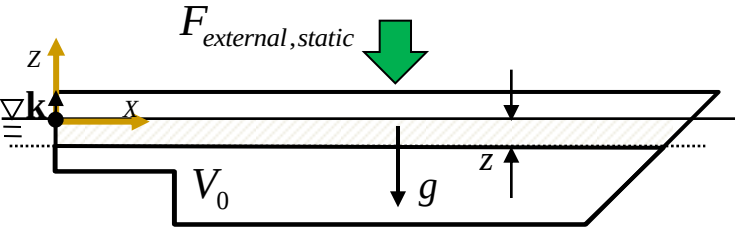
Nonlinearity

Nonlinearity of the nature → Nonlinear Mathematical Model → Numerical Method

Linearization

Linear Mathematical Model → Analytic Solution

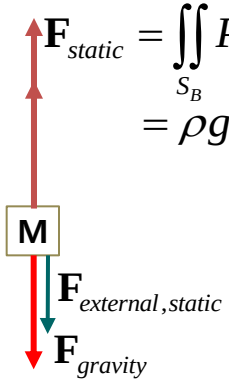
Ex) Heave Motion of a Ship – step 3



m : mass ρ : density of sea water
V_0 : submerged volume
S_B : submerged surface area
Awp : waterplane area

$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS = \rho g V_0 \mathbf{k} + \mathbf{F}_{addtional\,bouyancy}$$

additional buoyancy caused by additional displacement z



$$= -mg\mathbf{k}$$

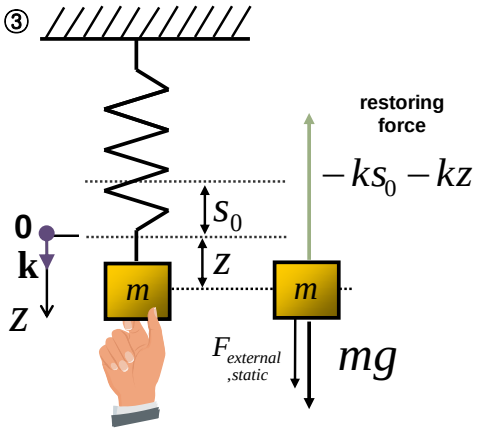
if, z is small

$$\mathbf{F}_{addtional\,bouyancy} = -\rho g A_{WP}\mathbf{z} = -k\mathbf{z}, \quad k = \rho g A_{WP}$$

$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= \mathbf{F}_{gravity} + \mathbf{F}_{static} + \mathbf{F}_{external,static}$$
$$= -mg\mathbf{k} + \rho g V_0 \mathbf{k} - \rho g A_{wp}\mathbf{z} + \mathbf{F}_{external,static}$$
$$= -\rho g A_{wp}\mathbf{z} + \mathbf{F}_{external,static}$$
$$= -k\mathbf{z} + \mathbf{F}_{external,static}$$
$$= 0 \ (\because \ddot{z} = 0)$$

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0 \mathbf{k}$

✓ Mass-Spring-Damper system



$$m\mathbf{z}'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} + \mathbf{F}_{external,static}$$
$$= -kz\mathbf{k} + \mathbf{F}_{external,static}$$
$$= 0 \quad (\because z'' = 0)$$

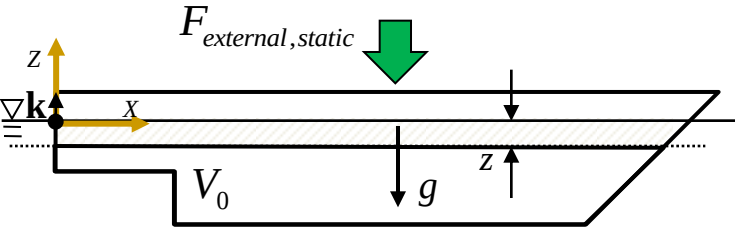
Nonlinearity

Nonlinearity of the nature → Nonlinear Mathematical Model → Numerical Method

Linearization

Linear Mathematical Model → Analytic Solution

Ex) Heave Motion of a Ship – step 4



m : mass ρ : density of sea water
V_0 : submerged volume
S_B : submerged surface area
Awp : waterplane area

$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS = \rho g V_0 \mathbf{k} + \mathbf{F}_{addtional\,bouyancy}$$

additional buoyancy caused by additional displacement z

M

$\mathbf{F}_{external,static}$

$\mathbf{F}_{gravity}$

$$= -mg\mathbf{k}$$

if, z is small

$$\mathbf{F}_{addtional\,bouyancy} = -\rho g A_{WP}\mathbf{z} = -k\mathbf{z}, \quad k = \rho g A_{WP}$$

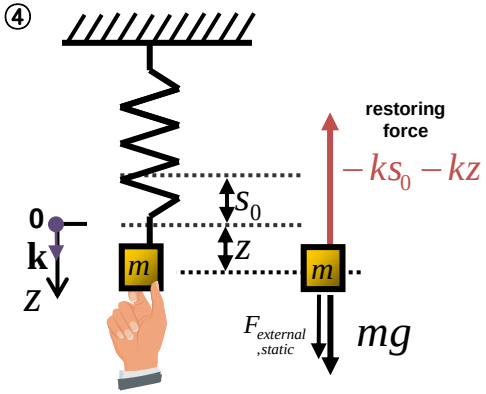
$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= \mathbf{F}_{gravity} + \mathbf{F}_{static} + \mathbf{F}_{external,static}$$
$$= -mg\mathbf{k} + \rho g V_0 \mathbf{k} - \rho g A_{wp}\mathbf{z} + \mathbf{F}_{external,static}$$
$$= -\rho g A_{wp}\mathbf{z} + \mathbf{F}_{external,static}$$
$$= -k\mathbf{z} + \mathbf{F}_{external,static}, \quad k = \rho g A_{WP}$$
$$= 0 \ (\because \ddot{z} = 0)$$

Linearized Restoring Force

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0 \mathbf{k}$

✓ Mass-Spring-Damper system

④



$$m\mathbf{z}'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} + \mathbf{F}_{external,static}$$
$$= -kz\mathbf{k} + \mathbf{F}_{external,static}$$

$$m\mathbf{z}'' + k\mathbf{z} = 0$$

Oscillation by the restoring force

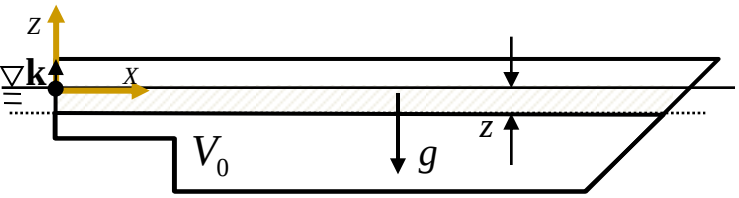
Nonlinearity

Nonlinearity of the nature → Nonlinear Mathematical Model → Numerical Method

Linearization

Linear Mathematical Model → Analytic Solution

Ex) Heave Motion of a Ship – step 4



m : mass ρ : density of sea water
V_0 : submerged volume
S_B : submerged surface area
Awp : waterplane area

$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS = \rho g V_0\mathbf{k} - \rho g A_{WP}\mathbf{z} = \rho g V_0\mathbf{k} - k\mathbf{z}$$

M

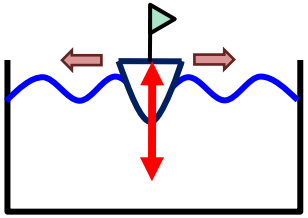
$$\mathbf{F}_{gravity} = -mg\mathbf{k}$$

$$m\ddot{\mathbf{z}} = \mathbf{F} = \mathbf{F}_{gravity} + \mathbf{F}_{static} = -mg\mathbf{k} + \rho g V_0\mathbf{k} - \rho g A_{wp}\mathbf{z} = -\rho g A_{wp}\mathbf{z} = -k\mathbf{z}$$

Ship will oscillate forever?

Energy is dissipated by radiation wave →

정수 중 선박의 강제 운동에 의해 발생한 힘



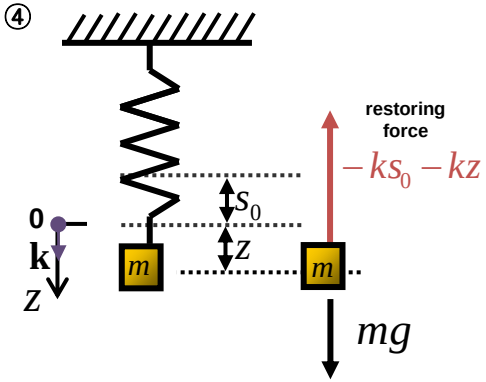
Radiation Force

$$\mathbf{F}_{radiation} = \iint_{S_B} P_{radiation}\mathbf{n}dS$$

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0\mathbf{k}$

✓ Mass-Spring-Damper system

④



$$m\mathbf{z}'' = \mathbf{F} = mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} = -kz\mathbf{k}$$

$$m\mathbf{z}'' + k\mathbf{z} = 0$$

Oscillation by the restoring force

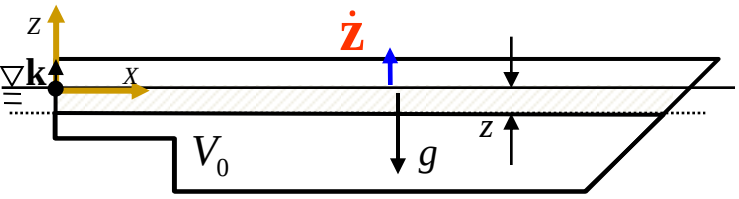
Nonlinearity

Nonlinearity of the nature → Nonlinear Mathematical Model → Numerical Method

Linearization

Linear Mathematical Model → Analytic Solution

Ex) Heave Motion of a Ship – step 5



m : mass ρ : density of sea water
V_0 : submerged volume
S_B : submerged surface area
Awp : waterplane area

$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS = \rho g V_0\mathbf{k} - \rho g A_{WP}\mathbf{z} = \rho g V_0\mathbf{k} - k\mathbf{z}$$

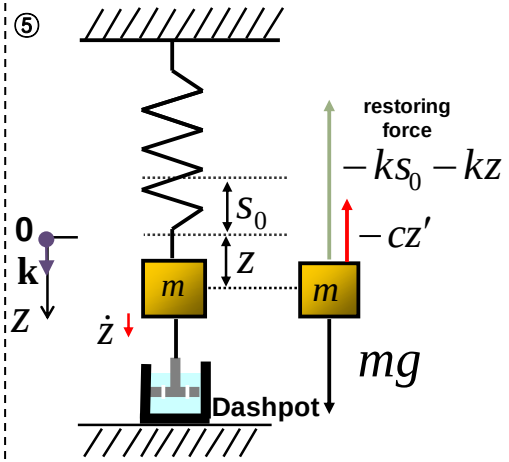
$$\mathbf{F}_{radiation} = -c\dot{\mathbf{z}}$$

$$\mathbf{F}_{gravity} = -mg\mathbf{k}$$

$$m\ddot{\mathbf{z}} = \mathbf{F} = \mathbf{F}_{gravity} + \mathbf{F}_{static} + \mathbf{F}_{radiation} = -mg\mathbf{k} + \rho g V_0\mathbf{k} - \rho g A_{wp}\mathbf{z} - c\dot{\mathbf{z}} = -\rho g A_{wp}\mathbf{z} - c\dot{\mathbf{z}} = -k\mathbf{z} - c\dot{\mathbf{z}}$$

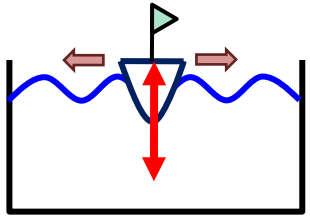
✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0\mathbf{k}$

✓ Mass-Spring-Damper system



$$mz'' = \mathbf{F} = mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - cz'\mathbf{k} = -kz\mathbf{k} - cz'\mathbf{k}$$

정수 중 선박의 강제 운동에 의해 발생한 힘



Radiation Force

$$\mathbf{F}_{radiation} = \iint_{S_B} P_{radiation}\mathbf{n}dS = -c\dot{\mathbf{z}}$$

opposite to velocity

c : damping coefficient

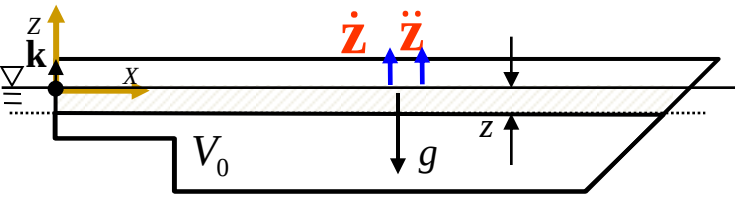
Nonlinearity

Nonlinearity of the nature → Nonlinear Mathematical Model → Numerical Method

Linearization

Linear Mathematical Model → Analytic Solution

Ex) Heave Motion of a Ship – step 5



m : mass ρ : density of sea water
V_0 : submerged volume
S_B : submerged surface area
A_{WP} : waterplane area

$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS = \rho g V_0\mathbf{k} - \rho g A_{WP}\mathbf{z} = \rho g V_0\mathbf{k} - k\mathbf{z}$$

M

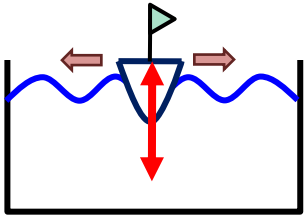
$$\mathbf{F}_{radiation} = -c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}}$$

$$\mathbf{F}_{gravity} = -mg\mathbf{k}$$

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0\mathbf{k}$

$$m\ddot{\mathbf{z}} = \mathbf{F} = \mathbf{F}_{gravity} + \mathbf{F}_{static} + \mathbf{F}_{radiation}$$
$$= -mg\mathbf{k} + \rho g V_0\mathbf{k} - \rho g A_{wp}\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}}$$
$$= -\rho g A_{wp}\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}}$$
$$= -k\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}}$$

정수 중 선박의 강제 운동에 의해 발생한 힘



Radiation Force

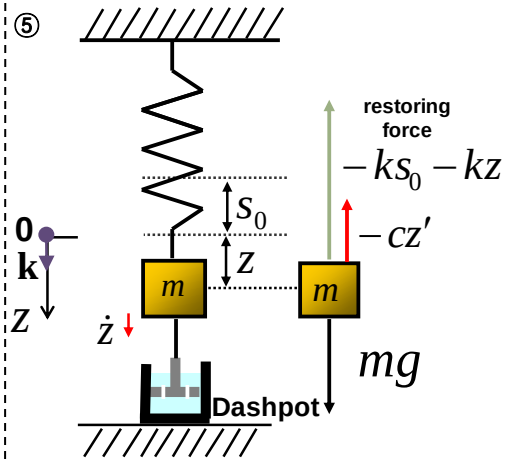
$$\mathbf{F}_{radiation} = \iint_{S_B} P_{radiation}\mathbf{n}dS = -c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}}$$

opposite to velocity

opposite to acceleration

c : damping coefficient
m_a : added mass

✓ Mass-Spring-Damper system



$$m\mathbf{z}'' = \mathbf{F} = mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - cz'\mathbf{k} = -kz\mathbf{k} - cz'\mathbf{k}$$

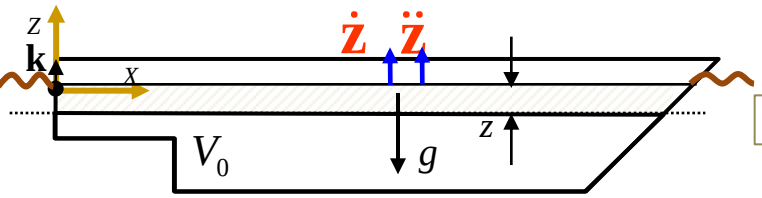
Nonlinearity

Nonlinearity of the nature → Nonlinear Mathematical Model → Numerical Method

Linearization

Linear Mathematical Model → Analytic Solution

Ex) Heave Motion of a Ship – step 6



m : mass ρ : density of sea water
V_0 : submerged volume
S_B : submerged surface area
Awp : waterplane area

$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS = \rho g V_0 \mathbf{k} - \rho g A_{WP}\mathbf{z} = \rho g V_0 \mathbf{k} - k\mathbf{z}$$

$$\mathbf{F}_{radiation} = -c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}}$$

$$\mathbf{F}_{gravity} = -mg\mathbf{k}$$

M

$$m\ddot{\mathbf{z}} = \mathbf{F}$$

$$= \mathbf{F}_{gravity} + \mathbf{F}_{static} + \mathbf{F}_{radiation} + \mathbf{F}_{exciting}$$

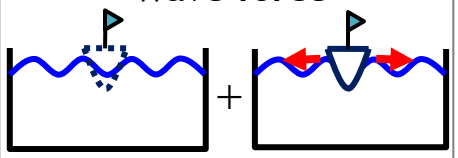
$$= -mg\mathbf{k} + \rho g V_0 \mathbf{k} - \rho g A_{wp}\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}} + \mathbf{F}_{exciting}$$

$$= -\rho g A_{wp}\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}} + \mathbf{F}_{exciting}$$

$$= -k\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}} + \mathbf{F}_{exciting}$$

c : damping coefficient
m_a : added mass

Wave force

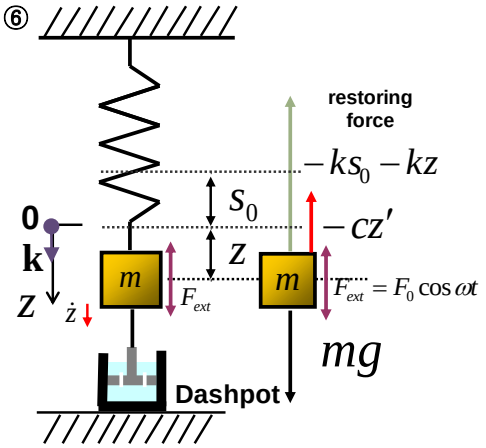


Froude-Kriloff Force Diffraction Force

$$\mathbf{F}_{wave\ exciting} = \iint_{S_B} P_{wave\ exciting}\mathbf{n}dS \quad \left(= \mathbf{F}_{exciting}\right)$$

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0 \mathbf{k}$

✓ Mass-Spring-Damper system



$$m\mathbf{z}'' = \mathbf{F}$$

$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F}_0 \cos \omega t$$

$$= -kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F}_0 \cos \omega t$$

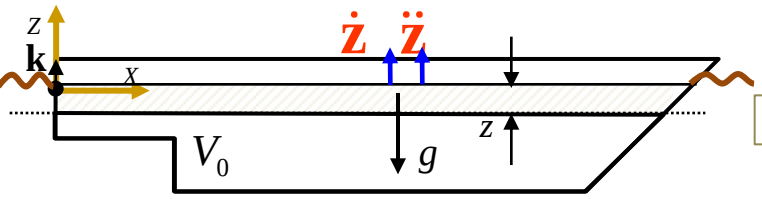
Nonlinearity

Nonlinearity of the nature → Nonlinear Mathematical Model → Numerical Method

Linearization

Linear Mathematical Model → Analytic Solution

Ex) Heave Motion of a Ship – step 6



m : mass
ρ : density of sea water
V_0 : submerged volume
S_B : submerged surface area
Awp : waterplane area

$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS = \rho g V_0 \mathbf{k} - \rho g A_{WP}\mathbf{z} = \rho g V_0 \mathbf{k} - k\mathbf{z}$$

$$\mathbf{F}_{radiation} = -c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}}$$

$$\mathbf{F}_{gravity} = -mg\mathbf{k}$$

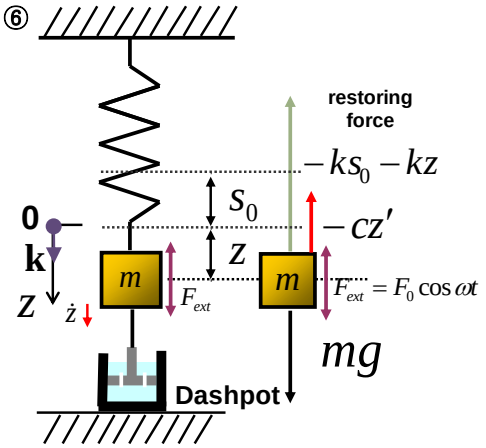
✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0 \mathbf{k}$

$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= \mathbf{F}_{gravity} + \mathbf{F}_{static} + \mathbf{F}_{radiation} + \mathbf{F}_{exciting}$$
$$= -mg\mathbf{k} + \rho g V_0 \mathbf{k} - \rho g A_{wp}\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}} + \mathbf{F}_{exciting}$$
$$= -\rho g A_{wp}\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}} + \mathbf{F}_{exciting}$$
$$= -k\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}} + \mathbf{F}_{exciting}$$

$$(m + m_a)\ddot{\mathbf{z}} + c\dot{\mathbf{z}} + k\mathbf{z} = \mathbf{F}_{exciting}$$

c : damping coefficient
m_a : added mass

✓ Mass-Spring-Damper system



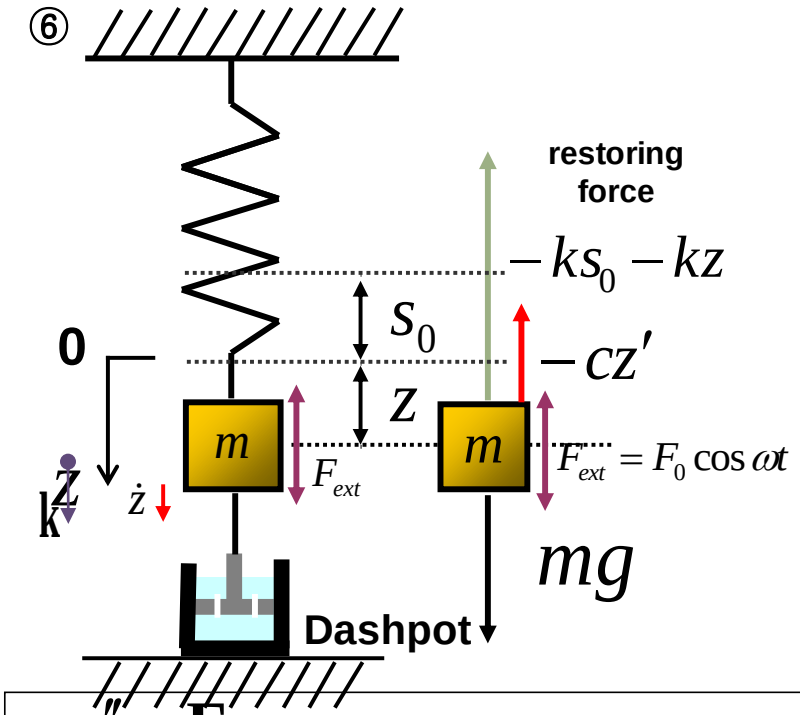
$$m\mathbf{z}'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F}_0 \cos \omega t$$
$$= -kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F}_0 \cos \omega t$$

$$m\mathbf{z}'' + c\mathbf{z}' + k\mathbf{z} = \mathbf{F}_0 \cos \omega t$$

Linear Model
Spring/Mass Systems: Driven Motion

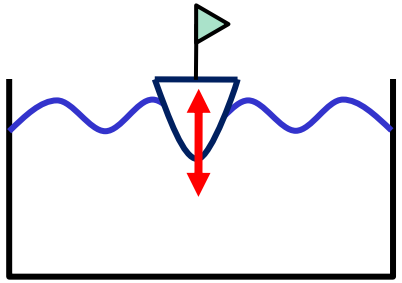
$$z = z(t)\ ,\ z'' = \frac{d^2 z}{dt^2}$$

✓ Mass-Spring-Damper system



$$mz'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F}_0 \cos \omega t$$
$$= -kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F}_0 \cos \omega t$$

$$mz'' + cz' + kz = \mathbf{F}_0 \cos \omega t$$



✓선박의 6자유도 운동 방정식

① Coordinate system 정의
(Global & Body-fixed coordinate)

② Newton's 2nd Law

$$\mathbf{M}\ddot{\mathbf{x}} = \sum \mathbf{F} = \mathbf{F}_{body} + \mathbf{F}_{surface}$$
$$= \mathbf{F}_{gravity} + \mathbf{F}_{static} + \mathbf{F}_{F.K} + \mathbf{F}_D + \mathbf{F}_R$$
$$= \mathbf{F}_{restoring} + \mathbf{F}_{exciting} - \mathbf{A}\ddot{\mathbf{x}} - \mathbf{B}\dot{\mathbf{x}}$$

$$(\mathbf{M} + \mathbf{A})\ddot{\mathbf{x}} + \mathbf{B}\dot{\mathbf{x}} = \mathbf{F}_{restoring} + \mathbf{F}_{exciting}$$

cf) 선형화 된 복원력 ($\mathbf{F}_{restoring} = -\mathbf{C}\mathbf{x}$)

$$(\mathbf{M} + \mathbf{A})\ddot{\mathbf{x}} + \mathbf{B}\dot{\mathbf{x}} + \mathbf{C}\mathbf{x} = \mathbf{F}_{exciting}$$

Identical

Spring/Mass Systems: Driven Motion

homogeneous $z = e^{\lambda t}$

$$mz'' + cz' + kz = F_0 \cos \omega t$$

- **Step1. homogeneous solution**

$$mz'' + cz' + kz = 0 \quad \text{try: } z = e^{\lambda t}$$

$$z = c_1 z_1 + c_2 z_2 \left(= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\right) \quad where,\ \lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

are you sure z_1, z_2 are linearly independent?

There could be three cases depending on the condition

Spring/Mass Systems: Driven Motion

homogeneous $z = e^{\lambda t}$

$$mz'' + cz' + kz = F_0 \cos \omega t$$

- **Step1. homogeneous solution**

are you sure z_1, z_2 are linearly independent?

$$mz'' + cz' + kz = 0 \quad \text{try: } z = e^{\lambda t}$$

$$z = c_1 z_1 + c_2 z_2 (= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}) \quad where,\ \lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

There could be three cases depending on the condition

cases	$c^2 - 4mk$	Root	
Case I	$c^2 - 4mk > 0$	Distinct Real Roots	λ_1, λ_2
Case II	$c^2 - 4mk = 0$	Repeated Real Roots	λ_1
Case III	$c^2 - 4mk < 0$	Conjugated Complex Roots	$\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$

Spring/Mass Systems: Driven Motion

homogeneous $z = e^{\lambda t}$

$$mz'' + cz' + kz = F_0 \cos \omega t$$

- **Step1. homogeneous solution**

$$mz'' + cz' + kz = 0 \quad \text{try: } z = e^{\lambda t}$$

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

cases	$c^2 - 4mk$	Root
Case I	$c^2 - 4mk > 0$	Distinct Real Roots $\lambda_1 \neq \lambda_2$

Solution
$z = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \Rightarrow z = e^{\lambda_1 t}$$

$$\lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \Rightarrow z = e^{\lambda_2 t}$$

Linearly independent

Linear combination

Recall the example

$c_1 e^t + c_2 e^{2t} = 0$

Satisfied only when $c_1 = c_2 = 0$

Linearly Independent

Spring/Mass Systems: Driven Motion

homogeneous $z = e^{\lambda t}$

$$mz'' + cz' + kz = F_0 \cos \omega t$$

- **Step1. homogeneous solution**

$$mz'' + cz' + kz = 0 \quad \text{try: } z = e^{\lambda t} \qquad \lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

cases	$c^2 - 4mk$	Root
Case II	$c^2 - 4mk = 0$	Repeated Real Roots λ_1

Solution

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \Rightarrow z_1 = e^{\lambda_1 t}$$

To create another linearly independent solution , multiply t to z_1 $\Rightarrow$ $z_2 = te^{\lambda_1 t}$

Reduction of order

Spring/Mass Systems: Driven Motion

homogeneous $z = e^{\lambda t}$

$$mz'' + cz' + kz = F_0 \cos \omega t$$

- **Step1. homogeneous solution**

$$mz'' + cz' + kz = 0 \quad \text{try: } z = e^{\lambda t} \qquad \lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

cases	$c^2 - 4mk$	Root
Case II	$c^2 - 4mk = 0$	Repeated Real Roots λ_1

Solution
$z = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}$

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a}$$

$z_1 = e^{\lambda_1 t}$

$z_2 = t e^{\lambda_2 t}$

Linearly independent

Linear combination

Spring/Mass Systems: Driven Motion

homogeneous $z = e^{\lambda t}$

$$mz'' + cz' + kz = F_0 \cos \omega t$$

- **Step1. homogeneous solution**

$$mz'' + cz' + kz = 0 \quad \text{try: } z = e^{\lambda t} \qquad \lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

cases	$c^2 - 4mk$	Root
Case III	$c^2 - 4mk < 0$	Conjugated Complex Roots $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$

Solution

$$\lambda_1 = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} + i\frac{\sqrt{4ac - b^2}}{2a} = \alpha + i\beta$$

$$\lambda_2 = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} = \underbrace{\frac{-b}{2a}}_{\alpha} - i\underbrace{\frac{\sqrt{4ac - b^2}}{2a}}_{\beta} = \alpha - i\beta$$

$$z_1 = e^{(\alpha + i\beta)t}$$

$$z_2 = e^{(\alpha - i\beta)t}$$

Linearly independent → Linear combination

$$z = C_1 z_1 + C_2 z_2$$

Spring/Mass Systems: Driven Motion

homogeneous $z = e^{\lambda t}$

$$mz'' + cz' + kz = F_0 \cos \omega t$$

- **Step1. homogeneous solution**

$$mz'' + cz' + kz = 0 \quad \text{try: } z = e^{\lambda t} \qquad \lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

cases	$c^2 - 4mk$	Root
Case III	$c^2 - 4mk < 0$	Conjugated Complex Roots $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$

Solution

Linear combination

$$z = C_1 e^{(\alpha+i\beta)t} + C_2 e^{(\alpha-i\beta)t}$$

Two solutions by the choices

$$C_1 = 1, C_2 = 1$$

$$z_1 = e^{(\alpha+i\beta)t} + e^{(\alpha-i\beta)t}$$

$$C_1 = 1, C_2 = -1$$

$$z_2 = e^{(\alpha+i\beta)t} - e^{(\alpha-i\beta)t}$$

Spring/Mass Systems: Driven Motion

homogeneous

$$z = e^{\lambda t}$$

Euler's formular

$$e^{it} = \cos t + i \sin t$$

$$mz'' + cz' + kz = F_0 \cos \omega t$$

- **Step1. homogeneous solution**

$$mz'' + cz' + kz = 0 \quad \text{try: } z = e^{\lambda t}$$

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

cases	$c^2 - 4mk$	Root
Case III	$c^2 - 4mk < 0$	Conjugated Complex Roots $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$

Solution

$$z_1 = e^{(\alpha+i\beta)t} + e^{(\alpha-i\beta)t} = e^{\alpha t}(\cos \beta t + i \sin \beta t) + e^{\alpha t}(\cos \beta t - i \sin \beta t) = 2e^{\alpha t} \cos \beta t$$

$$z_2 = e^{(\alpha+i\beta)t} - e^{(\alpha-i\beta)t} = e^{\alpha t}(\cos \beta t + i \sin \beta t) - e^{\alpha t}(\cos \beta t - i \sin \beta t) = 2e^{\alpha t} \sin \beta t$$

New fundamental set of solution

Spring/Mass Systems: Driven Motion

homogeneous $z = e^{\lambda t}$

$$mz'' + cz' + kz = F_0 \cos \omega t$$

- **Step1. homogeneous solution**

$$mz'' + cz' + kz = 0 \quad \text{try: } z = e^{\lambda t} \qquad \lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

cases	$c^2 - 4mk$	Root
Case III	$c^2 - 4mk < 0$	Conjugated Complex Roots $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$

Solution
$y = e^{\alpha t}(c_1 \cos \beta t + c_2 \cos \beta t)$

$z_1 = 2e^{\alpha t} \cos \beta t$

$z_2 = 2e^{\alpha t} \sin \beta t$

$e^{\alpha t} \cos \beta t$

$e^{\alpha t} \sin \beta t$

Linearly independent → Linear combination

Spring/Mass Systems: Driven Motion

homogeneous $z = e^{\lambda t}$

$$mz'' + cz' + kz = F_0 \cos \omega t$$

- **Step1. homogeneous solution**

$$mz'' + cz' + kz = 0 \qquad \textbf{try: } z = e^{\lambda t} \rightarrow a\lambda^2 + b\lambda + c = 0$$

Case 1 $c^2 - 4mk > 0$

$$z_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \left(\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} \right), (\lambda_1, \lambda_2 < 0)$$

Case 2 $c^2 - 4mk = 0$

$$z_h = (c_1 + c_2 \cdot t) \cdot e^{\frac{-c}{2m} \cdot t} \quad , \left(\frac{-c}{2m} < 0 \right)$$

Case 3 $c^2 - 4mk < 0$

$$z_h = e^{\frac{-c}{2m} t} \cdot \left(A \cdot \cos(\omega_0 t) + B \cdot \sin(\omega_0 t) \right) \quad , \left(\omega_0 = \sqrt{\left(\frac{k}{m}\right) - \frac{1}{4} \cdot \left(\frac{c}{m}\right)^2} = \frac{1}{2m} \sqrt{4mk - c^2} \right)$$

Spring/Mass Systems: Driven Motion

$$mz'' + cz' + kz = F_0 \cos \omega t$$

- **Step2. particular solution**

  By the method of Undetermined Coefficient, we choose that

$$z_p(t) = a\cos\omega t + b\sin\omega t$$

$$z_p'(t) = -a\omega\sin\omega t + b\omega\cos\omega t$$

$$z_p''(t) = -a\omega^2\cos\omega t - b\omega^2\sin\omega t$$

$y'' + ay' + by = r(x)$

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n (n = 0,1,\cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k\cos\omega x$ $k\sin\omega x$	$K\cos\omega x + M\sin\omega x$
$ke^{\alpha x}\cos\omega x$ $ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(K\cos\omega x + M\sin\omega x)$

$$m\left(-\omega^2 a\cos\omega t - \omega^2 b\sin\omega t\right) + c\left(-\omega a\sin\omega t + \omega b\cos\omega t\right) + k\left(a\cos\omega t + b\sin\omega t\right) = F_0\cos\omega t$$

Spring/Mass Systems: Driven Motion

$$mz'' + cz' + kz = F_0 \cos \omega t$$

$$z_p(t) = a\cos \omega t + b \sin \omega t$$

$$m\left(-\omega^2 a\cos \omega t - \omega^2 b \sin \omega t\right) + c\left(-\omega a \sin \omega t + \omega b \cos \omega t\right) + k\left(a\cos \omega t + b \sin \omega t\right) = F_0 \cos \omega t$$

$$[(k - m\omega^2)a + \omega cb]\cos \omega t + [-\omega ca + (k - m\omega^2)b]\sin \omega t = F_0 \cos \omega t$$

$$\begin{cases} (k - m\omega^2)a + \omega cb = F_0 \\ -\omega ca + (k - m\omega^2)b = 0 \end{cases}$$

$$\therefore a = F_0 \frac{m(\omega^2{}_0 - \omega^2)}{m^2(\omega^2{}_0 - \omega^2)^2 + \omega^2 c^2}, \quad b = F_0 \frac{\omega c}{m^2(\omega^2{}_0 - \omega^2)^2 + \omega^2 c^2}$$

$$z_p(t) = a\cos \omega t + b \sin \omega t$$

Spring/Mass Systems: Driven Motion

$$mz'' + cz' + kz = F_0 \cos \omega t$$

- **Step3. general solution**

$$z(t) = z_h(t) + z_p(t)$$

$$z_h(t) = \begin{cases} c_1 e^{-(\frac{c}{2m} - \frac{1}{2m}\sqrt{c^2 - 4mk})t} + c_2 e^{-(\frac{c}{2m} + \frac{1}{2m}\sqrt{c^2 - 4mk})t} & ,(c^2 - 4mk > 0) \\ (c_1 + c_2 t) e^{-\frac{c}{2m}t} & ,(c^2 - 4mk = 0) \\ e^{-\frac{c}{2m}t}(A\cos \omega_0 t + B \sin \omega_0 t) = Ce^{-\frac{c}{2m}t} \cos(\omega_0 t - \delta) & ,(c^2 - 4mk < 0) \end{cases}$$

$$z_p(t) = a\cos \omega t + b \sin \omega t \quad , \left(a = F_0 \frac{m(\omega^2{}_0 - \omega^2)}{m^2(\omega^2{}_0 - \omega^2)^2 + \omega^2 c^2} , \quad b = F_0 \frac{\omega c}{m^2(\omega^2{}_0 - \omega^2)^2 + \omega^2 c^2} \right)$$

$$\text{as } t \to \infty,\ z_h(t) \to 0, \text{ so } \ z(t) \to z_p(t)$$

Spring/Mass Systems: Driven Motion

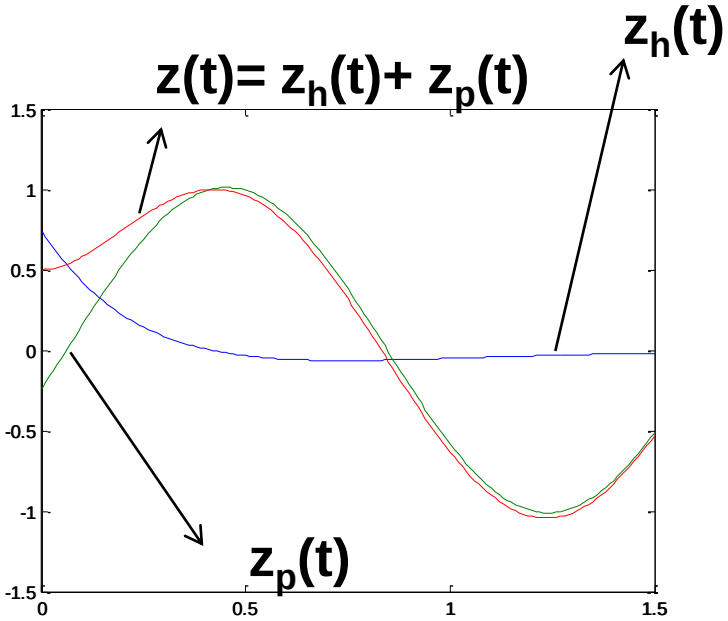
☑Transient and Steady-State Terms

$$mz'' + cz' + kz = F(t) = F_0 \cos \omega t$$

■ When *F* is periodic function, general solution have nonperiodic function $z_h(t)$ and periodic function $z_p(t)$

$z_h(t)$: transient solution

$z_p(t)$: steady-state solution



Comparison : example

$$y(t) = y_h(t) + y_p(t)$$

example*

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \sin 2t \quad y(0) = 1, \dot{y}(0) = 5$$

solution

1) Homogeneous Solution

Homogeneous

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 0$$

Try : $y_h(t) = e^{mt}$

$$(\ddot{m} + 3\dot{m} + 2)e^{mt} = 0$$

$$(m+1)(m+2) = 0$$

$$\therefore m = -1, m = -2$$

$$y_h(t) = e^{mt}$$

e^{-t} and e^{-2t} : linearly independent

$$\therefore y_h(t) = c_1 e^{-t} + c_2 e^{-2t}$$

! Notation changed in the example

$$z(t) \rightarrow y(t)$$

$$z'(t), z''(t) \rightarrow y'(t), y''(t)$$

$$z_{\text{transient}}, z_{\text{steady}} \rightarrow y_h(t), y_p(t)$$

$$z_{\text{zero-input}}, z_{\text{zero-state}} \rightarrow y_0(t), y_1(t)$$

*Zill & Cullen, Advanced Engineering Mathematics, 3rd Edition p204, example 5 modified

Comparison : example

$$y(t) = y_h(t) + y_p(t)$$

example

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \sin 2t \quad y(0) = 1, \dot{y}(0) = 5$$

Solution

2) Particular Solution

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \sin 2t$$

Try : $y_p(t) = A\sin 2t + B\cos 2t$

$$\ddot{y}_p = -4A\sin 2t - 4B\cos 2t$$

$$\dot{y}_p = 2A\cos 2t - 2B\sin 2t$$

L.H.S.: $-4(A\sin 2t + B\cos 2t) + 6(A\cos 2t - B\sin 2t) + 2(A\sin 2t + B\cos 2t) = (-2A - 6B)\sin 2t + (6A - 2B)\cos 2t$

R.H.S.: $\sin 2t$

L.H.S. = R.H.S.

$$\begin{aligned} -2A - 6B &= 1 \\ 6A - 2B &= 0 \end{aligned} \Rightarrow A = -\frac{1}{20}, B = -\frac{3}{20}$$

$$\therefore y_p = -\frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : example

$$y(t) = y_h(t) + y_p(t)$$

example

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \sin 2t \quad y(0) = 1, \dot{y}(0) = 5$$

solution

3) General Solution

$$y_h(t) = c_1 e^{-t} + c_2 e^{-2t}$$

$$y_p = -\frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Initial condition : $y(0) = 1, \dot{y}(0) = 5$

$$y(0) : c_1 + c_2 - \frac{3}{20} = 1$$

$$\dot{y}(0) : -c_1 - 2c_2 - \frac{2}{20} = 5$$

$$\Rightarrow \quad c_1 = \frac{37}{5}, \quad c_2 = -\frac{25}{4}$$

$$y_h(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t}, \quad y_p(t) = -\frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : example

example

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t \quad y(0)=1, \dot{y}(0)=5$$

$$y(t)=\frac{37}{5}e^{-t}-\frac{25}{4}e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

Zero Input

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=0$$
$$y(0)=1, \dot{y}(0)=5$$

Zero State

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t$$
$$y(0)=0, \dot{y}(0)=0$$

1) Homogeneous Solution

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=0$$
$$y_{0_h}(t)=e^{mt}$$
$$(m+1)(m+2)e^{mt}=0$$
$$y_{0_h}(t)=c_1e^{-t}+c_2e^{-2t}$$

$$\ddot{y}(t)-3\dot{y}(t)+2y(t)=0$$
$$y_{1_h}(t)=e^{mt}$$
$$(m+1)(m+2)e^{mt}=0$$
$$y_{1_h}(t)=c_1e^{-t}+c_2e^{-2t}$$

Comparison : example

example

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t \quad y(0)=1,\dot{y}(0)=5$$

$$y(t)=\frac{37}{5}e^{-t}-\frac{25}{4}e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

Zero Input

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=0$$
$$y(0)=1,\dot{y}(0)=5$$

Zero State

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t$$
$$y(0)=0,\dot{y}(0)=0$$

1) Homogeneous Solution

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=0 \qquad \ddot{y}(t)-3\dot{y}(t)+2y(t)=0$$

$$y_{0_h}(t)=c_1e^{-t}+c_2e^{-2t} \qquad y_{1_h}(t)=c_1e^{-t}+c_2e^{-2t}$$

2) Particular Solution

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=0 \qquad \ddot{y}(t)-3\dot{y}(t)+2y(t)=\sin 2t$$

$$y_{1_p}=A\sin 2t+B\cos 2t$$

L.H.S.: $(-2A-6B)\sin 2t+(6A-2B)\cos 2t$

R.H.S.: $\sin 2t$

L.H.S. = R.H.S. $\quad A=-\frac{1}{20},B=-\frac{3}{20}$

$$y_{0_p}(t)=0 \qquad y_{1_p}=-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

Comparison : example

예제

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t \quad y(0)=1,\dot{y}(0)=5$$

$$y(t)=\frac{37}{5}e^{-t}-\frac{25}{4}e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

	Zero Input $\ddot{y}(t)+3\dot{y}(t)+2y(t)=0$ $y(0)=1,\dot{y}(0)=5$	Zero State $\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t$ $y(0)=0,\dot{y}(0)=0$
1) Homogeneous Solution	$\ddot{y}(t)+3\dot{y}(t)+2y(t)=0$ $y_{0_h}(t)=c_1e^{-t}+c_2e^{-2t}$	$\ddot{y}(t)-3\dot{y}(t)+2y(t)=0$ $y_{1_h}(t)=c_1e^{-t}+c_2e^{-2t}$
2) Particular Solution	$\ddot{y}(t)+3\dot{y}(t)+2y(t)=0$ $y_{0_p}(t)=0$	$\ddot{y}(t)-3\dot{y}(t)+2y(t)=\sin 2t$ $y_{1_p}=-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$
3) General Solution	$y_{0_g}(t)=c_1e^{-t}+c_2e^{-2t}$	$y_{1_g}(t)=c_1e^{-t}+c_2e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$
Initial condition	$y(0)=1,\dot{y}(0)=5$	$y(0)=0\ ,\dot{y}(0)=0$
	$c_1+c_2=1$ $-c_1-2c_2=5$ ⇒ $c_1=7$ $c_2=-6$	$y(0):c_1+c_2-\frac{3}{20}=1$ $\dot{y}(0):-c_1-2c_2-\frac{2}{20}=5$ ⇒ $c_1=\frac{2}{5}$ $c_2=-\frac{1}{4}$

Comparison : example

example

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t \quad y(0)=1, \dot{y}(0)=5$$

$$y(t)=\frac{37}{5}e^{-t}-\frac{25}{4}e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

	Zero Input $\ddot{y}(t)+3\dot{y}(t)+2y(t)=0$ $y(0)=1, \dot{y}(0)=5$	Zero State $\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t$ $y(0)=0, \dot{y}(0)=0$
1) Homogeneous Solution	$\ddot{y}(t)+3\dot{y}(t)+2y(t)=0$	$\ddot{y}(t)-3\dot{y}(t)+2y(t)=0$
	$y_{0_h}(t)=c_1e^{-t}+c_2e^{-2t}$	$y_{1_h}(t)=c_1e^{-t}+c_2e^{-2t}$
2) Particular Solution	$\ddot{y}(t)+3\dot{y}(t)+2y(t)=0$	$\ddot{y}(t)-3\dot{y}(t)+2y(t)=\sin 2t$
	$y_{0_p}(t)=0$	$y_{1_p}=-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$
3) General Solution	$y_{0_g}(t)=c_1e^{-t}+c_2e^{-2t}$	$y_{1_g}(t)=c_1e^{-t}+c_2e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$
Initial condition	$y(0)=1, \dot{y}(0)=5$	$y(0)=0\ , \dot{y}(0)=0$
General Solution	$y_{0_g}(t)=7e^{-t}-6e^{-2t}$	$y_{1_g}(t)=\frac{2}{5}e^{-t}-\frac{1}{4}e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$

Comparison : example

example

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t \quad y(0)=1, \dot{y}(0)=5$$

$$y(t)=\frac{37}{5}e^{-t}-\frac{25}{4}e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

Zero Input

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=0$$
$$y(0)=1, \dot{y}(0)=5$$

Zero State

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t$$
$$y(0)=0, \dot{y}(0)=0$$

	Zero Input	Zero State
1) Homogeneous Solution	$y_{0_h}(t)=7e^{-t}-6e^{-2t}$	$y_{1_h}(t)=\frac{2}{5}e^{-t}-\frac{1}{4}e^{-2t}$
2) Particular Solution	$y_{0_p}(t)=0$	$y_{1_p}=-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$
3)General Solution	$y_{0_g}(t)=y_0(t)=7e^{-t}-6e^{-2t}$	$y_{1_g}(t)=y_1(t)=\frac{2}{5}e^{-t}-\frac{1}{4}e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$

Comparison : example

example

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t \quad y(0)=1,\dot{y}(0)=5$$

$$y(t)=\left(\frac{37}{5}e^{-t}-\frac{25}{4}e^{-2t}\right)+\left(\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t\right)$$

Zero Input

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=0$$
$$y(0)=1,\dot{y}(0)=5$$

Zero State

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t$$
$$y(0)=0,\dot{y}(0)=0$$

	Zero Input	Zero State
1) Homogeneous Solution	$y_{0_h}(t)=7e^{-t}-6e^{-2t}$	$y_{1_h}(t)=\frac{2}{5}e^{-t}-\frac{1}{4}e^{-2t}$
2) Particular Solution	$y_{0_p}(t)=(0)$	$y_{1_p}=-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$
3)General Solution	$y_{0_g}(t)=y_0(t)=(7e^{-t}-6e^{-2t})$	$y_{1_g}(t)=y_1(t)=\left(\frac{2}{5}e^{-t}-\frac{1}{4}e^{-2t}\right)+\left(\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t\right)$

$$\begin{array}{ccccc} y(t) & = & & + & \\ \| & & \| & & \| \\ y_0(t) & = & & + & \\ + & & + & & + \\ y_1(t) & = & & + & \end{array}$$

Comparison : example

example

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t \quad y(0)=1, \dot{y}(0)=5$$

$$y(t)=\frac{37}{5}e^{-t}-\frac{25}{4}e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

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$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=0$$
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$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t$$
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	Zero Input	Zero State
1) Homogeneous Solution	$y_{0_h}(t)=7e^{-t}-6e^{-2t}$	$y_{1_h}(t)=\frac{2}{5}e^{-t}-\frac{1}{4}e^{-2t}$
2) Particular Solution	$y_{0_p}(t)=0$	$y_{1_p}=-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$
3)General Solution	$y_{0_g}(t)=y_0(t)=7e^{-t}-6e^{-2t}$	$y_{1_g}(t)=y_1(t)=\frac{2}{5}e^{-t}-\frac{1}{4}e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$

$$\begin{array}{ccccc} y(t) & = & (\frac{37}{5}e^{-t}-\frac{25}{4}e^{-2t}) & + & (-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t) \\ || & & || & & || \\ y_0(t) & = & (7e^{-t}-6e^{-2t}) & + & (0) \\ + & & + & & + \\ y_1(t) & = & (\frac{2}{5}e^{-t}-\frac{1}{4}e^{-2t}) & + & (-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t) \end{array}$$

Comparison : example-proof

$$y(t) = y_0(t) + y_1(t)$$

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = u(t) \quad , u(t) \neq 0 \quad \cdots(1)$$

$$y(0) = a, \dot{y}(0) = b \qquad , a \neq 0, b \neq 0 \quad \cdots(2)$$

Zero Input solution : $y_0(t)$

$$m\ddot{y}_0(t) + c\dot{y}_0(t) + ky_0(t) = 0$$

$$y_0(0) = a, \dot{y}_0(0) = b$$

Zero state solution : $y_1(t)$

$$m\ddot{y}_1(t) + c\dot{y}_1(t) + ky_1(t) = u(t)$$

$$y_1(0) = 0, \dot{y}_1(0) = 0$$

assum. : $y(t) = y_0(t) + y_1(t)$

$\therefore y(t) = y_0(t) + y_1(t)$

$y_t \rightarrow$ (1) :

L.H.S.:

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t)$$

$$= m(\ddot{y}_0(t) + \ddot{y}_1(t)) + c(\dot{y}_0(t) + \dot{y}_1(t)) + k(y_0(t) + y_1(t))$$

$$= \left[m\ddot{y}_0(t) + c\dot{y}_0(t) + ky_0(t)\right] + \left[m\ddot{y}_1(t) + c\dot{y}_1(t) + ky_1(t)\right]$$

$$= 0 + u(t)$$

R.H.S.: $u(t)$

$\therefore$ L.H.S.=R.H.S “(1) satisfied”

$y_t \rightarrow$ (2) :

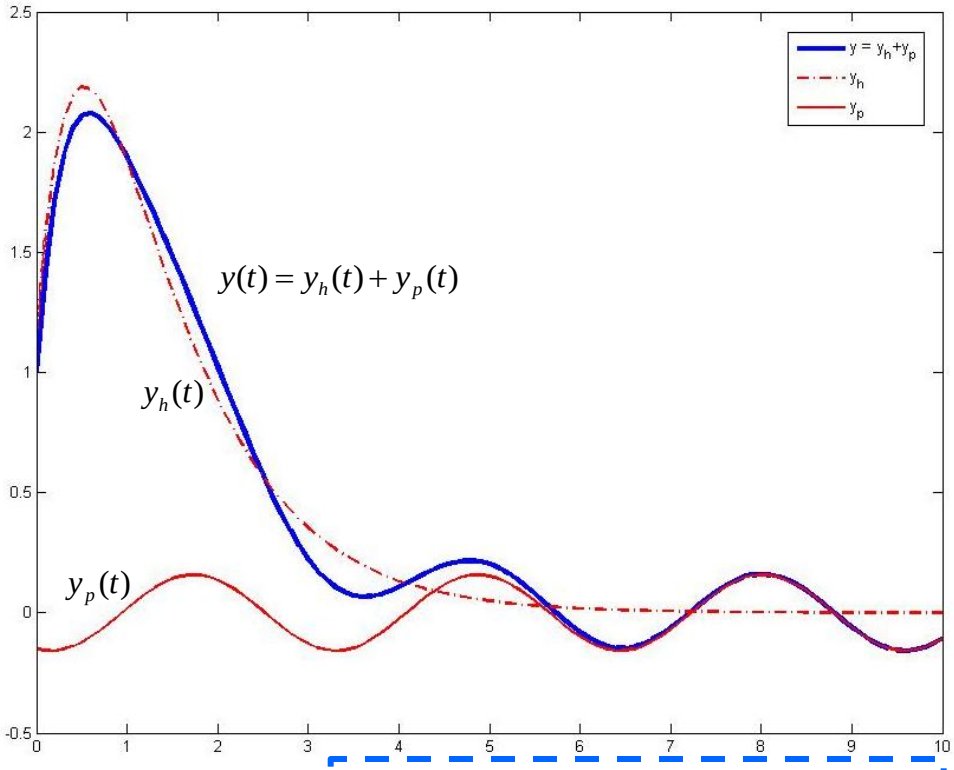
$$y(0) = y_0(0) + y_1(0) = a + 0$$

$$\dot{y}(0) = \dot{y}_0(0) + \dot{y}_1(0) = b + 0$$

$\therefore y(0) = a$ “(2) satisfied”

$\dot{y}(0) = b$

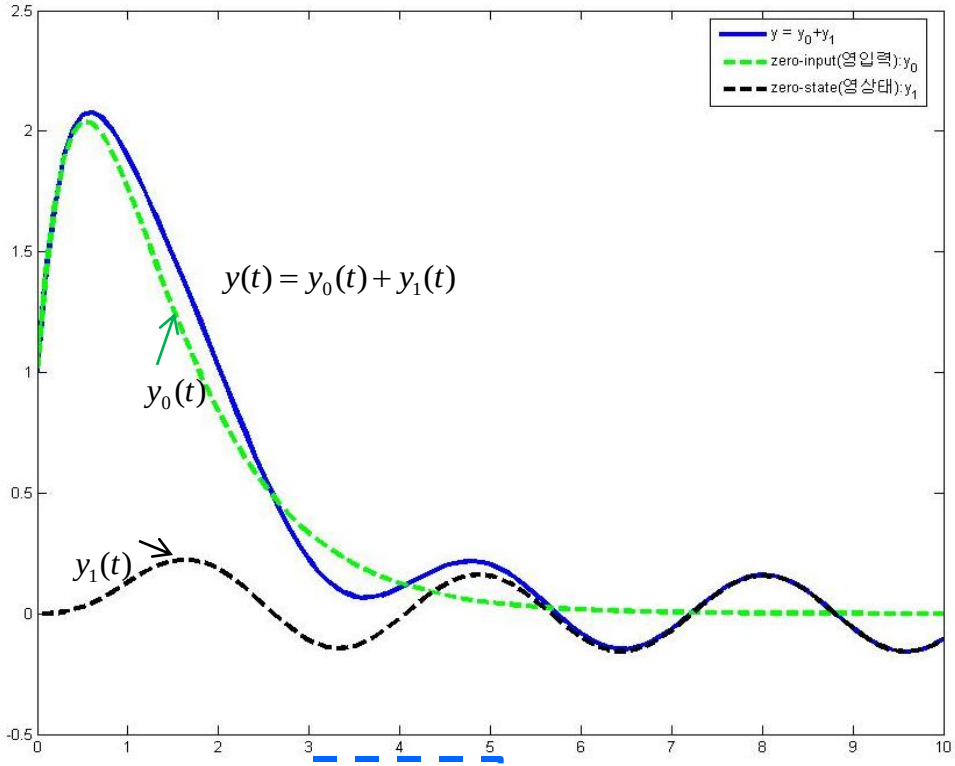
Comparison : graph



$$y(t) = y_h(t) + y_p(t)$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$y(t) = y_h(t) + y_p(t)$$

$$\| $$

$$y_0(t) + y_1(t)$$

Zero input solution
$u(t)=0$

Zero state solution
$y(0)=0, \dot{y}(0)=0$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

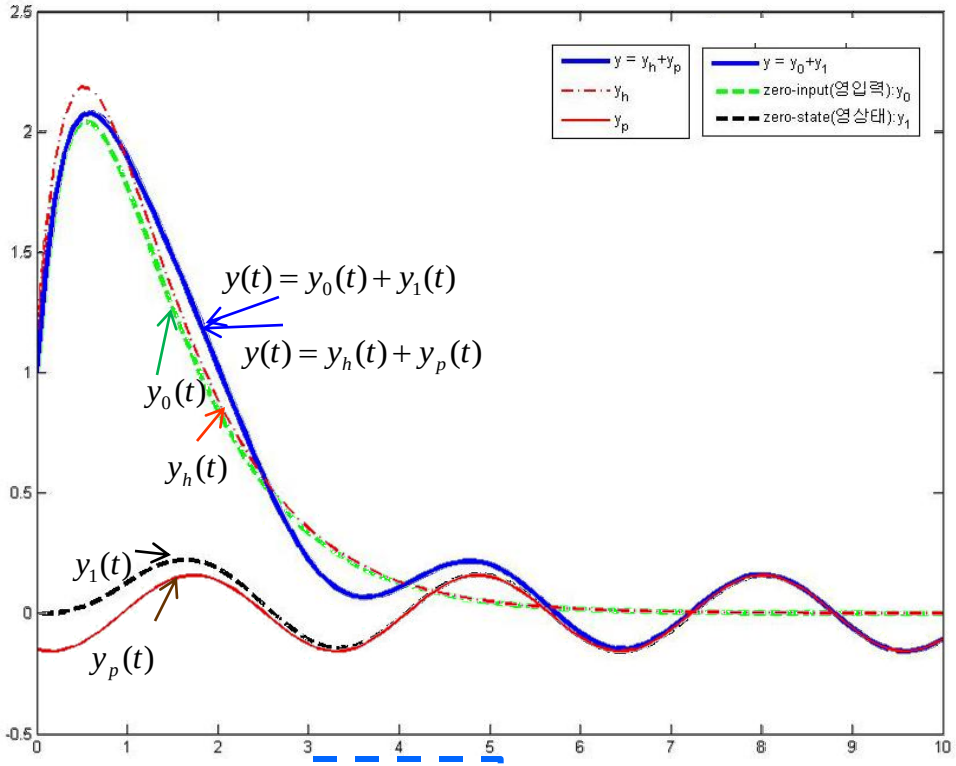
$$\|$$

$$y_0(t) = 7e^{-t} - 6e^{-2t}$$

$$+$$

$$y_1(t) = \frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$y(t) = y_h(t) + y_p(t)$$

$$\| $$

$$y_0(t) + y_1(t)$$

Zero input solution
$u(t)=0$

Zero state solution
$y(0)=0, \dot{y}(0)=0$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

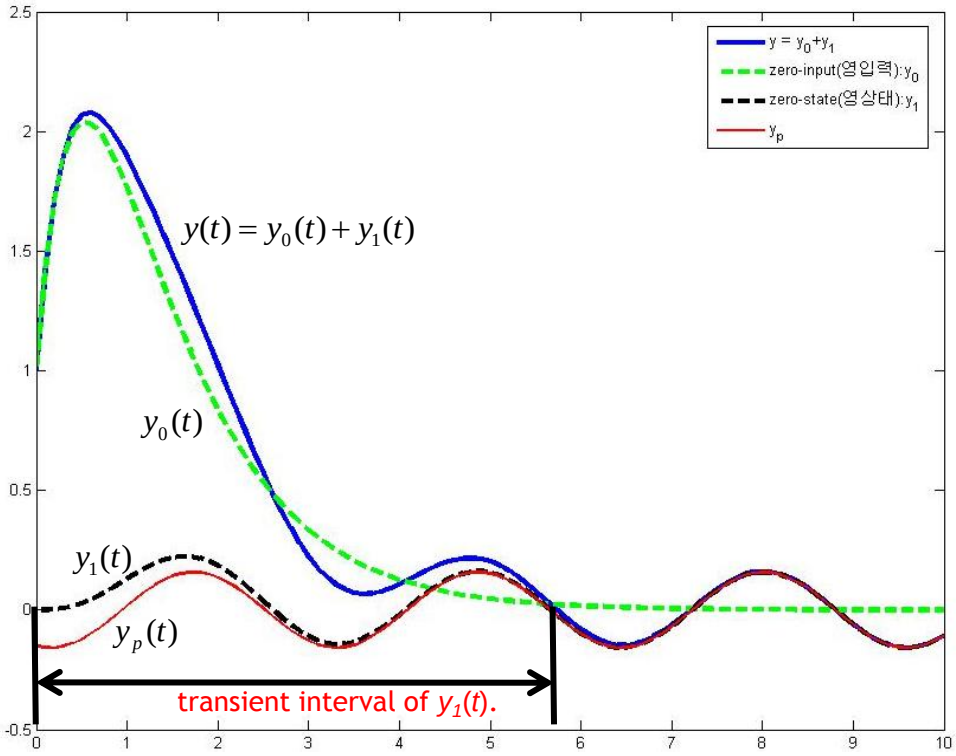
$$\|$$

$$y_0(t) = 7e^{-t} - 6e^{-2t}$$

$$+$$

$$y_1(t) = \frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$y(t) = y_h(t) + y_p(t)$$

$$\| \qquad \| \qquad \|$$

Zero Input solution
$u(t)=0$

$$y_0(t) = y_{0_h}(t) + y_{0_p}(t)$$

$$+ \qquad + \qquad +$$

Zero state solution
$y(0)=0, \dot{y}(0)=0$

$$y_1(t) = y_{1_h}(t) + y_{1_p}(t)$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

$$y_0(t) = 7e^{-t} - 6e^{-2t}$$

$$y_1(t) = \frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

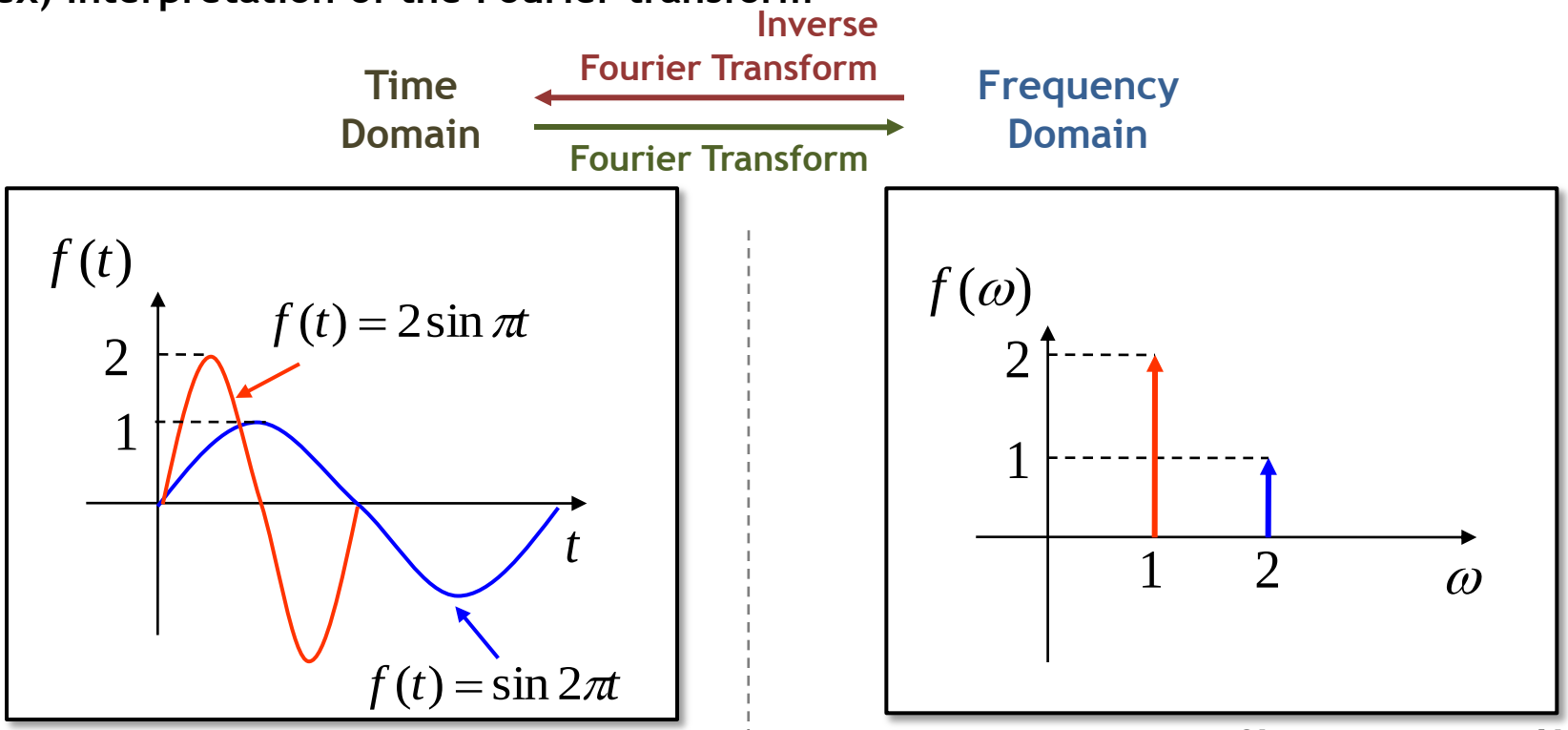
Reference slides

Fourier Transform

Fourier Transform

Transform between time domain and frequency domain.

ex) Interpretation of the Fourier transform



Frequency와 Amplitude로 표현됨

=> 시계열의 운동 복원 가능

Laplace Transforms (2) : Properties

Definition of the Laplace Transform
The Inverse Transform
Transforms of Derivatives
Translation Theorems
Additional Operational Properties
The Dirac Delta Function
Systems of Linear Differential Equations

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Definition of the Laplace Transform

☑Introduction

- **Differentiation and integration are transform**
  - **Transform a function into another function**

$$\frac{d}{dx}x^2 = 2x,\ \int x^2 dx = \frac{1}{3}x^3 + c,\ \int_0^3 x^2 dx = 9$$

- **These two transform possess the linearity property**

$$\frac{d}{dx}[\alpha f(x) + \beta g(x)] = \alpha f'(x) + \beta g'(x)$$

$$\int [\alpha f(x) + \beta g(x)]dx = \alpha \int f'(x)dx + \int \beta g'(x)dx$$

$$\int_a^b [\alpha f(x) + \beta g(x)]dx = \alpha \int_a^b f'(x)dx + \int_a^b \beta g'(x)dx$$

Definition of the Laplace Transform

☑Basic Definition

$$\int_0^\infty K(s,t)\,f(t)dt = \lim_{b\to\infty}\int_0^b K(s,t)\,f(t)dt$$

If the limit exists, the integral is exist (or convergent)
If the limit does not exists, the integral does not exist (or divergent)

If we choose *K(s,t)=e^-st*

Definition 4.1 **Laplace Transform**

Let f **be a function for** $t \geq 0$**, then the integral**

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

Is said to be the Laplace transform of f**, provided the integral converges**

Result of Laplace transform is function of *s*, so we can express it as

$$\mathcal{L}\{f(t)\} = F(s) \qquad \mathcal{L}\{g(t)\} = G(s) \qquad \mathcal{L}\{H(t)\} = h(s)$$

Definition of the Laplace Transform

☑Some examples

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}dt = \left.\frac{-e^{st}}{s}\right|_0^\infty = \frac{1}{s} \qquad s > 0$$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}tdt = \left.\frac{-te^{-st}}{s}\right|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt = \frac{1}{s}\mathcal{L}\{1\} = \frac{1}{s}(\frac{1}{s}) = \frac{1}{s^2} \quad s > 0$$

$$\mathcal{L}\{e^{-3t}\} = \int_0^\infty e^{-st}e^{-3t}tdt = \int_0^\infty e^{-(s+3)t}dt = \left.\frac{-e^{-(s+3)}}{s+3}\right|_0^\infty = \frac{1}{s+3} \qquad s > -3$$

Definition of the Laplace Transform

☑Some examples

$$\mathcal{L}\{\sin 2t\} = \int_0^\infty e^{-st}\sin 2t dt = \left.\frac{-e^{-st}\sin 2t}{s}\right|_0^\infty + \frac{2}{s}\int_0^\infty e^{-st}\cos 2t dt$$

$$= \frac{2}{s}\int_0^\infty e^{-st}\cos 2t dt \quad (s>0)$$

$$= \frac{2}{s}\left[\left.\frac{-e^{-st}\cos 2t}{s}\right|_0^\infty - \frac{2}{s}\underbrace{\int_0^\infty e^{-st}\sin 2t dt}_{\mathcal{L}\{\sin 2t\}}\right] = \frac{2}{s^2} - \frac{4}{s^2}\mathcal{L}\{\sin 2t\}$$

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}, \quad s>0$$

Definition of the Laplace Transform

☑$\mathcal{L}$ is a Linear Transform

$$\int_0^\infty e^{-st}[\alpha f(t)+\beta g(t)]dt = \alpha\int_0^\infty e^{-st} f(t)dt + \beta\int_0^\infty e^{-st} g(t)dt$$

Whenever both integral converges for s>c

$$\mathcal{L}\{\alpha f(t)+\beta g(t)\} = \alpha\mathcal{L}\{f(t)\}+\beta\mathcal{L}\{g(t)\} = \alpha F(s)+\beta G(s)$$

→ Linearity

For example,

$$\mathcal{L}\{1+5t\} = \mathcal{L}\{1\}+5\mathcal{L}\{t\} = \frac{1}{s}+\frac{5}{s^2}$$

$$\mathcal{L}\{4e^{-3t}-10\sin 2t\} = 4\mathcal{L}\{e^{-3t}\}-10\mathcal{L}\{\sin 2t\} = \frac{4}{s+3}-\frac{20}{s^2+4}$$

Definition of the Laplace Transform

☑$\mathscr{L}$ is a Linear Transform

Theorem 4.1

Transforms of Some Basic Functions

$$(a)\ \mathscr{L}\{1\} = \frac{1}{s}$$

$$(b)\ \mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, n = 1,2,3,\ldots \qquad (c)\ \mathscr{L}\{e^{at}\} = \frac{1}{s-a}$$

$$(d)\ \mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2} \qquad (e)\ \mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}$$

$$(f)\ \mathscr{L}\{\sinh kt\} = \frac{k}{s^2-k^2} \qquad (g)\ \mathscr{L}\{\cosh kt\} = \frac{s}{s^2-k^2}$$

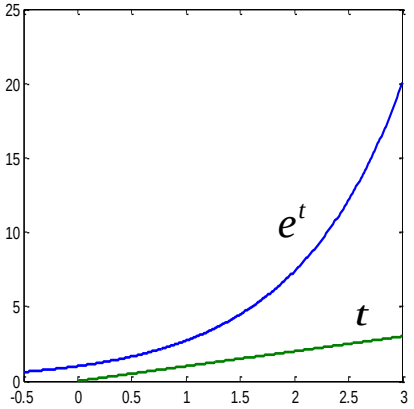
Definition of the Laplace Transform

☑Sufficient Conditions for Existence of ℒ{f(t)}

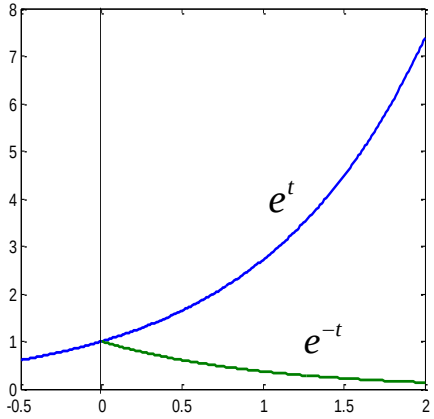
Definition 4.2 **Exponential Order**

A function f is said to be of exponential order *c* if there exist constants *c*, *M>0*, and *T* >0 such that $|f(t)| \le Me^{ct}$ for all *t* >*T*

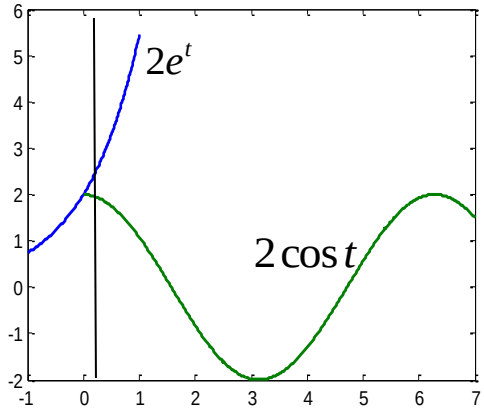
$f(t)=t, \quad f(t)=e^{-t}, \quad f(t)=2\cos t$ **Are all of exponential order *c* =1 for *t* >0**



$|t| \le e^t$



$|e^{-t}| \le e^t$



$|2\cos t| \le 2e^t$

Definition of the Laplace Transform

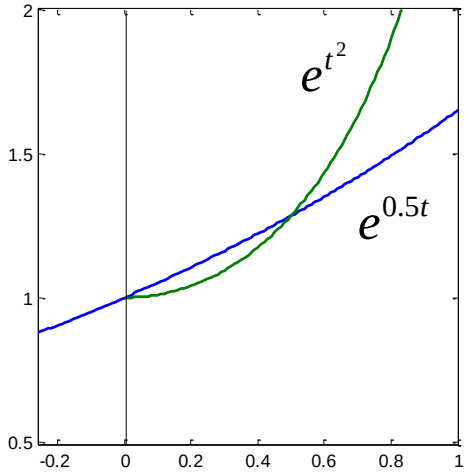
☑Sufficient Conditions for Existence of ℒ{f(t)}

Definition 4.2 **Exponential Order**

A function f is said to be of exponential order *c* if there exist constants *c*, *M>0*, and *T* >0 such that $|f(t)| \le Me^{ct}$ for all *t* >*T*

$f(t) = e^{t^2}$ Is not of exponential order

Since it grows faster than
any positive linear power of *e* for *t* > *c* >0



$|e^{t^2}| > e^{0.5t}$

Theorem 4.2 **Sufficient Conditions for Existence**

If $f(t)$ is piecewise continuous on the interval [0,∞) and of exponential order *c*, then ℒ{f(t)} exists for *s*>*c*.

Definition of the Laplace Transform

☑ Example 5

Transform of a Piecewise-Continuous Function

Evaluate

$$\mathcal{L}\{f(t)\} \text{ for } f(t) = \begin{cases} 0, & 0 \le t < 3 \\ 2, & t \ge 3 \end{cases}$$

solution)

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^3 e^{-st}(0)\,dt + \int_3^\infty e^{-st}(2)\,dt$$

$$= -\frac{2e^{-st}}{s}\Big|_3^\infty$$

$$= \frac{2e^{-3s}}{s}, \quad s > 0$$

Inverse Laplace Transforms

☑The Inverse Problem

Theorem 4.3

Some Inverse Transform

$$(a)\ 1=\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$(b)\ t^n=\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}, n=1,2,3,\ldots \qquad (c)\ e^{at}=\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}$$

$$(d)\ \sin kt=\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} \qquad (e)\ \cos kt=\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\}$$

$$(f)\ \sinh kt=\mathcal{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\} \qquad (g)\ \cosh kt=\mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\}$$

Inverse Transforms

☑ $\mathcal{L}^{-1}$ is a Linear Transform

$$\mathcal{L}^{-1}\{\alpha F(s)+\beta G(s)\}=\alpha\,\mathcal{L}^{-1}\{F(s)\}+\beta\,\mathcal{L}^{-1}\{G(s)\}$$

☑ Partial Fractions

■ Distinct linear factors

$$\frac{\alpha s^2+\beta s+\gamma}{(s+a)(s+b)(s+c)}=\frac{A}{(s+a)}+\frac{B}{(s+b)}+\frac{C}{(s+c)}$$

$$\mathcal{L}^{-1}\{\frac{\alpha s^2+\beta s+\gamma}{(s+a)(s+b)(s+c)}\}=A\mathcal{L}^{-1}\{\frac{1}{(s+a)}\}+B\mathcal{L}^{-1}\{\frac{1}{(s+b)}\}+C\,\mathcal{L}^{-1}\{\frac{1}{(s+c)}\}$$

Inverse Transforms

☑ Example 3
Partial Fractions and Linearity

Evaluate

$$\mathcal{L}^{-1}\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\}$$

solution)

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s+4}$$

$$= \frac{A(s-2)(s+4)+B(s-1)(s+4)+C(s-1)(s-2)}{(s-1)(s-2)(s+4)}$$

$$s^2+6s+9$$
$$= A(s-2)(s+4)+B(s-1)(s+4)+C(s-1)(s-2)$$

set $s=1,\ s=2,\ s=-4$ respectively

$$16 = A(-1)(5),\ 25 = B(1)(6)\ 1 = C(-5)(-6)$$

$$A=-\frac{16}{5},\ B=\frac{25}{6},\ C=\frac{1}{30}$$

Inverse Transforms

☑ Example 3
Partial Fractions and Linearity

Evaluate

$$\mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$$

solution)

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s+4}$$

$$A=-\frac{16}{5},\ B=\frac{25}{6},\ C=\frac{1}{30}$$

$$\therefore \frac{s^2+6s+9}{(s-1)(s-2)(s+4)}$$

$$=-\frac{16}{5}\frac{1}{s-1}+\frac{25}{6}\frac{1}{s-2}+\frac{1}{30}\frac{1}{s+4}$$

$$\mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$$

$$=-\frac{16}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}+\frac{25}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}+\frac{1}{30}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

Inverse Transforms

☑ Example 3
Partial Fractions and Linearity

Evaluate

$$\mathcal{L}^{-1}\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\}$$

solution)

$$\mathcal{L}^{-1}\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\}$$

$$=-\frac{16}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}+\frac{25}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}+\frac{1}{30}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$=-\frac{16}{5}e^{t}+\frac{25}{6}e^{2t}+\frac{1}{30}e^{-4t}$$

Transforms of Derivatives

☑Transform of a Derivative

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)dt$$

$$= -f(0) + s\mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

$$\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-st} f''(t)dt = e^{-st} f'(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f'(t)dt$$

$$= -f'(0) + s\mathcal{L}\{f'(t)\}$$

$$= s[sF(s) - f(0)] - f'(0)$$

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

$$\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2 f(0) - sf'(0) - f''(0)$$

Transforms of Derivatives

☑Transform of a Derivative

Theorem 4.4 — **Transform of a Derivative**

If $f, f', \ldots, f^{(n-1)}$ **are continuous on [0,∞) and are of exponential order and if** $f^{(n)}(t)$ **is piecewise-continuous on [0, ∞), then**

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

Where $F(s) = \mathcal{L}\{f(t)\}$

Transforms of Derivatives

☑Solving linear ODEs

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = g(t)$$

Linear differential equation with constant coefficient

$$y(0) = y_0,\ y'(0) = y_1, \ldots,\ y^{(n-1)}(0) = y_{n-1},$$

$$a_0, a_1, \ldots, a_n \text{ and } y_0, y_1, \ldots, y_{n-1} \text{ are constant}$$

⇩ **Laplace transform**

$$a_n \mathcal{L}\{\frac{d^n y}{dt^n}\} + a_{n-1} \mathcal{L}\{\frac{d^{n-1} y}{dt^{n-1}}\} + \cdots + a_0 \mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$

From theorem 4.4

$$a_n[s^n Y(s) - s^{n-1} y(0) - \ldots - y^{(n-1)}(0)]$$

$$+ a_{n-1}[s^{n-1} Y(s) - s^{n-2} y(0) - \ldots - y^{(n-2)}(0)] + \cdots + a_0 Y(s) = G(s)$$

Algebraic equation in Y(s)

Transforms of Derivatives

☑Solving linear ODEs

$$a_n[s^nY(s)-s^{n-1}y(0)-\ldots-y^{(n-1)}(0)]$$
$$+a_{n-1}[s^{n-1}Y(s)-s^{n-2}y(0)-\ldots-y^{(n-2)}(0)]+\cdots+a_0Y(s)=G(s)$$

$$P(s)Y(s)=Q(s)+G(s),$$

$$P(s)=a_ns^n+a_{n-1}s^{n-1}+\cdots+a_0$$

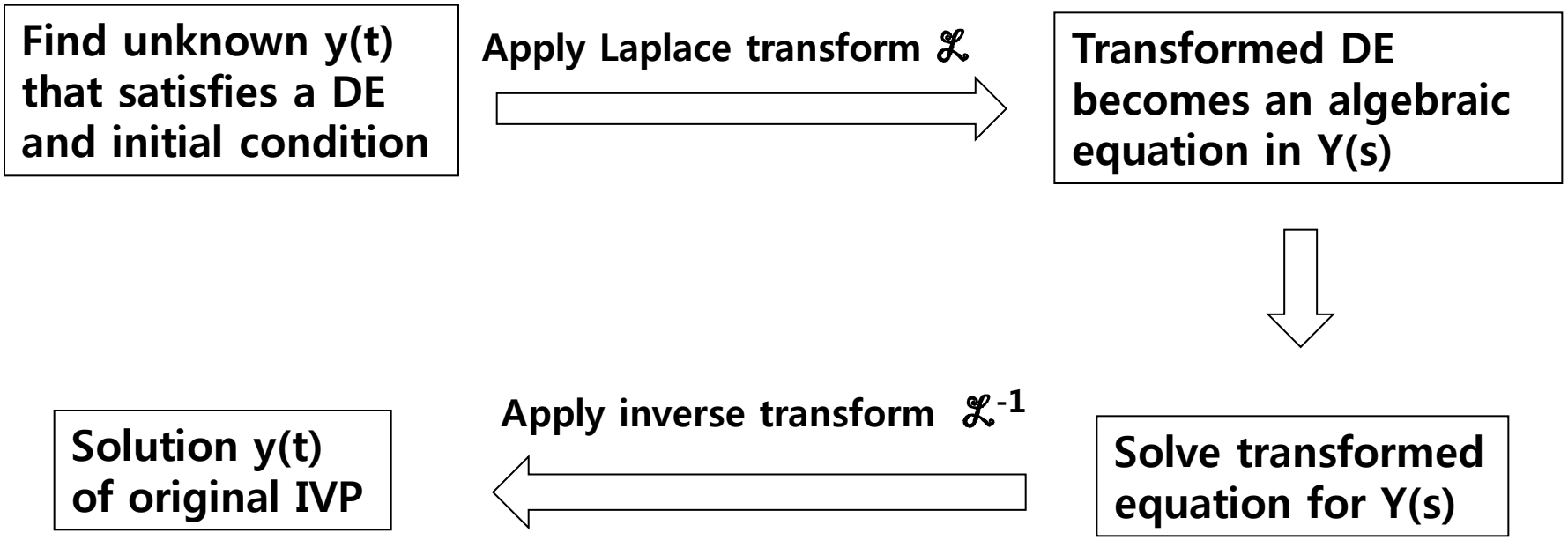
※ P(s) is the same as the nth degree auxiliary polynomial in Section 3.3(Homogeneous Linear equations with constant coefficient)

$$Y(s)=\frac{Q(s)}{P(s)}+\frac{G(s)}{P(s)}$$

$$y(t)=\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{Q(s)}{P(s)}\right\}+\mathcal{L}^{-1}\left\{\frac{G(s)}{P(s)}\right\}$$

Transforms of Derivatives

☑Solving linear ODEs



Transforms of Derivatives

☑Example 4 Solving a First-Order IVP

Use the Laplace transform to solve the IVP

$$\frac{dy}{dt}+3y=13\sin 2t, \quad y(0)=6$$

Solution)

take Laplace transform of each member of DE:

$$\mathcal{L}\left\{\frac{dy}{dt}\right\}+3\mathcal{L}\{y\}=13\mathcal{L}\{\sin 2t\}$$

$$sY(s)-y(0)+3Y(s)=13\frac{2}{s^2+4}, \qquad (s+3)Y(s)=6+\frac{26}{s^2+4}$$

$$Y(s)=\frac{6}{(s+3)}+\frac{26}{(s+3)(s^2+4)}=\frac{6s^2+50}{(s+3)(s^2+4)}$$

Transforms of Derivatives

☑Example 4 Solving a First-Order IVP

Use the Laplace transform to solve the IVP

$$\frac{dy}{dt}+3y=13\sin 2t, \quad y(0)=6$$

$$Y(s)=\frac{6s^2+50}{(s+3)(s^2+4)}=\frac{A}{s+3}+\frac{Bs+C}{s^2+4}$$

$$6s^2+50=A(s^2+4)+(Bs+C)(s+3)$$

$$=(A+B)s^2+(3B+C)s+4A+3C$$

$$\begin{cases} A+B=6 \\ 3B+C=0 \\ 4A+3C=50 \end{cases} \qquad A=8,\ B=-2,\ C=6 \qquad \therefore Y(s)=\frac{8}{s+3}+\frac{-2s+6}{s^2+4}$$

Transforms of Derivatives

☑Example 4 Solving a First-Order IVP

Use the Laplace transform to solve the IVP

$$\frac{dy}{dt} + 3y = 13\sin 2t, \quad y(0) = 6$$

$$Y(s) = \frac{8}{s+3} + \frac{-2s+6}{s^2+4}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 8\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$\therefore y(t) = 8e^{3t} - 2\cos 2t + 3\sin 2t$$

Transforms of Derivatives

☑Example 5 Solving a Second-Order IVP

solve

$$y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1,\ y'(0) = 5$$

Solution)

take Laplace transform of each member of DE:

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} - 3\mathcal{L}\left\{\frac{dy}{dt}\right\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-4t}\}$$

$$s^2Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+4}$$

$$(s^2 - 3s + 2)Y(s) = s + 2 + \frac{1}{s+4}$$

Transforms of Derivatives

☑Example 5 Solving a Second-Order IVP

solve

$$y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1,\ y'(0) = 5$$

$$(s^2 - 3s + 2)Y(s) = s + 2 + \frac{1}{s+4}$$

$$Y(s) = \frac{s+2}{(s^2 - 3s + 2)} + \frac{1}{(s^2 - 3s + 2)(s+4)} = \frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)}$$

$$= \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4}$$

Transforms of Derivatives

☑Example 5 Solving a Second-Order IVP

solve

$$y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1,\ y'(0) = 5$$

$$Y(s) = \frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4}$$

$$s^2 + 6s + 9 = A(s-2)(s+4) + B(s-1)(s+4) + C(s-1)(s-2)$$

Substituting s=1, s=2, s=-4 respectively,

$$16 = A(-1)(5), \quad 4 + 12 + 9 = B(1)(6), \quad 16 - 24 + 9 = C(-5)(-6)$$

$$A = -\frac{16}{5}, \qquad B = \frac{25}{6}, \qquad C = \frac{1}{30}$$

Transforms of Derivatives

☑Example 5 Solving a Second-Order IVP

solve

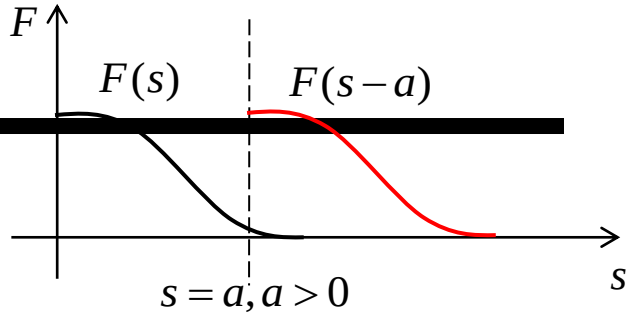
$$y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1,\ y'(0) = 5$$

$$Y(s) = -\frac{16}{15}\frac{1}{s-1} + \frac{25}{6}\frac{1}{s-2} + \frac{1}{30}\frac{1}{s+4}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\frac{16}{15}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{25}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{30}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$\therefore y(t) = -\frac{16}{15}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$$

Translation on the s-axis



☑Translation on the s-axis

Theorem 4.6 **First Translation Theorem**

If $\mathcal{L}\{f(t)\} = F(s)$ **and a is any real number, then**

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

Proof)

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)dt = \int_0^\infty e^{-(s-a)t} f(t)dt = F(s-a)$$

- **Inverse form of Theorem 4.6**

$$\mathcal{L}^{-1}\{F(s-a)\} = \mathcal{L}^{-1}\{F(s)\big|_{s\to s-a}\} = e^{at} f(t)$$

Translation on the s-axis

☑Example 1 Using the First Translation Theorem

Evaluate

$$(a)\ \mathcal{L}\{e^{5t}t^3\} \quad \text{and} \quad (b)\ \mathcal{L}\{e^{-2t}\cos 4t\}$$

Solution)

$$(a)\ \mathcal{L}\{e^{5t}t^3\} = \mathcal{L}\{t^3\}_{s\to s-5} = \frac{3!}{s^4}\bigg|_{s\to s-5} = \frac{6}{(s-5)^4}$$

$$(b)\ \mathcal{L}\{e^{-2t}\cos 4t\} = \mathcal{L}\{\cos 4t\}_{s\to s-(-2)} = \frac{s}{s^2+16}\bigg|_{s\to s+2} = \frac{s+2}{(s+2)^2+16}$$

Translation on the s-axis

☑ Example 2

Partial Fractions and Completing the Square

Evaluate

$$(a)\ \mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} \quad \text{and} \quad (b)\ \mathcal{L}^{-1}\left\{\frac{s/2+5/3}{s^2+4s+6}\right\}$$

Solution)

$$(a)\ \frac{2s+5}{(s-3)^2} = \frac{A}{s-3} + \frac{B}{(s-3)^2}$$

Multiply $(s-3)^2$ **both side**

$$2s+5 = A(s-3)+B$$

Substitute s=3 gives B=11 and s=4 gives A=2

$$\therefore \mathcal{L}\left\{\frac{2s+5}{(s-3)^2}\right\} = 2\mathcal{L}\left\{\frac{1}{s-3}\right\} + 11\mathcal{L}\left\{\frac{1}{(s-3)^2}\right\}$$

Translation on the s-axis

☑ Example 2

Partial Fractions and Completing the Square

Evaluate

$$(a)\ \mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} \quad \text{and} \quad (b)\ \mathcal{L}^{-1}\left\{\frac{s/2+5/3}{s^2+4s+6}\right\}$$

$$\mathcal{L}\left\{\frac{2s+5}{(s-3)^2}\right\} = 2\mathcal{L}\left\{\frac{1}{s-3}\right\} + 11\mathcal{L}\left\{\frac{1}{(s-3)^2}\right\}$$

$$= 2e^{3t} + 11\mathcal{L}\left\{\frac{1}{s^2}\bigg|_{s\to s-3}\right\} = 2e^{3t} + 11e^{3t}t$$

Translation on the s-axis

☑ Example 2

Partial Fractions and Completing the Square

Evaluate

$$(a)\ \mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} \quad \text{and} \quad (b)\ \mathcal{L}^{-1}\left\{\frac{s/2+5/3}{s^2+4s+6}\right\}$$

$$(b)\ \frac{s/2+5/3}{s^2+4s+6} = \frac{s/2+5/3}{(s+2)^2+2} = \frac{(1/2)(s+2)+2/3}{(s+2)^2+2}$$

$$= \frac{1}{2}\frac{s+2}{(s+2)^2+2} + \frac{2}{3}\frac{1}{(s+2)^2+2}$$

Translation on the s-axis

☑ Example 2

Partial Fractions and Completing the Square

Evaluate

$$(a)\ \mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} \quad \text{and} \quad (b)\ \mathcal{L}^{-1}\left\{\frac{s/2+5/3}{s^2+4s+6}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s/2+5/3}{s^2+4s+6}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+2}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+2}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\left.\frac{s}{s^2+2}\right|_{s\to s+2}\right\} + \frac{2}{3\sqrt{2}}\mathcal{L}^{-1}\left\{\left.\frac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right|_{s\to s+2}\right\}$$

$$= \frac{1}{2}e^{-2t}\cos\sqrt{2}t + \frac{\sqrt{2}}{3}e^{-2t}\sin\sqrt{2}t$$

Translation on the s-axis

☑Example 3 Initial-Value Problem

Solve

$$y'' - 6y' + 9y = t^2 e^{3t}, \quad y(0) = 2, \quad y'(0) = 17$$

---

solution) $\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{t^2 e^{3t}\}$

$$s^2 Y(s) - sy(0) - y'(0) - 6[sY(s) - y(0)] + 9Y(s) = \frac{2}{(s-3)^3}$$

$$(s^2 - 6s + 9)Y(s) = 2s + 17 - 12 + \frac{2}{(s-3)^3}$$

$$(s-3)^2 Y(s) = 2s + 5 + \frac{2}{(s-3)^3}$$

$$Y(s) = \frac{2s+5}{(s-3)^2} + \frac{2}{(s-3)^5}$$

Translation on the s-axis

☑ Example 3 Initial-Value Problem

Solve

$$y'' - 6y' + 9y = t^2 e^{3t}, \quad y(0) = 2, \quad y'(0) = 17$$

$$Y(s) = \frac{2s+5}{(s-3)^2} + \frac{2}{(s-3)^5} = \frac{2(s-3)+11}{(s-3)^2} + \frac{2}{(s-3)^5}$$

$$= \frac{2}{s-3} + \frac{11}{(s-3)^2} + \frac{2}{(s-3)^5}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + 11\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} + \frac{2}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{(s-3)^5}\right\}$$

$$= 2e^{3t} + 11te^{3t} + \frac{1}{12}t^4 e^{3t}$$

Translation on the s-axis

☑Example 4 Initial-Value Problem

Solve

$$y'' + 4y' + 6y = 1 + e^{-t}, \quad y(0) = 0, \quad y'(0) = 0$$

---

solution) $\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} = \mathcal{L}\{1\} + \mathcal{L}\{e^{-t}\}$

$$s^2Y(s) - sy(0) - y'(0) + 4[sY(s) - y(0)] + 6Y(s) = \frac{1}{s} + \frac{1}{s+1}$$

$$(s^2 + 4s + 6)Y(s) = \frac{2s+1}{s(s+1)}$$

$$Y(s) = \frac{2s+1}{s(s+1)(s^2+4s+6)} = \frac{1/6}{s} + \frac{1/3}{s+1} - \frac{s/2 + 5/3}{s^2+4s+6}$$

Translation on the s-axis

☑Example 4 Initial-Value Problem

Solve

$$y'' + 4y' + 6y = 1 + e^{-t}, \quad y(0) = 0, \quad y'(0) = 0$$

$$Y(s) = \frac{1/6}{s} + \frac{1/3}{s+1} - \frac{s/2 + 5/3}{s^2 + 4s + 6} = \frac{1/6}{s} + \frac{1/3}{s+1} - \frac{\frac{1}{2}(s+2) + \frac{2}{3}}{(s+2)^2 + 2}$$

$$y(t) = \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{(s+2)}{(s+2)^2 + 2}\right\} - \frac{2}{3\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s+2)^2 + 2}\right\}$$

$$= \frac{1}{6} + \frac{1}{3}e^{-t} - \frac{1}{2}e^{-2t}\cos\sqrt{2}t - \frac{\sqrt{2}}{3}e^{-2t}\sin\sqrt{2}t$$

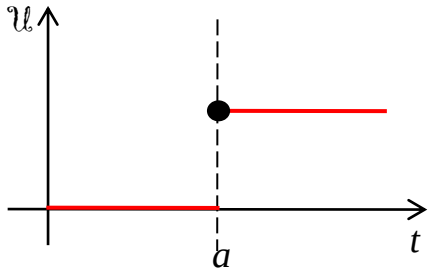
Translation on the t-axis

☑Unit step Function

Definition 4.3 **Unit Step Function**

The unit step function $\mathcal{U}(t-a)$ is defined to be

$$\mathcal{U}(t-a)=\begin{cases}0, & 0\le t<a\\ 1, & t\ge a\end{cases}$$



$$f(t)=\begin{cases}g(t), & 0\le t<a\\ h(t), & t\ge a\end{cases} \quad \Leftrightarrow \quad f(t)=g(t)-g(t)\mathcal{U}(t-a)+h(t)\mathcal{U}(t-a)$$

$$f(t)=\begin{cases}0 & 0\le t<a\\ g(t), & a\le t<b\\ 0, & t\ge b\end{cases} \quad \Leftrightarrow \quad f(t)=g(t)\{\mathcal{U}(t-a)-\mathcal{U}(t-b)\}$$

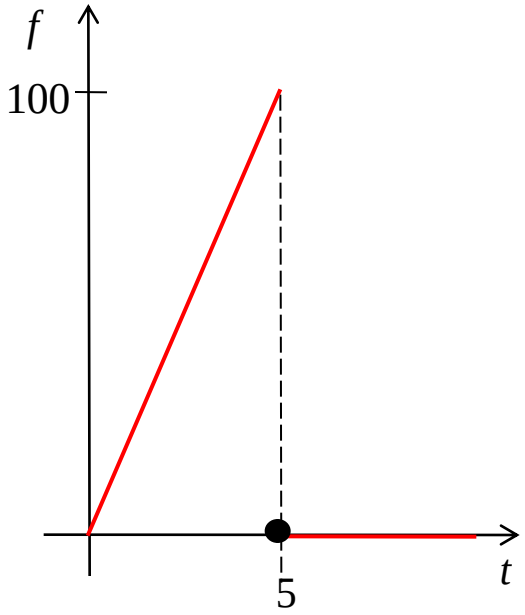
Translation on the t-axis

☑Example 5 A Piecewise-Defined Function

Express $f(t)=\begin{cases}20t, & 0\le t\le 5\\ 0, & t\ge 5\end{cases}$ **In terms of unit step functions. graph**

Solution)

$$f(t)=20t-20t\,\mathcal{U}(t-5)$$



☑Translation on the t-axis

Theorem 4.7 — **Second Translation Theorem**

If $\mathcal{L}\{f(t)\} = F(s)$ and a >0, then

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$$

Proof)

$$\mathcal{L}\{f(t-a)\,\mathcal{U}(t-a)\} = \int_0^a e^{-st} f(t-a)\underbrace{\mathcal{U}(t-a)}_{0 \text{ for } 0\le t<a} dt + \int_a^\infty e^{-st} f(t-a)\underbrace{\mathcal{U}(t-a)}_{1 \text{ for } t\ge a} dt$$

$$= \int_a^\infty e^{-st} f(t-a)dt$$

$$= \int_a^\infty e^{-s(v+a)} f(v)\,dv$$

$$= e^{-as}\int_0^\infty e^{-sv} f(v)dv = e^{-as}\mathcal{L}\{f(t)\} = e^{-as}F(s)$$

Translation on the t-axis

☑Translation on the t-axis

Theorem 4.7 — **Second Translation Theorem**

If $\mathcal{L}\{f(t)\} = F(s)$ **and a >0, then**

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$$

If $f(t-a) = 1$, then $F(s) = 1/s$. So

$$\mathcal{L}\{\mathcal{U}(t-a)\} = \frac{e^{-as}}{s}$$

☑Inverse form of theorem 4.7

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a)$$

Translation on the t-axis

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a)\cdots(15)$$

☑Example 6 Using Formula (15)

Evaluate $(a)\ \mathcal{L}^{-1}\left\{\frac{1}{s-4}e^{-2s}\right\}$ *and* $(b)\ \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}e^{-\pi s/2}\right\}$

Solution) **(a)** $a=2, F(s)=\frac{1}{(s-4)}, \mathcal{L}^{-1}\{F(s)\} = f(t) = e^{4t}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-4}e^{-2s}\right\} = e^{4(t-2)}\mathcal{U}(t-2)$$

(b) $a=\frac{\pi}{2}, F(s)=\frac{s}{s^2+9}, \mathcal{L}^{-1}\{F(s)\} = f(t) = \cos 3t$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}e^{-\pi s/2}\right\} = \cos 3\left(t-\frac{\pi}{2}\right)\mathcal{U}\left(t-\frac{\pi}{2}\right)$$

Translation on the t-axis

☑Alternate Form of Theorem 4.7

Theorem 4.7 — **Second Translation Theorem**

If $\mathcal{L}\{f(t)\} = F(s)$ **and a >0, then**

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$$

$$\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = \int_a^\infty e^{-st}g(t)dt = \int_a^\infty e^{-s(u+a)}g(u+a)du$$

$$\therefore \mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$$

"Alternate Form of Theorem 4.7"

Translation on the t-axis

☑ Example 7

Second Translation Theorem-Alternative form

Evaluate $\mathcal{L}\{\cos t\, \mathcal{U}(t-\pi)\}$.

Solution)

$$\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}\cdots(16)$$

$$g(t) = \cos t,\ a = \pi, \text{then},\ g(t+\pi) = \cos(t+\pi) = -\cos t$$

$$\mathcal{L}\{\cos t\, \mathcal{U}(t-\pi)\} = -e^{\pi s}\mathcal{L}\{\cos t\} = -\frac{s}{s^2+1}e^{-\pi s}$$

Translation on the t-axis

☑Example 8 An Initial-Value Problem

Solve

$$y' + y = f(t), \quad y(0) = 5, \quad \text{where } f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\cos t, & t \ge \pi \end{cases}$$

Solution)

We can rewrite $f(t)$ **as** $f(t) = 3\cos t\, \mathcal{U}(t-\pi)$

$$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 3\mathcal{L}\{\cos t\, \mathcal{U}(t-\pi)\}$$

$$sY(s) - y(0) + Y(s) = -3\frac{s}{s^2+1}e^{-\pi s}$$

$$(s+1)Y(s) = 5 - 3\frac{s}{s^2+1}e^{-\pi s}$$

$$Y(s) = \frac{5}{(s+1)} - 3\frac{s}{(s+1)(s^2+1)}e^{-\pi s}$$

Translation on the t-axis

☑Example 8 An Initial-Value Problem

Solve

$$y' + y = f(t), \quad y(0) = 5, \quad \text{where } f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\cos t, & t \ge \pi \end{cases}$$

$$Y(s) = \frac{5}{(s+1)} - 3\frac{s}{(s+1)(s^2+1)}e^{-\pi s}$$

$$\frac{s}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1} = \frac{(A+B)s^2 + (B+C)s + A + C}{(s+1)(s^2+1)}$$

$$A+B=0,\ B+C=1,\ A+C=0 \qquad A = -\frac{1}{2},\ B = \frac{1}{2},\ C = \frac{1}{2}$$

$$\frac{s}{(s+1)(s^2+1)} = -\frac{1}{2}\frac{1}{s+1} + \frac{1}{2}\frac{s+1}{s^2+1}$$

Translation on the t-axis

☑Example 8 An Initial-Value Problem

Solve

$$y' + y = f(t), \quad y(0) = 5, \quad \text{where } f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\cos t, & t \ge \pi \end{cases}$$

$$Y(s) = \frac{5}{(s+1)} - \frac{3}{2}\left[-\frac{1}{s+1}e^{-\pi s} + \frac{1}{s^2+1}e^{-\pi s} + \frac{s}{s^2+1}e^{-\pi s}\right]$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}e^{-\pi s}\right\} = e^{-(t-\pi)}\mathcal{U}(t-\pi), \quad \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}e^{-\pi s}\right\} = \sin(t-\pi)\mathcal{U}(t-\pi)$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}e^{-\pi s}\right\} = \cos(t-\pi)\mathcal{U}(t-\pi)$$

Translation on the t-axis

☑Example 8 An Initial-Value Problem

Solve

$$y' + y = f(t), \quad y(0) = 5, \quad \text{where } f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\cos t, & t \ge \pi \end{cases}$$

$$y(t) = 5\mathcal{L}^{-1}\left\{\frac{1}{(s+1)}\right\} - \frac{3}{2}\left[-\mathcal{L}^{-1}\left\{\frac{1}{s+1}e^{-\pi s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}e^{-\pi s}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}e^{-\pi s}\right\}\right]$$

$$= 5e^{-t} + \frac{3}{2}e^{-(t-\pi)}\mathcal{U}(t-\pi) - \frac{3}{2}\sin(t-\pi)\mathcal{U}(t-\pi) - \frac{3}{2}\cos(t-\pi)\mathcal{U}(t-\pi)$$

$$= 5e^{-t} + \frac{3}{2}\left[e^{-(t-\pi)} + \sin(\pi - t) - \cos(\pi - t)\right]\mathcal{U}(t-\pi)$$

$$= 5e^{-t} + \frac{3}{2}\left[e^{-(t-\pi)} + \sin t + \cos t\right]\mathcal{U}(t-\pi)$$

Translation on the t-axis

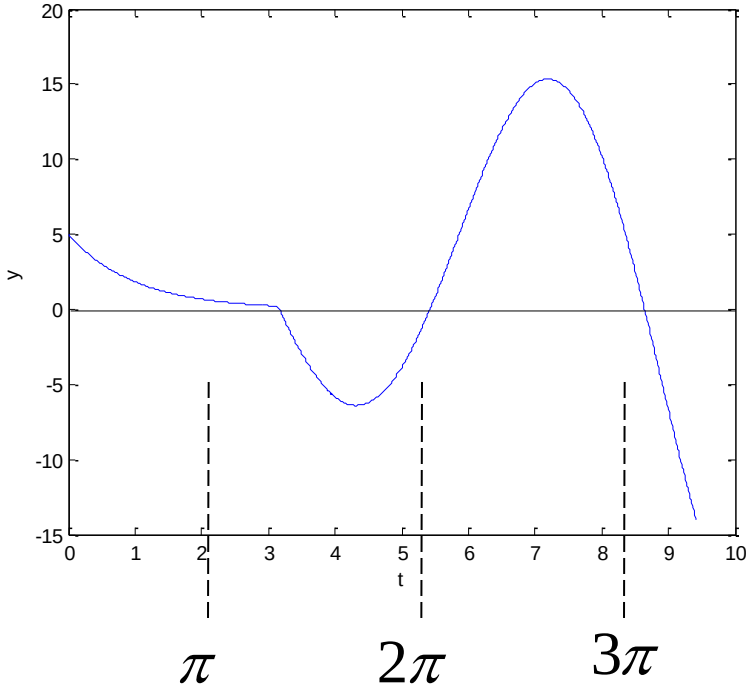
☑ Example 8 An Initial-Value Problem

Solve

$$y' + y = f(t), \quad y(0) = 5, \quad \text{where } f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\cos t, & t \ge \pi \end{cases}$$

$$y(t) = 5e^{-t} + \frac{3}{2}\left[e^{-(t-\pi)} + \sin t + \cos t\right]\mathcal{U}(t-\pi)$$

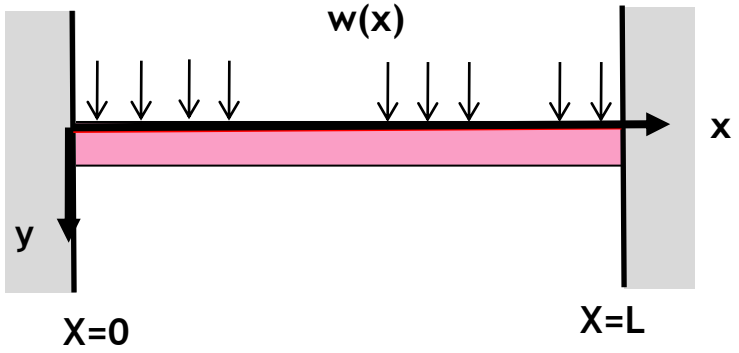
$$y(t) = \begin{cases} 5e^{-t} & , 0 \le t < \pi \\ 5e^{-t} + \frac{3}{2}e^{-(t-\pi)} + \frac{3}{2}\sin t + \frac{3}{2}\cos t & , t \ge \pi \end{cases}$$



Translation on the t-axis

☑ Beams

$$EI\frac{d^4y}{dx^4} = w(x)$$



When w(x) is piecewise defined,
it is useful to use Laplace transform.

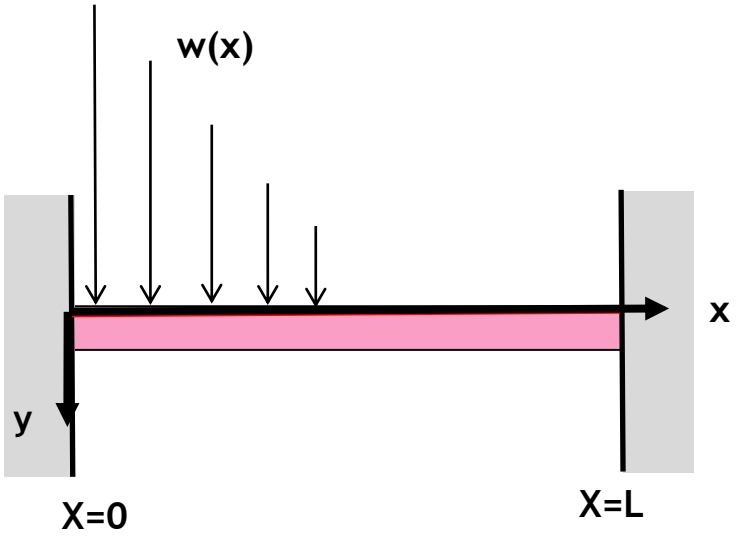
But, in order to use Laplace transform,
we have to assume that y(x) and w(x) are defined on (0,∞)

And this is a boundary-value problem.

Translation on the t-axis

☑Example 9 A Boundary-Value Problem

Find the deflection of the beam when the load is given by



$$w(x)=\begin{cases} w_0(1-\frac{2}{L}x), & 0<x<\frac{L}{2} \\ 0, & \frac{L}{2}<x<L, \end{cases}$$

Where w_0 is constant

Solution) **Boundary conditions are**

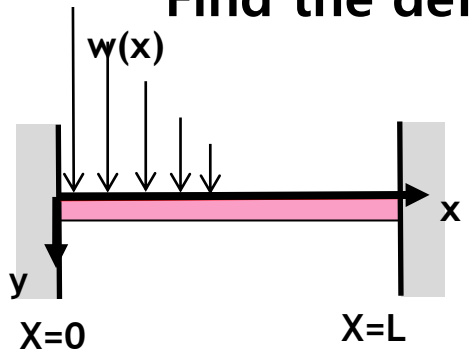
$$y(0)=0, y'(0)=0, y(L)=0, y'(L)=0$$

$$w(x)=w_0(1-\frac{2}{L}x)-w_0(1-\frac{2}{L}x)\mathcal{U}(x-\frac{L}{2})=\frac{2w_0}{L}\left[\frac{L}{2}-x+(x-\frac{L}{2})\mathcal{U}(x-\frac{L}{2})\right]$$

Translation on the t-axis

☑Example 9 A Boundary-Value Problem

Find the deflection of the beam when the load is given by



$$w(x)=\begin{cases} w_0(1-\frac{2}{L}x), & 0<x<\frac{L}{2} \\ 0, & \frac{L}{2}<x<L, \end{cases}$$

w_0 is constant

B.C.: $y(0)=0, y'(0)=0, y(L)=0, y'(L)=0$

$$w(x)=\frac{2w_0}{L}\left[\frac{L}{2}-x+(x-\frac{L}{2})\mathcal{U}(x-\frac{L}{2})\right]$$

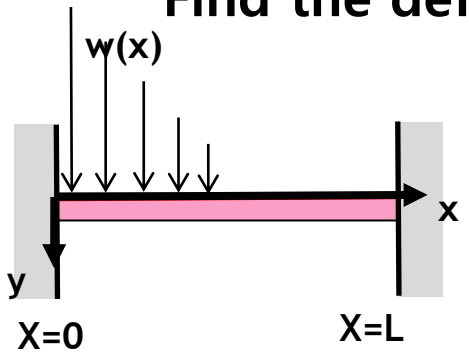
$$EI(s^4Y(s)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0))=\frac{2w_0}{L}\left[\frac{L/2}{s}-\frac{1}{s^2}+\frac{1}{s^2}e^{-Ls/2}\right]$$

$$s^4Y(s)-sy''(0)-y'''(0)=\frac{2w_0}{EIL}\left[\frac{L/2}{s}-\frac{1}{s^2}+\frac{1}{s^2}e^{-Ls/2}\right]$$

Translation on the t-axis

☑Example 9 A Boundary-Value Problem

Find the deflection of the beam when the load is given by



$$w(x)=\begin{cases} w_0\left(1-\frac{2}{L}x\right), & 0<x<\frac{L}{2} \\ 0, & \frac{L}{2}<x<L, \end{cases}$$

w_0 is constant

B.C.: $y(0)=0, y'(0)=0, y(L)=0, y'(L)=0$

$$s^4Y(s)-sy''(0)-y'''(0)=\frac{2w_0}{EIL}\left[\frac{L/2}{s}-\frac{1}{s^2}+\frac{1}{s^2}e^{-Ls/2}\right]$$

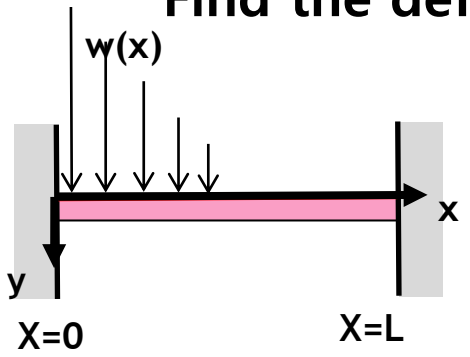
Let $c_1=y''(0),\ c_2=y'''(0)$ **then,**

$$Y(s)=\frac{c_1}{s^3}+\frac{c_2}{s^4}+\frac{2w_0}{EIL}\left[\frac{L/2}{s^5}-\frac{1}{s^6}+\frac{1}{s^6}e^{-Ls/2}\right]$$

Translation on the t-axis

☑Example 9 A Boundary-Value Problem

Find the deflection of the beam when the load is given by



$$w(x)=\begin{cases} w_0\left(1-\frac{2}{L}x\right), & 0<x<\frac{L}{2} \\ 0, & \frac{L}{2}<x<L, \end{cases}$$

w_0 is constant

B.C.: $y(0)=0, y'(0)=0, y(L)=0, y'(L)=0$

$$Y(s)=\frac{c_1}{s^3}+\frac{c_2}{s^4}+\frac{2w_0}{EIL}\left[\frac{L/2}{s^5}-\frac{1}{s^6}+\frac{1}{s^6}e^{-Ls/2}\right]$$

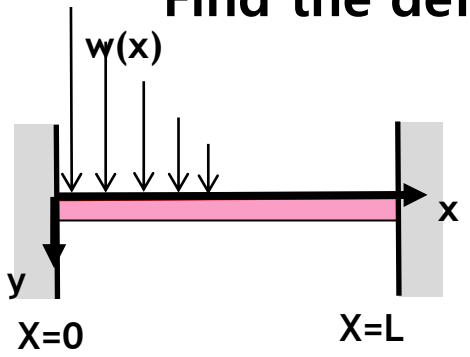
$$y(x)=\frac{c_1}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\}+\frac{c_2}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\}$$

$$+\frac{2w_0}{EIL}\left[\frac{L}{2\cdot 4!}\mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\}-\frac{1}{5!}\mathcal{L}^{-1}\left\{\frac{5!}{s^6}\right\}+\frac{1}{5!}\mathcal{L}^{-1}\left\{\frac{5!}{s^6}e^{-Ls/2}\right\}\right]$$

Translation on the t-axis

☑Example 9 A Boundary-Value Problem

Find the deflection of the beam when the load is given by



$$w(x)=\begin{cases} w_0\left(1-\frac{2}{L}x\right), & 0<x<\frac{L}{2} \\ 0, & \frac{L}{2}<x<L, \end{cases}$$

w_0 is constant

B.C.: $y(0)=0, y'(0)=0, y(L)=0, y'(L)=0$

$$y(x)=\frac{c_1}{2}x^2+\frac{c_2}{6}x^3+\frac{w_0}{60EIL}\left[\frac{5L}{2}x^4-x^5+\left(x-\frac{L}{2}\right)^5 u\left(x-\frac{L}{2}\right)\right]$$

According to Boundary condition $y(L)=0, y'(L)=0$

$$c_1\frac{L^2}{2}+c_2\frac{L^3}{6}+\frac{49w_0L^4}{1920EI}=0, \quad c_1L+c_2\frac{L^2}{2}+\frac{85w_0L^4}{960EI}=0$$

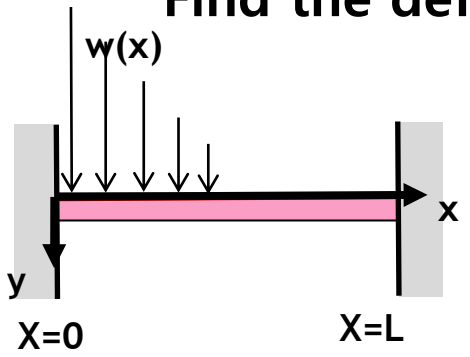
$$c_1=\frac{23w_0L^2}{960EI}$$

$$c_2=\frac{-9w_0L^2}{40EI}$$

Translation on the t-axis

☑Example 9 A Boundary-Value Problem

Find the deflection of the beam when the load is given by



$$w(x)=\begin{cases} w_0\left(1-\frac{2}{L}x\right), & 0<x<\frac{L}{2} \\ 0, & \frac{L}{2}<x<L, \end{cases}$$

w_0 is constant

B.C.: $y(0)=0, y'(0)=0, y(L)=0, y'(L)=0$

$$y(x)=\frac{c_1}{2}x^2+\frac{c_2}{6}x^3+\frac{w_0}{60EIL}\left[\frac{5L}{2}x^4-x^5+\left(x-\frac{L}{2}\right)^5\mathcal{U}\left(x-\frac{L}{2}\right)\right]$$

$$c_1=\frac{23w_0L^2}{960EI},\quad c_2=\frac{-9w_0L^2}{40EI}$$

$$\therefore y(x)=\frac{23w_0L^2}{1920EI}x^2-\frac{3w_0L}{80EI}x^3+\frac{w_0}{60EIL}\left[\frac{5L}{2}x^4-x^5+\left(x-\frac{L}{2}\right)^5\mathcal{U}\left(x-\frac{L}{2}\right)\right]$$

Derivatives of Transform

☑ Multiplying a Function by t^n

$$\frac{d}{ds}F(s)=\frac{d}{ds}\int_0^\infty e^{-st}f(t)dt=\int_0^\infty \frac{\partial}{\partial s}[e^{-st}f(t)]dt=-\int_0^\infty e^{-st}tf(t)dt=-\mathcal{L}\{tf(t)\}$$

$$\mathcal{L}\{tf(t)\}=-\frac{d}{ds}F(s)$$

Similarly, $\mathcal{L}\{t^2 f(t)\}=\mathcal{L}\{t\cdot tf(t)\}=-\mathcal{L}\{tf(t)\}$

$$=-\frac{d}{ds}(-\frac{d}{ds}\mathcal{L}\{f(t)\})=\frac{d^2}{ds^2}\mathcal{L}\{f(t)\}$$

The proceeding two cases suggest the general result

Theorem 4.8 **Derivatives of Transform**

If $\mathcal{L}\{f(t)\}=F(s)$ **and n=1,2,3,..., then**

$$\mathcal{L}\{t^n f(t)\}=(-1)^n\frac{d^n}{ds^n}F(s)$$

Derivatives of Transform

☑Example 1 Using Theorem 4.8

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

Evaluate $\mathcal{L}\{t \sin kt\}$

Solution)

$$f(t) = \sin kt,\ F(s) = \frac{k}{s^2 + k^2},\ n = 1$$

$$\mathcal{L}\{t \sin kt\} = -\frac{d}{ds}\mathcal{L}\{\sin kt\} = -\frac{d}{ds}\left(\frac{k}{s^2 + k^2}\right) = \frac{2ks}{(s^2 + k^2)^2}$$

Derivatives of Transform

☑Example 2 An Initial-Value Problem

Solve

$$x'' + 16x = \cos 4t, \quad x(0) = 0, \quad x'(0) = 1$$

Solution)

$$\mathcal{L}\{x''\} + 16\mathcal{L}\{x\} = \mathcal{L}\{\cos 4t\}$$

$$s^2 X(s) - sx(0) - x'(0) + 16sX(s) - 16x(0) = \frac{s}{s^2 + 16}$$

$$(s^2 + 16s)X(s) = 1 + \frac{s}{s^2 + 16}$$

$$X(s) = \frac{1}{s^2 + 16s} + \frac{s}{(s^2 + 16)^2}$$

Derivatives of Transform

☑Example 2 An Initial-Value Problem

Solve

$$x'' + 16x = \cos 4t, \quad x(0) = 0, \quad x'(0) = 1$$

$$X(s) = \frac{1}{s^2 + 16s} + \frac{s}{(s^2 + 16)^2}$$

In Example 1,

$$\mathcal{L}\{t \sin kt\} = \frac{2ks}{(s^2 + k^2)^2} \Rightarrow \mathcal{L}^{-1}\left\{\frac{2ks}{(s^2 + k^2)^2}\right\} = t \sin kt$$

$$x(t) = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{4}{s^2 + 16}\right\} + \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{8s}{(s^2 + 16)^2}\right\} = \frac{1}{4}\sin 4t + \frac{1}{8}t \sin 4t$$

Transform of Integrals

☑ Convolution

$$f * g = \int_0^t f(\tau) g(t-\tau) d\tau$$ **Ref : Properties of convolution**

Theorem 4.9 **Convolution Theorem**

If $f(t)$ and $g(t)$ are piecewise continuous on [0,∞) and of exponential order, then

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$$

Proof)

Additional Operational Properties

☑ Example 3 Transform of a Convolution

Evaluate

$$\mathcal{L}\left\{\int_0^t e^{\tau}\sin(t-\tau)d\tau\right\}$$

Solution)

$$f(t)=e^t,\ \ g(t)=\sin t$$

$$\mathcal{L}\left\{\int_0^t e^{\tau}\sin(t-\tau)d\tau\right\}=F(s)G(s)=\mathcal{L}\{e^t\}\mathcal{L}\{\sin t\}$$

$$=\frac{1}{s-1}\cdot\frac{1}{s^2+1}=\frac{1}{(s-1)(s^2+1)}$$

Transform of Integrals

Theorem 4.9 — **Convolution Theorem**

If $f(t)$ and $g(t)$ are piecewise continuous on [0,∞) and of exponential order, then

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$$

☑ Inverse Form of Theorem 4.9

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f * g$$

Transform of Integrals

☑Example 4 Inverse Transform as a Convolution

Evaluate $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+k^2)^2}\right\}$

Solution) Let $F(s)=G(s)=\dfrac{1}{(s^2+k^2)^2}$

$$f(t)=g(t)=\frac{1}{k}L^{-1}\left\{\frac{k}{s^2+k^2}\right\}=\frac{1}{k}\sin kt$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\}=\frac{1}{k^2}\int_0^t \sin k\tau \sin k(t-\tau)d\tau$$

$$=\frac{1}{k^2}\int_0^t \frac{1}{2}[\cos(k\tau-kt+k\tau)-\cos(k\tau+kt-k\tau)]d\tau$$

$$=\frac{1}{2k^2}\int_0^t [\cos(2k\tau-kt)-\cos(kt)]d\tau$$

Transform of Integrals

☑Example 4 Inverse Transform as a Convolution

Evaluate $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+k^2)^2}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} = \frac{1}{2k^2}\int_0^t [\cos(2k\tau - kt) - \cos(kt)]d\tau$$

$$= \frac{1}{2k^2}\left[\frac{1}{2k}\sin k(2\tau - t) - \tau\cos kt\right]_0^t$$

$$= \frac{\sin kt - kt\cos kt}{2k^3}$$

Transform of Integrals

☑ Transform of an integral

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$$ **Convolution Theorem**

When $g(t) = 1$, $G(s) = 1/s$ **So,**

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s} \cdots (7)$$

Inverse of (7)

$$\int_0^\infty f(\tau)d\tau = \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} \cdots (8)$$

By (8)

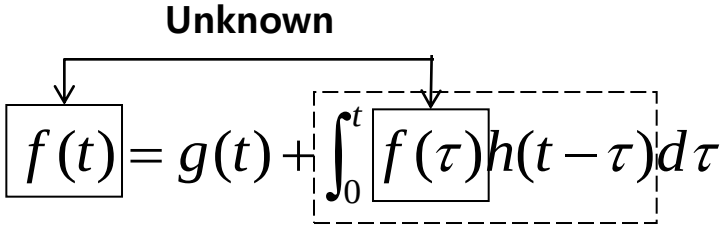
$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin\tau\, d\tau = 1 - \cos t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t (1-\cos\tau)\, d\tau = t - \sin t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \int_0^t (\tau - \sin\tau)\, d\tau = \frac{t^2}{2} - 1 + \cos t$$ **And so on.**

Transform of Integrals

☑ Volterra integral Equation



: Volterra integral equation for f(t)

Additional Operational Properties

☑Example 5 An Integral Equation

Solve $f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau) e^{t-\tau} d\tau$ for $f(t)$

Solution) $h(t-\tau) = e^{t-\tau}$, $h(t) = e^t$

$$\mathcal{L}\{f(t)\} = 3\mathcal{L}\{t^2\} - \mathcal{L}\{e^{-t}\} - \mathcal{L}\left\{\int_0^t f(\tau) e^{t-\tau} d\tau\right\}$$

$$F(s) = 3\frac{2!}{s^3} - \frac{1}{s+1} - F(s)\cdot\frac{1}{s-1}$$

$$F(s) = \frac{6}{s^3} - \frac{6}{s^4} + \frac{1}{s} - \frac{2}{s+1}$$

Additional Operational Properties

☑Example 5 An Integral Equation

Solve $f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau) e^{t-\tau} d\tau$ for $f(t)$

Solution)

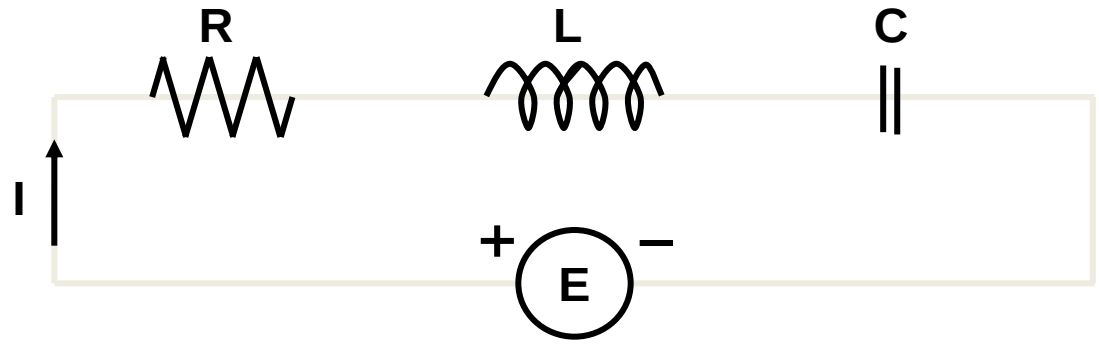
$$F(s) = \frac{6}{s^3} - \frac{6}{s^4} + \frac{1}{s} - \frac{2}{s+1}$$

$$f(t) = 3\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\} - \mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= 3t^2 - t^3 + 1 - 2e^{-t}$$

Transform of Integrals

☑ Series Circuits



$$L\frac{di}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = E(t)$$

Integrodifferential equation

In chapter 3.8 we differentiate equation to eliminate integral, but by Using Laplace transform, we don't have to differentiate equation

☑Example 6 An Integrodifferential Equation

Determine the current $i(t)$ in a single-loop LRC-circuit When L=0.1[h], R=2[Ω], C=0.1[f], $i(0)=0$ and the impressed voltage is

$$E(t)=120t-120t\,\mathcal{U}(t-1)$$

Solution)

$$0.1\frac{di}{dt}+2i+10\int_0^t i(\tau)d\tau=120t-120t\,\mathcal{U}(t-1)$$

$$0.1\mathcal{L}\left\{\frac{di}{dt}\right\}+2\mathcal{L}\{i\}+10\mathcal{L}\left\{\int_0^t i(\tau)d\tau\right\}=120\mathcal{L}\{t\}-120\mathcal{L}\{t\,\mathcal{U}(t-1)\}$$

$$0.1sI(s)+2I(s)+10\frac{I(s)}{s}=120\left[\frac{1}{s^2}-\frac{1}{s^2}e^{-s}-\frac{1}{s}e^{-s}\right]$$

☑Example 6 An Integrodifferential Equation

Determine the current $i(t)$ in a single-loop LRC-circuit When L=0.1[h], R=2[Ω], C=0.1[f], $i(0)=0$ and the impressed voltage is

$$E(t)=120t-120t\,\mathcal{U}(t-1)$$

$$0.1sI(s)+2I(s)+10\frac{I(s)}{s}=120\left[\frac{1}{s^2}-\frac{1}{s^2}e^{-s}-\frac{1}{s}e^{-s}\right]$$

$$(s^2+20s+100)I(s)=1200\left[\frac{1}{s}-\frac{1}{s}e^{-s}-e^{-s}\right]$$

$$I(s)=1200\left[\frac{1}{s(s+10)^2}-\frac{1}{s(s+10)^2}e^{-s}-\frac{1}{(s+10)^2}e^{-s}\right]$$

☑Example 6 An Integrodifferential Equation

Determine the current $i(t)$ in a single-loop LRC-circuit When L=0.1[h], R=2[Ω], C=0.1[f], $i(0)=0$ and the impressed voltage is

$$E(t)=120t-120t\,\mathcal{U}(t-1)$$

$$I(s)=1200\left[\frac{1}{s(s+10)^2}-\frac{1}{s(s+10)^2}e^{-s}-\frac{1}{(s+10)^2}e^{-s}\right]$$

By partial fraction,

$$I(s)=1200\left[\frac{1/100}{s}-\frac{1/100}{s+10}-\frac{1/10}{(s+10)^2}\right.$$
$$\left.-\frac{1/100}{s}e^{-s}+\frac{1/100}{s+10}e^{-s}+\frac{1/10}{(s+10)^2}e^{-s}-\frac{1}{(s+10)^2}e^{-s}\right]$$

Transform of Integrals

☑Example 6 An Integrodifferential Equation

Determine the current $i(t)$ in a single-loop LRC-circuit When L=0.1[h], R=2[Ω], C=0.1[f], $i(0)=0$ and the impressed voltage is

$$E(t)=120t-120t\,\mathcal{U}(t-1)$$

$$I(s)=1200\left[\frac{1/100}{s}-\frac{1/100}{s+10}-\frac{1/10}{(s+10)^2}\right.$$
$$\left.-\frac{1/100}{s}e^{-s}+\frac{1/100}{s+10}e^{-s}+\frac{1/10}{(s+10)^2}e^{-s}-\frac{1}{(s+10)^2}e^{-s}\right]$$

$$i(t)=12[1-\mathcal{U}(t-1)]-12[e^{-10t}-e^{-10(t-1)}\mathcal{U}(t-1)]$$
$$-120te^{-10t}-1080(t-1)e^{-10(t-1)}\mathcal{U}(t-1)$$

Transform of a Periodic Function

☑ Periodic function

$$f(t+T) = f(t) \ (\text{period}: T,\ T > 0)$$

Theorem 4.10 — **Transform of a Periodic Function**

If $f(t)$ is piecewise continuous on [0,∞), of exponential order, and periodic with period *T*, then

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)dt$$

Proof) $$\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)dt + \int_T^\infty e^{-st} f(t)dt$$

Let *t=u+T*

$$\int_T^\infty e^{-st} f(t)dt = \int_0^\infty e^{-s(u+T)} f(u+T)du = e^{-sT} \int_0^\infty e^{-su} f(u)du = e^{-sT}\mathcal{L}\{f(t)\}$$

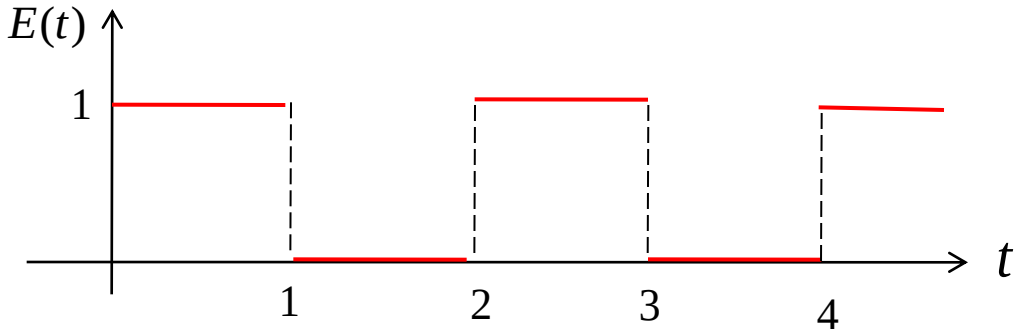
Therefore $$\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)dt + e^{-st}\mathcal{L}\{f(t)\}$$

Solving for $\mathcal{L}\{f(t)\}$ Proves the theorem

Transform of a Periodic Function

☑Example 7 Transform of a Periodic Function

Find the Laplace transform of the periodic function shown in Figure



('Square wave')

Solution)

$$T = 2, \quad E(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{cases} \quad \text{on } 0 \le t < 2$$

$$\mathcal{L}\{E(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} E(t)dt = \frac{1}{1-e^{-2s}} \int_0^1 e^{-st} dt$$

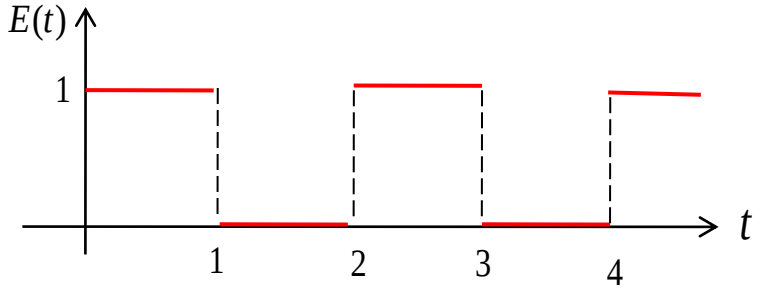
$$= \frac{1}{1-e^{-2s}} \frac{1-e^{-s}}{s} = \frac{1}{s(1+e^{-s})}$$

Transform of a Periodic Function

☑Example 8 A Periodic Impressed Voltage

$$L\frac{di}{dt}+Ri=E(t)$$

Determine the $i(t)$ **When** $i(0)=0$ and $E(t)$ is shown in figure



('Square wave')

Solution)

$$L\mathcal{L}\left\{\frac{di}{dt}\right\}+R\mathcal{L}\{i\}=\mathcal{L}\{E(t)\}$$

$$LsI(s)+RI(s)=\frac{1}{s(1+e^{-s})} \quad or \quad I(s)=\frac{1/L}{s(s+R/L)}\cdot\frac{1}{1+e^{-s}}$$

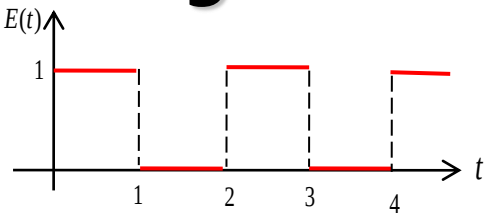
In example 7

$$\mathcal{L}\{E(t)\}=\frac{1}{s(1+e^{-s})}$$

Transform of a Periodic Function

☑ Example 8 A Periodic Impressed Voltage

$$L\frac{di}{dt}+Ri=E(t) \qquad E(t)=\begin{cases}1, & 0\le t<1\\ 0, & 1\le t<2\end{cases} \qquad E(t+2)=E(t)$$



Determine the $i(t)$ When $i(0)=0$

$$\frac{1}{1+x}=1-x+x^2-x^3+\cdots \xrightarrow{x=e^{-s}} \frac{1}{1+e^{=s}}=1-e^{=s}+e^{=2s}-e^{=3s}+\cdots$$

$$\frac{1}{s(s+R/L)}=\frac{L/R}{s}-\frac{L/R}{s+R/L}$$

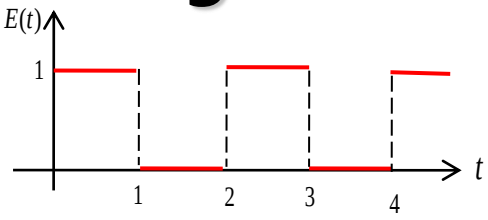
$$I(s)=\frac{1/L}{s(s+R/L)}\cdot\frac{1}{1+e^{-s}}=\frac{1}{R}\left(\frac{1}{s}-\frac{1}{s+R/L}\right)(1-e^{=s}+e^{=2s}-e^{=3s}+\cdots)$$

$$=\frac{1}{R}\left(\frac{1}{s}-\frac{e^{=s}}{s}+\frac{e^{=2s}}{s}-\frac{e^{=3s}}{s}+\cdots\right)-\frac{1}{R}\left(\frac{1}{s+R/L}-\frac{e^{=s}}{s+R/L}+\frac{e^{=2s}}{s+R/L}-\cdots\right)$$

Transform of a Periodic Function

☑ Example 8 A Periodic Impressed Voltage

$$L\frac{di}{dt}+Ri=E(t) \qquad E(t)=\begin{cases}1, & 0\le t<1\\ 0, & 1\le t<2\end{cases} \qquad E(t+2)=E(t)$$

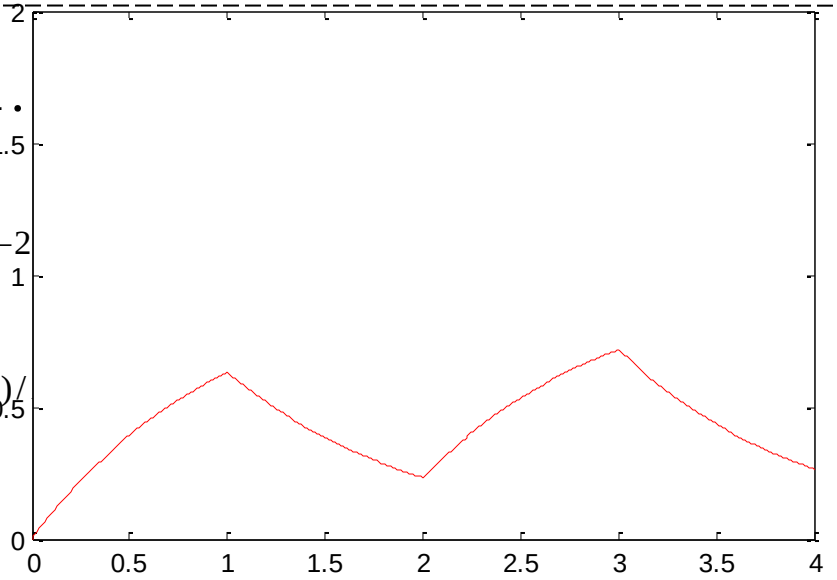


Determine the $i(t)$ When $i(0)=0$

$$i(t)=\frac{1}{R}\left(1-\mathcal{U}(t-1)+\mathcal{U}(t-2)-\mathcal{U}(t-3)+\cdots\right.$$

$$-\frac{1}{R}\left(e^{-Rt/L}-e^{-R(t-1)/L}\mathcal{U}(t-1)+e^{-R(t-2}\right.$$

$$i(t)=\frac{1}{R}\left(1-e^{-Rt/L}\right)+\frac{1}{R}\sum_{n=1}^{\infty}(-1)^n(1-e^{-R(t-n)/}$$



when $R=1, L=1$ *and* $0\le t<4$

$$i(t)=1-e^{-t}-\left(1-e^{-(t-1)}\right)\mathcal{U}(t-1)+\left(1-e^{-(t-2)}\right)\mathcal{U}(t-2)-\left(1-e^{-(t-3)}\right)\mathcal{U}(t-3)$$

The Dirac Delta Function

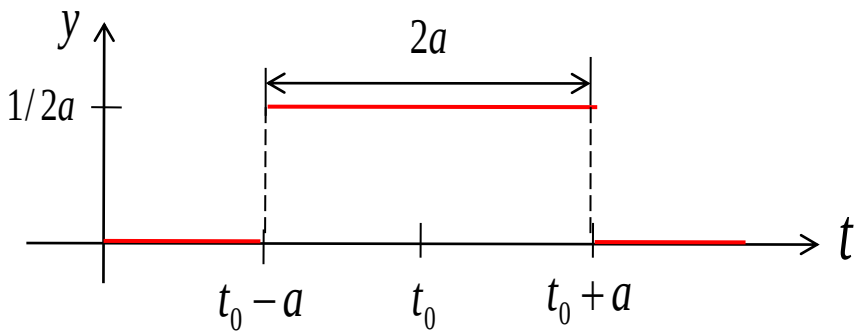
☑ Unit Impulse

- External force of large magnitude that acts only for a very short period of time

$$\delta_a(t-t_0)=\begin{cases}0, & 0\le t<t_0-a\\ \dfrac{1}{2a}, & t_0-a\le t<t_0+a\\ 0, & t\ge t_0+a\end{cases}$$

$$\int_0^\infty \delta_a(t-t_0)dt=1$$

->'Unit' impulse



For small ***a***, $\delta_a(t-t_0)$ have large magnitude.

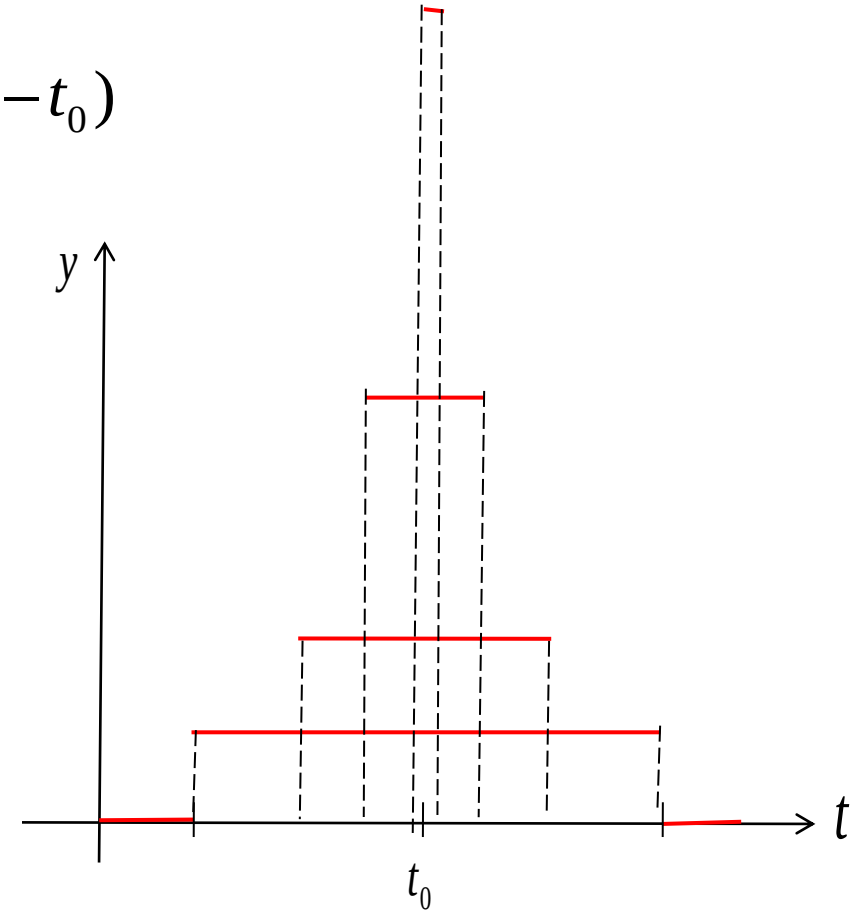
The Dirac Delta Function

☑ The Dirac Delta Function

$$\delta(t-t_0)=\lim_{a\to 0}\delta_a(t-t_0)$$

$$(i)\ \delta(t-t_0)=\begin{cases}\infty, & t=t_0\\ 0, & t\neq t_0\end{cases}$$

$$(ii)\ \int_0^x \delta(t-t_0)dt=1$$



☑ The Dirac Delta Function

$$\delta(t-t_0)=\lim_{a\to 0}\delta_a(t-t_0)$$

Theorem 4.11 — **Transform of the Dirac Delta Function**

For $t_0 > 0$

$$\mathcal{L}\{\delta(t-t_0)\}=e^{-st_0}$$

Proof) $$\delta_a(t-t_0)=\frac{1}{2a}[\mathcal{U}(t-(t_0-a))-\mathcal{U}(t-(t_0+a))]$$

$$\mathcal{L}\{\delta_a(t-t_0)\}=\frac{1}{2a}\left[\frac{e^{-s(t_0-a)}}{s}-\frac{e^{-s(t_0+a)}}{s}\right]=e^{-st_0}\left(\frac{e^{sa}-e^{-sa}}{2sa}\right)$$

$$\mathcal{L}\{\delta(t-t_0)\}=\lim_{a\to 0}\mathcal{L}\{\delta_a(t-t_0)\}=\lim_{a\to 0}e^{-st_0}\left(\frac{e^{sa}-e^{-sa}}{2sa}\right)=e^{-st_0}$$

When $t_0 = 0$, $$\mathcal{L}\{\delta(t)\}=1$$

The Dirac Delta Function

☑Example 1 Two Initial-Value Problems

Solve $y'' + y = 4\delta(t - 2\pi)$ **subject to**

$(a)\ y(0) = 1,\ y'(0) = 0$ $\qquad$ $(b)\ y(0) = 0,\ y'(0) = 0$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = 4\mathcal{L}\{\delta(t - 2\pi)\}$$

$$s^2Y(s) - sy(0) - y'(0) + sY(s) - y(0) = 4e^{-2\pi s}$$

$$(a)\quad s^2Y(s) - s + Y(s) = 4e^{-2\pi s}\ \ or\ \ Y(s) = \frac{s}{s^2 + 1} + \frac{4e^{-2\pi s}}{s^2 + 1}$$

$$y(t) = \cos t + 4\sin(t - 2\pi)\mathcal{U}(t - 2\pi)$$

$$y(t) = \begin{cases} \cos t & 0 \le t < 2\pi \\ \cos t + 4\sin t & t \ge 2\pi \end{cases}$$

The Dirac Delta Function

☑Example 1 Two Initial-Value Problems

Solve $y'' + y = 4\delta(t - 2\pi)$ **subject to**

$(a)\ y(0) = 1,\ y'(0) = 0$ $\qquad$ $(b)\ y(0) = 0,\ y'(0) = 0$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = 4\mathcal{L}\{\delta(t - 2\pi)\}$$

$$s^2Y(s) - sy(0) - y'(0) + sY(s) - y(0) = 4e^{-2\pi s}$$

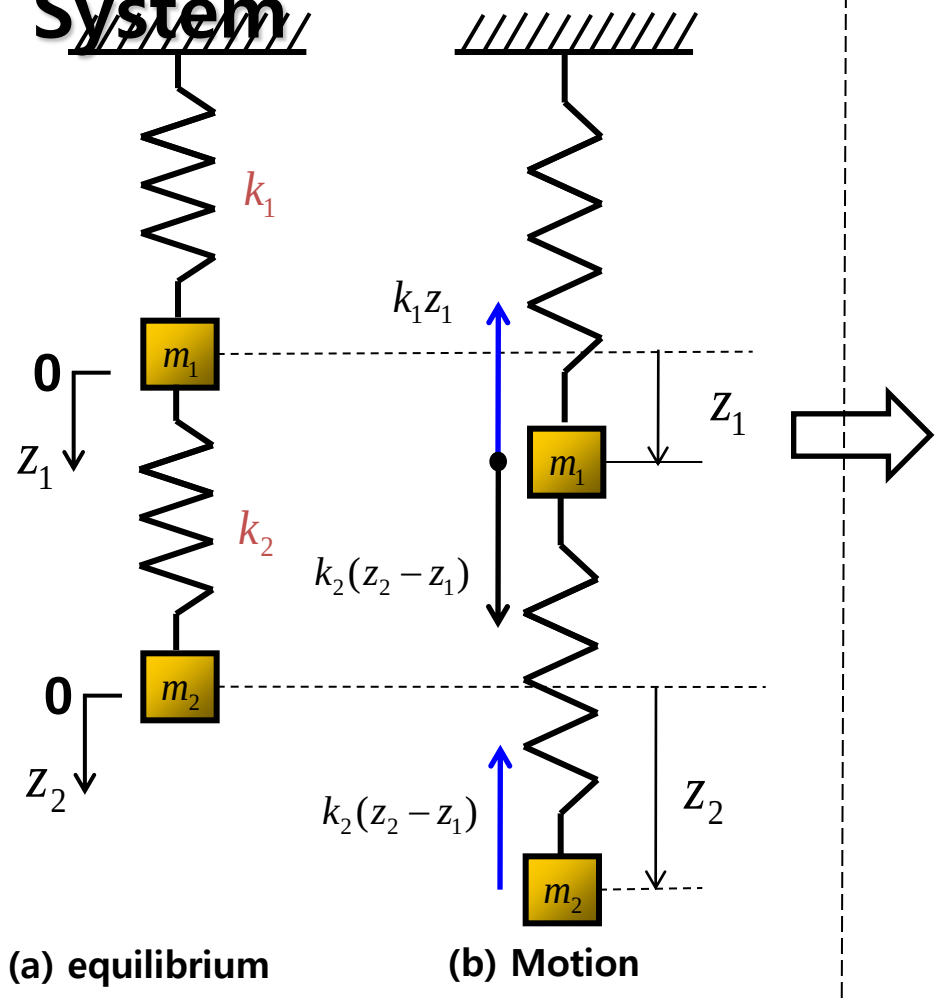
(b) $\quad s^2Y(s) + Y(s) = 4e^{-2\pi s}$ *or* $Y(s) = \dfrac{4e^{-2\pi s}}{s^2 + 1}$

$$y(t) = 4\sin(t - 2\pi)\mathcal{U}(t - 2\pi)$$

$$y(t) = \begin{cases} 0 & 0 \le t < 2\pi \\ 4\sin t & t \ge 2\pi \end{cases}$$

Systems of Linear Differential Equations

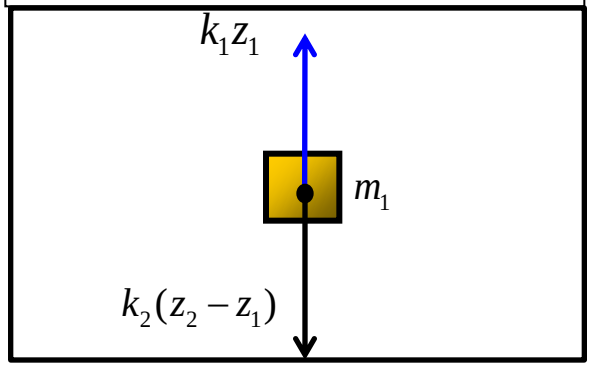
☑Coupled Spring/Mass System



(a) equilibrium (b) Motion

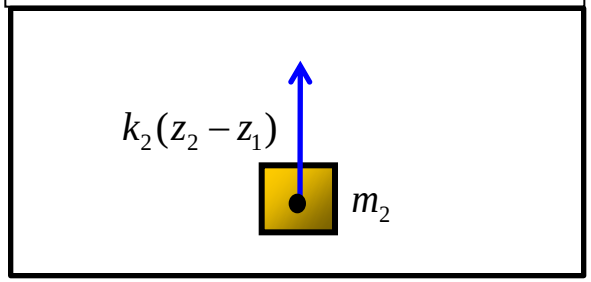
From Newton's 2nd raw,

Free body diagram for m_1



$$m_1 \frac{d^2 z_1}{dt^2} = -k_1 z_1 + k_2(z_2 - z_1)$$

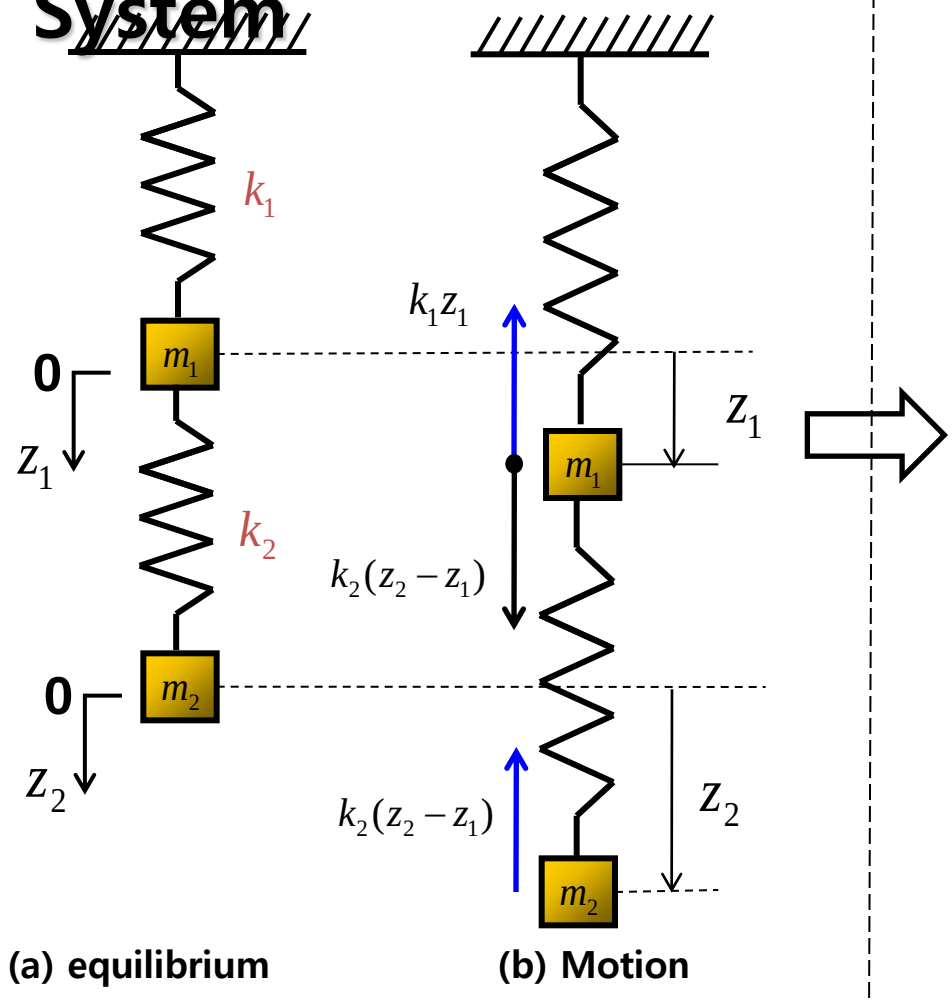
Free body diagram for m_2

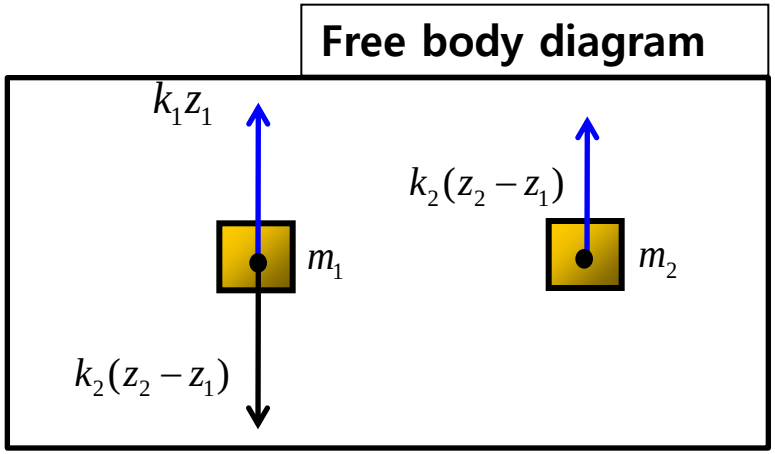


$$m_2 \frac{d^2 z_2}{dt^2} = -k_2(z_2 - z_1)$$

Systems of Linear Differential Equations

☑ Coupled Spring/Mass System





System of linear 2nd order equations

$$\begin{cases} m_1 \dfrac{d^2 z_1}{dt^2} = -k_1 z_1 + k_2 (z_2 - z_1) \\ m_2 \dfrac{d^2 z_2}{dt^2} = -k_2 (z_2 - z_1) \end{cases}$$

Systems of Linear Differential Equations

☑Example 1 Example 4 of Section 3.11 Revisited

Use the Laplace transform to solve

$$x_1'' + 10x_1 - 4x_2 = 0$$
$$4x_1 + x_2'' + 4x_2 = 0$$

Subject to

$$x_1(0) = 0,\ x_1'(0) = 1$$
$$x_2(0) = 0,\ x_2'(0) = -1$$

$$s^2X_1(s) - sx_1(0) - x_1'(0) + 10X_1(s) - 4X_2(s) = 0$$
$$-4X_1(s) + s^2X_2(s) - sx_2(0) - x_2'(0) + 4X_2(s) = 0$$

$$\begin{aligned}(s^2+10)X_1(s) - \quad & 4X_2(s) = 1 \\ -4X_1(s) + (s^2+4)X_2(s) &= -1\end{aligned} \quad \cdots(1)$$

$$X_1(s) = \frac{s^2}{(s^2+2)(s^2+12)} = -\frac{1/5}{s^2+2} + \frac{6/5}{s^2+12}$$

$$X_2(s) = -\frac{s^2+6}{(s^2+2)(s^2+12)} = -\frac{2/5}{s^2+2} - \frac{3/5}{s^2+12}$$

Systems of Linear Differential Equations

☑Example 1 Example 4 of Section 3.11 Revisited

Use the Laplace transform to solve

$$x_1'' + 10x_1 - 4x_2 = 0$$
$$4x_1 + x_2'' + 4x_2 = 0$$

Subject to

$$x_1(0) = 0,\ x_1'(0) = 1$$
$$x_2(0) = 0,\ x_2'(0) = -1$$

$$X_1(s) = \frac{s^2}{(s^2+2)(s^2+12)} = -\frac{1/5}{s^2+2} + \frac{6/5}{s^2+12}$$

$$x_1(t) = -\frac{1}{5\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\right\} + \frac{6}{5\sqrt{12}}\mathcal{L}^{-1}\left\{\frac{\sqrt{12}}{s^2+12}\right\}$$

$$= -\frac{\sqrt{2}}{10}\sin\sqrt{2}t + \frac{\sqrt{3}}{5}\sin 2\sqrt{3}t$$

Systems of Linear Differential Equations

☑Example 1 Example 4 of Section 3.11 Revisited

Use the Laplace transform to solve

$$x_1'' + 10x_1 - 4x_2 = 0$$
$$4x_1 + x_2'' + 4x_2 = 0$$

Subject to

$$x_1(0) = 0,\ x_1'(0) = 1$$
$$x_2(0) = 0,\ x_2'(0) = -1$$

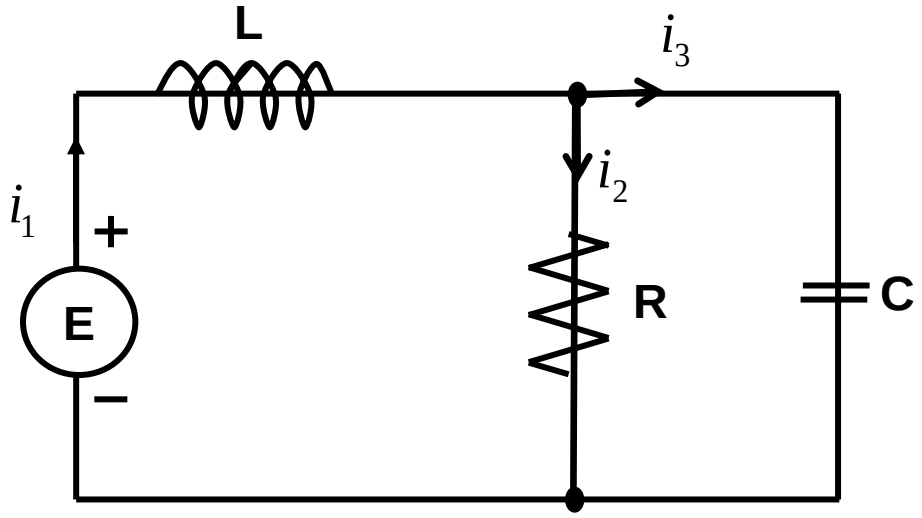
$$X_2(s) = -\frac{s^2+6}{(s^2+2)(s^2+12)} = -\frac{2/5}{s^2+2} - \frac{3/5}{s^2+12}$$

$$x_2(t) = -\frac{2}{5\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\right\} - \frac{3}{5\sqrt{12}}\mathcal{L}^{-1}\left\{\frac{\sqrt{12}}{s^2+12}\right\}$$

$$= -\frac{\sqrt{2}}{5}\sin\sqrt{2}t - \frac{\sqrt{3}}{10}\sin 2\sqrt{3}t$$

Systems of Linear Differential Equations

☑ Networks



$$L\frac{di_1}{dt} + Ri_2 = E(t)$$

$$RC\frac{di_2}{dt} + i_2 - i_1 = 0$$

☑Example 2 An Electrical Network

Solve the system under the condition $E(t)=60[\mathrm{V}],\ L=1[h],\ R=50[\Omega]$

$C=10^{-4}[\mathrm{f}]$

$$L\frac{di_1}{dt}+Ri_2=E(t)$$

$$RC\frac{di_2}{dt}+i_2-i_1=0$$

Subject to $i_1(0)=0,\ i_2(0)=0$

$$\frac{di_1}{dt}+50i_2=60 \qquad sI_1(s)+50I_2(s)=\frac{60}{s}$$

$$50(10^{-4})\frac{di_2}{dt}+i_2-i_1=0 \qquad -200I_1(s)+(s+200)I_2(s)=0$$

$$I_1(s)=\frac{60s+12{,}000}{s(s+100)^2}=\frac{6/5}{s}-\frac{6/5}{s+100}-\frac{60}{(s+100)^2}$$

$$I_2(s)=\frac{12{,}000}{s(s+100)^2}=\frac{6/5}{s}-\frac{6/5}{s+100}-\frac{120}{(s+100)^2}$$

Systems of Linear Differential Equations

☑Example 2 An Electrical Network

Solve the system under the condition $E(t) = 60[\mathrm{V}],\ L = 1[h],\ R = 50[\Omega]$

$$C = 10^{-4}[\mathrm{f}]$$

$$L\frac{di_1}{dt} + Ri_2 = E(t)$$

$$RC\frac{di_2}{dt} + i_2 - i_1 = 0$$

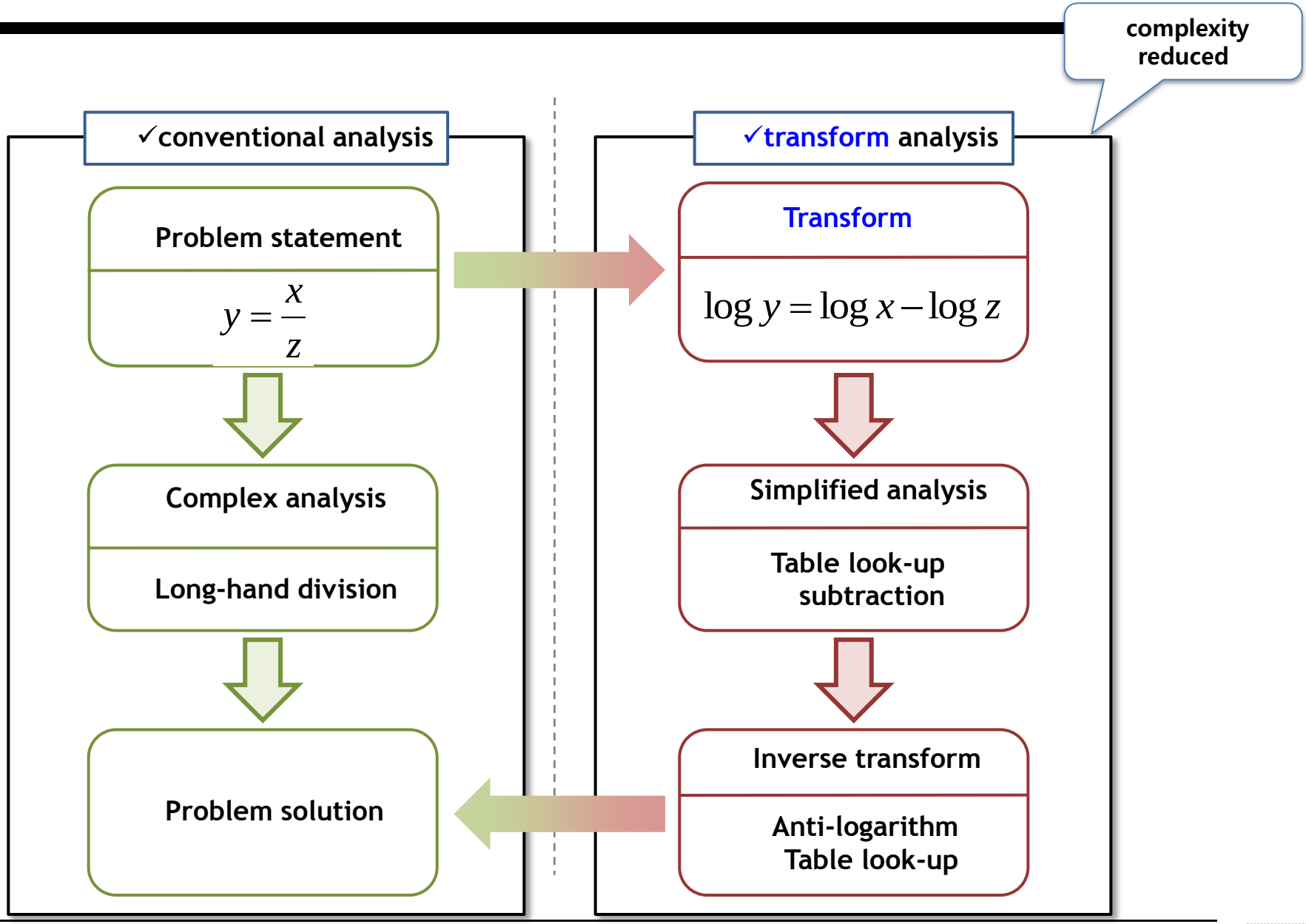
Subject to $i_1(0) = 0,\ i_2(0) = 0$

$$i_1(t) = \frac{6}{5} - \frac{6}{5}e^{-100t} - 60te^{-100t}$$

$$i_2(t) = \frac{6}{5} - \frac{6}{5}e^{-100t} - 120te^{-100t}$$

5-1. TRANSFORMS

What is Transform ?



What kind of Transform ?

$$y = \frac{x}{z} \xrightarrow{\log} \log y = \log x - \log z$$

$$y = x^2 \xrightarrow{\frac{d}{dx}} \frac{d}{dx} y = 2x$$

$$y = x^2 \xrightarrow{\int dx} \int y dx = \frac{1}{3} x^3 + c$$

Laplace Transform

$$f(t) \xrightarrow{\int_0^\infty e^{-st}\, dt} \int_0^\infty e^{-st} f(t) dt = \mathcal{L}\{f(t)\}$$

Fourier Transform

$$s(t) \xrightarrow{\int_{-\infty}^\infty e^{-i\omega t}\, dt} \int_{-\infty}^\infty e^{-i\omega t} s(t) dt = S(\omega)$$

Ref. Fourier Transform

Laplace Transform

Definition

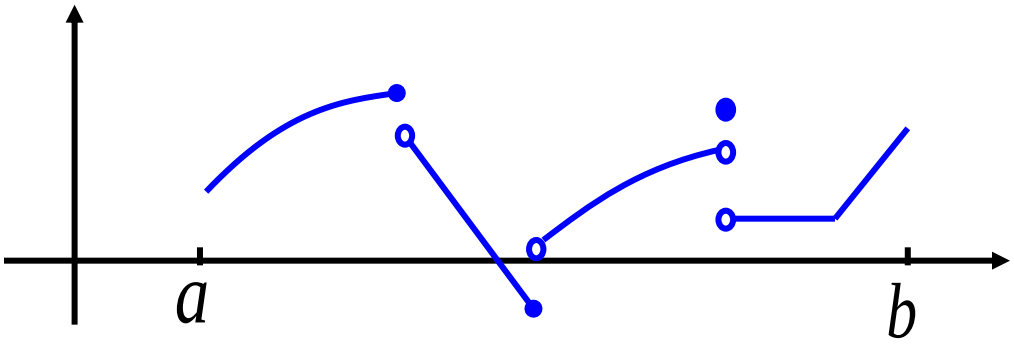
$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$,provided the integral converges

sufficient conditions for existence

$f(t)$ **: be piecewise continuous on** $[0, \infty)$

Piecewise Continuous Function

- **A function has a value on a finite interval [*a*, *b*].**
- **It has finite limits as *t* approaches either endpoints of a interval.**

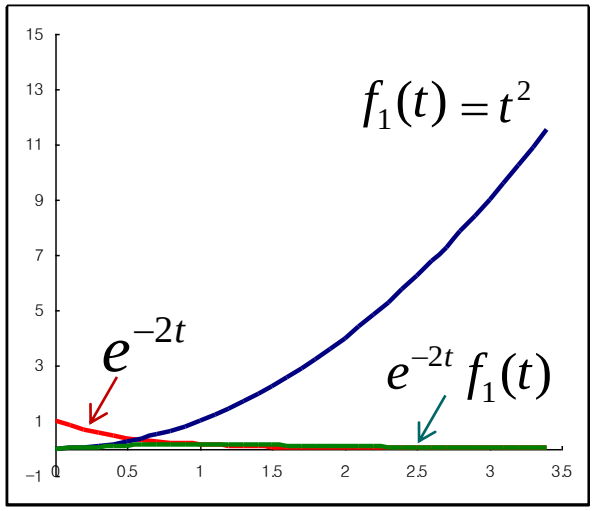


Laplace Transform

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$ **Definition**

sufficient conditions for existence

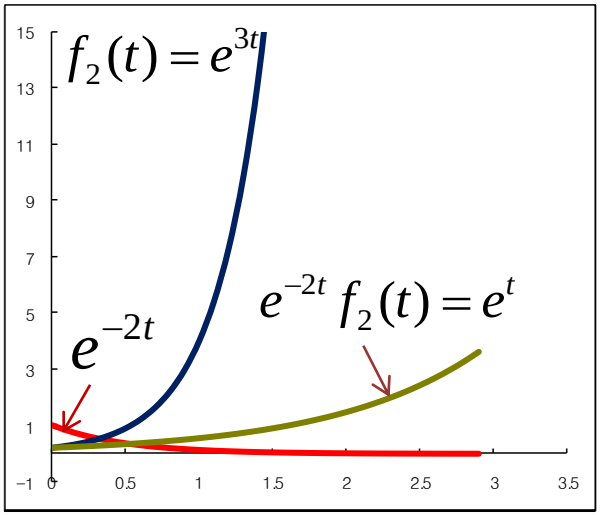
$f(t)$ **: be of exponential order for** $t > T$



t^2 **Increase as** $t \to \infty$

$e^{-st}t^2$ **Converge to 0 as** $t \to \infty$

$\int_0^\infty e^{-st}t^2 dt$ ➔ **The integral exists**



e^{3t} **Increase as** $t \to \infty$

$e^{-st}e^{3t}$ **Diverge to infinite as** $t \to \infty$

$\int_0^\infty e^{-st}e^{3t} dt$ ➔ **The integral does not exist**

Laplace Transform

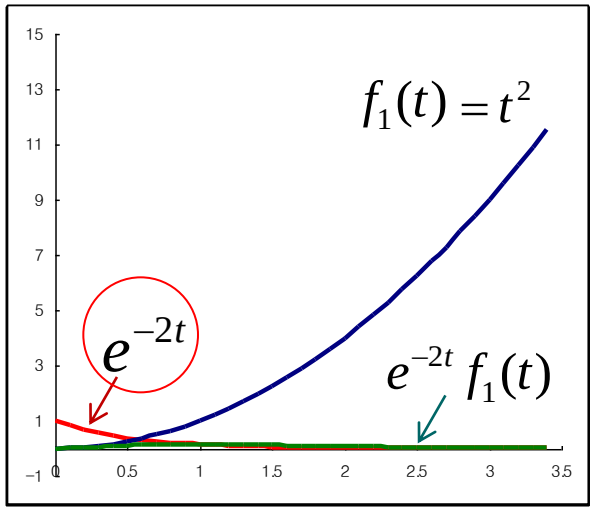
$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

Definition

sufficient conditions for existence

$f(t)$: **be of exponential order for** $t > T$

Playing an important role in cooling $f(t)$ **down to be convergent on [0,∞)**



t^2 **Increase as** $t \to \infty$

$e^{-st}t^2$ **Converge to 0 as** $t \to \infty$

$\int_0^\infty e^{-st}t^2 dt$ ➔ **The integral exists**

Let $g(t)=Me^{kt}$ and prove the existence of Laplace transform

$$\int_0^\infty e^{-st} g(t)dt = \int_0^\infty e^{-st} Me^{kt} dt = \int_0^\infty Me^{-(s-k)t} dt$$

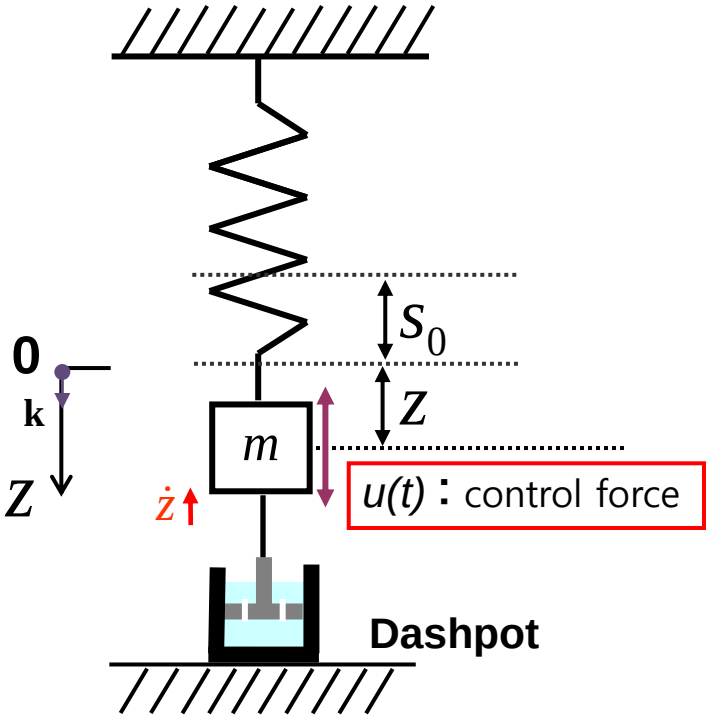
$$= \left[-\frac{M}{s-k} e^{-(s-k)t} \right]_0^\infty = \begin{cases} \frac{M}{s-k}, & (s > k) \\ \infty, & (s \le k) \end{cases}$$

A function of which absolute value is smaller than exponential function $g(t)$ has Laplace transform.

$$|f(t)| \le Me^{kt}$$

The Laplace transform exists for all $s>k$.

Dynamic System Modeling : Linear ODE

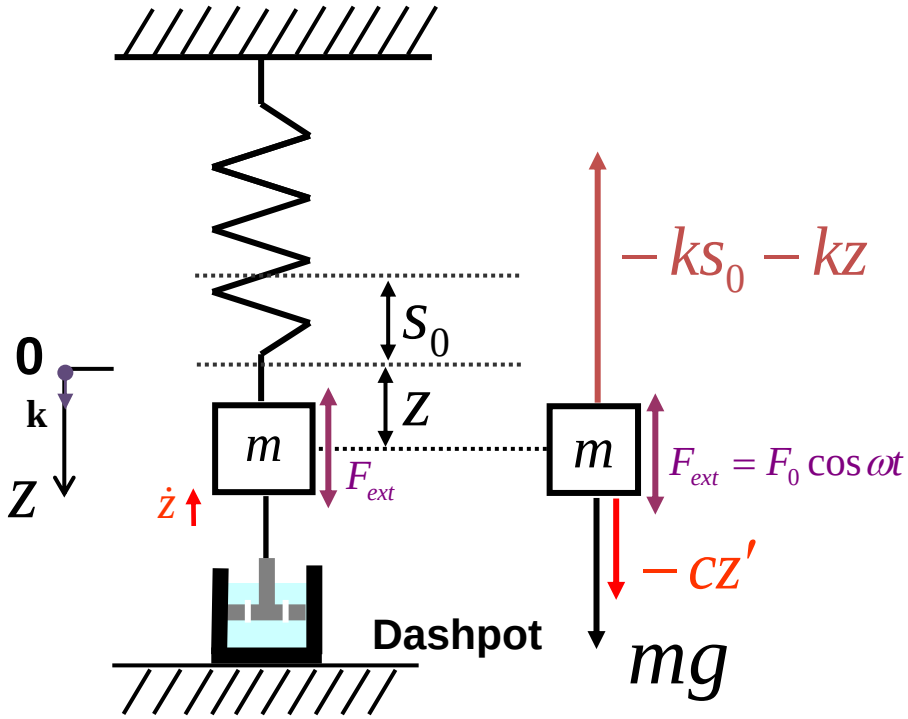


$$mz'' = F$$

$$= mg - k(s_0 + z) - cz' + u(t)$$

$$= -kz - cz' + u(t)$$

$$mz'' + cz' + kz = u(t)$$



$$m\mathbf{z}'' = \mathbf{F}$$

$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F}_0 \cos \omega t$$

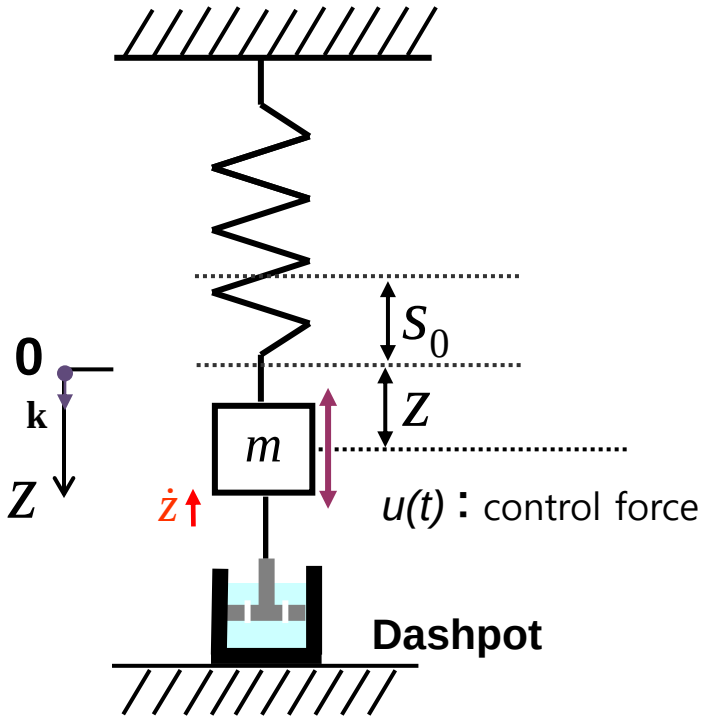
$$= -kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F}_0 \cos \omega t$$

Physical Phenomenon

Mathematical Equation

$$m\mathbf{z}'' + c\mathbf{z}' + k\mathbf{z} = \mathbf{F}_0 \cos \omega t$$

Dynamic System Modeling : Linear ODE



$$mz'' = F$$

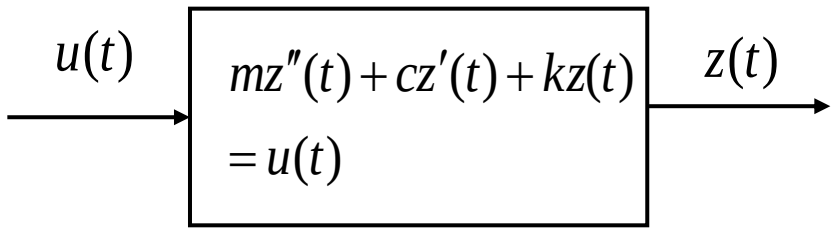
$$= mg - k(s_0 + z) - cz' + u(t)$$

$$= -kz - cz' + u(t)$$

$$mz'' + cz' + kz = u(t)$$

$$mz''(t) + cz'(t) + kz(t) = u(t)$$

Input(control force): *u(t)* Output: *z(t)*



Method to find the solution of the 2nd-order O.D.E

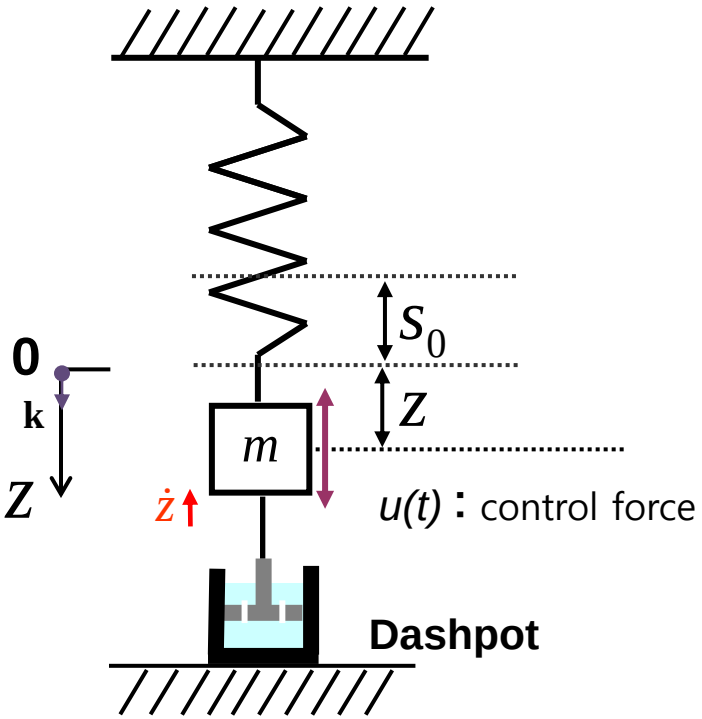
Laplace Transform

$$\mathcal{L}\{z(t)\}$$

$$z(t) = e^{\lambda t}$$

5-2. Solution of Linear ODE by Laplace Transform

Solution of Linear ODE by Laplace Transform



$$mz'' = F$$
$$= mg - k(s_0 + z) - cz' + u(t)$$
$$= -kz - cz' + u(t)$$

$$mz'' + cz' + kz = u(t)$$

Laplace Transform

Find unknown z(t) that satisfies a D.E. and initial condition

Laplace transform $\mathcal{L}$

Transformed DE becomes an algebraic equation in Z(s)

Solve transformed algebraic equation for Z(s)

inverse transform $\mathcal{L}^{-1}$

Solution z(t) of original D.E

Solution of Linear ODE by Laplace Transform

$$mz''(t)+cz'(t)+kz(t)=u(t)$$

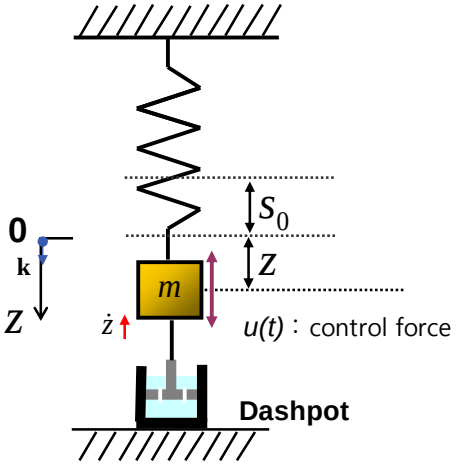
Laplace Transform

$$\mathcal{L}(mz''(t)+cz'(t)+kz(t))=\mathcal{L}(u(t))$$

Linearity

$$\mathcal{L}(mz''(t))+\mathcal{L}(cz'(t))+\mathcal{L}(kz(t))=\mathcal{L}(u(t))$$

$$m[s^2Z(s)-sz(0)-z'(0)]+c[sZ(s)-z(0)]+kZ(s)=U(s)$$



$$mz''=F$$
$$=mg-k(s_0+z)-cz'+u(t)$$
$$=-kz-cz'+u(t)$$

$$mz''+cz'+kz=u(t)$$

$$\mathcal{L}\{u(t)\}=U(s)$$

$$\mathcal{L}\{z(t)\}=Z(s)$$

Integrating by Part

$$\int_a^b u(x)v'(x)dx=\left[u(x)v(x)\right]_a^b-\int_a^b u'(x)v(x)dx$$

$$\mathcal{L}\{z'(t)\}=\int_0^\infty e^{-st}z'(t)dt=e^{-st}z(t)\Big|_0^\infty-\int_0^\infty(-s)e^{-st}z(t)dt=-z(0)+s\mathcal{L}\{z(t)\}$$

$$\therefore \mathcal{L}\{z'(t)\}=sZ(s)-z(0)$$

$$\mathcal{L}\{z''(t)\}=\int_0^\infty e^{-st}z''(t)dt=e^{-st}z'(t)\Big|_0^\infty+s\int_0^\infty e^{-st}z'(t)dt=-z'(0)+s\mathcal{L}\{z'(t)\}$$

$$=s[sZ(s)-z(0)]-z'(0)$$

$$\therefore \mathcal{L}\{z''(t)\}=s^2Z(s)-sz(0)-z'(0)$$

Solution of Linear ODE by Laplace Transform

$$mz''(t)+cz'(t)+kz(t)=u(t)$$

D.E.

Laplace Transform

$$\mathcal{L}(mz''(t)+cz'(t)+kz(t))=\mathcal{L}(u(t))$$

Linearity

$$\mathcal{L}(mz''(t))+\mathcal{L}(cz'(t))+\mathcal{L}(kz(t))=\mathcal{L}(u(t))$$

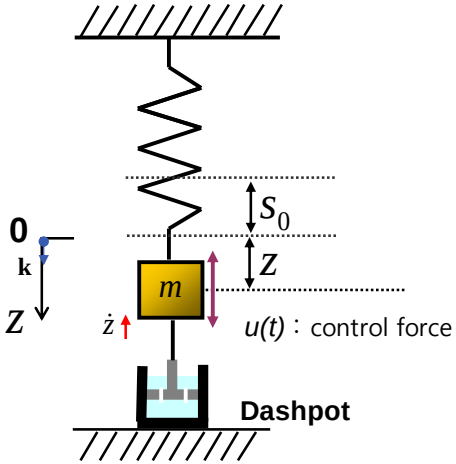
$$m[s^2Z(s)-sz(0)-z'(0)]+c[sZ(s)-z(0)]+kZ(s)=U(s)$$

$$ms^2Z(s)-msz(0)-mz'(0)+csZ(s)-cz(0)+kZ(s)=U(s)$$

$$(ms^2+cs+k)Z(s)=msz(0)+mz'(0)+cz(0)+U(s)$$

$$Z(s)=\frac{1}{(ms^2+cs+k)}[\overline{Q(s)}+U(s)]$$

algebraic equation



$$mz''=F$$
$$=mg-k(s_0+z)-cz'+u(t)$$
$$=-kz-cz'+u(t)$$

$$mz''+cz'+kz=u(t)$$

Solution of Linear ODE by Laplace Transform

$$mz''(t)+cz'(t)+kz(t)=u(t)$$

Laplace Transform

$$\mathcal{L}(mz''(t)+cz'(t)+kz(t))=\mathcal{L}(u(t))$$

$$(ms^2+cs+k)Z(s)=msz(0)+mz'(0)+cz(0)+U(s)$$

$$Z(s)=\frac{1}{(ms^2+cs+k)}[Q(s)+U(s)]$$

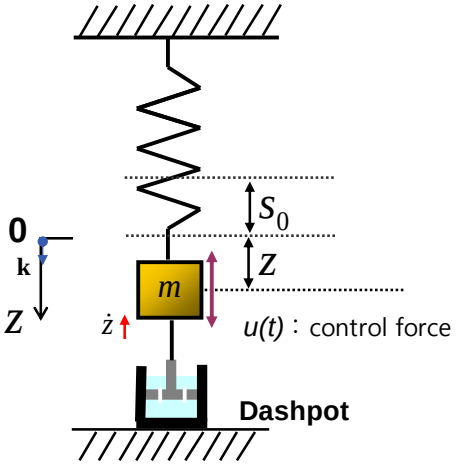
$$Z(s)=W(s)[Q(s)+U(s)]=W(s)Q(s)+W(s)U(s)$$

$$\boxed{Z(s)=W(s)Q(s)+W(s)U(s)}$$

$W(s)$: **transfer function**

$W(s)Q(s)$: **the effect on the response that is due to the initial condition**

$W(s)U(s)$: **the effect on the response that is due to the input function**



$$mz'' = F$$
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$$= -kz - cz' + u(t)$$

$$mz''+cz'+kz=u(t)$$

Solution of Linear ODE by Laplace Transform

$$mz''(t)+cz'(t)+kz(t)=u(t)$$

Laplace Transform

$$\mathcal{L}(mz''(t)+cz'(t)+kz(t))=\mathcal{L}(u(t))$$

$$Z(s)=W(s)Q(s)+W(s)U(s)$$

$$\therefore z(t)=\mathcal{L}^{-1}\{W(s)Q(s)\}+\mathcal{L}^{-1}\{W(s)U(s)\}=z_0(t)+z_1(t)$$

If the input $u(t)=0$

$$Z(s)=W(s)Q(s)$$

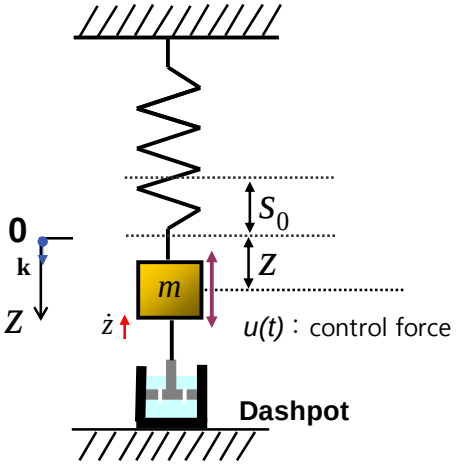
$$\therefore z(t)=z_0(t)=\mathcal{L}^{-1}\{W(s)Q(s)\}$$ **Zero-input response**

If all the initial conditions are zero, $Q(s)=msz(0)+mz'(0)+cz(0)=0$

$$Z(s)=W(s)U(s)$$

$$\therefore z(t)=z_1(t)=\mathcal{L}^{-1}\{W(s)U(s)\}$$ **Zero-state response**

→ **To see the response due to control force only**



$$mz''=F$$
$$=mg-k(s_0+z)-cz'+u(t)$$
$$=-kz-cz'+u(t)$$

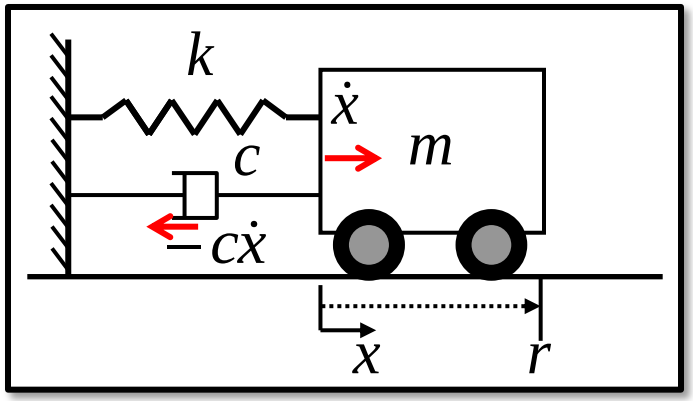
$$mz''+cz'+kz=u(t)$$

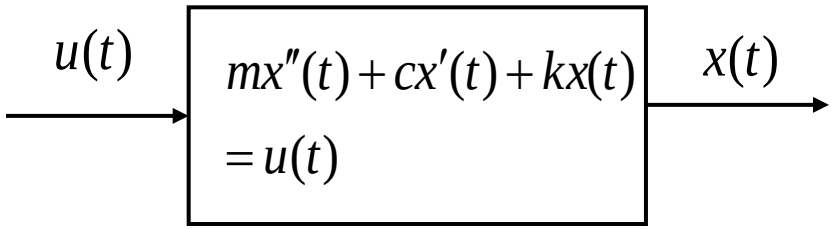
Table of Laplace Transform

$f(t)$	$L(f)$	$f(t)$	$L(f)$
1	$\frac{1}{s}$	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
t	$\frac{1}{s^2}$	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
t^2	$\frac{2!}{s^3}$	$\cosh at$	$\frac{s}{s^2-a^2}$
t^n	$\frac{n!}{s^{n+1}}$	$\sinh at$	$\frac{a}{s^2-a^2}$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$	$e^{at}\cos \omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
e^{at}	$\frac{1}{s-a}$	$e^{at}\sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$

5-3. Proportional, Integral, Derivative Controller(PID Controller)

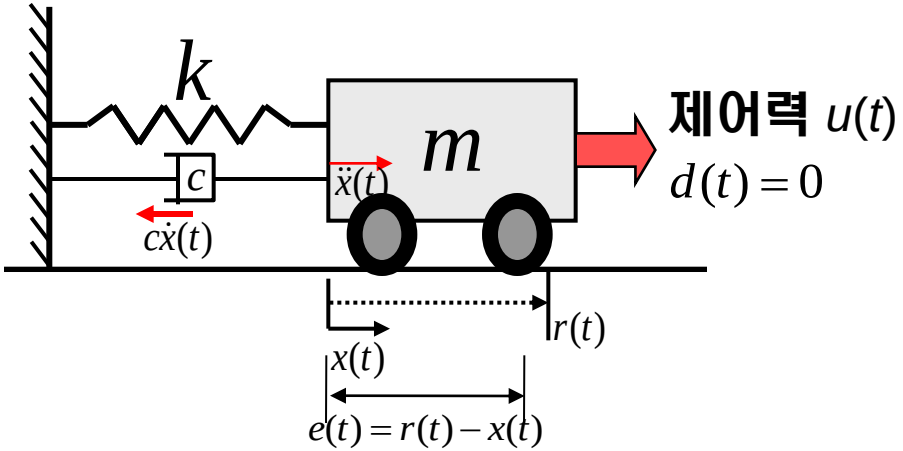
Mass-Spring-Damper Linear Mechanical System*





- ☑ 수레와 지면의 사이의 마찰을 무시함
- ☑ 초기 상태
  - ■ 수레의 초기 위치 : $x=0$
  - ■ 수레의 초기 속도 : $x'=0$
  - ■ 수레의 초기 가속도 : $x''=0$
- ☑ 목표
  - ■ 수레의 거동 특성을 변경시키는 것
  - ■ 수레의 위치를 r로 유지시키는 것
  - ■ 수레의 위치를 변화시키기 위해서 수레에 가하는 제어력 u의 크기를 결정해야 함

Proportional, Integral, Derivative Controller(PID Controller)



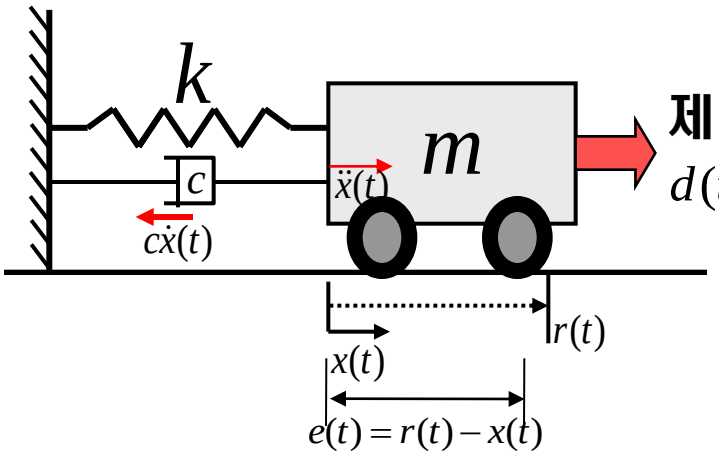
$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=u(t)$$

목표 값[$r(t)$]과 현재 상태[$x(t)$]의 차이[$e(t)$]를 이용하여 제어 입력[$u(t)$]를 계산

분류	제어력 계산 법	특성
Proportional	$u(t)=K_p e(t)$	✓ 제어 대상에 따라 목표 값에 수렴하지 못할 수 있음
Integral	$u(t)=K_i\int_0^t e(t)dt$	✓ 제어 대상이 목표 값에 수렴함 ✓ 오차의 누적을 이용하기 때문에 오차가 0인 순간에도 제어력이 생성될 수 있음
Derivative	$u(t)=K_d\dot{e}(t)$	✓ 목표 값과는 상관 없이 제어 시스템의 Damping 특성을 변경함

1) Proportional Controller

Proportional Controller(비례 제어기)



목적
- 시스템의 특성을 변경 시킴
- 수레의 위치를 목표 값에 도달하게 함

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=u(t)$$

$$u(t)=K_p e(t)$$

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=K_p e(t)$$

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=K_p\left(r(t)-x(t)\right)$$

$$m\ddot{x}(t)+c\dot{x}(t)+\left(k+K_p\right)x(t)=K_p r(t)$$

$$x(t)=x_p+K_p\cdot Be^{\alpha t}+K_p\cdot Ce^{\beta t}$$

해: $x(t)=x_p+K_p\cdot Be^{\alpha t}+K_p\cdot Ce^{\beta t}$

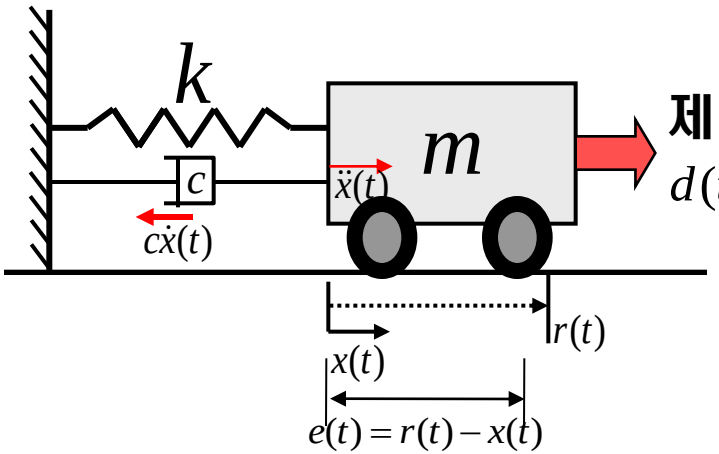
$$\alpha,\ \beta=\frac{-c\pm\sqrt{c^2-4mK^*}}{2m}$$

m, c, k는 **항상 양수이므로**
α,β**의 실수부인** -c/2m**은 음수이다.**

where $k+K_p=K^*$, $\alpha,\ \beta=\frac{-c\pm\sqrt{c^2-4mK^*}}{2m}$, $A=\frac{1}{K^*}$,

$$B=-\frac{c+\sqrt{c^2-4mK^*}}{2K^*\sqrt{c^2-4mK^*}},\quad C=\frac{c-\sqrt{c^2-4mK^*}}{2K^*\sqrt{c^2-4mK^*}}$$

- Proportional Controller(비례 제어기)



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$$u(t)=K_p e(t)$$

해: $x(t)=x_p+K_p\cdot Be^{\alpha t}+K_p\cdot Ce^{\beta t}$

$$m\ddot{x}(t)+c\dot{x}(t)+\left(k+K_p\right)x(t)=K_p r(t)$$

$r(t)$**가 시간에 관한 함수가 아닌 상수** r **이라면**
Particular Solution x_p**를** a_0**로 가정하고 대입**

$$\left(k+K_p\right)a_0=K_p r$$

$$a_0=\frac{K_p}{k+K_p}r$$

$$x_p=\frac{K_p}{k+K_p}r$$

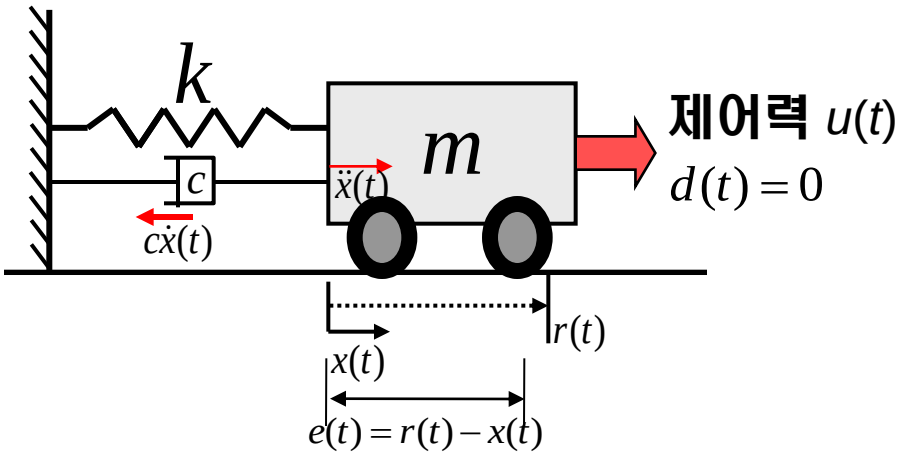
$$x(t)=\frac{K_p}{k+K_p}r+K_p\cdot Be^{\alpha t}+K_p\cdot Ce^{\beta t}$$

$$e(t)=r(t)-x(t)$$

$$=r-\frac{K_p}{k+K_p}r+K_p\cdot Be^{\alpha t}+K_p\cdot Ce^{\beta t}$$

$$=\frac{k}{k+K_p}r+K_p\cdot Be^{\alpha t}+K_p\cdot Ce^{\beta t}$$

- Proportional Controller(비례 제어기)



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$$m\ddot{x}(t)+c\dot{x}(t)+\left(k+K_p\right)x(t)=K_p r(t)$$

$$r(t)=r$$

$$e(t)=\frac{k}{k+K_p}r+K_p\cdot Be^{\alpha t}+K_p\cdot Ce^{\beta t}$$

α, β의 실수가 0보다 작으므로

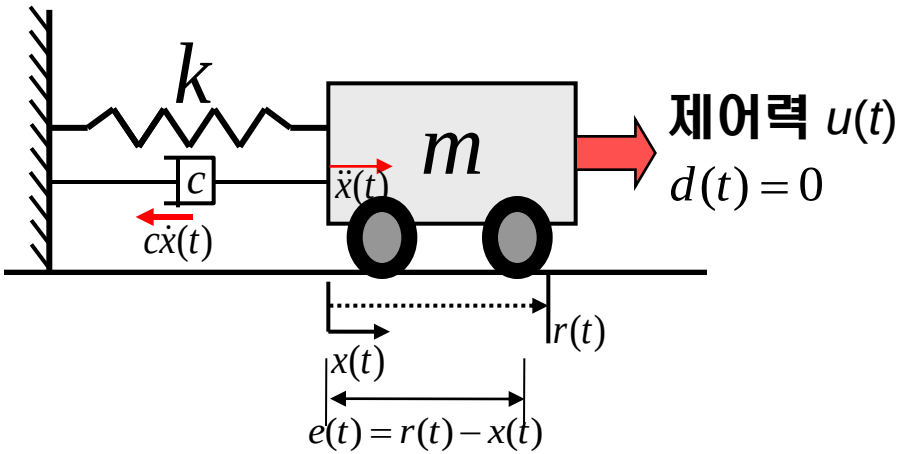
$$t\to\infty \text{ 일 때}$$

$$e(t)=\frac{k}{k+K_p}r+K_p\cdot Be^{\alpha t}+K_p\cdot Ce^{\beta t}$$

$$e(t)=\frac{k}{k+K_p}r$$

$x(t)$는 $r(t)$에 수렴하지 못함

- Proportional Controller(비례 제어기)



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해: $x(t)=x_p+K_p\cdot Be^{\alpha t}+K_p\cdot Ce^{\beta t}$

$$m\ddot{x}(t)+c\dot{x}(t)+(k+K_p)x(t)=K_p r(t)$$

*r(t)*가 시간에 관한 함수 *qt*라면

Particular Solution x_p를 a_0t+a_1로 가정하고 대입

$$ca_0+(k+K_p)a_0t+(k+K_p)a_1=K_pqt$$

$$ca_0+(k+K_p)a_1+(k+K_p)a_0t=K_pqt$$

계수 비교법

$$(k+K_p)a_0=K_pq \quad\Rightarrow\quad a_0=\frac{K_p}{k+K_p}q$$

$$ca_0+(k+K_p)a_1=0 \quad\Rightarrow\quad a_1=-\frac{K_pc}{(k+K_p)^2}q$$

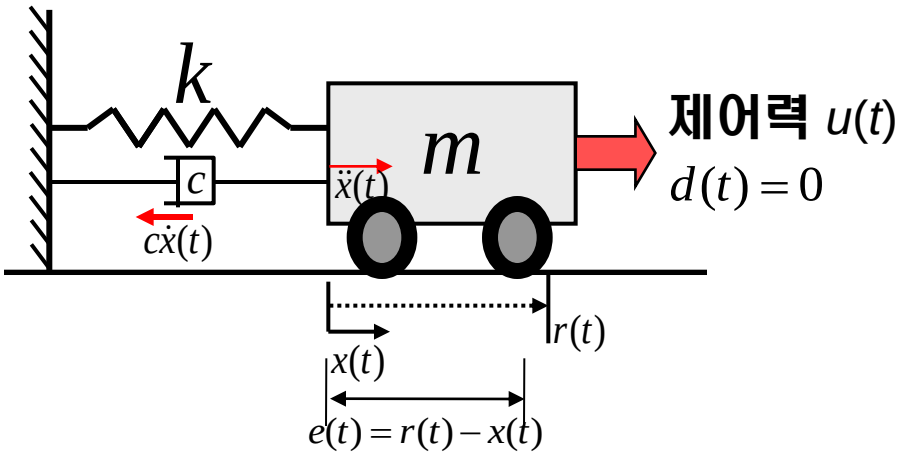
$$x_p(t)=\frac{K_p}{k+K_p}qt-\frac{K_pc}{(c+K_p)^2}q$$

$$x(t)=\frac{K_p}{k+K_p}qt-\frac{K_pc}{(c+K_p)^2}q+K_p\cdot Be^{\alpha t}+K_p\cdot Ce^{\beta t}$$

$$e(t)=r(t)-x(t)$$

$$=qt-\frac{K_p}{k+K_p}qt+\frac{K_pc}{(c+K_p)^2}q-K_p\cdot Be^{\alpha t}-K_p\cdot Ce^{\beta t}$$

- Proportional Controller(비례 제어기)



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$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=u(t)$$

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해: $x(t)=x_p+K_p\cdot Be^{\alpha t}+K_p\cdot Ce^{\beta t}$

$$m\ddot{x}(t)+c\dot{x}(t)+\left(k+K_p\right)x(t)=K_p r(t)$$

$$r(t)=qt$$

$$e(t)=qt-\frac{K_p}{k+K_p}qt+\frac{K_p c}{\left(c+K_p\right)^2}q-K_p\cdot Be^{\alpha t}-K_p\cdot Ce^{\beta t}$$

α, β의 실수가 0보다 작으므로

$t\to\infty$ **일 때**

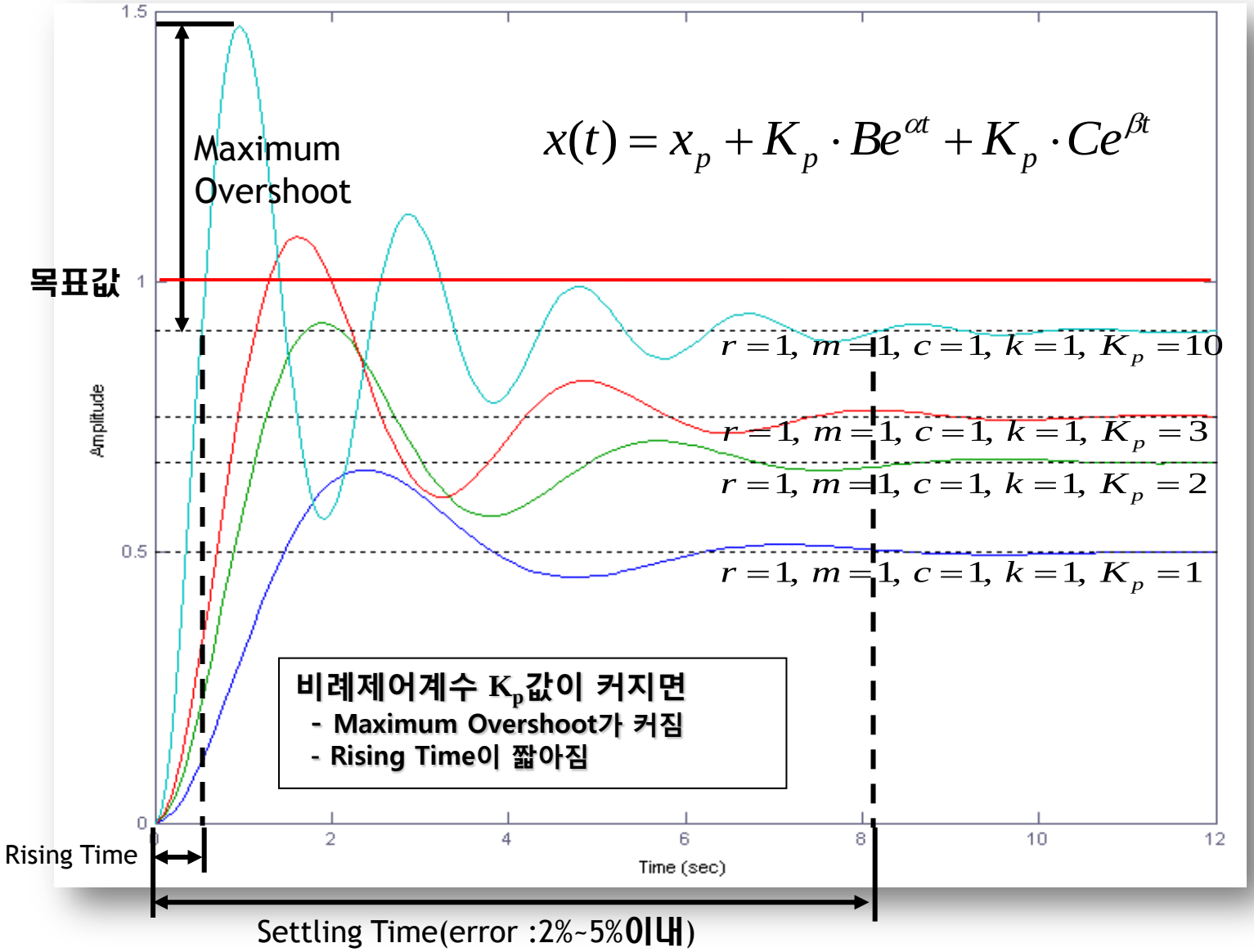
$$e(t)=qt-\frac{K_p}{k+K_p}qt+\frac{K_p c}{\left(c+K_p\right)^2}q-K_p\cdot Be^{\alpha t}-K_p\cdot Ce^{\beta t}$$

$$e(t)=\frac{k}{k+K_p}qt+\frac{K_p c}{\left(c+K_p\right)^2}$$

→ $x(t)$는 $r(t)$에 수렴하지 못함

- Proportional Controller(비례 제어기)

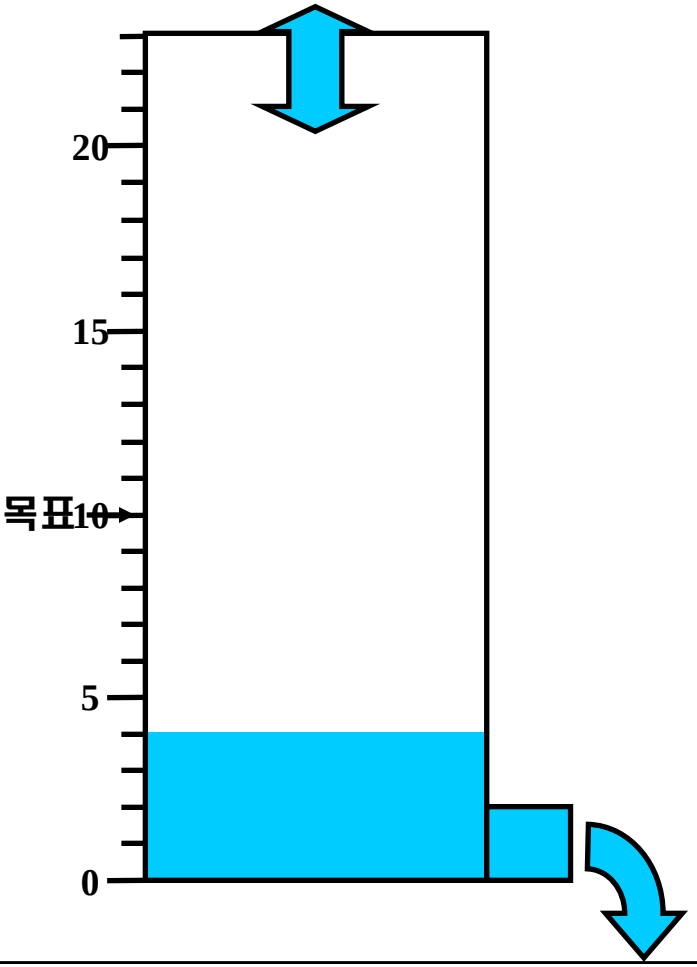
$$m\ddot{x}(t)+c\dot{x}(t)+\left(k+K_p\right)x(t)=K_p r(t)$$



Example of Proportional Controller(비례 제어기)

$$K_p e(t) = u(t)$$

$$u(t) = K_p e(t) = 1 \cdot e(t) \text{ m}^3/\text{min}$$



문제 : 목표 수위를 유지하기 위하여 수조에 물 몇m^3/min을 넣어야 하는가?
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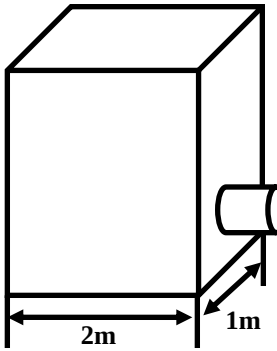


다음과 같이 가정한다.

$e(t)$ = 목표 물의 높이$r(t)$ - 현재 물의 높이$c(t)$

1분당 넣는 물의 양 $u(t)$: $K_p e(t)$

$K_p = 1, r = 10$

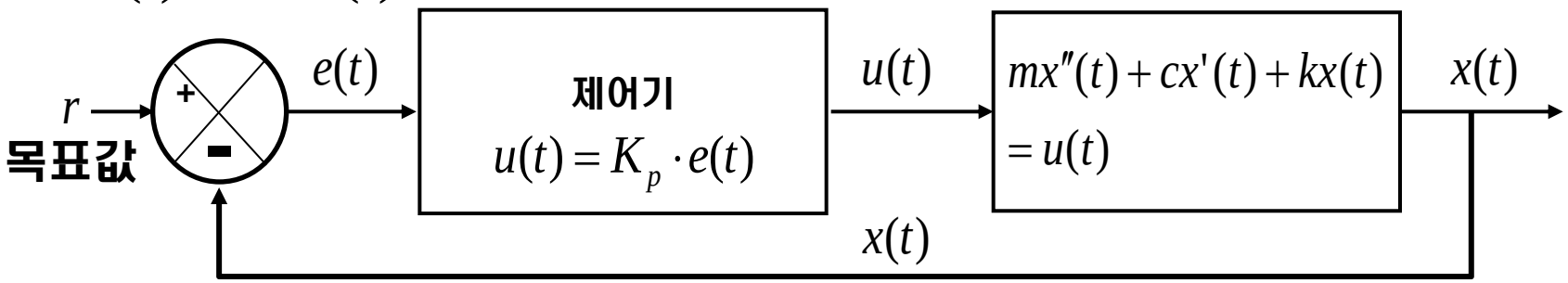
분	$c(t)$	r	$e(t)$	$u(t)=K_p e(t)$	물 증가량$(u(t)-2))$	수위 증가 (물증가량/밑넓이)

Formulation of Linear Control System by Laplace Transform

- Proportional Controller(비례 제어기)

$$mx''(t)+cx'(t)+kx(t)=u(t) \Leftarrow u(t)=K_p e(t)=K_p\left(r-x(t)\right)$$

$$e(t)=r-x(t)$$



Laplace Transform

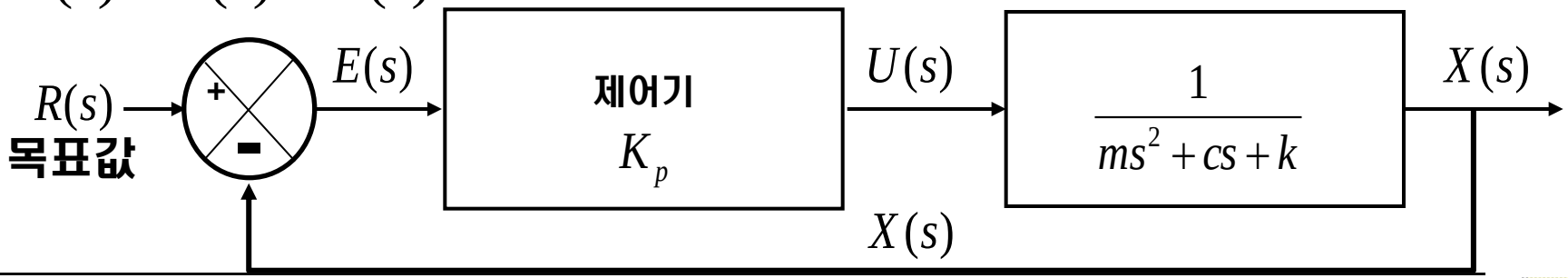
$$U(s)=K_p \cdot E(s)$$

$$K_p \cdot E(s)=U(s)$$

$$X(s)=\frac{1}{\left(ms^2+cs+k\right)}\left[U(s)\right]$$

$$U(s)\cdot\frac{1}{ms^2+cs+k}=X(s)$$

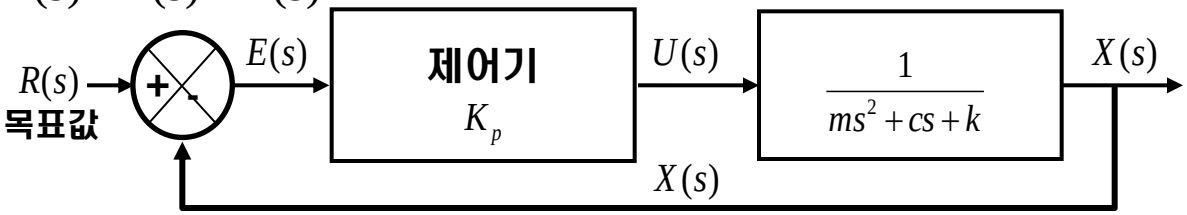
$$E(s)=R(s)-X(s)$$



Formulation of Linear Control System by Laplace Transform

- Proportional Controller(비례 제어기)

$$E(s) = R(s) - X(s)$$



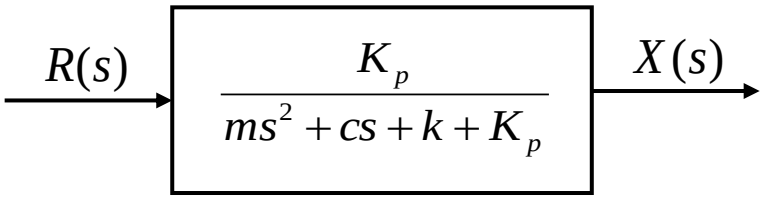
$$mx''(t) + cx'(t) + (k + K_p)x(t) = K_p r$$

전체 거동을 살펴보기 위하여
폐루프 전달함수를 이용하여
미분 방정식을 풀어보자

$$E(s) = R(s) - X(s)$$

$$\{R(s) - X(s)\}K_p = U(s)$$

$$\{R(s) - X(s)\}K_p \frac{1}{ms^2 + cs + k} = X(s)$$



$$X(s) = \frac{R(s)K_p}{ms^2 + cs + k + K_p} = \frac{rK_p}{s(ms^2 + cs + k + K_p)}$$

$$R(s) = \mathcal{L}(r(t)) = \frac{r}{s}, (\because r(t) = r)$$

$$x(t) = \mathcal{L}^{-1}(X(s)) = \mathcal{L}^{-1}\left(\frac{rK_p}{s(ms^2 + cs + k + K_p)}\right) = rK_p \times \mathcal{L}^{-1}\left(\frac{1}{s(ms^2 + cs + k + K_p)}\right)$$

Formulation of Linear Control System by Laplace Transform

- Proportional Controller(비례 제어기)

$$x(t) = \mathcal{L}^{-1}\left(X(s)\right) = rK_p \times \mathcal{L}^{-1}\left(\frac{1}{s(ms^2 + cs + k + K_p)}\right)$$

$$\frac{1}{s(ms^2 + cs + k + K_p)} = \frac{1}{s(ms^2 + cs + K^*)}, \quad k + K_p = K^*$$

$$= \frac{1}{ms(s-\alpha)(s-\beta)}, \quad \alpha,\ \beta = \frac{-c \pm \sqrt{c^2 - 4mK*}}{2m}$$

$$= \frac{A}{s} + \frac{B}{s-\alpha} + \frac{C}{s-\beta}$$

$$A = \frac{1}{K^*},\ B = -\frac{c + \sqrt{c^2 - 4mK^*}}{2K^* \cdot \sqrt{c^2 - 4mK^*}},\ C = \frac{c - \sqrt{c^2 - 4mK^*}}{2K^* \cdot \sqrt{c^2 - 4mK^*}}$$

cf. 2nd ODE의 해

$$mx''(t) + cx'(t) + (k + K_p)x(t) = K_p r$$

$$x(t) = x_h(t) + x_p(t)$$

$$= Ae^{\lambda_1 t} + Be^{\lambda_2 t} + C$$

$$\mathcal{L}^{-1}\left(X(s)\right) = rK_p \times L^{-1}\left(\frac{A}{s} + \frac{B}{s-\alpha} + \frac{C}{s-\beta}\right)$$

$$= rK_p \times (A + Be^{\alpha t} + Ce^{\beta t})$$

Ref: Inverse Laplace Transforms

☑The Inverse Problem

Theorem 4.3

Some Inverse Transform

$$(a)\ 1=\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$

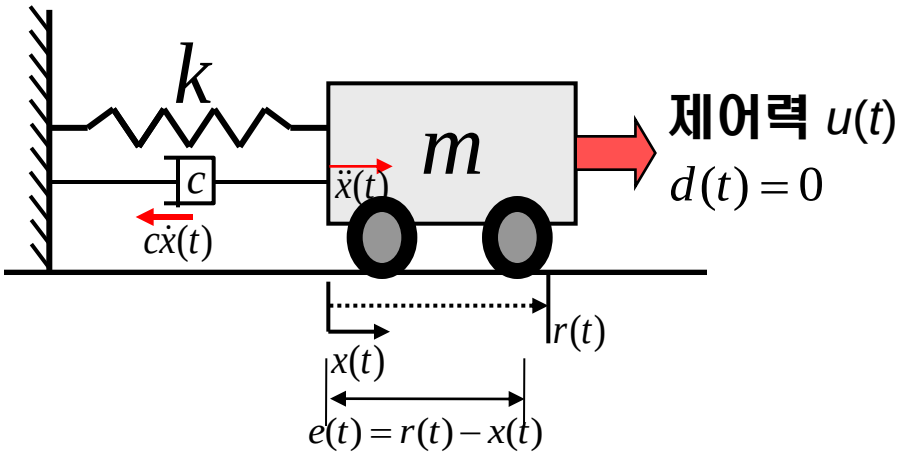
$$(b)\ t^n=\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}, n=1,2,3,\ldots \qquad (c)\ e^{at}=\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}$$

$$(d)\ \sin kt=\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} \qquad (e)\ \cos kt=\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\}$$

$$(f)\ \sinh kt=\mathcal{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\} \qquad (g)\ \cosh kt=\mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\}$$

2) Integral Controller

Integral Controller(적분 제어기)



목적
- 시스템의 특성을 변경 시킴
- 수레의 위치를 목표 값에 도달하게 함

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=u(t)$$

$$u(t)=K_i\int_0^t e(t)dt$$

해: $x(t)=x_p+C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}+C_3e^{\lambda_3 t}$

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=K_i\int_0^t e(t)dt$$

양변을 미분

$$m\dddot{x}(t)+c\ddot{x}(t)+k\dot{x}(t)=K_ie(t)$$

$$m\dddot{x}(t)+c\ddot{x}(t)+k\dot{x}(t)=K_i(r(t)-x(t))$$

$$m\dddot{x}(t)+c\ddot{x}(t)+k\dot{x}(t)+K_ix(t)=K_ir(t)$$

$$x(t)=x_p+C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}+C_3e^{\lambda_3 t}$$

Integral Controller(적분 제어기)

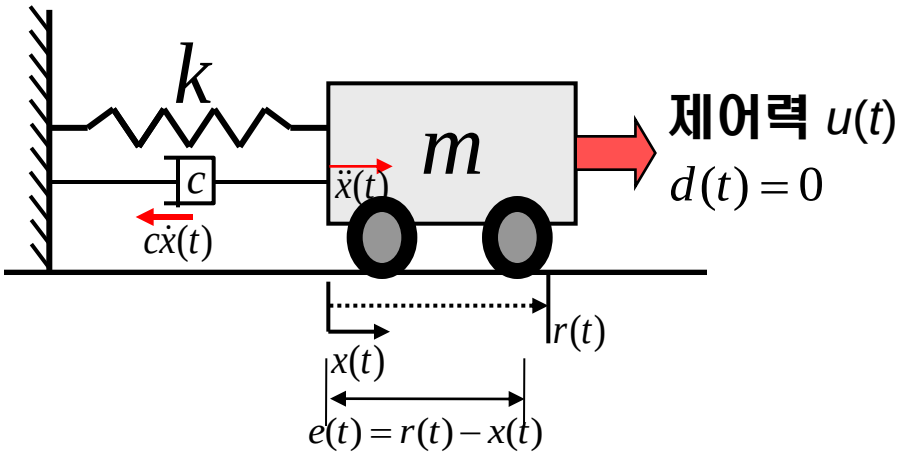
$$ax^3 + bx^2 + cx + d = 0$$

$$[x_1] = \frac{\sqrt[3]{-2b^3+9abc-27a^2d+\sqrt{4(-b^2+3ac)^3+(-2b^3+9abc-27a^2d)^2}}}{3\sqrt[3]{2}a}$$
$$-\frac{\sqrt[3]{2}(-b^2+3ac)}{3a\sqrt[3]{-2b^3+9abc-27a^2d+\sqrt{4(-b^2+3ac)^3+(-2b^3+9abc-27a^2d)^2}}} - \frac{b}{3a}$$

$$[x_2] = -\frac{(1-\sqrt{3}i)\cdot\sqrt[3]{-2b^3+9abc-27a^2d+\sqrt{4(-b^2+3ac)^3+(-2b^3+9abc-27a^2d)^2}}}{6\sqrt[3]{2}a}$$
$$+\frac{(1+\sqrt{3}i)(-b^2+3ac)}{3\sqrt[3]{4}a\cdot\sqrt[3]{-2b^3+9abc-27a^2d+\sqrt{4(-b^2+3ac)^3+(-2b^3+9abc-27a^2d)^2}}} - \frac{b}{3a}$$

$$[x_3] = -\frac{(1+\sqrt{3}i)\cdot\sqrt[3]{-2b^3+9abc-27a^2d+\sqrt{4(-b^2+3ac)^3+(-2b^3+9abc-27a^2d)^2}}}{6\sqrt[3]{2}a}$$
$$+\frac{(1-\sqrt{3}i)(-b^2+3ac)}{3\sqrt[3]{4}a\cdot\sqrt[3]{-2b^3+9abc-27a^2d+\sqrt{4(-b^2+3ac)^3+(-2b^3+9abc-27a^2d)^2}}} - \frac{b}{3a}$$

Integral Controller(적분 제어기)



$$m\dddot{x}(t)+c\ddot{x}(t)+k\dot{x}(t)+K_i x(t)=K_i r(t)$$

$r(t)$가 시간에 관한 함수가 아닌 상수 r 이라면
Particular Solution x_p를 a_0로 가정하고 대입

$$K_i a_0 = K_i r$$

$$a_0 = r$$

$$x_p = r$$

목적
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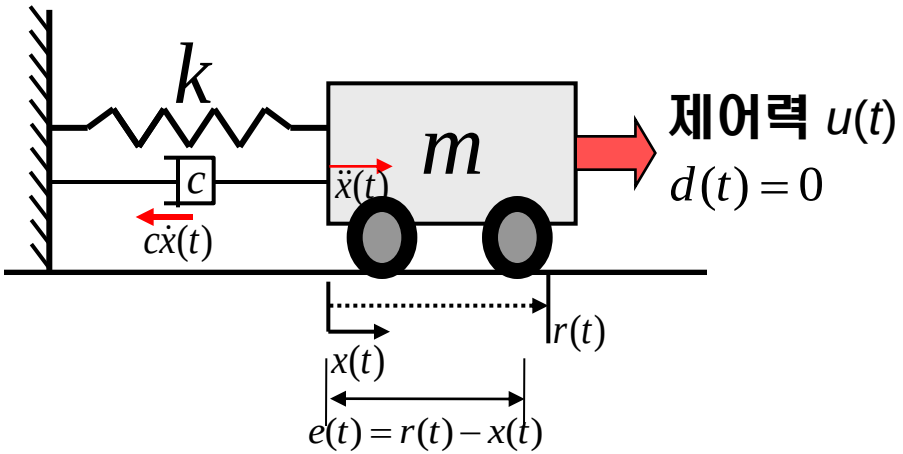
$$x(t)=r+C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}+C_3e^{\lambda_3 t}$$

$$\begin{aligned}e(t)&=r(t)-x(t)\\&=r-r-C_1e^{\lambda_1 t}-C_2e^{\lambda_2 t}-C_3e^{\lambda_3 t}\\&=-C_1e^{\lambda_1 t}-C_2e^{\lambda_2 t}-C_3e^{\lambda_3 t}\end{aligned}$$

Integral Controller(적분 제어기)



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$$m\dddot{x}(t)+c\ddot{x}(t)+k\dot{x}(t)+K_ix(t)=K_ir(t)$$

$$r(t)=r$$

$$e(t)=-C_1e^{\lambda_1 t}-C_2e^{\lambda_2 t}-C_3e^{\lambda_3 t}$$

$\lambda_1, \lambda_2, \lambda_3$ 의 실수가 0보다 작다면

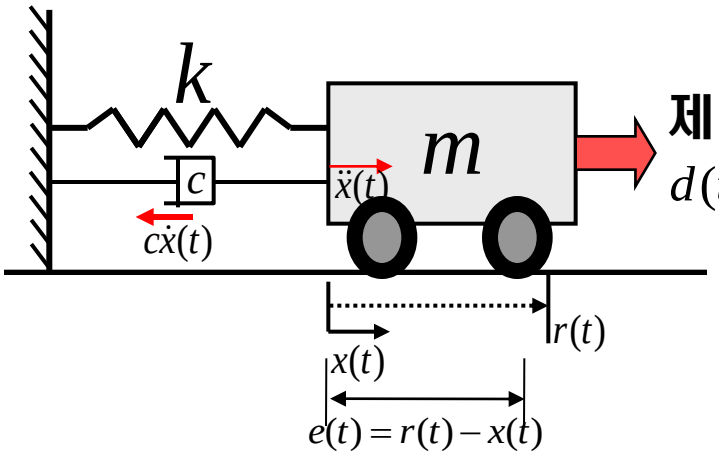
$t\to\infty$ 일 때

$$e(t)=-C_1e^{\lambda_1 t}-C_2e^{\lambda_2 t}-C_3e^{\lambda_3 t}$$

$$e(t)=0$$

$x(t)$는 $r(t)$에 수렴

Integral Controller(적분 제어기)



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$$m\dddot{x}(t)+c\ddot{x}(t)+k\dot{x}(t)+K_ix(t)=K_ir(t)$$

*r(t)*가 시간에 관한 함수 *qt*라면

Particular Solution x_p를 a_0t+a_1로 가정하고 대입

$$x_p(t)=qt-\frac{k}{K_i}q$$

$$ka_0+K_ia_0t+K_ia_1=K_iqt$$

$$ka_0+K_ia_1+K_ia_0t=K_iqt$$

$$x(t)=qt-\frac{k}{K_i}q+C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}+C_3e^{\lambda_3 t}$$

계수 비교법

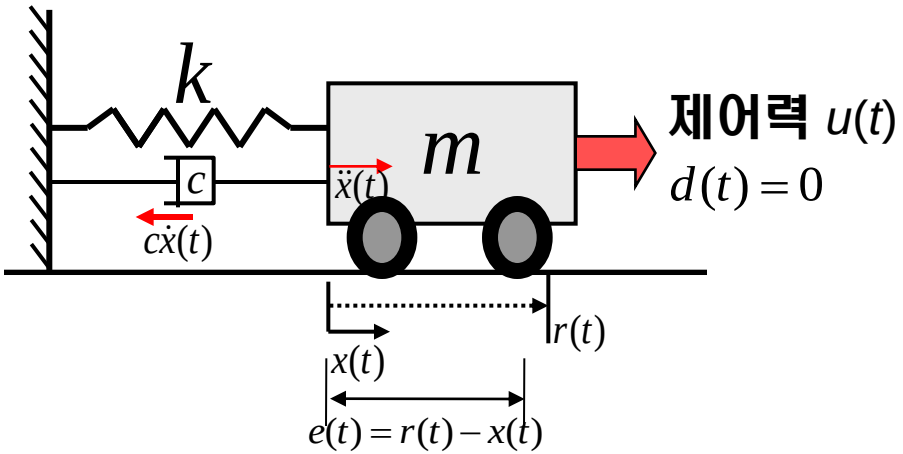
$$K_ia_0=K_iq \quad\Rightarrow\quad a_0=q$$

$$ka_0+K_ia_1=0 \quad\Rightarrow\quad a_1=-\frac{k}{K_i}q$$

$$e(t)=r(t)-x(t)$$

$$=qt-qt+\frac{k}{K_i}q-C_1e^{\lambda_1 t}-C_2e^{\lambda_2 t}-C_3e^{\lambda_3 t}$$

Integral Controller(적분 제어기)



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$$m\dddot{x}(t)+c\ddot{x}(t)+k\dot{x}(t)+K_ix(t)=K_ir(t)$$

$$r(t)=qt$$

$$e(t)=qt-qt+\frac{k}{K_i}q-C_1e^{\lambda_1 t}-C_2e^{\lambda_2 t}-C_3e^{\lambda_3 t}$$

$$=\frac{k}{K_i}q-C_1e^{\lambda_1 t}-C_2e^{\lambda_2 t}-C_3e^{\lambda_3 t}$$

λ_1, λ_2, λ_3 의 실수가 0보다 작다면

$t\to\infty$ 일 때

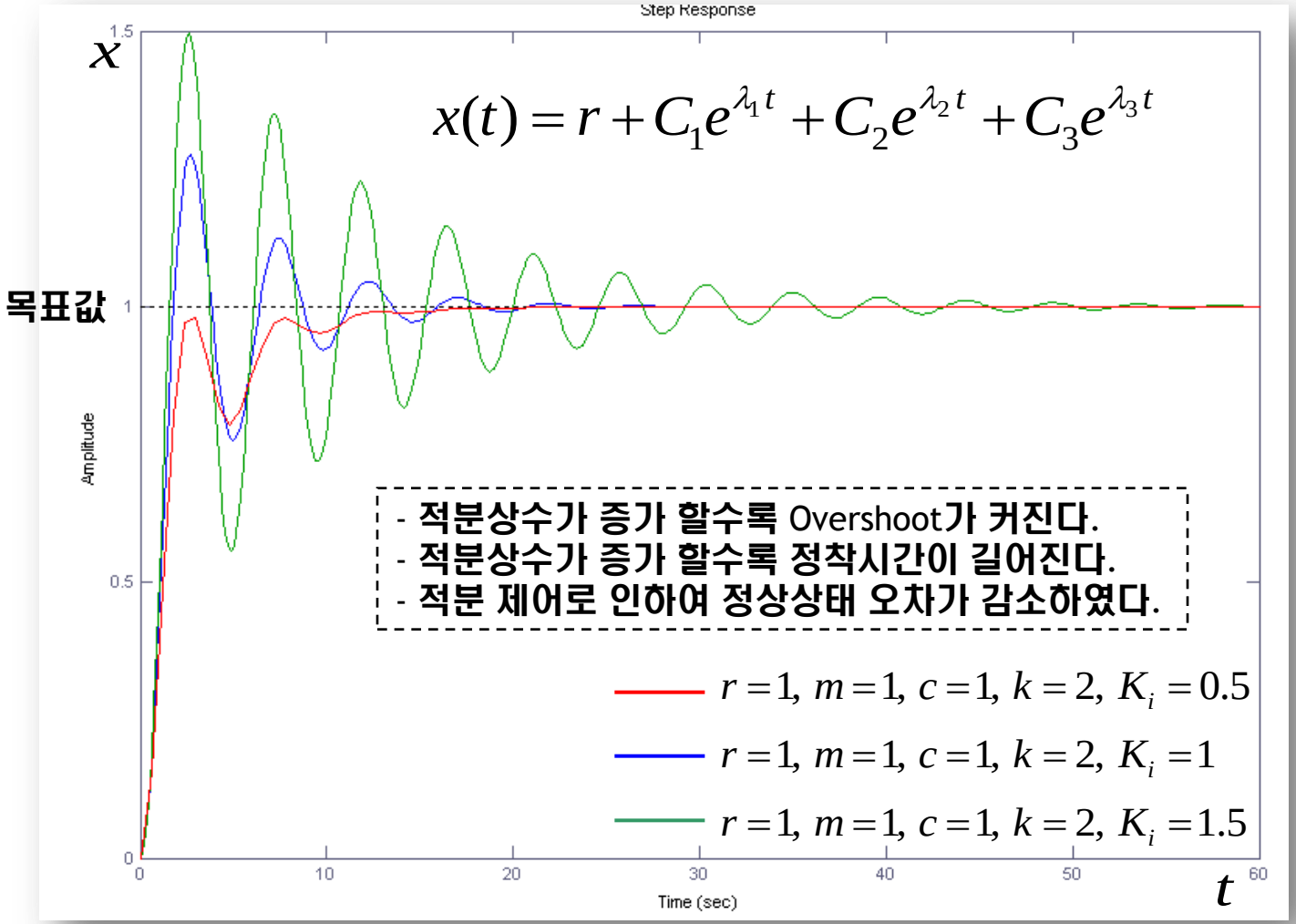
$$e(t)=\frac{k}{K_i}q-C_1e^{\lambda_1 t}-C_2e^{\lambda_2 t}-C_3e^{\lambda_3 t}$$

$$e(t)=\frac{k}{K_i}q$$

$x(t)$는 $r(t)$에 수렴하지 않음

Integral Controller(적분 제어기)

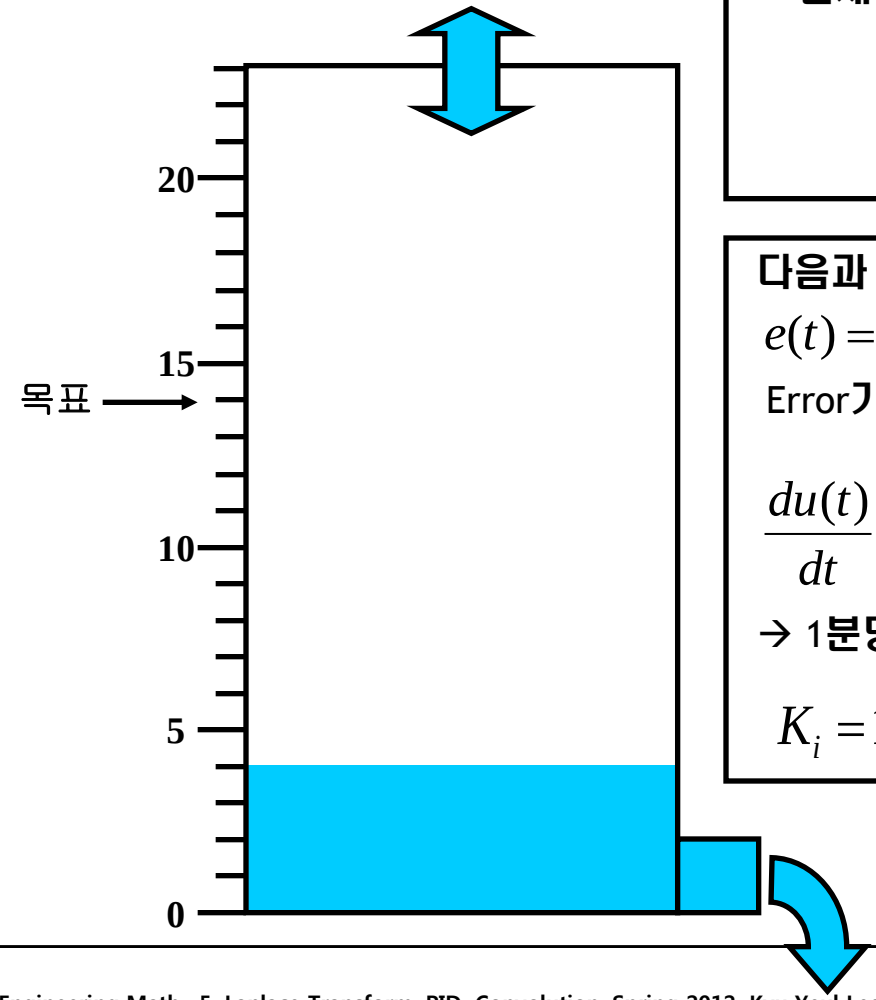
$$m\dddot{x}(t)+c\ddot{x}(t)+k\dot{x}(t)+K_i x(t)=K_i r(t)$$



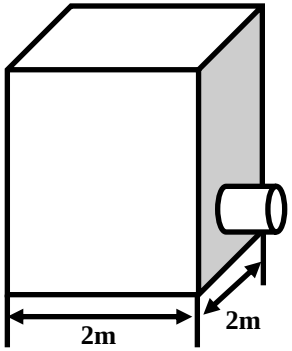
Example of Integral Controller(적분 제어기)

$$u(t) = K_i \int_0^t e(t)dt$$

$$u(t) = K_i \int_0^t e(t)dt = \int_0^t e(t)dt \ \text{m}^3/\text{min}$$



문제 : 목표 수위를 유지하기 위하여 수조에 물 몇m^3/min을 넣어야 하는가?
Given: - 수조의 밑 넓이 $4m^2$
- 물이 수조에서 $2m^3$/min씩 흘러나간다.
- 계산시 소수점 이하는 버린다.



다음과 같이 가정한다.

$e(t)$ = 목표 물의 높이$r(t)$ - 현재 물의 높이$c(t)$

Error가 크면 현재 출력과 단위시간 이후 출력의 차이를 크게 한다.

$$\frac{du(t)}{dt} = K_i e(t)$$

→ 1분당 넣는 물의 양 $u(t)$: $K_i \int_0^t e(t)dt$

$$K_i = 1,\ r = 14$$

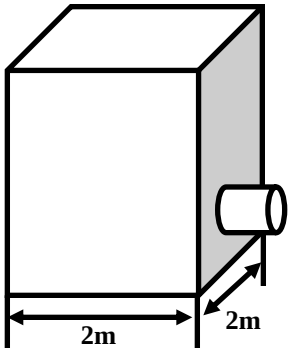
$e(t)$ = 목표 물의 높이$r(t)$ - 현재 물의 높이$c(t)$

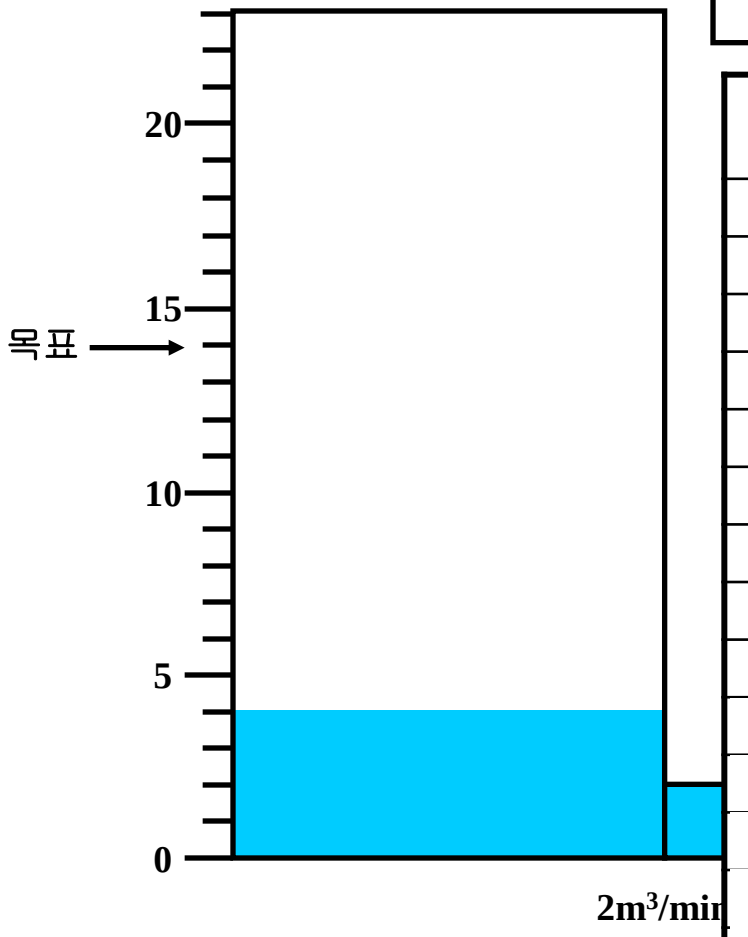
$$u(t) = K_i \int_0^t e(t)dt = \int_0^t e(t)dt \quad \text{m}^3/\text{min}$$

문제 : 목표 수위를 유지하기 위하여 수조에 1분당 물 몇m^3를 넣어야 하는가?

Given: - 수조의 밑 넓이 $4m^2$

- 물이 수조에서 분당 $2m^3$씩 흘러나간다.

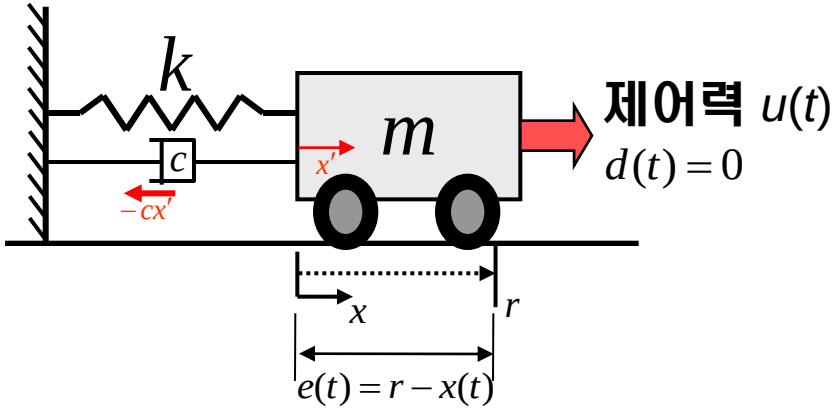




분	$c(t)$	r	$e(t)$	$K_i \int_0^t e(t)dt$	물 증가량$(u(t)-2))$	수위 증가 (물증가량/밑넓이)

3) Derivative Controller

Derivative Controller(미분 제어기)



목적
- 시스템의 특성을 변경 시킴
- 수레의 위치를 목표 값에 도달하게 함

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=u(t)$$

$$u(t)=K_d\dot{e}(t)$$

해: $x(t)=x_p+C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}$

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=K_d\dot{e}(t)$$

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=K_d(\dot{r}(t)-\dot{x}(t))$$

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=K_d\dot{r}(t)-K_d\dot{x}(t)$$

$$m\ddot{x}(t)+(c+K_d)\dot{x}(t)+kx(t)=K_d\dot{r}(t)$$

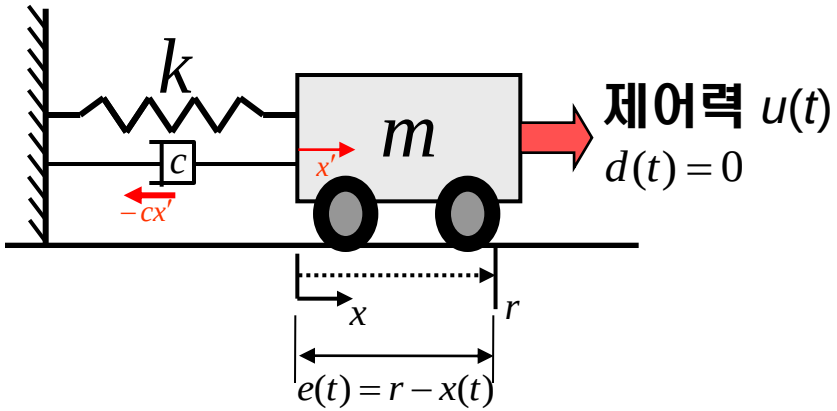
$$x(t)=x_p+C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}$$

where $c+K_d=K^*$,

$$\alpha,\beta=\frac{-K^*\pm\sqrt{(K^*)^2-4mk}}{-2m}$$

m, *c*, *k*는 항상 양수이고 K_d가 양수라면 α,β의 실수부인 $-K^*/2m$은 음수이다.

Derivative Controller(미분 제어기)



목적
- 시스템의 특성을 변경 시킴
- 수레의 위치를 목표 값에 도달하게 함

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=u(t)$$

$$u(t)=K_d\dot{e}(t)$$

해: $x(t)=x_p+C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}$

$$m\ddot{x}(t)+(c+K_d)\dot{x}(t)+kx(t)=K_d\dot{r}(t)$$

$r(t)$가 시간에 관한 함수가 아닌 상수 r 이라면
Particular Solution x_p를 a_0로 가정하고 대입

$$ka_0=0$$

$$a_0=0$$

$$x_p=0$$

$$x(t)=0+C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}$$

$$r(t)-x(t)=r-C_1e^{\lambda_1 t}-C_2e^{\lambda_2 t}$$

λ_1, λ_2 의 실수가 0보다 작기 때문에
$t\to\infty$ 일 때

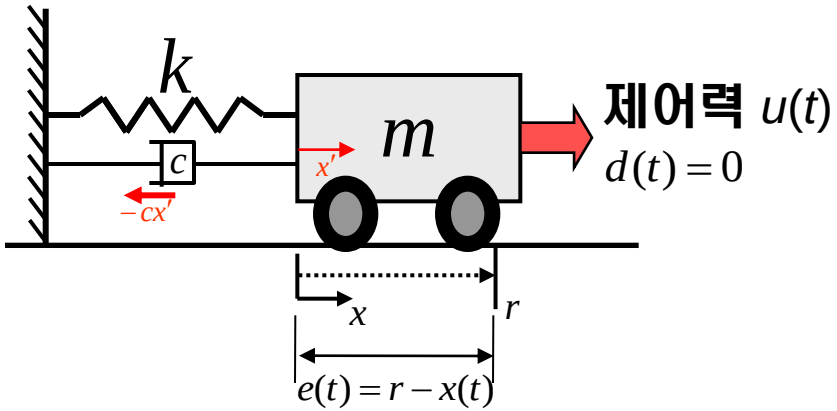
$$e(t)=r(t)-x(t)$$

$$=r-C_1e^{\lambda_1 t}-C_2e^{\lambda_2 t}$$

$$e(t)=r$$

$x(t)$는 $r(t)$에 수렴하지 않음

Derivative Controller(미분 제어기)



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$$m\ddot{x}(t)+(c+K_d)\dot{x}(t)+kx(t)=K_d\dot{r}(t)$$

*r(t)*가 시간에 관한 함수 *qt* 라면

Particular Solution x_p를 a_0t+a_1로 가정하고 대입

$$(c+K_d)a_0+ka_0t+ka_1=K_dq$$

$$(c+K_d)a_0+ka_1+ka_0t=K_dq$$

계수 비교법

$$ka_0=0 \quad\Rightarrow\quad a_0=0$$

$$(c+K_d)a_0+ka_1=K_dq \quad\Rightarrow\quad a_1=\frac{K_d}{k}q$$

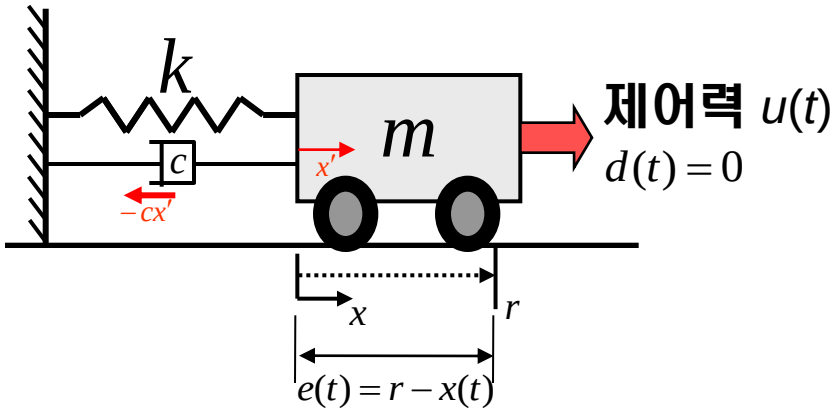
$$x_p(t)=\frac{K_d}{k}q$$

$$x(t)=\frac{K_d}{k}q+C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}$$

$$e(t)=r(t)-x(t)$$

$$=qt-\frac{K_d}{k}q-C_1e^{\lambda_1 t}-C_2e^{\lambda_2 t}$$

Derivative Controller(미분 제어기)



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$$m\ddot{x}(t)+\left(c+K_d\right)\dot{x}(t)+kx(t)=K_d\dot{r}(t)$$

$$r(t)=qt$$

$$e(t)=qt-\frac{K_d}{k}q-C_1e^{\lambda_1 t}-C_2e^{\lambda_2 t}$$

λ_1, λ_2 의 실수가 0보다 작기 때문에

$t\to\infty$ 일 때

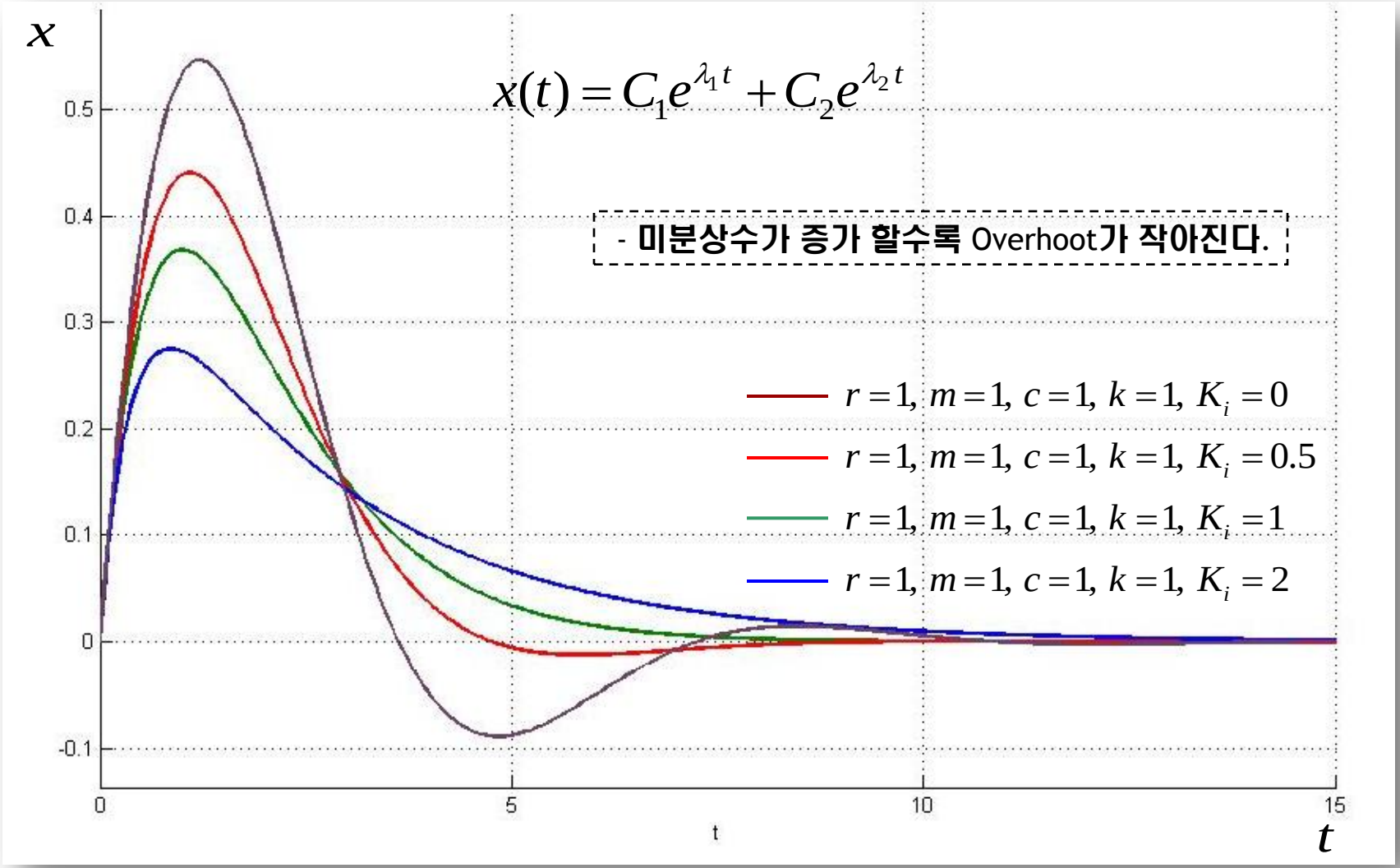
$$e(t)=qt-\frac{K_d}{k}q-C_1e^{\lambda_1 t}-C_2e^{\lambda_2 t}$$

$$e(t)=qt-\frac{K_d}{k}q$$

$x(t)$는 $r(t)$에 수렴하지 않음

Derivative Controller(미분 제어기)

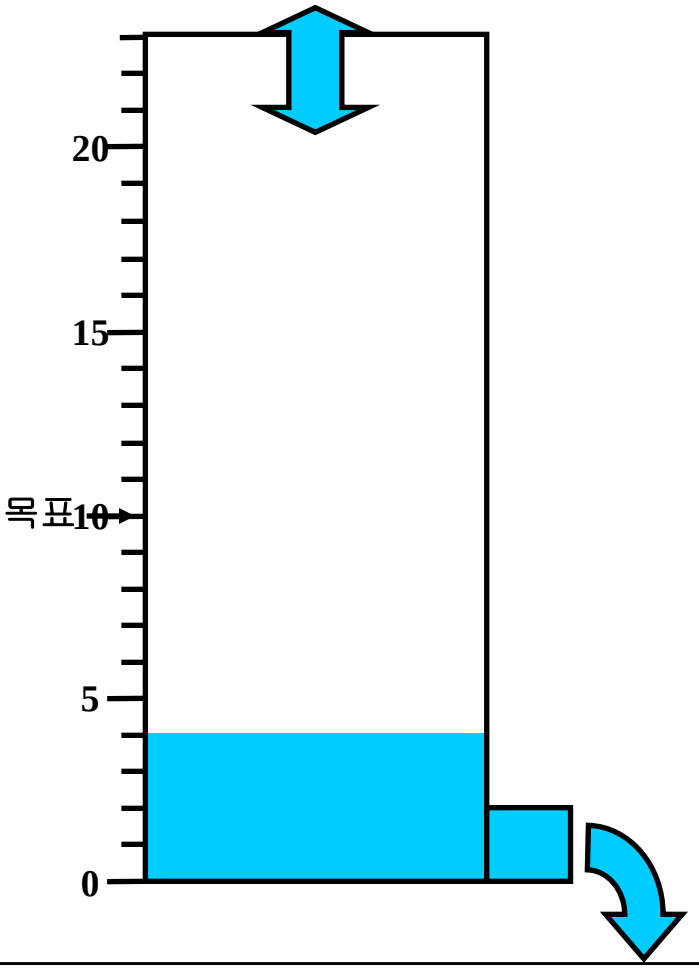
$$m\ddot{x}(t)+(c+K_d)\dot{x}(t)+kx(t)=K_d\dot{r}(t)$$



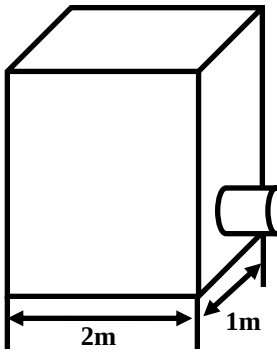
Example of Derivative Controller(미분 제어기)

$$K_p e(t) = u(t)$$

$$u(t) = K_p e(t) = 1 \cdot e(t) \text{ m}^3/\text{min}$$



문제 : 목표 수위를 유지하기 위하여 수조에 물 몇m^3/min을 넣어야 하는가?
Given: - 수조의 밑 넓이 $2m^2$
- 물이 수조에서 $2m^3$/min씩 흘러나간다.



다음과 같이 가정한다.

$e(t)$ = 목표 물의 높이$r(t)$ - 현재 물의 높이$c(t)$

1분당 넣는 물의 양 $u(t)$: $K_p e(t)$

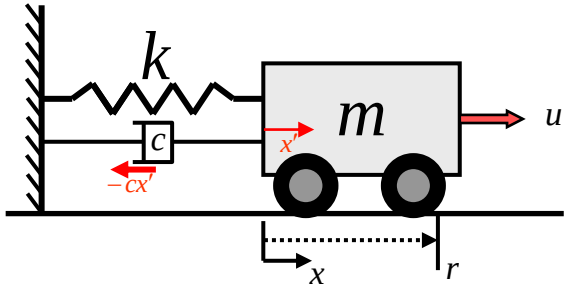
$K_p = 1, r = 10$

분	$c(t)$	r	$e(t)$	$u(t)=K_p e(t)$	물 증가량$(u(t)-2))$	수위 증가 (물증가량/밑넓이)

4) Proportional & Derivative (PD)Controller

Mass-Spring-Damper Linear Mechanical System

- Proportional-Derivative Control (PD제어)



$$m\ddot{x} = F$$

$$m\ddot{x} = -kx - c\dot{x} + u$$

$$m\ddot{x} + c\dot{x} + kx = u$$

$$K_p e(t) + K_d \dot{e}(t) = u(t)$$

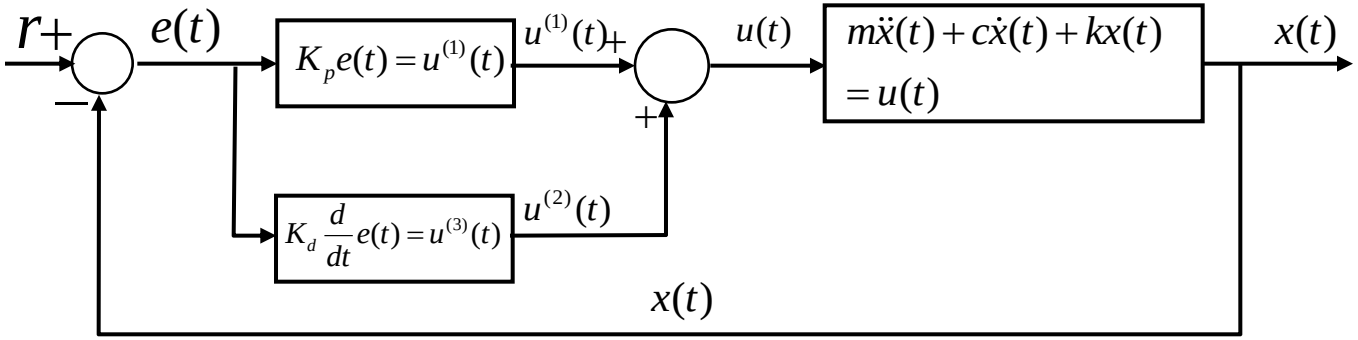
$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = K_p e + K_d \dot{e}$$

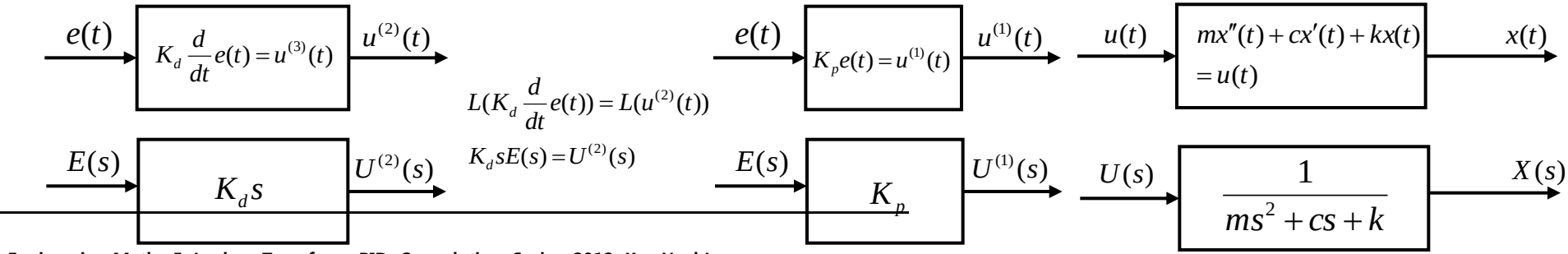
$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = K_p (r - x) + K_d (\dot{r} - \dot{x})$$

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = K_p r - K_p x - K_d \dot{x}$$

$$m\ddot{x}(t) + (c + K_d)\dot{x}(t) + (k + K_p)x(t) = K_p r$$

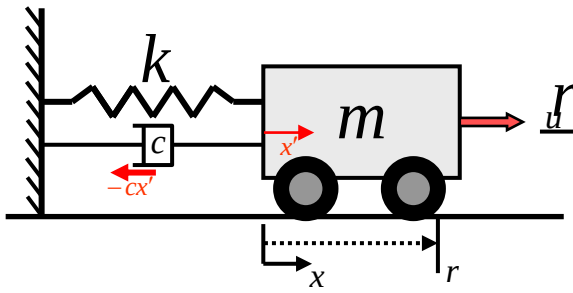
$$r - x(t) = e(t)$$

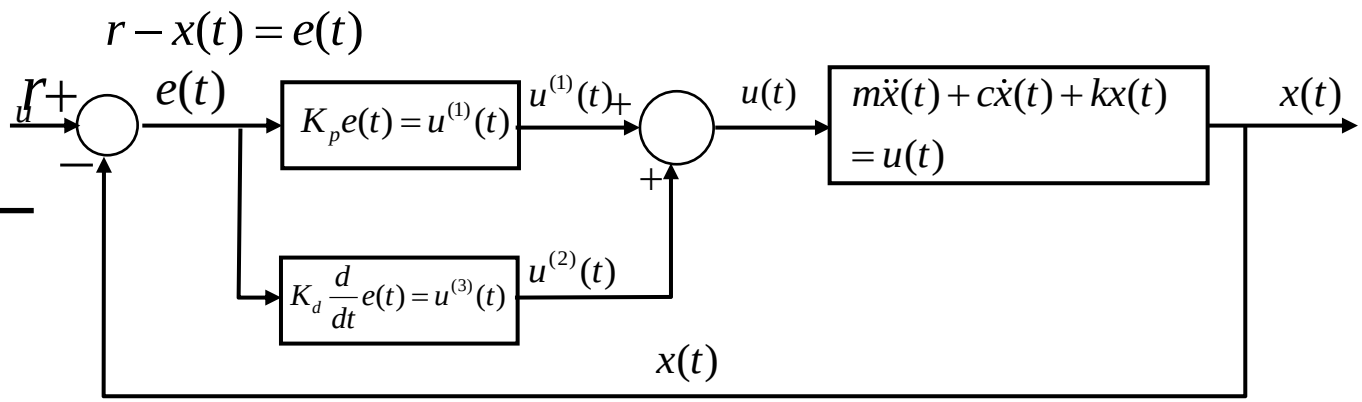




Mass-Spring-Damper Linear Mechanical System

- Proportional-Derivative Control (PD제어)



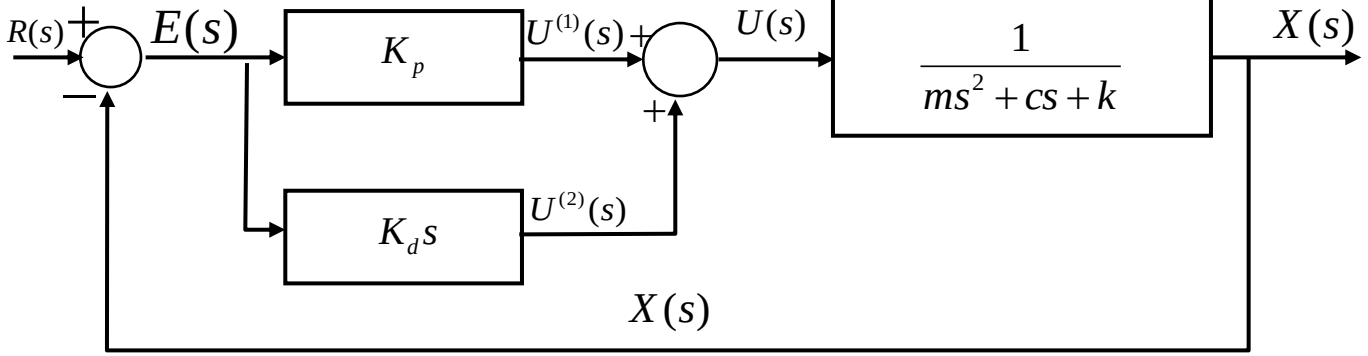


Laplace Transform

$$R(s) - X(s) = E(s)$$

$$E(s) \times (K_p + K_d s) = U(s)$$

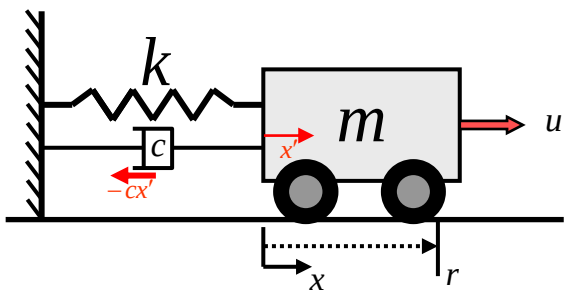
$$U(s) \times \frac{1}{ms^2 + cs + k} = X(s)$$



Mass-Spring-Damper Linear Mechanical System

- Proportional-Derivative Control (PD제어)

$$m\ddot{x}(t)+(c+K_d)\dot{x}(t)+(k+K_p)x(t)=K_p r$$

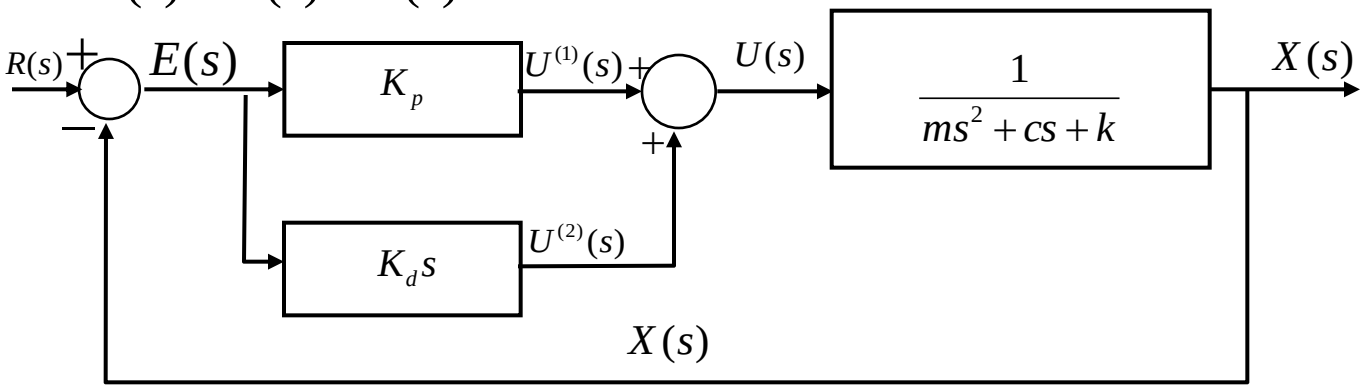


$$m\ddot{x}=F$$

$$m\ddot{x}=-kx-c\dot{x}+u$$

$$m\ddot{x}+c\dot{x}+kx=u$$

$$R(s)-X(s)=E(s)$$



$$R(s)-X(s)=E(s)$$

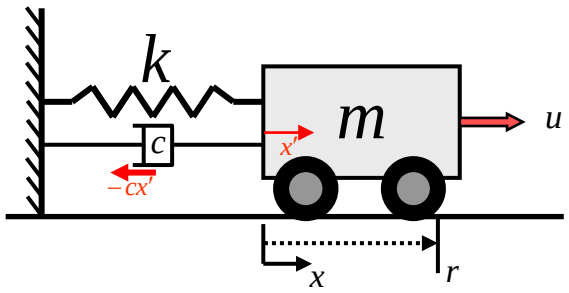
$$E(s)\times(K_p+K_d s)=U(s)$$

$$U(s)\times\frac{1}{ms^2+cs+k}=X(s)$$

$$\{R(s)-X(s)\}\times(K_p+K_d s)\times\frac{1}{ms^2+cs+k}=X(s)$$

Mass-Spring-Damper Linear Mechanical System

- Proportional-Derivative Control (PD제어)

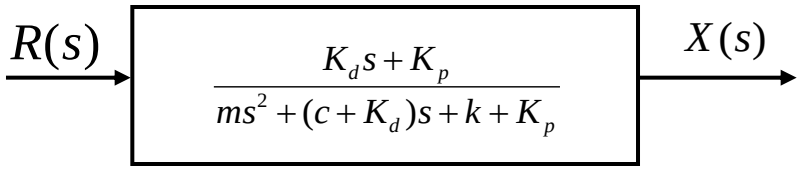


$$m\ddot{x} = F$$

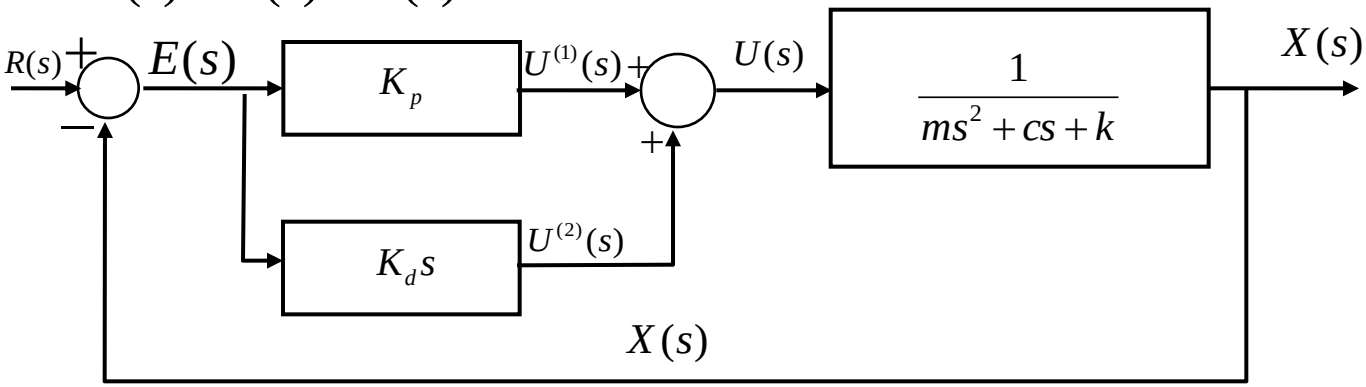
$$m\ddot{x} = -kx - c\dot{x} + u$$

$$m\ddot{x} + c\dot{x} + kx = u$$

$$m\ddot{x}(t) + (c + K_d)\dot{x}(t) + (k + K_p)x(t) = K_p r$$



$$R(s) - X(s) = E(s)$$

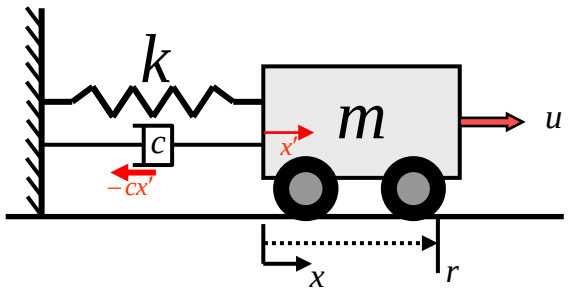


$$\{R(s) - X(s)\} \times (K_p + K_d s) \times \frac{1}{ms^2 + cs + k} = X(s)$$

$$\Rightarrow \quad \frac{R(s)(K_d s + K_p)}{ms^2 + (c + K_d)s + k + K_p} = X(s)$$

Mass-Spring-Damper Linear Mechanical System

- Proportional-Derivative Control (PD제어)



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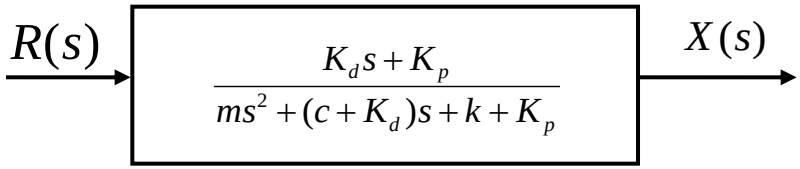
$$m\ddot{x}(t) + (c + K_d)\dot{x}(t) + (k + K_p)x(t) = K_p r$$



$$X(s) = \frac{R(s)(K_d s + K_p)}{ms^2 + (c + K_d)s + k + K_p} = \frac{r(K_d s + K_p)}{s(ms^2 + (c + K_d)s + k + K_p)}$$

$$L^{-1}(X(s)) = L^{-1}\left(\frac{r(K_d s + K_p)}{s(ms^2 + (c + K_d)s + k + K_p)}\right)$$

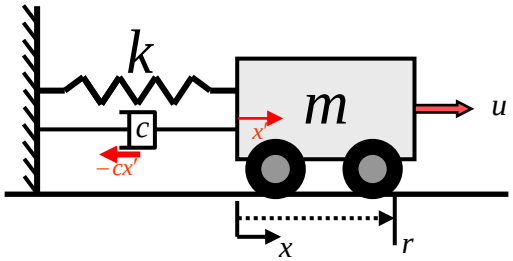
MATLAB을 이용하여 $C_1 \sim C_3$, $\lambda_1 \sim \lambda_3$ 를 구하였음

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + C_3 e^{\lambda_3 t}$$

Mass-Spring-Damper Linear Mechanical System

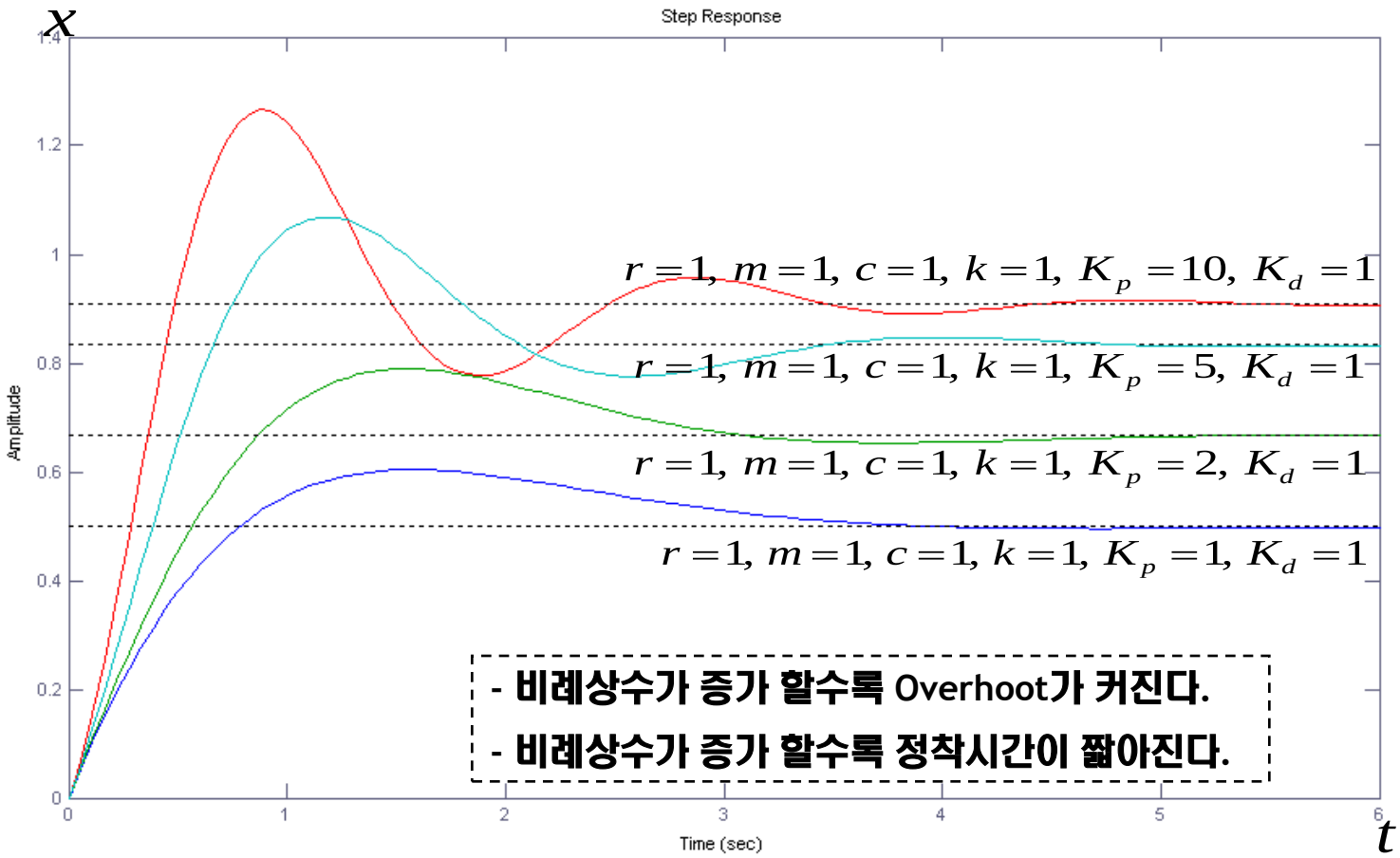
- Proportional-Derivative Control (PD제어)

$$m\ddot{x}(t)+(c+K_d)\dot{x}(t)+(k+K_p)x(t)=K_p r$$



$$R(s) \rightarrow \frac{K_d s + K_p}{ms^2 + (c+K_d)s + k + K_p} \rightarrow X(s)$$

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + C_3 e^{\lambda_3 t}$$



Mass-Spring-Damper Linear Mechanical System

- Proportional-Derivative Control (PD제어)

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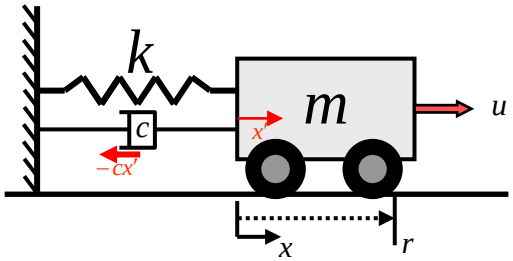

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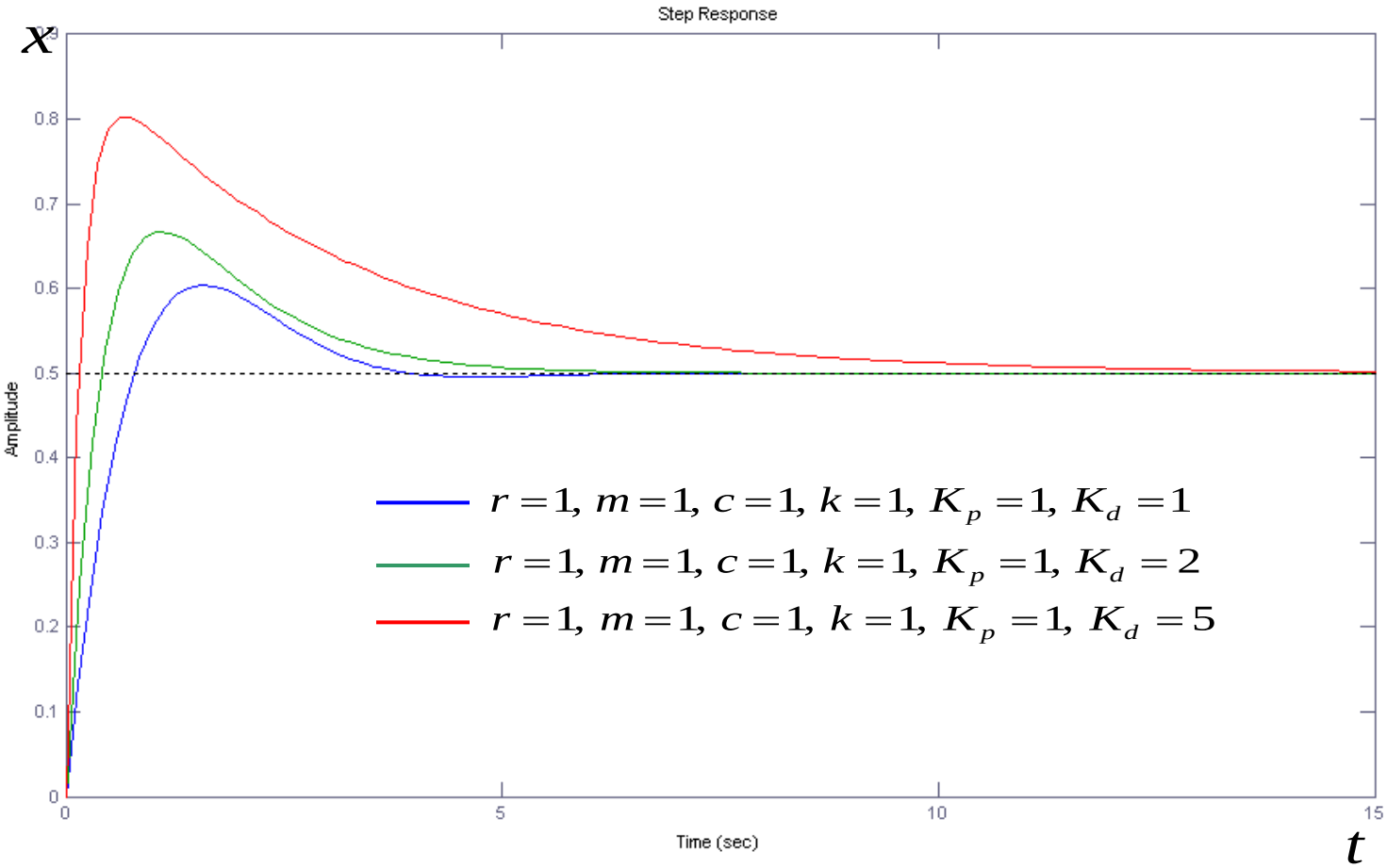
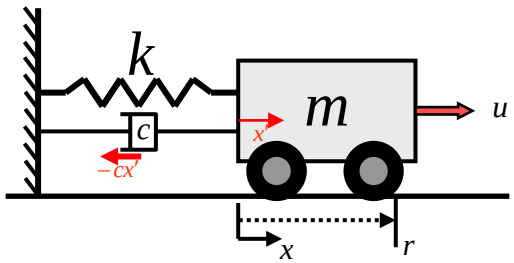
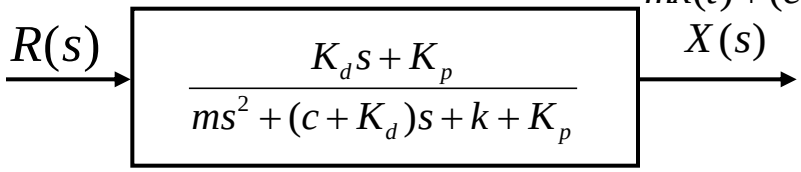


Mass-Spring-Damper Linear Mechanical System

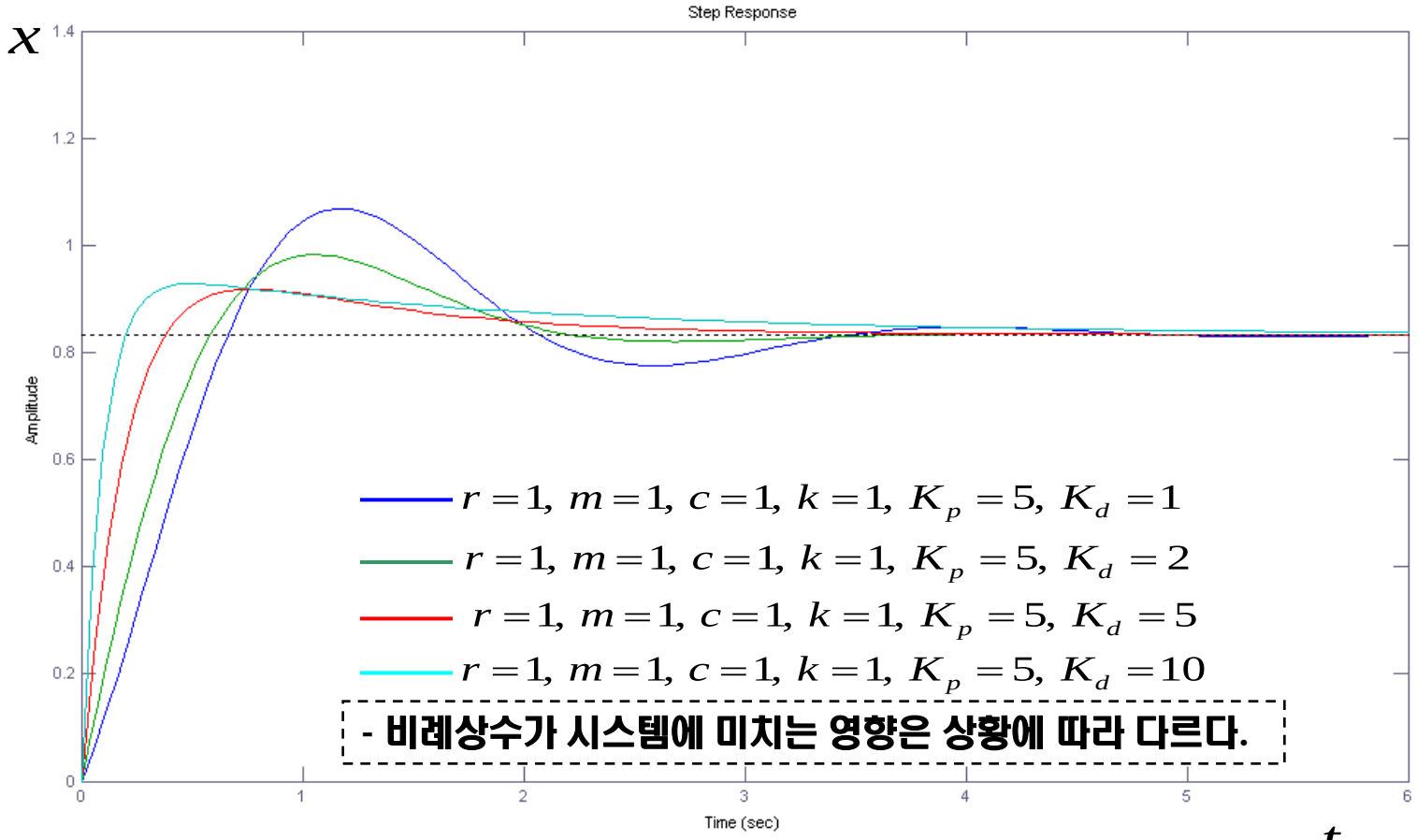
- Proportional-Derivative Control (PD제어)



$$m\ddot{x}(t)+(c+K_d)\dot{x}(t)+(k+K_p)x(t)=K_p r$$



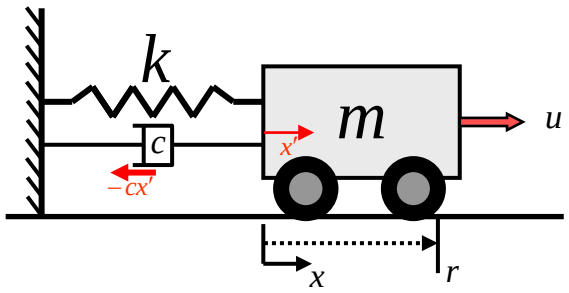
$$x(t)=C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}+C_3e^{\lambda_3 t}$$



5) PROPORTIONAL, INTEGRAL, DERIVATIVE CONTROLLER(PID CONTROLLER)

Mass-Spring-Damper Linear Mechanical System

- Proportional-Integral-Derivative Control (PID제어)



$$m\ddot{x} = F$$

$$m\ddot{x} = -kx - c\dot{x} + u$$

$$m\ddot{x} + c\dot{x} + kx = u$$

$$K_i\int_0^t e(\tau)d\tau + K_p e(t) + K_d e'(t) = u(t)$$

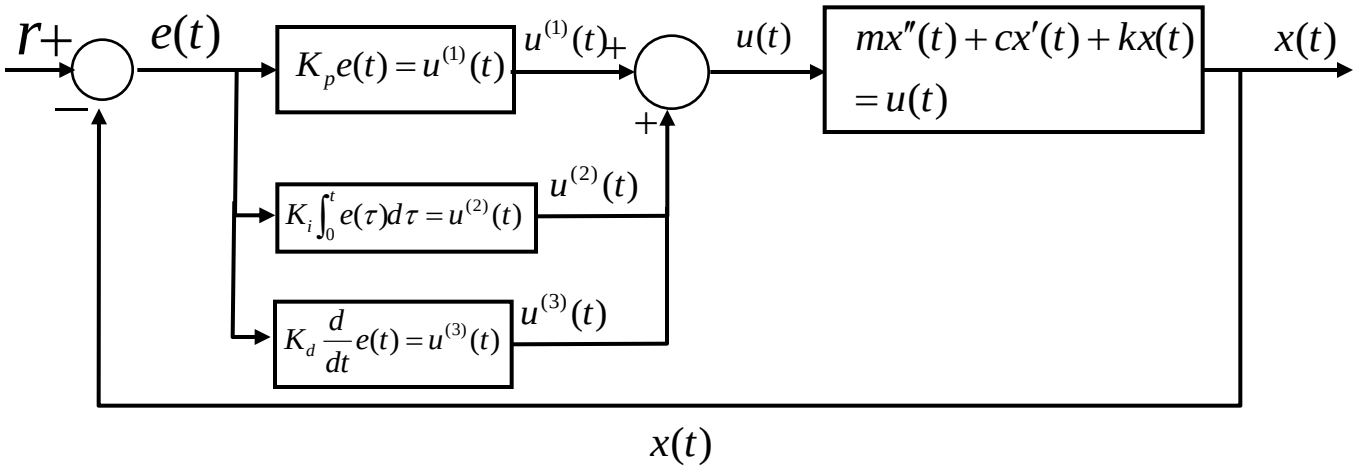
$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = K_i\int_0^t e(\tau)d\tau + K_p e + K_d\dot{e}$$

$$m\dddot{x}(t) + c\ddot{x}(t) + k\dot{x}(t) = K_i(r - x) + K_p(\dot{r} - \dot{x}) + K_d(\ddot{r} - \ddot{x})$$

$$m\dddot{x}(t) + c\ddot{x}(t) + k\dot{x}(t) = K_i r - K_i x - K_p\dot{x} - K_d\ddot{x}$$

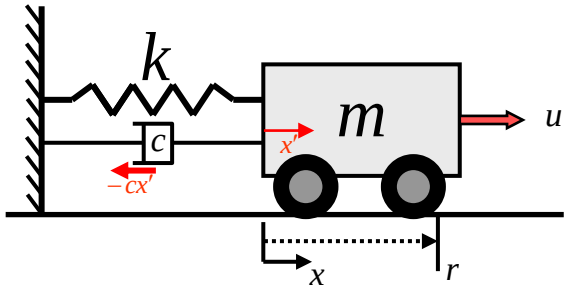
$$m\dddot{x}(t) + (c + K_d)\ddot{x}(t) + (k + K_p)\dot{x}(t) + K_i x = K_i r$$

$$r - x(t) = e(t)$$



Mass-Spring-Damper Linear Mechanical System

- Proportional-Integral-Derivative Control (PID제어)



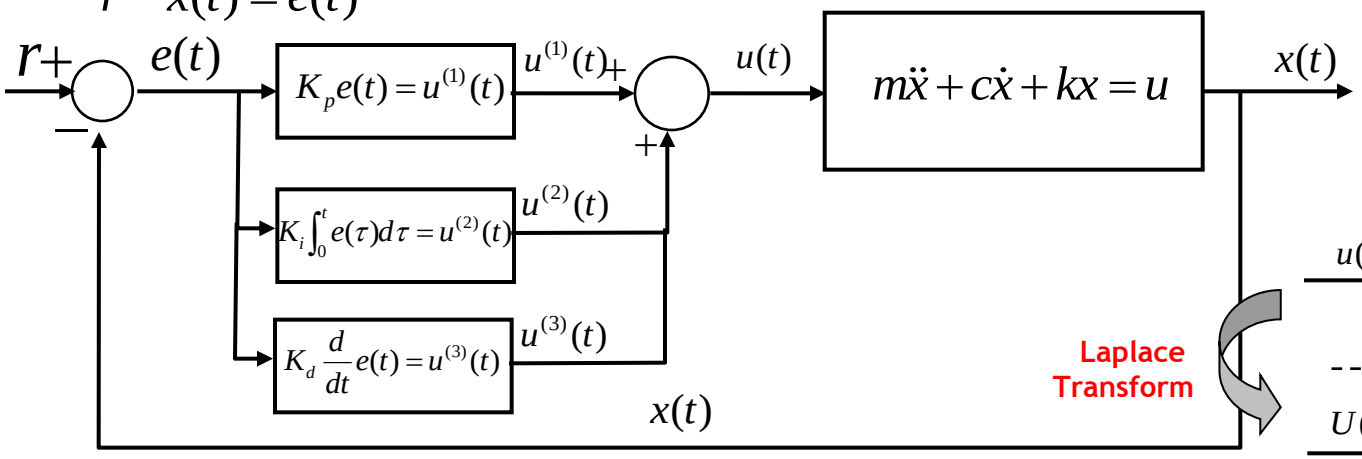
$$m\ddot{x} = F$$

$$m\ddot{x} = -kx - c\dot{x} + u$$

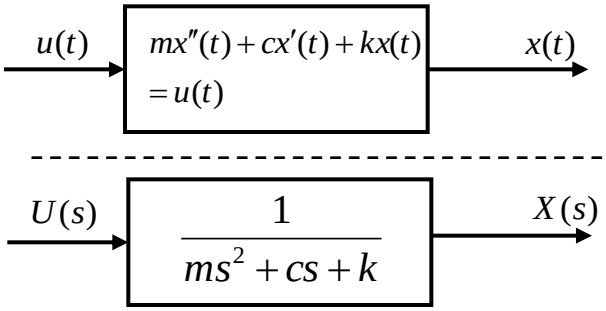
$$m\ddot{x} + c\dot{x} + kx = u$$

$$m\dddot{x}(t) + (c + K_d)\ddot{x}(t) + (k + K_p)\dot{x}(t) + K_i x = K_i r$$

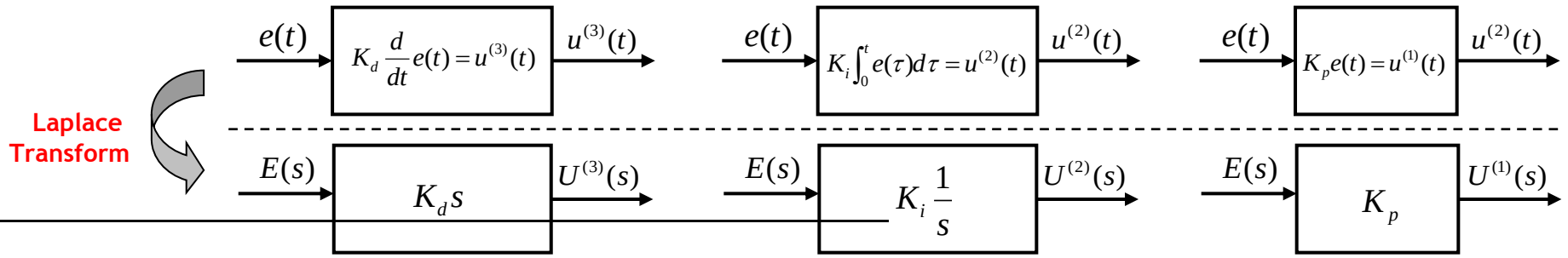
$$r - x(t) = e(t)$$



Laplace Transform

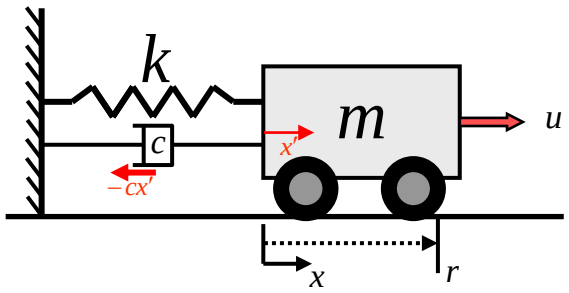


Laplace Transform



Mass-Spring-Damper Linear Mechanical System

- Proportional-Integral-Derivative Control (PID제어)



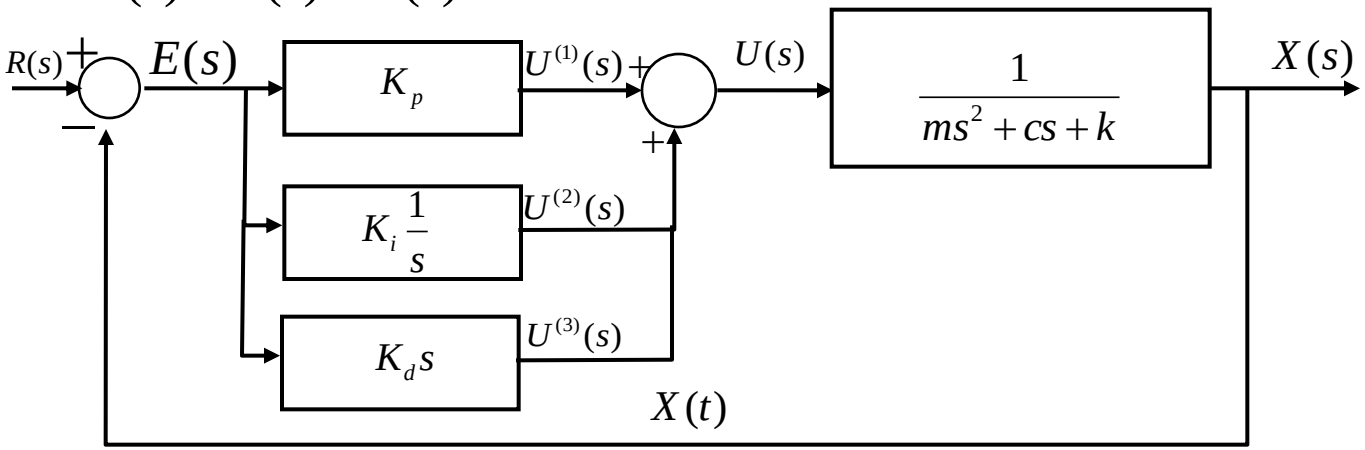
$$m\ddot{x} = F$$

$$m\ddot{x} = -kx - c\dot{x} + u$$

$$m\ddot{x} + c\dot{x} + kx = u$$

$$m\dddot{x}(t) + (c + K_d)\ddot{x}(t) + (k + K_p)\dot{x}(t) + K_i x = K_i r$$

$$R(s) - X(s) = E(s)$$



$$R(s) - X(s) = E(s)$$

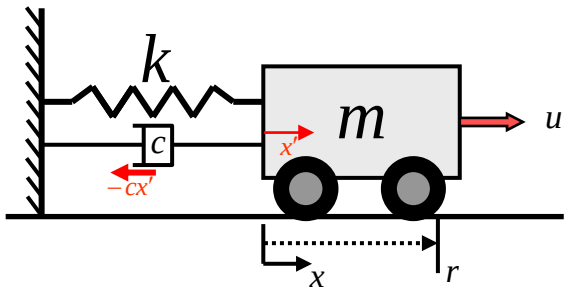
$$E(s) \times (K_p + K_i \frac{1}{s} + K_d s) = U(s)$$

$$U(s) \times \frac{1}{ms^2 + cs + k} = X(s)$$

$$\{R(s) - X(s)\} \times (K_p + K_i \frac{1}{s} + K_d s) \times \frac{1}{ms^2 + cs + k} = X(s)$$

Mass-Spring-Damper Linear Mechanical System

- Proportional-Integral-Derivative Control (PID제어)

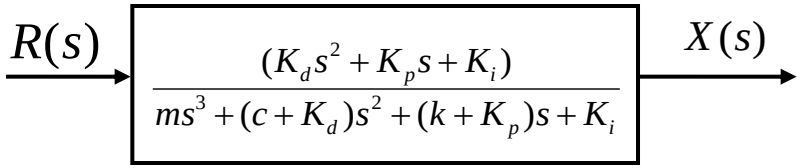


$$m\ddot{x} = F$$

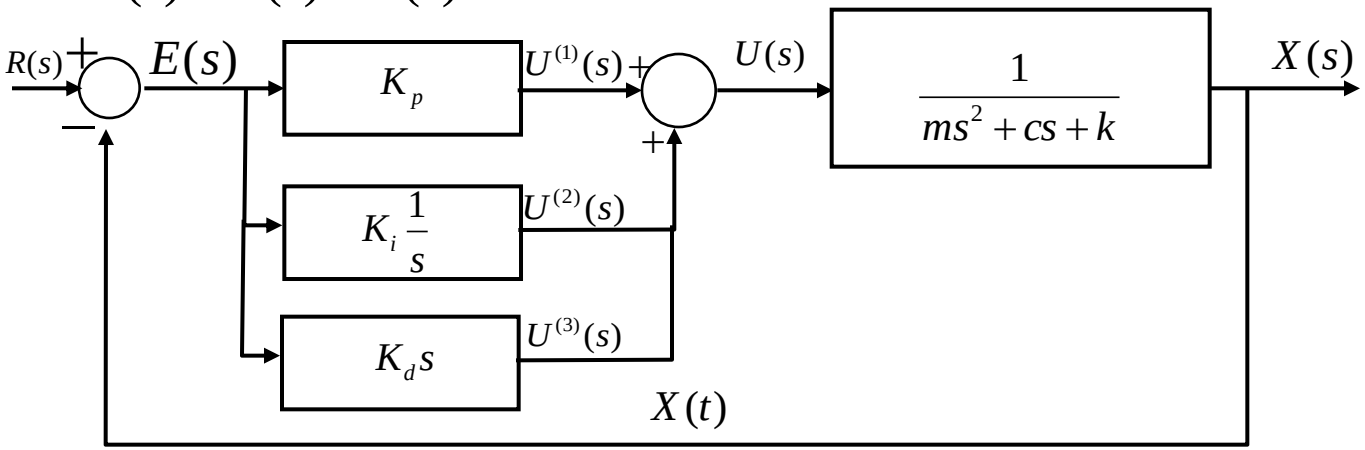
$$m\ddot{x} = -kx - c\dot{x} + u$$

$$m\ddot{x} + c\dot{x} + kx = u$$

$$m\dddot{x}(t) + (c + K_d)\ddot{x}(t) + (k + K_p)\dot{x}(t) + K_i x = K_i r$$



$$R(s) - X(s) = E(s)$$

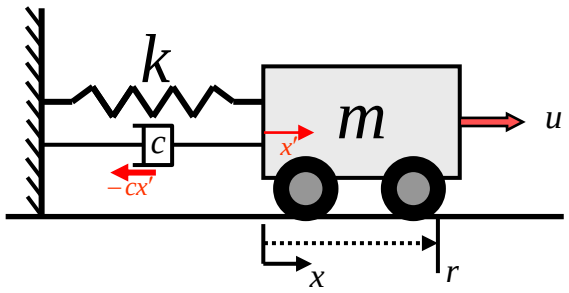


$$\{R(s) - X(s)\} \times (K_p + K_i \frac{1}{s} + K_d s) \times \frac{1}{ms^2 + cs + k} = X(s)$$

$$\Rightarrow \quad \frac{R(s)(K_d s^2 + K_p s + K_i)}{ms^3 + (c + K_d)s^2 + (k + K_p)s + K_i} = X(s)$$

Mass-Spring-Damper Linear Mechanical System

- Proportional-Integral-Derivative Control (PID제어)

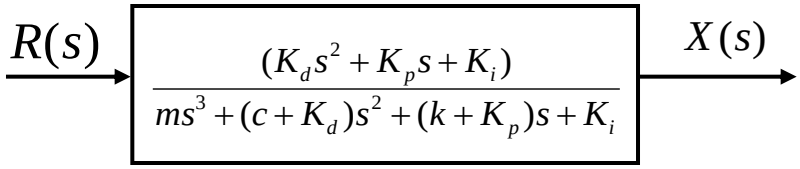


$$m\ddot{x} = F$$

$$m\ddot{x} = -kx - c\dot{x} + u$$

$$m\ddot{x} + c\dot{x} + kx = u$$

$$m\dddot{x}(t) + (c + K_d)\ddot{x}(t) + (k + K_p)\dot{x}(t) + K_i x = K_i r$$



$$X(s) = \frac{R(s)(K_d s^2 + K_p s + K_i)}{ms^3 + (c + K_d)s^2 + (k + K_p)s + K_i} = \frac{r(K_d s^2 + K_p s + K_i)}{s(ms^3 + (c + K_d)s^2 + (k + K_p)s + K_i)}$$

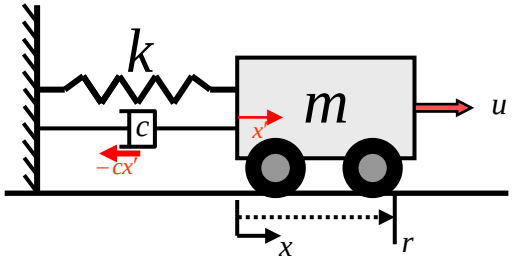
$$L^{-1}(X(s)) = L^{-1}\left(\frac{r(K_d s^2 + K_p s + K_i)}{s(ms^3 + (c + K_d)s^2 + (k + K_p)s + K_i)}\right)$$

MATLAB을 이용하여 $C_1 \sim C_4$, $\lambda_1 \sim \lambda_4$ 를 구하였음

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + C_3 e^{\lambda_3 t} + C_4 e^{\lambda_4 t}$$

Mass-Spring-Damper Linear Mechanical System

- Proportional-Integral-Derivative Control (PID제어)

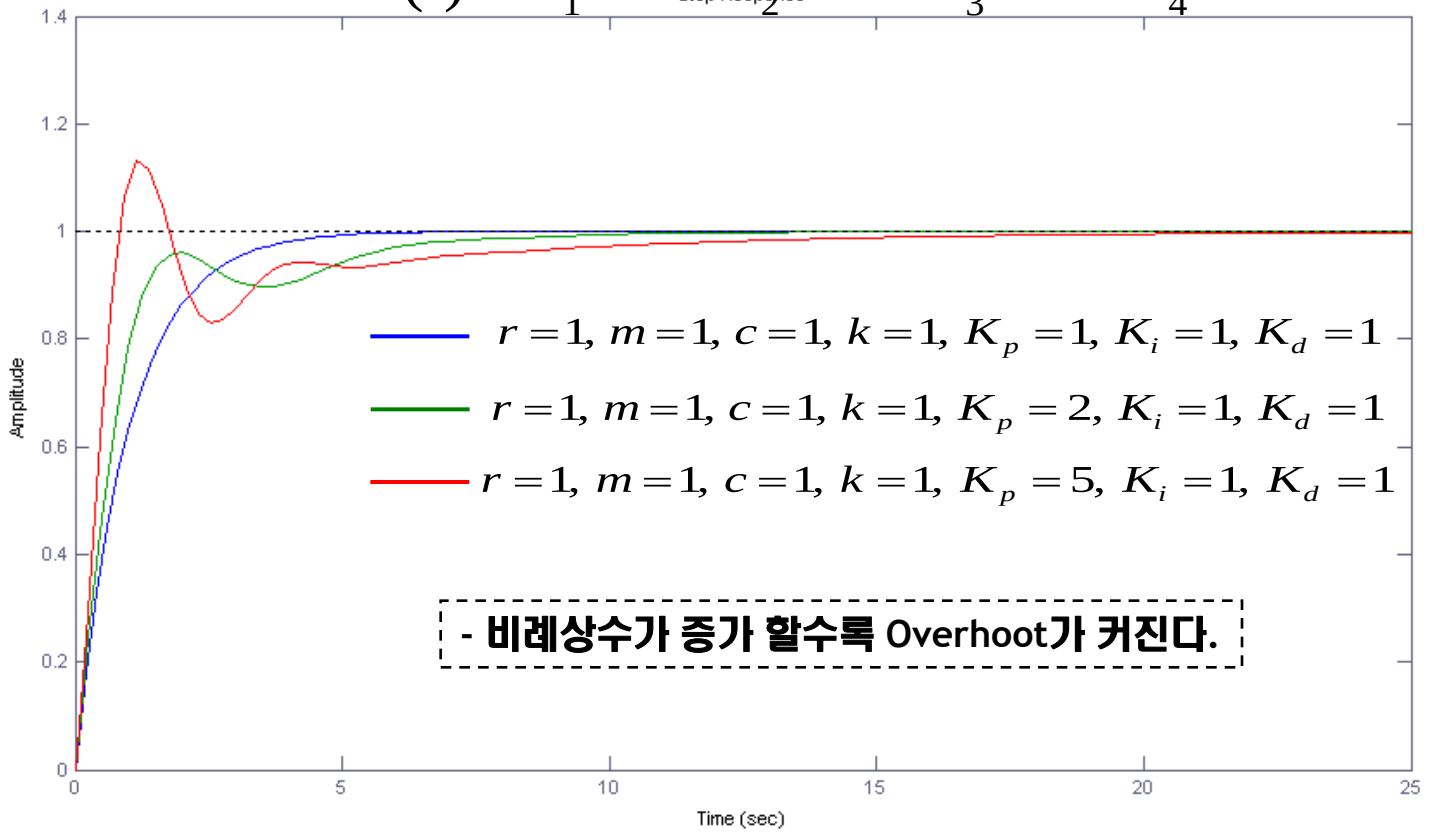


$$m\dddot{x}(t)+(c+K_d)\ddot{x}(t)+(k+K_p)\dot{x}(t)+K_i x=K_i r$$

$$R(s) \rightarrow \frac{(K_d s^2+K_p s+K_i)}{ms^3+(c+K_d)s^2+(k+K_p)s+K_i} \rightarrow X(s)$$

x

$$x(t)=C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}+C_3e^{\lambda_3 t}+C_4e^{\lambda_4 t}$$

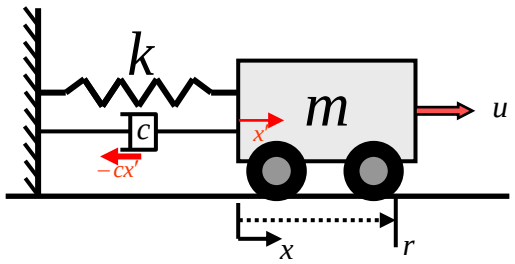


- 비례상수가 증가 할수록 Overhoot가 커진다.

t

Mass-Spring-Damper Linear Mechanical System

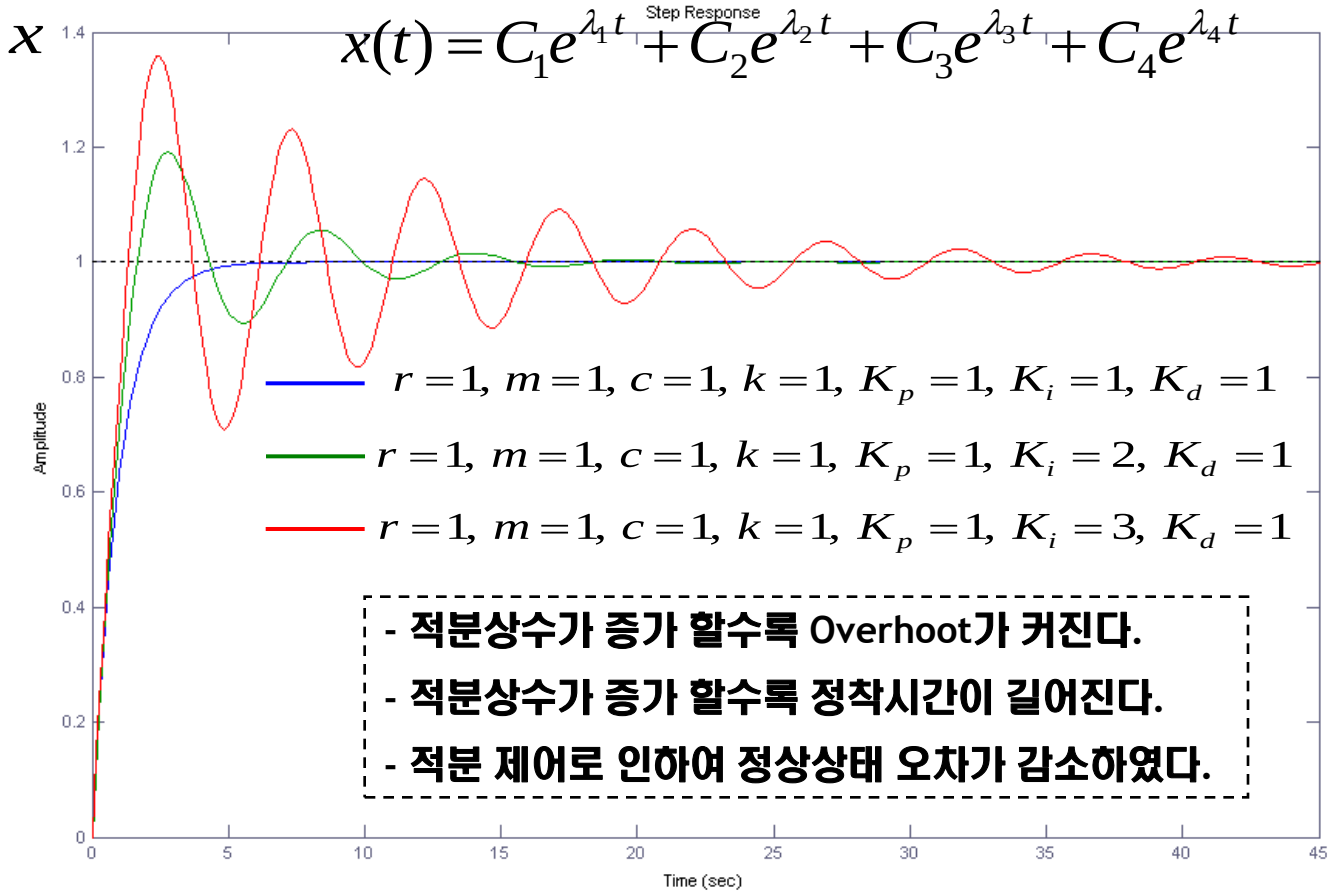
- Proportional-Integral-Derivative Control (PID제어)



$$m\dddot{x}(t)+(c+K_d)\ddot{x}(t)+(k+K_p)\dot{x}(t)+K_i x=K_i r$$

$$R(s) \rightarrow \frac{(K_d s^2+K_p s+K_i)}{ms^3+(c+K_d)s^2+(k+K_p)s+K_i} \rightarrow X(s)$$

$$x(t)=C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}+C_3e^{\lambda_3 t}+C_4e^{\lambda_4 t}$$



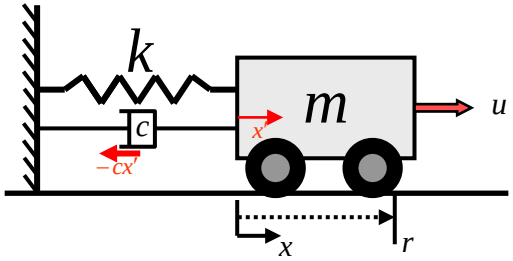
x

t

- 적분상수가 증가 할수록 Overhoot가 커진다.
- 적분상수가 증가 할수록 정착시간이 길어진다.
- 적분 제어로 인하여 정상상태 오차가 감소하였다.

Mass-Spring-Damper Linear Mechanical System

- Proportional-Integral-Derivative Control (PID제어)

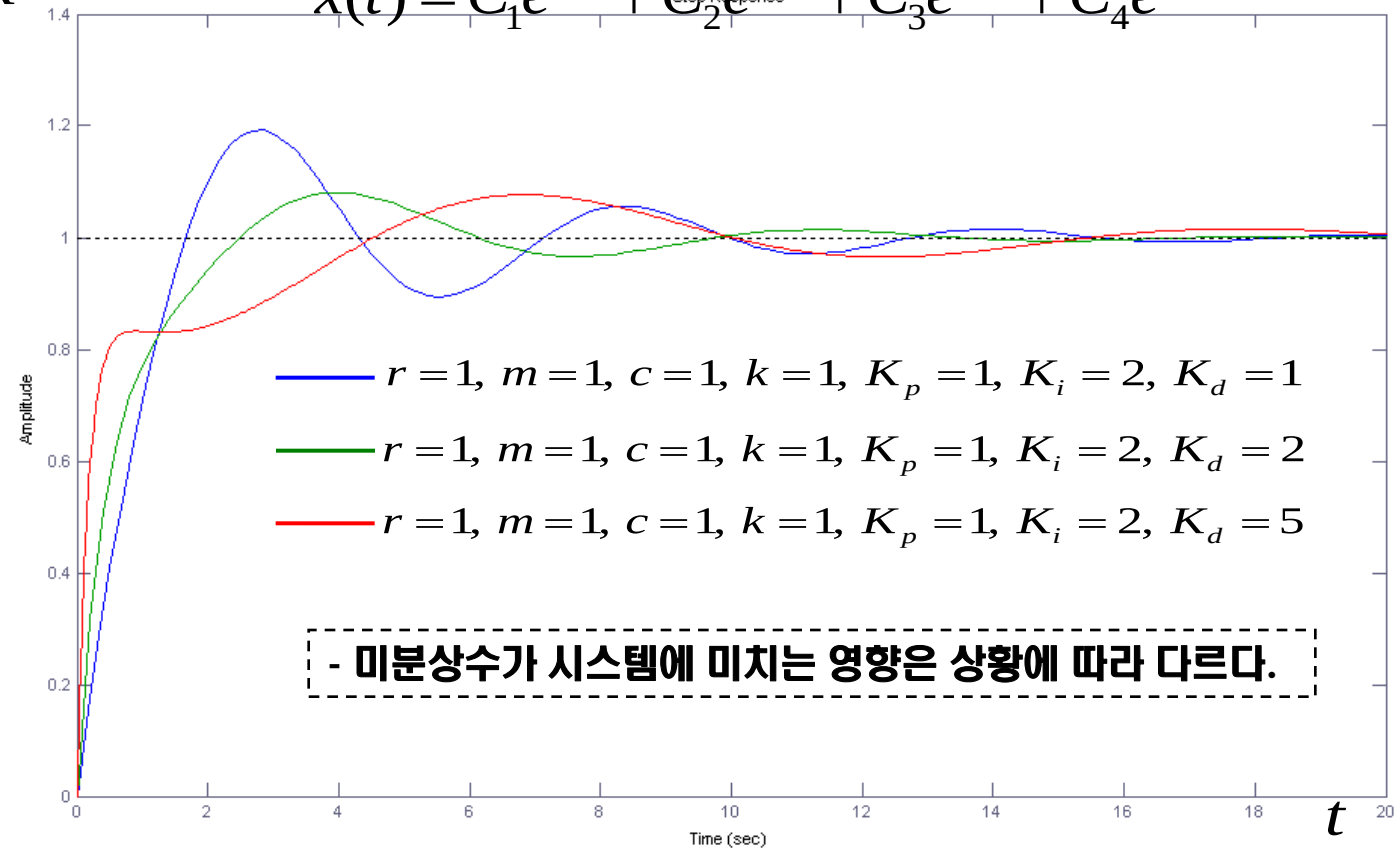


$$m\dddot{x}(t)+(c+K_d)\ddot{x}(t)+(k+K_p)\dot{x}(t)+K_i x=K_i r$$

$$R(s) \rightarrow \boxed{\frac{(K_d s^2+K_p s+K_i)}{ms^3+(c+K_d)s^2+(k+K_p)s+K_i}} \rightarrow X(s)$$

x

$$x(t)=C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}+C_3e^{\lambda_3 t}+C_4e^{\lambda_4 t}$$



t

- 미분상수가 시스템에 미치는 영향은 상황에 따라 다르다.

Reference slides

Properties of Convolution

Property of Convolution

Commutative law of convolution

$$(f * g)(t) = \int_0^t f(\tau) g(t-\tau) d\tau$$

replace $\tau = t - \tau'$

Differentiate with τ $\qquad 1 = -\dfrac{d\tau'}{d\tau} \Rightarrow d\tau = -d\tau'$

if τ **Varies 0 ~ t ,** τ' **varies t ~ 0**

$$\begin{aligned}
(f * g)(t) &= \int_0^t f(\tau) g(t-\tau) d\tau \\
&= \int_t^0 f(t-\tau') g(\tau')(-d\tau') \\
&= -\int_t^0 f(t-\tau') g(\tau') d\tau' \\
&= \int_0^t f(t-\tau') g(\tau') d\tau' \\
&= \int_0^t g(\tau') f(t-\tau') d\tau' = (g * f)(t)
\end{aligned}$$

Property of Convolution

Associative law of convolution

$$\{(f*g)*v\}(t) = \int_0^t (f*g)(\tau)v(t-\tau)d\tau$$

$$= \int_0^t \int_0^\tau f(p)g(\tau-p)dpv(t-\tau)d\tau$$

$$= \int_0^t \int_0^\tau f(p)g(\tau-p)v(t-\tau)dpd\tau$$

$$= \int_0^t \int_{-p}^{t-p} f(p)g(k)v(t-(p+k))dkdp$$

$$= \int_0^t \int_0^{t-p} f(p)g(k)v(t-p-k))dkdp$$

$$= \int_0^t f(p)\int_0^{t-p} g(k)v(t-p-k))dkdp$$

$$= \int_0^t f(p)\cdot(g*v)(t-p)dp$$

$$= \{f*(g*v)\}(t)$$

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

$$(f*g)(\tau) = \int_0^\tau f(p)g(\tau-p)dp$$

$$\tau - p = k,\ \ 1 = \frac{dk}{d\tau},\ \ d\tau = dk$$

$$if\ \ \tau\ is\ 0 \to t,\ \ k\ is\ -p \to t-p$$

Property of Convolution

Distributive law of convolution

$$f*(g_1+g_2)(t)=\int_0^t f(\tau)\cdot(g_1+g_2)(t-\tau)d\tau$$

$$=\int_0^t f(\tau)\cdot\{g_1(t-\tau)+g_2(t-\tau)\}d\tau$$

$$=\int_0^t f(\tau)\cdot g_1(t-\tau)+f(\tau)\cdot g_2(t-\tau)d\tau$$

$$=\int_0^t f(\tau)\cdot g_1(t-\tau)d\tau+\int_0^t f(\tau)\cdot g_2(t-\tau)d\tau$$

$$=f*g_1(t)+f*g_2(t)$$

Convolution Integral (1)

✓ **Definition :** $x(t) = \int_{-\infty}^{\infty} F(\tau) \cdot h(t-\tau)\, d\tau = F(t) * h(t)$

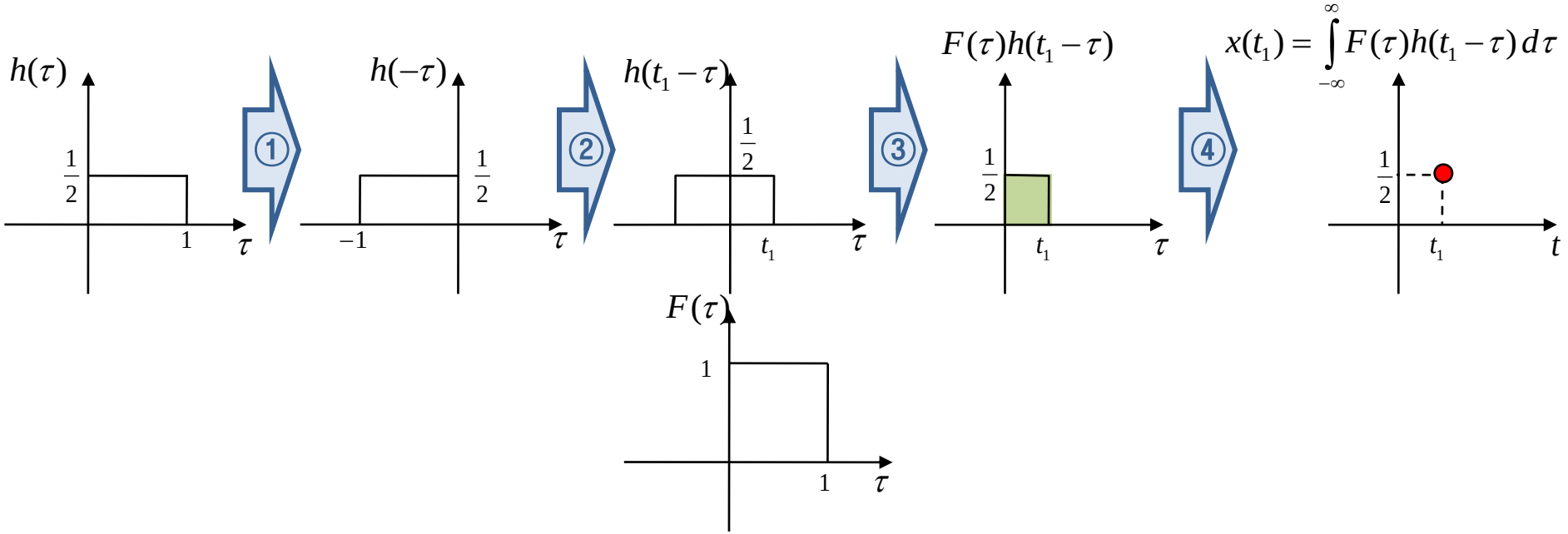
✓**Graphical Technique**

① **Folding : mirror image about ordinate axis**

② **Displacement : Shift by t**

③ **Multiplication : Multiply two functions**

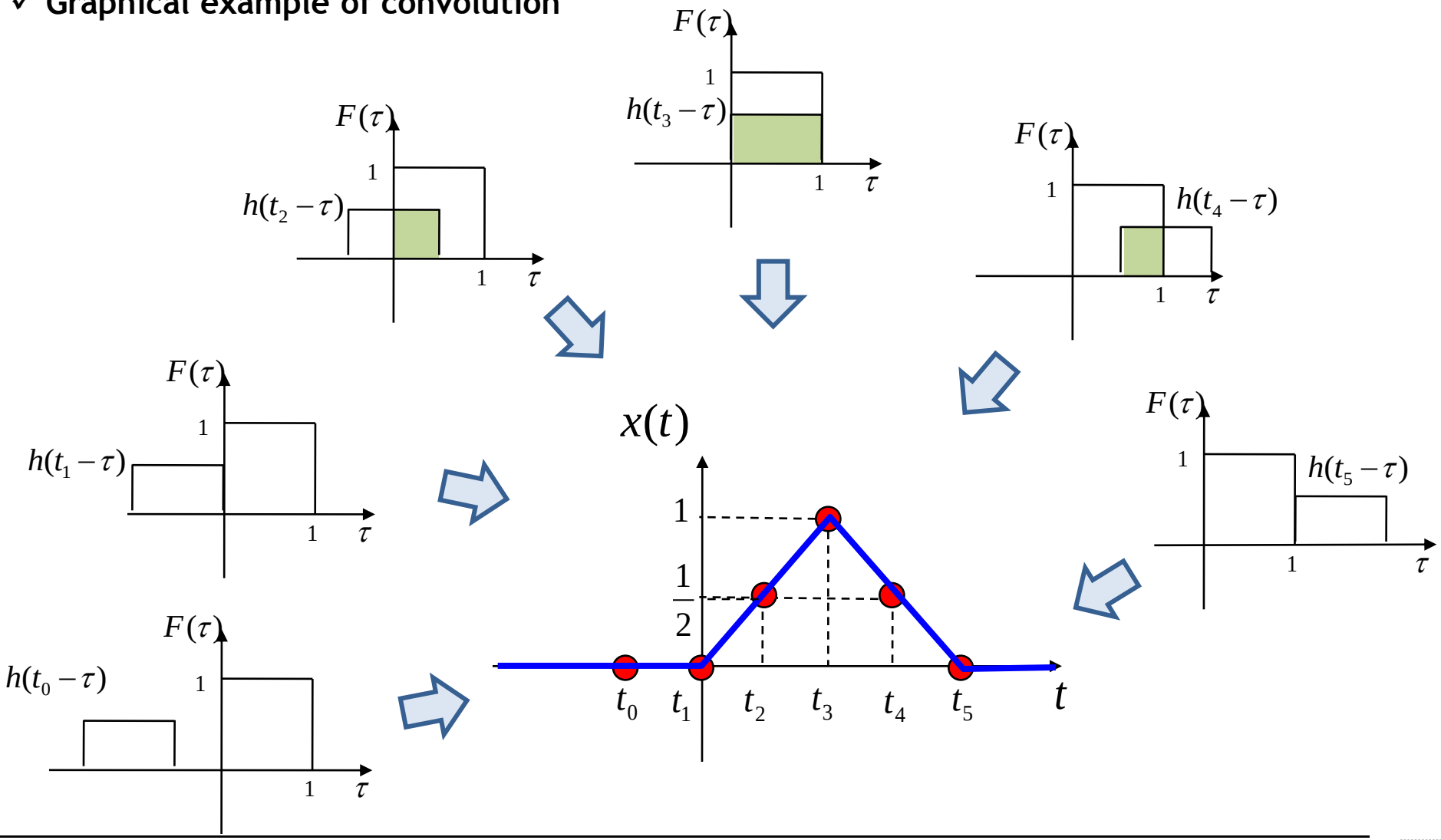
④ **Integration**



Convolution Integral (2)

$$x(t) = \int_{-\infty}^{\infty} F(\tau) \cdot h(t-\tau)\, d\tau = F(t) * h(t)$$

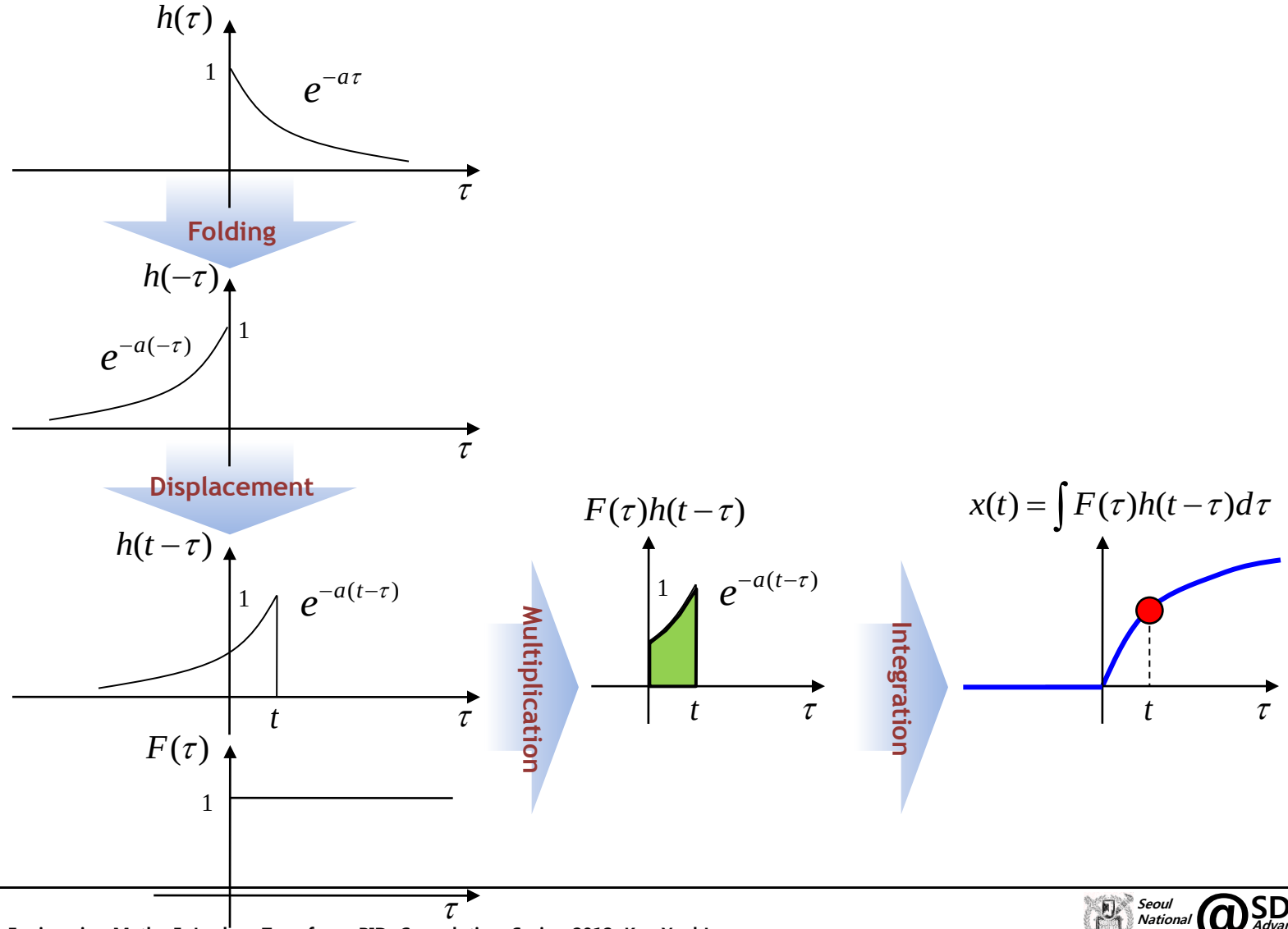
✓ Graphical example of convolution



Convolution Integral (3)

$$x(t) = \int_{-\infty}^{\infty} F(\tau) \cdot h(t-\tau)\, d\tau = F(t) * h(t)$$

✓ Another example of convolution



1) Rao,Singiresu S., Mechanical Vibrations 4th edition, Pearson Prentice Hall, 2004, pp.312-324
2) Thomson,William T., Theory of Vibration with Applications 2nd edition, Prentice Hall, 1981,pp.92-1

Transient Vibration[1)2)]

: The response to a suddenly applied nonperiodic excitation

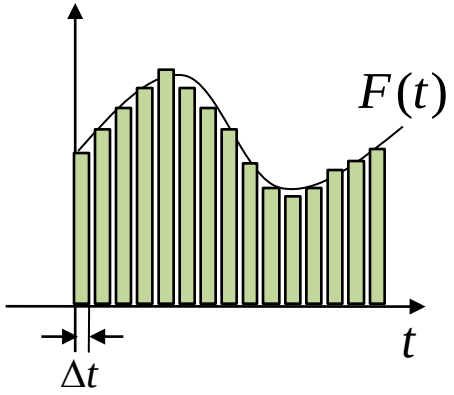
Impulse

1) Rao,Singiresu S., Mechanical Vibrations 4th edition, Pearson Prentice Hall, 2004, pp.312-324
2) Thomson,William T., Theory of Vibration with Applications 2nd edition, Prentice Hall, 1981,pp.92-10

✓ **Implulse : the time integral of the force**

$$\hat{F} = \int F(t)dt \quad (F(t) : \text{force})$$

$$= \sum (force) \times (Time)$$



✓ **Impulse and Initial condition**

(Impulse) = (Mass) X (Velocity change)

$$\hat{F} = F(t) \cdot dt = m \cdot \{v(t+dt) - v(t)\}$$

Impulse acting on the mass at t=0

$$\hat{F} = m \cdot \{v(0+dt) - v(0)\} = m \cdot \{\dot{x}(0) - 0\} = m \cdot \dot{x}(0) \quad \Rightarrow \quad \therefore \dot{x}(0) = \frac{\hat{F}}{m} \left(= \frac{\text{Impulse}}{\text{Mass}} \right)$$

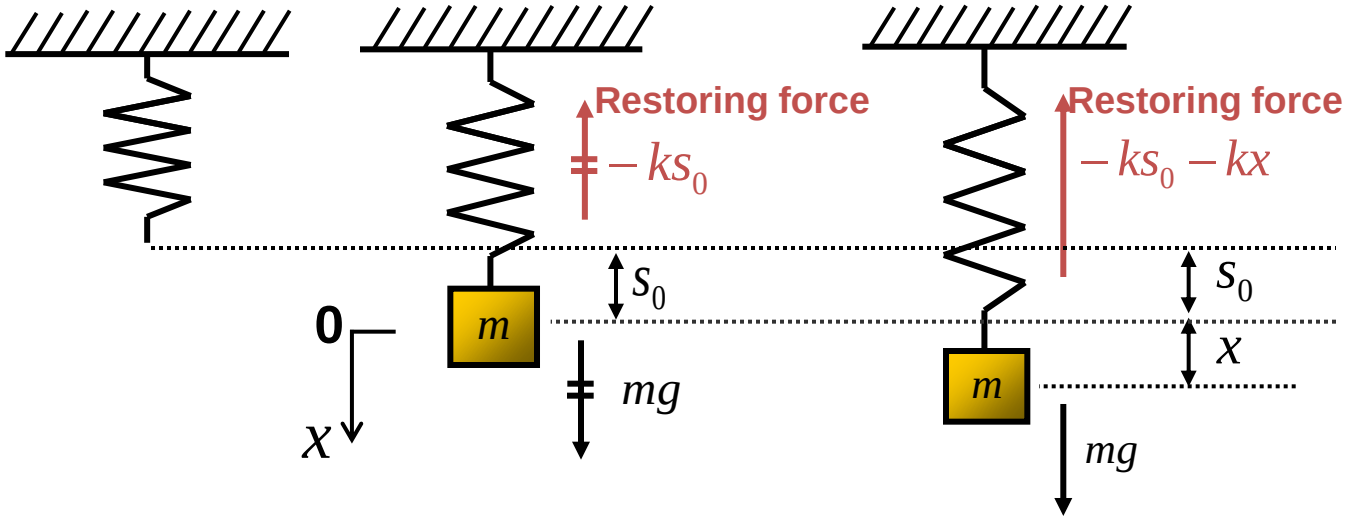
(Impulse가 가해진 직후의 속도) (Impulse가 가해지기 전에 정지해 있었음)

1) Rao,Singiresu S., Mechanical Vibrations 4th edition, Pearson Prentice Hall, 2004, pp.312-324
2) Thomson,William T., Theory of Vibration with Applications 2nd edition, Prentice Hall, 1981,pp.92-100

Impulse Response & Impulse Response Function (IRF)

(Q) Find the response of a free undamped spring-mass system excited by an impulse $\hat{1}$

※ Erwin Kreyszig, Advanced Engineering Mathematics 9th edition, Wiley, Ch 2.4



$$m\ddot{x} = F$$
$$= mg - ks_0 = 0$$
$$\therefore mg = ks_0$$
$$(\because \ddot{x} = 0)$$

: Static equilibrium

$$m\ddot{x} = F$$
$$= mg - ks_0 - kx$$
$$= -kx$$
$$\therefore m\ddot{x} + kx = 0$$

1) Rao,Singiresu S., Mechanical Vibrations 4th edition, Pearson Prentice Hall, 2004, pp.312-324
2) Thomson,William T., Theory of Vibration with Applications 2nd edition, Prentice Hall, 1981,pp.92-100

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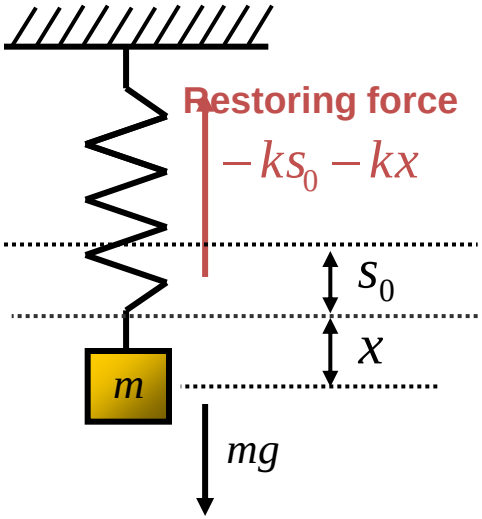


① $m\ddot{x} + kx = 0$ (Initial condition : $x(0) = 0,\ \dot{x}(0) = \frac{\hat{F}}{m}$)

② $x(t) = A\cos\omega t + B\sin\omega t \quad \left(\omega = \sqrt{\frac{k}{m}}\right)$

③ $x(0) = A = 0$

$\dot{x}(t) = B\omega\cos\omega t$

$\dot{x}(0) = B\omega = \hat{F}/m$

④ $x(t) = \frac{\hat{F}}{m\omega}\sin\omega t$

Let $h(t) = \frac{1}{m\omega}\sin\omega t$

⑤ Impulse Response → $x(t) = \hat{F}\cdot h(t)$

Impulse

Impulse Response Function
(단위 impulse에 대한 응답 함수)

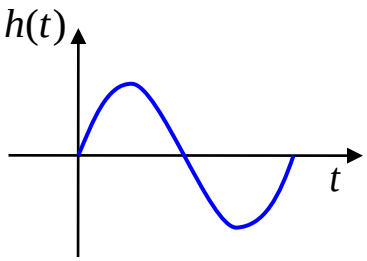
1) Rao,Singiresu S., Mechanical Vibrations 4th edition, Pearson Prentice Hall, 2004, pp.312-324
2) Thomson,William T., Theory of Vibration with Applications 2nd edition, Prentice Hall, 1981,pp.92-100

Impulse Response & Impulse Response Function (IRF)

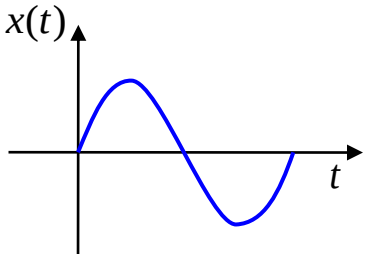
Impulse Response → $x(t) = \hat{F} \cdot h(t)$

- $\hat{F}$: Impulse
- $h(t)$: Impulse Response Function (단위 impulse에 대한 응답 함수)

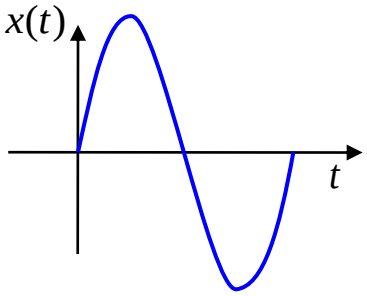
ex)



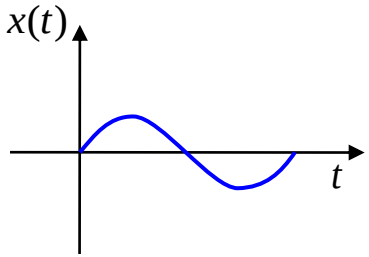
① $\hat{F} = 1$



② $\hat{F} = 2$

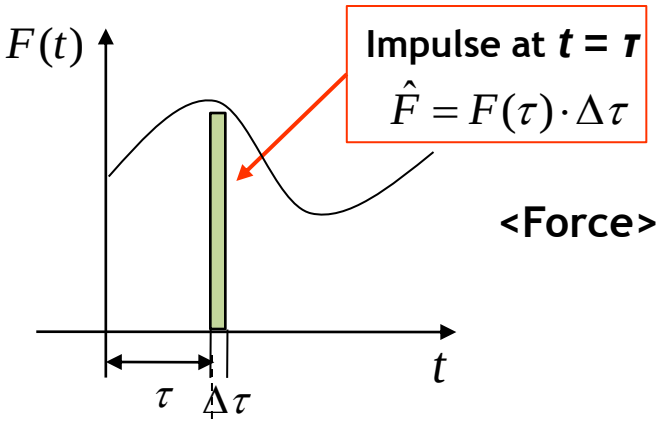


③ $\hat{F} = 0.5$



1) Rao,Singiresu S., Mechanical Vibrations 4th edition, Pearson Prentice Hall, 2004, pp.312-324
2) Thomson,William T., Theory of Vibration with Applications 2nd edition, Prentice Hall, 1981,pp.92-100

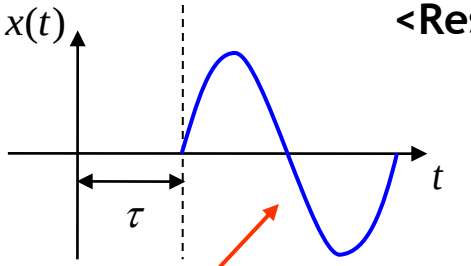
Arbitrary Excitation



Impulse at $t = \tau$

$$\hat{F} = F(\tau)\cdot\Delta\tau$$

<Force>

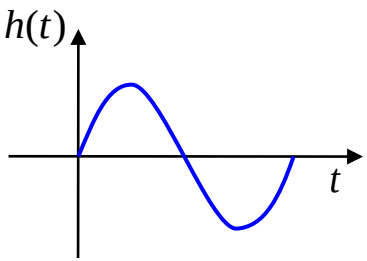


<Response>

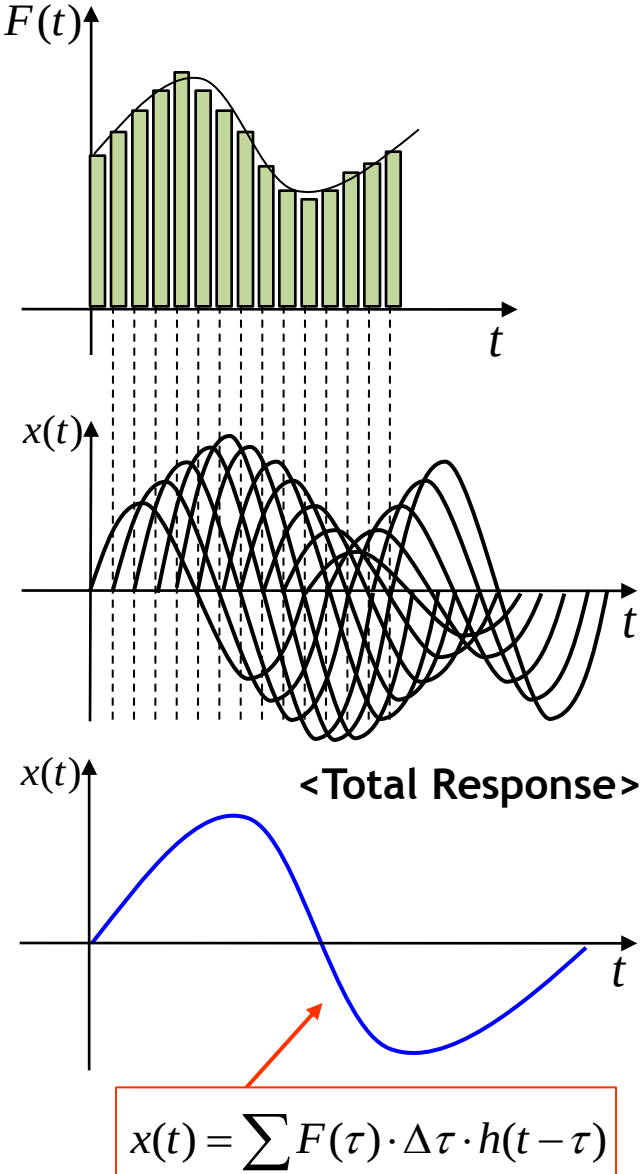
$$x(t) = \hat{F}\cdot h(t-\tau)$$

Displacement

$$= F(\tau)\cdot\Delta\tau\cdot h(t-\tau)$$



<Impulse Response Function>



<Total Response>

$$x(t) = \sum F(\tau)\cdot\Delta\tau\cdot h(t-\tau)$$

$$x(t) = \sum F(\tau)\cdot\Delta\tau\cdot h(t-\tau)$$

$$\Delta\tau \to 0$$

$$x(t) = \int_0^t F(\tau)\,h(t-\tau)\,d\tau$$

or

$$x(t) = \int_0^t F(t-\tau)\,h(\tau)\,d\tau$$

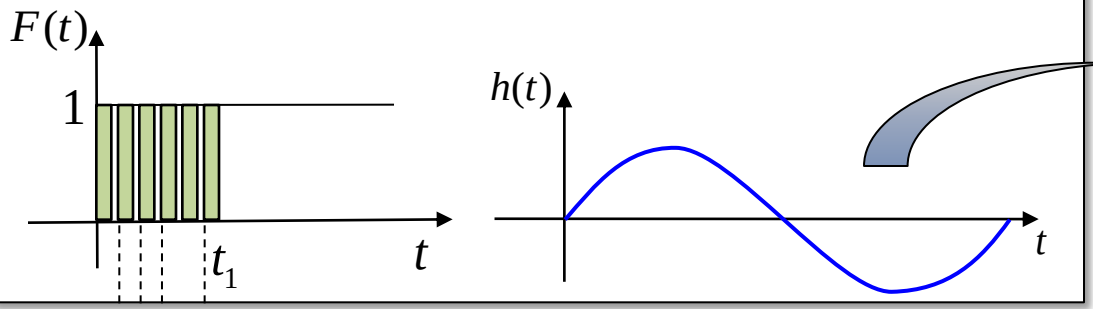
⇒ **Convolution Integral**

Summation(or superposition) of the response excited by the impulse between 0 and t

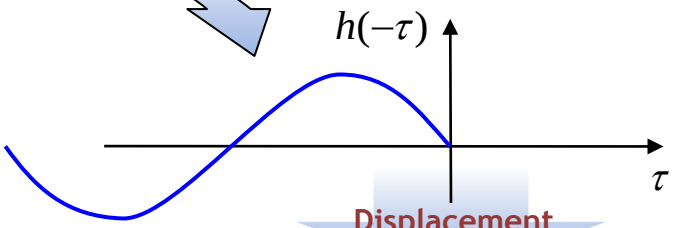
1) Rao,Singiresu S., Mechanical Vibrations 4th edition, Pearson Prentice Hall, 2004, pp.312-324
2) Thomson,William T., Theory of Vibration with Applications 2nd edition, Prentice Hall, 1981,pp.92-100

Arbitrary Excitation

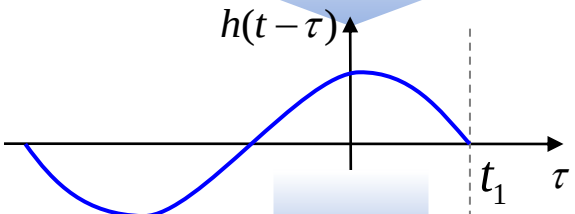
(Q) Find $x(t_1)$



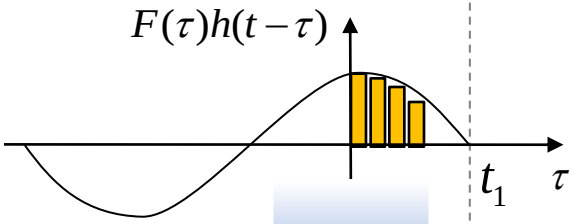
Folding



Displacement

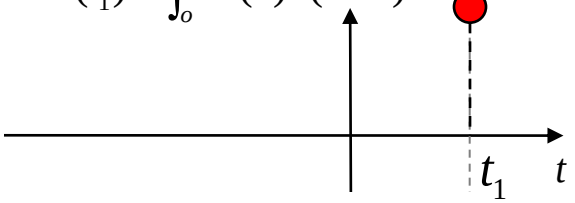


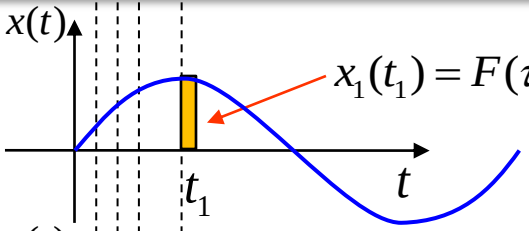
Multiplication



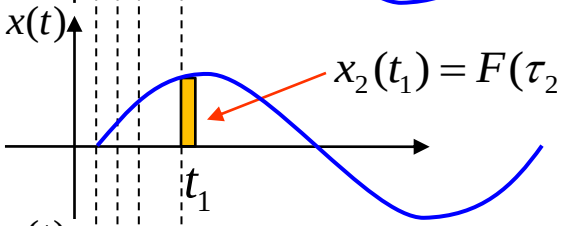
Integration

$$x(t_1) = \int_o^{t_1} F(\tau)h(t-\tau)$$

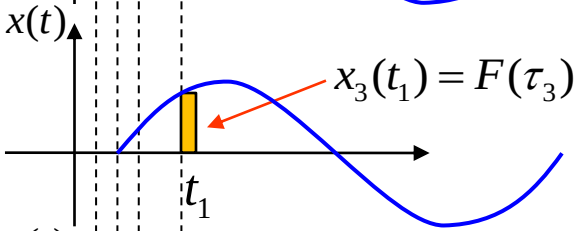




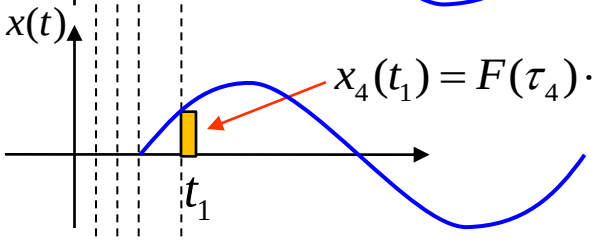
$$x_1(t_1) = F(\tau_1)\cdot\Delta\tau\cdot h(t_1-\tau_1)$$



$$x_2(t_1) = F(\tau_2)\cdot\Delta\tau\cdot h(t_1-\tau_2)$$



$$x_3(t_1) = F(\tau_3)\cdot\Delta\tau\cdot h(t_1-\tau_3)$$



$$x_4(t_1) = F(\tau_4)\cdot\Delta\tau\cdot h(t_1-\tau_4)$$

$$\therefore x(t_1) = \sum x_i(t_1)$$

1) Rao,Singiresu S., Mechanical Vibrations 4th edition, Pearson Prentice Hall, 2004, pp.312-324
2) Thomson,William T., Theory of Vibration with Applications 2nd edition, Prentice Hall, 1981,pp.92-100

Arbitrary Excitation

$$x(t) = \int_0^t F(\tau)\,h(t-\tau)\,d\tau$$
$$= \int_0^t F(t-\tau)\,h(\tau)\,d\tau$$

$F(\tau)$:시간 τ 에서 입력의 세기

$h(t-\tau)$:단위 입력에 대한 시간($t-\tau$) 후의 응답

$F(\tau)\cdot h(t-\tau)$

시간 τ 에서의 입력에 대한 시간($t-\tau$) 후의 응답
즉, 시간 τ 에서의 입력에 대한 시간 t 에서의 응답

예) 시간 0, 1, 2, 3, … , 7, 8, 9, 10에서 각각 입력(τ) 가 있을 때, 시간$=10$ 에서의 응답$x(10)$

$$x(10) = \sum_{\tau=0}^{10} F(\tau)\,h(10-\tau)$$

$$x(10) = \sum_{\tau=0}^{10} F(10-\tau)\,h(\tau)$$

τ	$F(\tau)\cdot h(t-\tau)$	$F(t-\tau)\cdot h(\tau)$
0	$F(0)\cdot h(10-0)=F(0)\cdot h(10)$	$F(10-0)\cdot h(0)=F(10)\cdot h(0)$
1	$F(1)\cdot h(10-1)=F(1)\cdot h(9)$	$F(10-1)\cdot h(1)=F(9)\cdot h(1)$
2	$F(2)\cdot h(10-2)=F(2)\cdot h(8)$	$F(10-2)\cdot h(2)=F(8)\cdot h(2)$
3	$F(3)\cdot h(10-3)=F(3)\cdot h(7)$	$F(10-3)\cdot h(3)=F(7)\cdot h(3)$
⋮		서로 같다
7	$F(7)\cdot h(10-7)=F(7)\cdot h(3)$	$F(10-3)\cdot h(3)=F(3)\cdot h(7)$
8	$F(8)\cdot h(10-8)=F(8)\cdot h(2)$	$F(10-8)\cdot h(8)=F(2)\cdot h(8)$
9	$F(9)\cdot h(10-9)=F(9)\cdot h(1)$	$F(10-9)\cdot h(9)=F(1)\cdot h(9)$
10	$F(10)\cdot h(10-10)=F(10)\cdot h(0)$	$F(10-10)\cdot h(10)=F(0)\cdot h(10)$

CHAPTER 6. STURM-LIOUVILLE BVP(BOUNDARY VALUE PROBLEM)

Review

▪ **Review**

Linear Equations	General solutions
$y' + \alpha y = 0$	$y = c_1 e^{-\alpha x}$
$y'' + \alpha^2 y = 0 \quad \alpha > 0$	$y = c_1 \cos \alpha x + c_2 \sin \alpha x$
$y'' - \alpha^2 y = 0 \quad \alpha > 0$	$\begin{cases} y = c_1 e^{-\alpha x} + c_2 e^{\alpha x}, or & \leftarrow \text{When } x \text{ is an infinite or half finite interval} \\ y = c_1 \cosh \alpha x + c_2 \sinh \alpha x) & \leftarrow \text{When } x \text{ is a finite interval} \end{cases}$

Cauchy-Euler Equation	General solutions $x > 0$
$x^2 y'' + xy' - \alpha^2 y = 0 \quad \alpha \geq 0$	$\begin{cases} y = c_1 x^{-\alpha} + c_2 x^{\alpha}, \alpha \neq 0 \\ y = c_1 + c_2 \ln x, \quad \alpha = 0 \end{cases}$

Linear Equations

▪Review

Parametric Bessel equation $\nu = 0$ **General solutions** $x > 0$

$$x^2 y'' + y' + \alpha^2 x^2 y = 0$$

$$y = c_1 J_0(\alpha x) + c_2 Y_0(\alpha x)$$

Legendre's equation

$n = 0,1,2,\ldots$

Particular solutions are polynomials

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$y = P_0(x) = 1,$$

$$y = P_1(x) = x,$$

$$y = P_2(x) = \frac{1}{2}(3x^2 - 1),$$

$$\vdots$$

▪Eigenvalues and Eigenfunctions

Recall example 2 of section 3.9

$$y''+\lambda y=0, \quad y(0)=0, \quad y(L)=0$$

When $\lambda > 0$ **(Case III)**

Write $\lambda = \alpha^2, \ \alpha > 0$

Then roots of auxiliary equation is

$$m_1 = i\alpha, \ m_2 = -i\alpha$$

$$y = c_1 \cos \alpha x + c_2 \sin \alpha x$$

$$y(0)=0 \Rightarrow c_1 = 0$$

$$y(L)=0 \Rightarrow c_2 = 0 \quad \textbf{or} \quad \alpha L = n\pi, \quad \lambda_n = \alpha_n{}^2 = \left(\frac{n\pi}{L}\right)^2$$

Eigenvalues

(nontrivial solution) $y_n = c_2 \sin \dfrac{n\pi}{L} x$

Eigenfunctions

▪**Eigenvalues and Eigenfunctions**

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$$

Eigenvalues

$$\lambda_n = \alpha_n{}^2 = \left(\frac{n\pi}{L}\right)^2$$

Eigenfunctions

$$y_n = c_2 \sin\frac{n\pi}{L}x$$

It is important to recognize the set of functions generated by this B.V.P the orthogonal set of functions on the interval $(0, L)$ ***used as the basis for the Fourier sine series***

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0 \longrightarrow$$ **the Fourier cosine series**

☑ Example 1 Eigenvalues and Eigenfunctions

It is left as an exercise to show, by considering the three possible cases for the parameter λ (zero, negative, or positive; that is, $\lambda = 0,\ \lambda = -\alpha^2 < 0,\ \alpha > 0,\ and\ \lambda = \alpha^2 > 0,\ \alpha > 0$) that the eigenvalues and eigenfunctions for the boundary-value problem

$$y'' + \lambda y = 0,\ \ y'(0) = 0,\ \ y'(L) = 0$$

are, respectively, $\lambda_n = \alpha_n{}^2 = n^2\pi^2 / L^2,\ n = 0,1,2,\cdots,$ $y = c_1 \cos(n\pi x / L),\ c_1 \neq 0.$ $\lambda_0 = 0$ is an eigenvalue for this BVP and $y = 1$ is the corresponding eigenfunction. The latter comes from solving $y'' = 0$ subject to the same boundary conditions $y'(0) = 0,\ y'(L) = 0$. Note also that $y = 1$ can be incorporated into the family $y = \cos(n\pi x / L)$ by permitting $n = 0$. The set $\{\cos(n\pi x / L)\},$ $n = 0,1,2,3,\cdots,$ is orthogonal on the interval [0,L].

6-1. Sturm-Liouville BVP(Boundary Value Problem)

Sturm-Liouville Problem

▪**Regular Sturm-Liouville Problem** **B.V.P**

Solve : $\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$

Subject to: $A_1 y(a) + B_1 y'(a) = 0$

$A_2 y(b) + B_2 y'(b) = 0$

p, q, r, r' **real-valued functions continuous on an interval** $[a, b]$

$r(x) > 0,\ p(x) > 0$ **for every** x **in the interval** $[a, b]$

A_1, B_1 **are not both zero**

A_2, B_2 **are not both zero**

Special case

$$p(x) = 1, q(x) = 0, r(x) = 1$$

$$A_1 = 1, B_1 = 0, A_2 = 1, B_2 = 0, a = 0, b = L$$

$$\Rightarrow y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$$

$$A_1 = 0, B_1 = 1, A_2 = 0, B_2 = 1, a = 0, b = L$$

$$\Rightarrow y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0$$

Sturm-Liouville Problem : Homogeneous Boundary Value problem

-> Trivial solution y=0

-> goal : find nontrivial solution y (Eigenvalues, Eigenfunctions)

Sturm-Liouville Problem

▪**Regular Sturm-Liouville Problem** **B.V.P**

Solve : $\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$

Subject to: $A_1 y(a) + B_1 y'(a) = 0$

$A_2 y(b) + B_2 y'(b) = 0$

p, q, r, r' **real-valued functions continuous on an interval** $[a, b]$

$r(x) > 0,\ p(x) > 0$ **for every** x **in the interval** $[a, b]$

A_1, B_1 **are not both zero**

A_2, B_2 **are not both zero**

"Homogeneous"

⇨ **Homogeneous D.E.+**

Homogeneous B/C

Nonhomogeneous B/C

$A_1 y(a) + B_1 y'(a) = C_2, C_2 : nonzero$

Sturm-Liouville Problem

▪**Regular Sturm-Liouville Problem** **B.V.P**

Solve : $\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$

Subject to: $A_1 y(a) + B_1 y'(a) = 0$

$A_2 y(b) + B_2 y'(b) = 0$

p, q, r, r' **real-valued functions continuous on an interval** $[a, b]$

$r(x) > 0,\ p(x) > 0$ **for every** x **in the interval** $[a, b]$

A_1, B_1 **are not both zero**

A_2, B_2 **are not both zero**

"trivial solution is not our interest"

Homogeneous B.V.P always possesses the trivial solution $y = 0$

Sturm-Liouville Problem

Theorem 12.3

Properties of the Regular Sturm-Liouville Problem

(a) There exist an infinite number of real eigenvalues that can be arranged in increasing order $\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots \lambda_n \to \infty$ such that $n \to \infty$ as

(b) For each eigenvalues there is only one eigenfunction (except for nonzero constant multiples)

(c) Eigenfunctions corresponding to different eigenvalues are linearly independent

(d) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ on interval $[a,b]$

Solve : $$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$$

Subject to: $$A_1 y(a) + B_1 y'(a) = 0$$
$$A_2 y(b) + B_2 y'(b) = 0$$

Sturm-Liouville Problem

(d) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ on interval $[a,b]$

Proof of (d) $\int_a^b p(x)y_m(x)y_n(x)dx = 0, \quad \lambda_m \neq \lambda_n$

$$\frac{d}{dx}[r(x)y_m'] + [q(x) + \lambda_m p(x)]y_m = 0 \cdots (1)$$

$$\frac{d}{dx}[r(x)y_n'] + [q(x) + \lambda_n p(x)]y_n = 0 \cdots (2)$$

$$(1)\times y_n - (2)\times y_m : \; y_n \frac{d}{dx}[r(x)y_m'] - y_m \frac{d}{dx}[r(x)y_n'] + (\lambda_m - \lambda_n)p(x)y_n y_m = 0$$

$$(\lambda_n - \lambda_m)p(x)y_n y_m = y_n \frac{d}{dx}[r(x)y_m'] - y_m \frac{d}{dx}[r(x)y_n']$$

$$= y_n \frac{d}{dx}[r(x)y_m'] + \underbrace{[r(x)y_m']\frac{d}{dx}y_n}_{r(x)y_n'y_m'} - y_m \frac{d}{dx}[r(x)y_n'] - \underbrace{[r(x)y_n']\frac{d}{dx}y_m}_{-r(x)y_n'y_m'}$$

$$= \frac{d}{dx}\left(y_n[r(x)y_m']\right) - \frac{d}{dx}\left(y_m[r(x)y_n']\right)$$

Sturm-Liouville Problem

(d) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ on interval $[a,b]$

Proof of (d) $\int_a^b p(x)y_m(x)y_n(x)dx = 0, \quad \lambda_m \neq \lambda_n$

Integrating $(\lambda_n - \lambda_m)p(x)y_n y_m = \frac{d}{dx}\left(y_n[r(x)y_m']\right) - \frac{d}{dx}\left(y_m[r(x)y_n']\right)$

$$(\lambda_n - \lambda_m)\int_a^b p(x)y_n y_m dx = \int_a^b \left(\frac{d}{dx}\left(y_n[r(x)y_m']\right) - \frac{d}{dx}\left(y_m[r(x)y_n']\right)\right)dx$$

$$= y_n(b)[r(b)y_m'(b)] - y_n(a)[r(a)y_m'(a)]$$

$$-\left(y_m(b)[r(b)y_n'(b)] - y_m(a)[r(a)y_n'(a)]\right)$$

$$= r(b)[y_m'(b)y_n(b) - y_m(b)y_n'(b)]$$

$$-r(a)[y_m'(a)y_n(a) - y_m(a)y_n'(a)]$$

Boundary Condition

$$A_1y(a) + B_1y'(a) = 0 \longrightarrow A_1y_m(a) + B_1y_m'(a) = 0 \cdots (3)$$

$$\longrightarrow A_1y_n(a) + B_1y_n'(a) = 0 \cdots (4)$$

$$A_2y(b) + B_2y'(b) = 0 \longrightarrow A_2y_m(b) + B_2y_m'(b) = 0 \cdots (5)$$

$$\longrightarrow A_2y_n(b) + B_2y_m'(b) = 0 \cdots (6)$$

Sturm-Liouville Problem

(d) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ on interval $[a,b]$

Proof of (d) $\int_a^b p(x)y_m(x)y_n(x)dx = 0, \quad \lambda_m \neq \lambda_n$

$$(\lambda_n - \lambda_m)\int_a^b p(x)y_n y_m dx = r(b)[y_m'(b)y_n(b) - y_m(b)y_n'(b)] - r(a)[y_m'(a)y_n(a) - y_m(a)y_n'(a)]$$

Boundary Condition

$$A_1 y_m(a) + B_1 y_m'(a) = 0 \cdots (3)$$
$$A_1 y_n(a) + B_1 y_n'(a) = 0 \cdots (4)$$

$$\Rightarrow \begin{bmatrix} y_m(a) & y_m'(a) \\ y_n(a) & y_n'(a) \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As A_1, B_1 are not both zero

$$\det \begin{bmatrix} y_m(a) & y_m'(a) \\ y_n(a) & y_n'(a) \end{bmatrix} = y_m(a)y_n'(a) - y_m'(a)y_n(a) = 0$$

Sturm-Liouville Problem

(d) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ on interval $[a,b]$

Proof of (d) $\int_a^b p(x)y_m(x)y_n(x)dx = 0, \quad \lambda_m \neq \lambda_n$

$$(\lambda_n - \lambda_m)\int_a^b p(x)y_n y_m dx = r(b)[y_m'(b)y_n(b) - y_m(b)y_n'(b)] - r(a)[y_m'(a)y_n(a) - y_m(a)y_n'(a)]$$

Boundary Condition

$$A_2 y_m(b) + B_2 y_m'(b) = 0 \cdots (5)$$
$$A_2 y_n(b) + B_2 y_n'(b) = 0 \cdots (6)$$

$$\Rightarrow \begin{bmatrix} y_m(b) & y_m'(b) \\ y_n(b) & y_n'(b) \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As A_2, B_2 are not both zero

$$\det \begin{bmatrix} y_m(b) & y_m'(b) \\ y_n(b) & y_n'(b) \end{bmatrix} = y_m(b)y_n'(b) - y_m'(b)y_n(b) = 0$$

Sturm-Liouville Problem

(d) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ on interval $[a,b]$

Proof of (d) $\int_a^b p(x)y_m(x)y_n(x)dx = 0, \quad \lambda_m \neq \lambda_n$

$$(\lambda_n - \lambda_m)\int_a^b p(x)y_n y_m dx = r(b)[y_m'(b)y_n(b) - y_m(b)y_n'(b)] - r(a)[y_m'(a)y_n(a) - y_m(a)y_n'(a)]$$

zero, zero

From Boundary Condition:

$$y_m(a)y_n'(a) - y_m'(a)y_n(a) = 0$$

$$y_m(b)y_n'(b) - y_m'(b)y_n(b) = 0$$

$$\therefore (\lambda_n - \lambda_m)\int_a^b p(x)y_n y_m dx = 0$$

$$\int_a^b p(x)y_n y_m dx = 0, \quad \lambda_n \neq \lambda_m$$

Sturm-Liouville Problem

(d) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ on interval $[a,b]$

Proof of (d) $\int_a^b p(x)y_m(x)y_n(x)dx = 0, \quad \lambda_m \neq \lambda_n$

$$(\lambda_n - \lambda_m)\int_a^b p(x)y_n y_m dx = r(b)[y'_m(b)y_n(b) - y_m(b)y'_n(b)] \quad (y_n(b) \to \text{zero})$$
$$- r(a)[y'_m(a)y_n(a) - y_m(a)y'_n(a)] \quad (y_m(a) \to \text{zero})$$

From Boundary Condition:

$$y_m(a)y'_n(a) - y'_m(a)y_n(a) = 0$$
$$y_m(b)y'_n(b) - y'_m(b)y_n(b) = 0$$

$$\therefore (\lambda_n - \lambda_m)\int_a^b p(x)y_n y_m dx = 0$$

Orthogonal relation

$$\int_a^b p(x)y_n y_m dx = 0, \quad \lambda_n \neq \lambda_m$$

Sturm-Liouville Problem

☑ Example 2 A Regular Sturm-Liouville Problem

Slove the boundary-value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) + y'(1) = 0.$$

$\lambda = 0$ *and* $\lambda = -\alpha^2 < 0$, *where* $\alpha > 0$,
the trivial solution $y = 0$

$\lambda = \alpha^2 > 0, \quad \alpha > 0,$
the general solution of $y'' + \alpha^2 y = 0$
is $y = c_1 \cos \alpha x + c_2 \sin \alpha x$

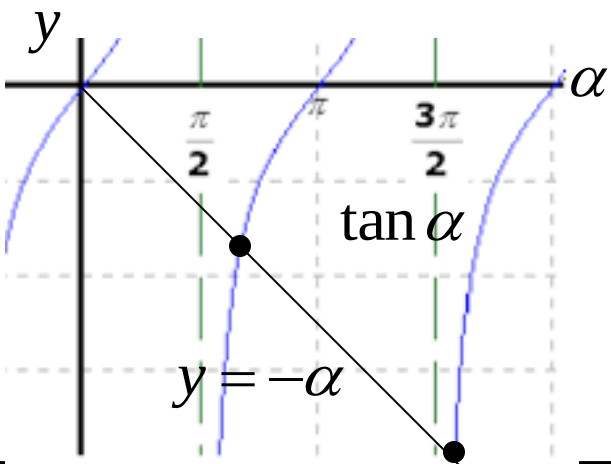
$y(0) = c_1 = 0 \quad \therefore y = c_2 \sin \alpha x$

The second boundary condition $y(1) + y'(1) = 0$ **is satisfied if**

$$c_2 \sin\alpha + c_2 \alpha \cos\alpha = c_2(\sin\alpha + \alpha\cos\alpha) = 0$$

Choosing $c_2 \neq 0$ **, we see that the last equation is equivalent to** $\tan\alpha = -\alpha$

The eigenvalues of problem are then $\lambda_n = \alpha_n^2$**, where** α_n **,** $n = 1,2,3,\cdots$ **, are the consecutive positive roots** $\alpha_1, \alpha_2, \alpha_3, \cdots$ **of** $\tan\alpha = -\alpha$



Sturm-Liouville Problem

☑ Example 2 A Regular Sturm-Liouville Problem

Slove the boundary-value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) + y'(1) = 0.$$

In general, the eigenfunctions of the problem are $\{\sin \alpha_n\}, \ n = 1,2,3,\cdots$

Regular Sturm-Liouville Problem

$\{\sin \alpha_n\}, \ n = 1,2,3,\cdots$ **is an orthogonal set with respect to the weight function**

$p(x) = 1$ **on the interval [0,1].**

▪**Regular Sturm-Liouville Problem** **B.V.P**

Solve : $\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$

Subject to: $A_1 y(a) + B_1 y'(a) = 0$

$A_2 y(b) + B_2 y'(b) = 0$

$$r(x) = 1, q(x) = 0, p(x) = 1$$

$$A_1 = 1, B_1 = 0, A_2 = 1, B_2 = 1$$

Orthogonal relation

$$\int_a^b p(x) y_n y_m dx = 0, \quad \lambda_n \neq \lambda_m$$

Sturm-Liouville Problem

☑ Example 2 A Regular Sturm-Liouville Problem

Slove the boundary-value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) + y'(1) = 0.$$

$\lambda = 0$ *and* $\lambda = -\alpha^2 < 0$, *where* $\alpha > 0$,
the trivial solution $y = 0$

$\lambda = \alpha^2 > 0, \quad \alpha > 0,$
the general solution of $y'' + \alpha^2 y = 0$
is $y = c_1 \cos \alpha x + c_2 \sin \alpha x$

$y(0) = c_1 = 0 \quad \therefore y = c_2 \sin \alpha x$

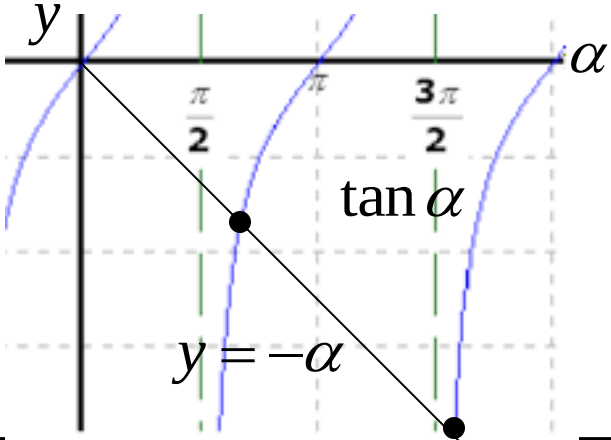
The second boundary condition $y(1) + y'(1) = 0$ **is satisfied if**

$$c_2 \sin \alpha + c_2 \alpha \cos \alpha = c_2 (\sin \alpha + \alpha \cos \alpha) = 0$$

Choosing $c_2 \neq 0$ **, we see that the last equation is equivalent to** $\tan \alpha = -\alpha$

$\alpha_1 = 2.0288, \ \alpha_2 = 4.9132, \ \alpha_3 = 7.9787, \ \alpha_4 = 11.0855,$
and the corresponding solutions are
$y_1 = \sin 2.0288x, \ y_2 = \sin 4.9132x, \ y_3 = \sin 7.9787x,$
$y_4 = \sin 11.0855x$

In general, the eigenfunctions of the problem are $\{\sin \alpha_n\}, \ n = 1,2,3,\cdots$



Sturm-Liouville Problem

Boundary Condition:

$$A_1 y(a) + B_1 y'(a) = 0$$
$$A_2 y(b) + B_2 y'(b) = 0$$

- **Regular Sturm-Liouville Problem**

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0 \quad [a,b]$$

$$(\lambda_n - \lambda_m)\int_a^b p(x) y_n y_m dx = r(b)[y_m'(b)y_n(b) - y_m(b)y_n'(b)] - r(a)[y_m'(a)y_n(a) - y_m(a)y_n'(a)]$$

Orthogonal relation

$$\int_a^b p(x) y_n y_m dx = 0, \quad \lambda_n \neq \lambda_m$$

should be zero for Orthogonal relation

- ***In some circumstances, we can prove the orthogonality of the solutions of*** $\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$ ***without the necessity of specifying a boundary condition at x=a and at x=b***

→ ***Singular Sturm-Liouville Problem***

Sturm-Liouville Problem

Boundary Condition:

$$A_1 y(a) + B_1 y'(a) = 0$$
$$A_2 y(b) + B_2 y'(b) = 0$$

- **Regular Sturm-Liouville Problem**

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0 \quad [a,b]$$

$$(\lambda_n - \lambda_m)\int_a^b p(x) y_n y_m dx = r(b)[y_m'(b)y_n(b) - y_m(b)y_n'(b)] - r(a)[y_m'(a)y_n(a) - y_m(a)y_n'(a)]$$

Orthogonal relation

$$\int_a^b p(x) y_n y_m dx = 0, \quad \lambda_n \neq \lambda_m$$

should be zero for Orthogonal relation

- **Singular Sturm-Liouville Problem**

If $r(a) = 0$ **then** $x = a$ **may be a singular and the equation**

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$$

may become unbounded as $x \to a \;, (a,b]$

however

$$r(b)[y_m'(b)y_n(b) - y_m(b)y_n'(b)] - r(a)[y_m'(a)y_n(a) - y_m(a)y_n'(a)]$$

zero

Orthogonal relation hold on $[a,b]$ ⇐ **dropped from the problem : no boundary condition at** $x = a$

Sturm-Liouville Problem

- **Regular Sturm-Liouville Problem**

$$\frac{d}{dx}[r(x)y']+[q(x)+\lambda p(x)]y=0 \quad [a,b]$$

Boundary Condition:

$$A_1 y(a)+B_1 y'(a)=0$$
$$A_2 y(b)+B_2 y'(b)=0$$

$$(\lambda_n-\lambda_m)\int_a^b p(x)y_n y_m dx = r(b)[y'_m(b)y_n(b)-y_m(b)y'_n(b)] - r(a)[y'_m(a)y_n(a)-y_m(a)y'_n(a)]$$

Orthogonal relation

$$\int_a^b p(x)y_n y_m dx=0, \quad \lambda_n \neq \lambda_m$$

should be zero for Orthogonal relation

- **Singular Sturm-Liouville Problem**

If $r(b)=0$ **then** $x=b$ **may be a singular and the equation**

$$\frac{d}{dx}[r(x)y']+[q(x)+\lambda p(x)]y=0$$

may become unbounded as $x \to b$ $,[a,b)$

however

zero

$$r(b)[y'_m(b)y_n(b)-y_m(b)y'_n(b)] - r(a)[y'_m(a)y_n(a)-y_m(a)y'_n(a)]$$

dropped from the problem
: no boundary condition at $x=b$

➡ **Orthogonal relation hold on** $[a,b]$

Sturm-Liouville Problem

$$\frac{d}{dx}[r(x)y']+[q(x)+\lambda p(x)]y=0 \quad [a,b]$$

Legendre's equation

$$(1-x^2)y''-2xy'+n(n+1)y=0$$

Orthogonal relation

$$\int_a^b p(x)y_n y_m dx=0, \quad \lambda_n \neq \lambda_m$$

▪Singular Sturm-Liouville Problem

example*) Legendre's equation is a Sturm-Liouville equation

$$\left[(1-x^2)y'\right]'+\lambda y=0 \quad \Leftrightarrow \quad \frac{(1-x^2)y''-2xy'+\lambda y=0}{r(x)} \qquad \lambda=n(n+1)$$

Since $r(\pm 1)=0$ **need no boundary conditions, but have a singular Sturm-Liouville problem on the interval** $-1\le x\le 1$ **. We know that , the Legendre polynomials** $P_n(x)$ **are solutions of the problem for** $n=0,1,2,\ldots(\lambda=0,1\cdot 2,2\cdot 3,\ldots)$

Hence these are the eigenfunctions. They are orthogonal on the interval

$$\int_{-1}^{1} p_m(x)p_n(x)dx=0, \quad (m\neq n)$$

Sturm-Liouville Problem

- **Regular Sturm-Liouville Problem**

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0 \quad [a,b]$$

Boundary Condition:

$$A_1 y(a) + B_1 y'(a) = 0$$
$$A_2 y(b) + B_2 y'(b) = 0$$

$$(\lambda_n - \lambda_m)\int_a^b p(x) y_n y_m dx = r(b)[y'_m(b)y_n(b) - y_m(b)y'_n(b)] - r(a)[y'_m(a)y_n(a) - y_m(a)y'_n(a)]$$

Orthogonal relation

$$\int_a^b p(x) y_n y_m dx = 0, \quad \lambda_n \neq \lambda_m$$

should be zero for Orthogonal relation

- **Periodic Sturm-Liouville Problem**

If $r(a) = r(b)$ then

$$r(b)[y'_m(b)y_n(b) - y_m(b)y'_n(b)] - r(a)[y'_m(a)y_n(a) - y_m(a)y'_n(a)]$$
$$= r(a)\left[(y'_m(b)y_n(b) - y'_m(a)y_n(a)) + (y_m(a)y'_n(a) - y_m(b)y'_n(b))\right]$$

$\therefore$ **Orthogonal relation hold on** $[a,b]$ **with** $y(a) = y(b),\ y'(a) = y'(b)$

Sturm-Liouville Problem

$$(\lambda_n - \lambda_m)\int_a^b p(x) y_n y_m dx$$

▪ **Sturm-Liouville Problem**

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0 \quad [a,b]$$

$$= r(b)[y'_m(b)y_n(b) - y_m(b)y'_n(b)] - r(a)[y'_m(a)y_n(a) - y_m(a)y'_n(a)]$$

▪ **By assuming the solution (y) are bounded on the closed interval [a,b] , then**

Orthogonal relation hold on.. [a,b]

$$\int_a^b p(x) y_n y_m dx = 0, \quad \lambda_n \neq \lambda_m \quad [a,b]$$

- **Regular** $r(x) \neq 0$ **with boundary Condition:** $A_1 y(a) + B_1 y'(a) = 0$, $A_2 y(b) + B_2 y'(b) = 0$
- **Singular** $r(a) = 0$ **without B/C at x=a** $A_2 y(b) + B_2 y'(b) = 0$

  $r(b) = 0$ **without B/C at x=b** $A_1 y(a) + B_1 y'(a) = 0$
- **Periodic** $r(a) = r(b)$ **with** $y(a) = y(b),\ y'(a) = y'(b)$

Sturm-Liouville Problem

▪Self-Adjoint Form

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$$

If the coefficient are continuous and $a(x) \neq 0$ for all x in some interval, then any second-order differential equation

$$a(x)y'' + b(x)y' + (c(x) + \lambda d(x))y = 0$$

can be recast into the so-called '**self-adjoint form**'.

Recall, ch. 2.3 integrating factor

$$a_1(x)y' + a_0(x)y = 0 \longrightarrow \frac{d}{dx}[\mu y] = 0$$

$$\mu = e^{\int p(x)dx},\ p(x) = \frac{a_0(x)}{a_1(x)}$$

Sturm-Liouville Problem

▪ **Self-Adjoint Form**

$$a(x)y'' + b(x)y' + (c(x) + \lambda d(x))y = 0$$

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$$

$$y'' + \frac{b(x)}{a(x)}y' + \left(\frac{c(x)}{a(x)} + \lambda\frac{d(x)}{a(x)}\right)y = 0$$

divided by $a(x)$

multiply $e^{\int \frac{b(x)}{a(x)}dx}$

$$e^{\int \frac{b(x)}{a(x)}dx}y'' + \frac{b(x)}{a(x)}e^{\int \frac{b(x)}{a(x)}dx}y' + \left(e^{\int \frac{b(x)}{a(x)}dx}\frac{c(x)}{a(x)} + \lambda e^{\int \frac{b(x)}{a(x)}dx}\frac{d(x)}{a(x)}\right)y = 0$$

$$\frac{d}{dt}\underbrace{\left[e^{\int \frac{b(x)}{a(x)}dx}y'\right]}_{r(x)} + \left(\underbrace{\frac{c(x)}{a(x)}e^{\int \frac{b(x)}{a(x)}dx}}_{q(x)} + \lambda\underbrace{\frac{d(x)}{a(x)}e^{\int \frac{b(x)}{a(x)}dx}}_{p(x)}\right)y = 0$$

Sturm-Liouville Problem

▪ **Self-Adjoint Form**

$$a(x)y'' + b(x)y' + (c(x) + \lambda d(x))y = 0$$

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$$

$$r(x) = e^{\int \frac{b(x)}{a(x)} dx}$$

$$q(x) = \frac{c(x)}{a(x)} e^{\int \frac{b(x)}{a(x)} dx}$$

$$p(x) = \frac{d(x)}{a(x)} e^{\int \frac{b(x)}{a(x)} dx}$$

Example) $3y'' + 6y' + \lambda y = 0$

$$y'' + \frac{6}{3}y' + \frac{\lambda}{3}y = 0$$

$$e^{\int \frac{b(x)}{a(x)} dx} = e^{\int 2dx} = e^{2x}$$

$$e^{2x}y'' + 2e^{2x}y' + \frac{\lambda}{3}e^{2x}y = 0$$

$$\frac{d}{dx}\left[e^{2x}y'\right] + \lambda \frac{e^{2x}}{3} y = 0$$

Sturm-Liouville Problem

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0$$

▪**Self-Adjoint Form**

$$\int_a^b p(x) y_n y_m dx = 0, \quad \lambda_n \neq \lambda_m \qquad [a,b]$$

Ex.)Parametric Bessel Series*

$$x^2 y'' + xy' + (\alpha^2 x^2 - n^2)y = 0, \quad n = 0,1,2,\ldots$$

General solution $y = c_1 J_n(\alpha x) + c_2 Y_n(\alpha x)$

$J_n(x)$ converges on $[0,\infty)$ when $n \geq 0$

$Y_n(x)$ converges on $(0,\infty)$

---

divided by x^2

$$y'' + \frac{1}{x}y' + (\alpha^2 - \frac{n^2}{x^2})y = 0 \longrightarrow xy'' + y' + (x\alpha^2 - \frac{n^2}{x})y = 0$$

multiply $e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

$x > 0$

$$\frac{d}{dx}[xy'] + (-\frac{n^2}{x} + \alpha^2 x)y = 0$$

$r(x)$ $q(x)$ λ $p(x)$

*Zill & Cullen, Advanced Engineering Mathematics, 3rd edition, John and Bartlett, 2006, p263

Sturm-Liouville Problem

$$\frac{d}{dx}[r(x)y']+[q(x)+\lambda p(x)]y=0$$
$$\int_a^b p(x)y_n y_m dx=0,\ \lambda_n\neq\lambda_m\ [a,b]$$

▪Self-Adjoint Form

• Singular $r(a)=0$: Orthogonal relation holds on without B/C at x=a

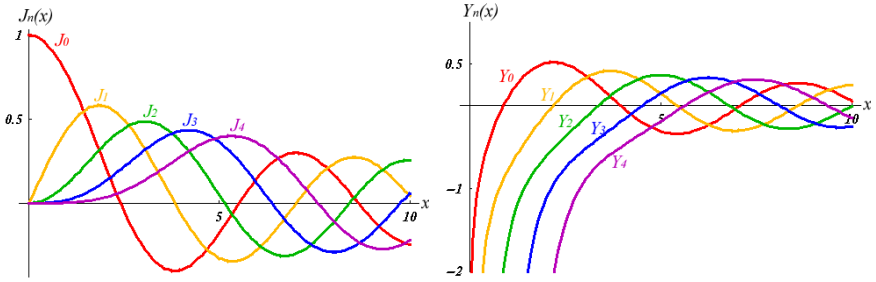
Ex.)Parametric Bessel Series*

$$x^2y''+xy'+(\alpha^2x^2-n^2)y=0,\quad n=0,1,2,\ldots$$

General solution $y=c_1J_n(\alpha x)+c_2Y_n(\alpha x)$ $J_n(x)$ converges on $[0,\infty)$ when $n\geq 0$

$Y_n(x)$ converges on $(0,\infty)$

$$\frac{d}{dx}\underbrace{[xy']}_{r(x)}+\left(-\underbrace{\frac{n^2}{x}}_{q(x)}+\underbrace{\alpha^2}_{\lambda}\underbrace{x}_{p(x)}\right)y=0$$



$r(0)=0$

only $J_n(\alpha x)$ is bounded at $x=0$

of the two solutions $J_n(\alpha x), Y_n(\alpha x)$

$(Y_n\to-\infty\ as\ x\to 0)$

Recall, singular Sturm-Liouville Problem, the set $\{J_n(\alpha_i x)\}, i=1,2,3\ldots$, is orthogonal with respect to the weight function $p(x)=x$ on an interval $[0,b]$

The orthogonality relation is

$$\int_0^b xJ_n(\alpha_i x)J_n(\alpha_j x)dx=0,\ \lambda_i\neq\lambda_j\ (\lambda=\alpha^2)$$

Sturm-Liouville Problem

▪**Self-Adjoint Form**

Ex.)Parametric Bessel Series*

$$\frac{d}{dx}[r(x)y']+[q(x)+\lambda p(x)]y=0 \quad [a,b]$$

Boundary Condition:

$$A_1 y(a)+B_1 y'(a)=0$$
$$A_2 y(b)+B_2 y'(b)=0$$

Orthogonal relation

$$\int_a^b p(x)y_n y_m dx=0,\ \ \lambda_n \neq \lambda_m$$

$$\frac{d}{dx}[xy']+(-\frac{n^2}{x}+\alpha^2 x)y=0$$

$r(x)$ $\quad$ $q(x)$ $\quad$ $\lambda\ p(x)$

$\{J_n(\alpha_i x)\}, i=1,2,3...,$: **orthogonal set**

orthogonal relation

$$\int_0^b xJ_n(\alpha_i x)J_n(\alpha_j x)dx=0,\ \ \lambda_i \neq \lambda_j\ (\lambda=\alpha^2)$$

Provided the α_i, and hence the eigenvalues $\lambda_i=\alpha_i^{\,2}$, $i=1,2,3,\cdots$ are defined by means of a boundary condition at $x=b$ of the type given in $A_2y(b)+B_2y'(b)=0$:

$$A_2 J_n(\alpha b)+B_2\alpha J_n'(\alpha b)=0$$

→ **Roots** $x_i(=\alpha_i b)$

→ **Eigenvalues** $\lambda_i=(\alpha_i)^2=\left(\frac{x_i}{b}\right)^2$

Sturm-Liouville Problem

▪**Self-Adjoint Form**

Ex.) Legendre's Equation*

$$\frac{d}{dx}[r(x)y'] + [q(x) + \lambda p(x)]y = 0 \quad [a,b]$$

Boundary Condition:

$$A_1 y(a) + B_1 y'(a) = 0$$

$$A_2 y(b) + B_2 y'(b) = 0$$

Orthogonal relation

$$\int_a^b p(x) y_n y_m dx = 0, \quad \lambda_n \neq \lambda_m$$

$$\left[(1-x^2)y'\right]' + \lambda y = 0 \quad \Leftrightarrow \quad (1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$q(x) = 0, \quad p(x) = 1$$

Legendre's D.E.
→ polynomial solutions $P_n(x)$ $\quad n = 0,1,2,\cdots$

$$\lambda = n(n+1)$$

As $P_n(x)$ is the only solutions of the equation that are bounded on the closed interval [-1,1], and $r(-1) = r(1) = 0$ (no boundary condition required) that the set $P_n(x)$ is orthogonal with respect to the weight function on [-1,1], The orthogonality relation is

$$\int_{-1}^{1} 1 \cdot P_m(x) P_n(x) dx = 0, \quad m \neq n$$

6-2. Fourier-Bessel Series, Fourier-Legendre Series

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Bessel and Legendre Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right)$$ ▪Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x$$ ▪Fourier Cosine Series

$$f(x) = \sum_{n=1}^{\infty}\left(b_n \sin\frac{n\pi}{p}x\right)$$ ▪Fourier Sine Series

Three ways of expanding a function in term of orthogonal set of functions

But such *expansion are by no means limited to orthogonal sets of trigonometric functions*

Fourier-Bessel Series

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x)$$

Recall, orthogonal relation

$$\int_0^b x J_n(\alpha_i x) J_n(\alpha_j x) dx = 0, \quad \lambda_i \neq \lambda_j \ (\lambda = \alpha^2)$$

Fourier-Legendre Series

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$$

Recall, orthogonal relation

$$\int_{-1}^{1} 1 \cdot P_m(x) P_n(x) dx = 0, \quad m \neq n$$

Fourier-Bessel Series

orthogonal relation

$$\int_0^b xJ_n(\alpha_i x)J_n(\alpha_j x)dx = 0, \quad \lambda_i \neq \lambda_j \ (\lambda = \alpha^2)$$

$$A_2 J_n(\alpha b) + B_2 \alpha J_n'(\alpha b) = 0$$

Definition 12.8

Fourier-Bessel Series

The Fourier-Bessel series of a function f defined on the interval $(0, b)$ is given by

1) α_i are defined by $J_n(\alpha_i b) = 0$ — **Case i**

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \quad , c_i = \frac{2}{b^2 J^2{}_{n+1}(\alpha_i b)} \int_0^b xJ_n(\alpha_i x) f(x)dx$$

2) α_i are defined by $hJ_n(\alpha b) + \alpha b J_n'(\alpha b) = 0$ — **Case ii**

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \quad , c_i = \frac{2\alpha_i^2}{(\alpha_i^2 b^2 - n^2 + h^2) J_n^2(\alpha_i b)} \int_0^b xJ_n(\alpha_i x) f(x)dx$$

3) α_i are defined by $J_n'(\alpha b) = 0$ — **Case iii**

$$f(x) = c_1 + \sum_{i=2}^{\infty} c_i J_n(\alpha_i x)$$

$$, c_1 = \frac{2}{b^2} \int_0^b xf(x)dx$$

$$, c_i = \frac{2}{b^2 J_0^2(\alpha_i b)} \int_0^b xJ_n(\alpha_i x) f(x)dx$$

Fourier-Bessel Series

Orthogonal series expansion of (generalized Fourier Series) $f(x)$

$$f(x)=\sum_{n=0}^{\infty}c_n\phi_n(x) \qquad c_n=\frac{\int_a^b f(x)w(x)\phi_n(x)dx}{\|\phi_n(x)\|^2}$$

$$=\frac{\int_a^b f(x)w(x)\phi_n(x)dx}{\int_a^b w(x)\phi^2{}_n(x)dx}$$

Bessel function orthogonal relation

$$\int_0^b xJ_n(\alpha_i x)J_n(\alpha_j x)dx=0,$$

$$\lambda_i\neq\lambda_j\ (\lambda=\alpha^2)$$

▪Fourier-Bessel Series

Parametric Bessel equation of order n

$$x^2y''+xy'+(\alpha^2x^2-n^2)y=0, \quad n=0,1,2,\ldots$$

Self-adjoint form

$$\frac{d}{dx}[xy']+(-\frac{n^2}{x}+\alpha^2x)y=0$$

The eigenvalues of the corresponding Sturm-Liouville problem are $\lambda=\alpha_i^2$.

The orthogonal series expansion of a function f defined on the interval $[0,b]$ in terms of this orthogonal set is $f(x)=\sum_{i=1}^{\infty}c_iJ_n(\alpha_i x)$

$$c_i=\frac{\int_0^b xJ_n(\alpha_i x)f(x)dx}{\|J_n(\alpha_i x)\|^2} \longleftarrow \|J_n(\alpha_i x)\|^2=\int_0^b xJ_n^2(\alpha_i x)dx$$

Fourier-Bessel Series

Orthogonal series expansion of (generalized Fourier Series) $f(x)$

$$f(x)=\sum_{n=0}^{\infty}c_n\phi_n(x) \qquad c_n=\frac{\int_a^b f(x)w(x)\phi_n(x)dx}{\|\phi_n(x)\|^2}$$

$$=\frac{\int_a^b f(x)w(x)\phi_n(x)dx}{\int_a^b w(x)\phi^2{}_n(x)dx}$$

Bessel function orthogonal relation

$$\int_0^b xJ_n(\alpha_i x)J_n(\alpha_j x)dx=0,$$

$$\lambda_i\neq\lambda_j\ (\lambda=\alpha^2)$$

▪Fourier-Bessel Series

Parametric Bessel equation of order n

$$x^2y''+xy'+(\alpha^2x^2-n^2)y=0, \quad n=0,1,2,\ldots$$

Self-adjoint form

$$\frac{d}{dx}[xy']+(-\frac{n^2}{x}+\alpha^2x)y=0$$

$$f(x)=\sum_{i=1}^{\infty}c_iJ_n(\alpha_i x) \qquad [0,b] \qquad \lambda=\alpha_i^2 \qquad c_i=\frac{\int_0^b xJ_n(\alpha_i x)f(x)dx}{\|J_n(\alpha_i x)\|^2}$$

▪Recall Differential Recurrence Relations*

$$\frac{d}{dx}\left[x^nJ_n(x)\right]=x^nJ_{n-1}(x)$$

$$\frac{d}{dx}\left[x^{-n}J_n(x)\right]=-x^{-n}J_{n+1}(x)$$

▪Derivatives

$$\frac{d}{dx}\left[x^{-n}J_n(x)\right]=-x^{-n}J_{n+1}(x)$$

$$-nx^{-n-1}J_n(x)+x^{-n}J_n'(x)=-x^{-n}J_{n+1}(x)$$

$$-nJ_n(x)+xJ_n'(x)=-xJ_{n+1}(x)$$

$$\therefore xJ_n'(x)=nJ_n(x)-xJ_{n+1}(x)$$

*Kreyszig E. ,Advanced Engineering Mathematics, 9th edition, Willey, 2006, p194

Fourier-Bessel Series

Orthogonal series expansion of (generalized Fourier Series) $f(x)$

$$f(x)=\sum_{n=0}^{\infty}c_n\phi_n(x) \qquad c_n=\frac{\int_a^b f(x)w(x)\phi_n(x)dx}{\|\phi_n(x)\|^2}=\frac{\int_a^b f(x)w(x)\phi_n(x)dx}{\int_a^b w(x)\phi^2{}_n(x)dx}$$

Bessel function orthogonal relation

$$\int_0^b xJ_n(\alpha_i x)J_n(\alpha_j x)dx=0,$$

$$\lambda_i\neq\lambda_j\ (\lambda=\alpha^2)$$

▪Fourier-Bessel Series

Parametric Bessel equation of order n

$$x^2y''+xy'+(\alpha^2x^2-n^2)y=0, \quad n=0,1,2,\ldots$$

Self-adjoint form

$$\frac{d}{dx}[xy']+(-\frac{n^2}{x}+\alpha^2x)y=0$$

$$f(x)=\sum_{i=1}^{\infty}c_iJ_n(\alpha_i x) \qquad [0,b] \qquad \lambda=\alpha_i^2 \qquad c_i=\frac{\int_0^b xJ_n(\alpha_i x)f(x)dx}{\|J_n(\alpha_i x)\|^2}$$

▪**Square Norm** $\|J_n(\alpha_i x)\|^2=\int_0^b xJ_n^2(\alpha_i x)dx$

$$\frac{d}{dx}[xy']+(\alpha^2x-\frac{n^2}{x})y=0$$

The value of the square norm depends on how the eigenvalues $\lambda=\alpha_i^2$ ***are defined***

Fourier-Bessel Series

▪ **Square Norm**

$$f(x)=\sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \qquad c_i=\frac{\int_0^b x J_n(\alpha_i x) f(x) dx}{\left\| J_n(\alpha_i x) \right\|^2}$$

$$\frac{d}{dx}[xy']+(\alpha^2 x-\frac{n^2}{x})y=0$$

The value of the square norm depends on how the eigenvalues $\lambda=\alpha_i^2$ *are defined*

To fine square norm $\left\| J_n(\alpha_i x) \right\|^2=\int_0^b x J_n^2(\alpha_i x) dx$

$$\frac{d}{dx}[xy']+(\alpha^2 x-\frac{n^2}{x})y=0 \xrightarrow[\text{Multiply by } 2xy']{}$$

$$2xy'\frac{d}{dx}[xy']+2xy'(\alpha^2 x-\frac{n^2}{x})y=0$$

$$\frac{d}{dx}[xy']^2+2y'(\alpha^2 x^2-n^2)y=0$$

$$\frac{d}{dx}[xy']^2+(\alpha^2 x^2-n^2)\frac{d}{dx}[y]^2=0$$

Fourier-Bessel Series

▪ **Square Norm**

$$f(x)=\sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \qquad c_i=\frac{\int_0^b x J_n(\alpha_i x) f(x) dx}{\left\| J_n(\alpha_i x) \right\|^2}$$

$$\frac{d}{dx}[xy']+(\alpha^2 x-\frac{n^2}{x})y=0$$

The value of the square norm depends on how the eigenvalues $\lambda=\alpha_i^2$ *are defined*

To fine square norm $\left\| J_n(\alpha_i x) \right\|^2=\int_0^b x J_n^2(\alpha_i x) dx$

$$\frac{d}{dx}[xy']^2+(\alpha^2 x^2-n^2)\frac{d}{dx}[y]^2=0$$

$$\frac{d}{dx}[xy']^2+\alpha^2 x^2\frac{d}{dx}[y]^2-n^2\frac{d}{dx}[y]^2=0$$

$$\frac{d}{dx}[xy']^2+\alpha^2 x^2\frac{d}{dx}[y]^2=n^2\frac{d}{dx}[y]^2$$

Fourier-Bessel Series

▪ **Square Norm**

$$f(x)=\sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \qquad c_i=\frac{\int_0^b x J_n(\alpha_i x) f(x) dx}{\left\| J_n(\alpha_i x) \right\|^2}$$

$$\frac{d}{dx}[xy']+(\alpha^2 x-\frac{n^2}{x})y=0$$

The value of the square norm depends on how the eigenvalues $\lambda=\alpha_i^2$ *are defined*

To fine square norm $\left\| J_n(\alpha_i x) \right\|^2=\int_0^b x J_n^2(\alpha_i x) dx$

$$\frac{d}{dx}[xy']^2+\alpha^2 x^2 \frac{d}{dx}[y]^2=n^2 \frac{d}{dx}[y]^2$$

Integration by part

$$\int_0^b \frac{d}{dx}[xy']^2 dx+\int_0^b \alpha^2 x^2 \frac{d}{dx}[y]^2 dx=\int_0^b n^2 \frac{d}{dx}[y]^2 dx$$

$$\int_0^b \frac{d}{dx}[xy']^2 dx+\left[\alpha^2 x^2 y^2\right]_0^b-\int_0^b \alpha^2 2xy^2 dx=\int_0^b n^2 \frac{d}{dx}[y]^2 dx$$

Fourier-Bessel Series

▪ **Square Norm**

$$f(x)=\sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \qquad c_i=\frac{\int_0^b x J_n(\alpha_i x) f(x) dx}{\left\| J_n(\alpha_i x) \right\|^2}$$

$$\frac{d}{dx}[xy']+(\alpha^2 x-\frac{n^2}{x})y=0$$

The value of the square norm depends on how the eigenvalues $\lambda=\alpha_i^2$ *are defined*

To fine square norm $\left\| J_n(\alpha_i x) \right\|^2=\int_0^b x J_n^2(\alpha_i x) dx$

$$\int_0^b \frac{d}{dx}[xy']^2 dx+\left[\alpha^2 x^2 y^2\right]_0^b-\int_0^b \alpha^2 2xy^2 dx=\int_0^b n^2 \frac{d}{dx}[y]^2 dx$$

$$\int_0^b \alpha^2 2xy^2 dx=\int_0^b \frac{d}{dx}[xy']^2 dx+\left[\alpha^2 x^2 y^2\right]_0^b-\int_0^b n^2 \frac{d}{dx}[y]^2 dx$$

$$2\alpha^2\int_0^b xy^2 dx=\left[(xy')^2\right]_0^b+\left[\alpha^2 x^2 y^2\right]_0^b-\left[\left(n^2 y^2\right)\right]_0^b$$

$$2\alpha^2\int_0^b xy^2 dx=\left[(xy')^2+\left(\alpha^2 x^2-n^2\right)y^2\right]_0^b$$

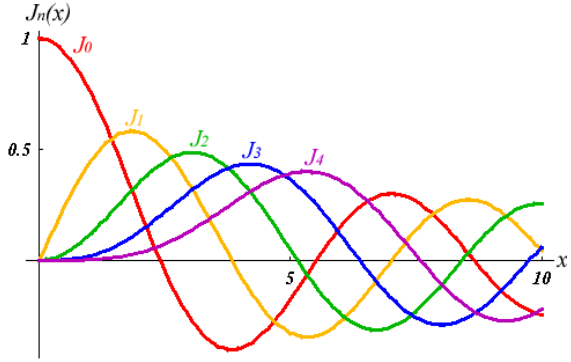
Fourier-Bessel Series

▪ **Square Norm**

$$f(x)=\sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \qquad c_i=\frac{\int_0^b xJ_n(\alpha_i x)f(x)dx}{\left\|J_n(\alpha_i x)\right\|^2}$$

$$\frac{d}{dx}[xy']+(\alpha^2 x-\frac{n^2}{x})y=0$$

The value of the square norm depends on how the eigenvalues $\lambda=\alpha_i^2$ *are defined*



To fine square norm $\left\|J_n(\alpha_i x)\right\|^2=\int_0^b xJ_n^2(\alpha_i x)dx$

$$2\alpha^2\int_0^b xy^2dx=\left[(xy')^2+\left(\alpha^2x^2-n^2\right)y^2\right]_0^b$$

since $y=J_n(\alpha x),\quad y'=\alpha J_n'(\alpha x),\quad J_n(0)=0$

$$2\alpha^2\int_0^b xJ_n^2(\alpha x)dx=\alpha^2b^2\left[J_n'(\alpha b)\right]^2+\left(\alpha^2b^2-n^2\right)\left[J_n(\alpha b)\right]^2$$

$$\therefore\int_0^b xJ_n^2(\alpha x)dx=\frac{1}{2\alpha^2}\left\{\alpha^2b^2\left[J_n'(\alpha b)\right]^2+\left(\alpha^2b^2-n^2\right)\left[J_n(\alpha b)\right]^2\right\}$$

Fourier-Bessel Series

▪ **Square Norm**

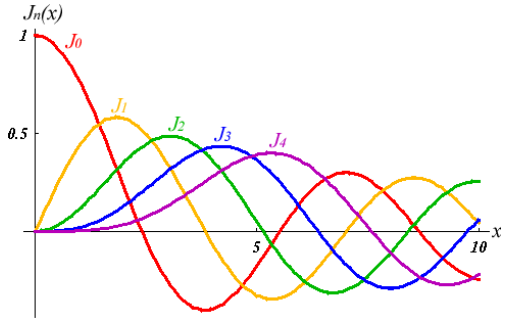
$$f(x)=\sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \qquad c_i=\frac{\int_0^b xJ_n(\alpha_i x)f(x)dx}{\left\|J_n(\alpha_i x)\right\|^2}$$

$$\frac{d}{dx}[xy']+(\alpha^2 x-\frac{n^2}{x})y=0$$

The value of the square norm depends on how the eigenvalues $\lambda=\alpha_i^2$ *are defined*

To fine square norm $\left\|J_n(\alpha_i x)\right\|^2=\int_0^b xJ_n^2(\alpha_i x)dx$

$$\therefore \int_0^b xJ_n^2(\alpha x)dx=\frac{1}{2\alpha^2}\left\{\alpha^2 b^2\left[J_n'(\alpha b)\right]^2+\left(\alpha^2 b^2-n^2\right)\left[J_n(\alpha b)\right]^2\right\}$$



Consider boundary condition

$$A_2 J_n(\alpha b)+\alpha B_2 J_n'(\alpha b)=0$$

$\lambda=\alpha^2$
Eigenvalues from this equation of condition according to the cases (for nontrivial solutions)

$$A_1 y(a)+B_1 y'(a)=0$$

$$A_2 y(b)+B_2 y'(b)=0$$

Case I $A_2=1, B_2=0$

Case II $A_2=h\geq 0,\ B_2=b$

Case III $h=0,\ n=0$

Fourier-Bessel Series

▪ **Square Norm**

$$f(x)=\sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \qquad c_i=\frac{\int_0^b x J_n(\alpha_i x) f(x)dx}{\left\|J_n(\alpha_i x)\right\|^2}$$

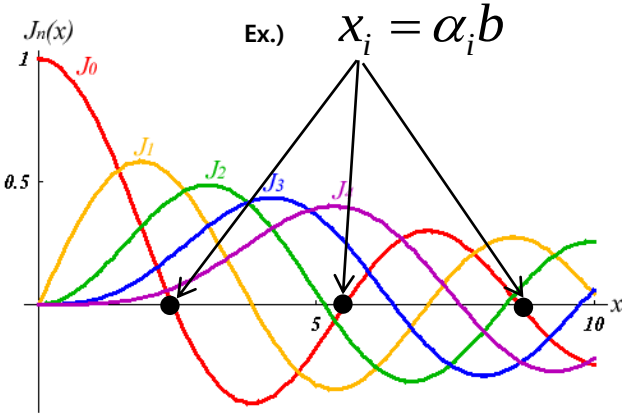
$$\frac{d}{dx}\left[xy'\right]+\left(\alpha^2 x-\frac{n^2}{x}\right)y=0$$

The value of the square norm depends on how the eigenvalues $\lambda=\alpha_i^2$ *are defined*

$$\left\|J_n(\alpha_i x)\right\|^2=\int_0^b x J_n^2(\alpha_i x)dx=\frac{1}{2\alpha_i^{\,2}}\left\{\alpha_i^{\,2}b^2\left[J_n'(\alpha_i b)\right]^2+\left(\alpha_i^{\,2}b^2-n^2\right)\left[J_n(\alpha_i b)\right]^2\right\}$$

Consider boundary condition $A_2 J_n(\alpha b)+\alpha B_2 J_n'(\alpha b)=0$

α_i **are defined from the root of** $J_n(\alpha_i b)=0$



Case I $A_2=1, B_2=0$

then $\left\|J_n(\alpha_i x)\right\|^2=\int_0^b x J_n^2(\alpha_i x)dx$

$$=\frac{1}{2\alpha_i^{\,2}}\left\{\alpha_i^{\,2}b^2\left[J_n'(\alpha_i b)\right]^2+\left(\alpha_i^{\,2}b^2-n^2\right)\left[J_n(\alpha_i b)\right]^2\right\}$$

zero

$$=\frac{b^2}{2}\left[J_n'(\alpha_i b)\right]^2=\frac{b^2}{2}\left[J_{n+1}(\alpha_i b)\right]^2$$

$$\because\ xJ_n'(x)=nJ_n(x)-xJ_{n+1}(x)$$

zero

Fourier-Bessel Series

▪ **Square Norm**

$$f(x)=\sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \qquad c_i=\frac{\int_0^b x J_n(\alpha_i x) f(x)dx}{\left\|J_n(\alpha_i x)\right\|^2}$$

$$\frac{d}{dx}\left[xy'\right]+\left(\alpha^2 x-\frac{n^2}{x}\right)y=0$$

The value of the square norm depends on how the eigenvalues $\lambda=\alpha_i^2$ *are defined*

$$\left\|J_n(\alpha_i x)\right\|^2=\int_0^b x J_n^2(\alpha_i x)dx=\frac{1}{2\alpha_i^2}\left\{\alpha_i^2 b^2\left[J_n'(\alpha_i b)\right]^2+\left(\alpha_i^2 b^2-n^2\right)\left[J_n(\alpha_i b)\right]^2\right\}$$

Consider boundary condition $A_2 J_n(\alpha b)+\alpha B_2 J_n'(\alpha b)=0$

α_i **are defined from the root of**

Case I $A_2=1, B_2=0 \longrightarrow J_n(\alpha_i b)=0$

then $\left\|J_n(\alpha_i x)\right\|^2=\int_0^b x J_n^2(\alpha_i x)dx$

$$=\frac{1}{2\alpha_i^2}\left\{\alpha_i^2 b^2\left[J_n'(\alpha_i b)\right]^2+\left(\alpha_i^2 b^2-n^2\right)\left[J_n(\alpha_i b)\right]^2\right\}$$

zero

$$=\frac{b^2}{2}\left[J_n'(\alpha_i b)\right]^2=\frac{b^2}{2}\left[J_{n+1}(\alpha_i b)\right]^2$$

Recall, ch.5.3
differential recurrence relations

$$\frac{d}{dx}\left[x^{-n}J_n(x)\right]=-x^{-n}J_{n+1}(x)$$

$$-nx^{-n-1}J_n(x)+x^{-n}J_n'(x)=-x^{-n}J_{n+1}(x)$$

$$-nJ_n(x)+xJ_n'(x)=-xJ_{n+1}(x)$$

$$\therefore xJ_n'(x)=nJ_n(x)-xJ_{n+1}(x)$$

zero

$$xJ_n'(\alpha b)=nJ_n(\alpha b)-xJ_{n+1}(\alpha b)$$

$$J_n'(\alpha b)=-J_{n+1}(\alpha b)$$

Fourier-Bessel Series

▪ **Square Norm**

$$f(x)=\sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \qquad c_i=\frac{\int_0^b xJ_n(\alpha_i x)f(x)dx}{\left\|J_n(\alpha_i x)\right\|^2}$$

$$\frac{d}{dx}\left[xy'\right]+\left(\alpha^2 x-\frac{n^2}{x}\right)y=0$$

The value of the square norm depends on how the eigenvalues $\lambda=\alpha_i^2$ *are defined*

$$\left\|J_n(\alpha_i x)\right\|^2=\int_0^b xJ_n^2(\alpha_i x)dx=\frac{1}{2\alpha^2}\left\{\alpha_i{}^2b^2\left[J_n'(\alpha_i b)\right]^2+\left(\alpha_i{}^2b^2-n^2\right)\left[J_n(\alpha_i b)\right]^2\right\}$$

Consider boundary condition $A_2J_n(\alpha b)+\alpha B_2J_n'(\alpha b)=0$

α_i **are defined from the root of**

Case II $A_2=h\geq 0,\ B_2=b \longrightarrow \alpha_i bJ_n'(\alpha_i b)=-hJ_n(\alpha_i b)$

then $\left\|J_n(\alpha_i x)\right\|^2=\int_0^b xJ_n^2(\alpha_i x)dx$

$$=\frac{1}{2\alpha_i{}^2}\left\{\alpha_i{}^2b^2\left[J_n'(\alpha_i b)\right]^2+\left(\alpha_i{}^2b^2-n^2\right)\left[J_n(\alpha_i b)\right]^2\right\}=\frac{1}{2\alpha_i{}^2}\left\{h^2\left[J_n(\alpha_i b)\right]^2+\left(\alpha_i{}^2b^2-n^2\right)\left[J_n(\alpha_i b)\right]^2\right\}$$

$$=\frac{\alpha_i{}^2b^2-n^2+h^2}{2\alpha_i^2}\left[J_n(\alpha_i b)\right]^2$$

Fourier-Bessel Series

▪ **Square Norm**

$$f(x)=\sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \qquad c_i=\frac{\int_0^b xJ_n(\alpha_i x)f(x)dx}{\left\|J_n(\alpha_i x)\right\|^2}$$

$$\frac{d}{dx}[xy']+(\alpha^2 x-\frac{n^2}{x})y=0$$

The value of the square norm depends on how the eigenvalues $\lambda=\alpha_i^2$ *are defined*

$$\left\|J_n(\alpha_i x)\right\|^2=\int_0^b xJ_n^2(\alpha_i x)dx=\frac{1}{2\alpha^2}\left\{\alpha_i{}^2 b^2\left[J_n'(\alpha_i b)\right]^2+\left(\alpha_i{}^2 b^2-n^2\right)\left[J_n(\alpha_i b)\right]^2\right\}$$

Consider boundary condition $A_2 J_n(\alpha b)+\alpha B_2 J_n'(\alpha b)=0$

α_i **are defined from the root of**

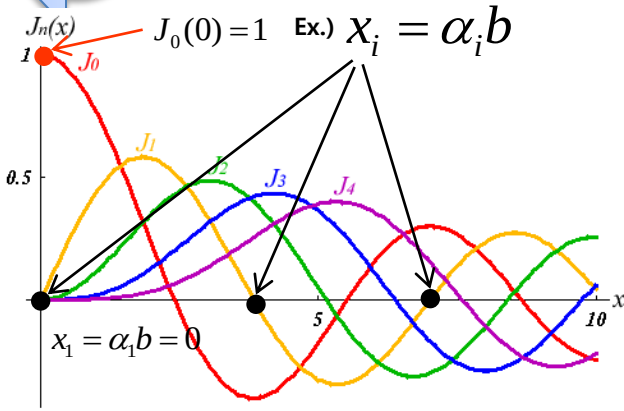
Case III $A_2=h=0,\ n=0 \longrightarrow J_0'(\alpha b)=0$

Then ***Case III-1)*** $\alpha_i=0$,$(or\ \alpha_1=0,\ \lambda=0)$

from $J_n'(\alpha b)=-J_{n+1}(\alpha b) \Rightarrow J_0'(\alpha_i b)=-J_1(\alpha_i b)=0$

$J_0(0)=1$: **nontrivial solution! so λ=0 is also eigenvalue**

$$\therefore \left\|J_0(0)\right\|^2=\left\|1\right\|^2=\int_0^b xJ_0^2(0)dx=\frac{b^2}{2}$$



Fourier-Bessel Series

▪**Square Norm**

$$f(x)=\sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \qquad c_i=\frac{\int_0^b x J_n(\alpha_i x) f(x)dx}{\left\|J_n(\alpha_i x)\right\|^2}$$

$$\frac{d}{dx}[xy']+(\alpha^2 x-\frac{n^2}{x})y=0$$

The value of the square norm depends on how the eigenvalues $\lambda=\alpha_i^2$ *are defined*

$$\left\|J_n(\alpha_i x)\right\|^2=\int_0^b x J_n^2(\alpha_i x)dx=\frac{1}{2\alpha^2}\left\{\alpha_i{}^2 b^2\left[J_n'(\alpha_i b)\right]^2+\left(\alpha_i{}^2 b^2-n^2\right)\left[J_n(\alpha_i b)\right]^2\right\}$$

Consider boundary condition $A_2 J_n(\alpha b)+\alpha B_2 J_n'(\alpha b)=0$

α_i **are defined from the root of**

Case III $A_2=h=0,\ n=0 \longrightarrow J_0'(\alpha b)=0$

Case III-2) $\alpha_i>0$

$$\left\|J_n(\alpha_i x)\right\|^2=\int_0^b x J_n^2(\alpha_i x)dx=\frac{\alpha_i^2 b^2-n^2+h^2}{2\alpha_i^2}\left[J_n(\alpha_i b)\right]^2=\frac{b^2}{2}\left[J_n(\alpha_i b)\right]^2$$

$$\left\|J_0(\alpha_i x)\right\|^2=\int_0^b x J_0^2(\alpha_i x)dx=\frac{b^2}{2}\left[J_0(\alpha_i b)\right]^2$$

Fourier-Bessel Series

▪ **Square Norm**

$$f(x)=\sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \qquad c_i=\frac{\int_0^b xJ_n(\alpha_i x)f(x)dx}{\left\|J_n(\alpha_i x)\right\|^2}$$

$$\frac{d}{dx}[xy']+(\alpha^2 x-\frac{n^2}{x})y=0$$

The value of the square norm depends on how the eigenvalues $\lambda=\alpha_i^2$ *are defined*

$$\left\|J_n(\alpha_i x)\right\|^2=\int_0^b xJ_n^2(\alpha_i x)dx=\frac{1}{2\alpha^2}\left\{\alpha_i{}^2 b^2\left[J_n'(\alpha_i b)\right]^2+\left(\alpha_i{}^2 b^2-n^2\right)\left[J_n(\alpha_i b)\right]^2\right\}$$

Consider boundary condition $A_2 J_n(\alpha b)+\alpha B_2 J_n'(\alpha b)=0$

α_i **are defined from the root of**

Case III $A_2=h=0,\ n=0 \longrightarrow J_0'(\alpha b)=0$

then

$$\left\|J_n(\alpha_i x)\right\|^2=\int_0^b xJ_n^2(\alpha_i x)dx=\frac{\alpha_i^2 b^2-n^2+h^2}{2\alpha_i^2}\left[J_n(\alpha_i b)\right]^2=\frac{b^2}{2}\left[J_n(\alpha_i b)\right]^2$$

$$\left\|J_0(\alpha_i x)\right\|^2=\int_0^b xJ_0^2(\alpha_i x)dx=\frac{b^2}{2}\left[J_0(\alpha_i b)\right]^2$$

Fourier-Bessel Series

Fourier-Bessel Series

The Fourier-Bessel series of a function f defined on the interval $(0,b)$ is given by

1) α_i are defined by $J_n(\alpha_i b)=0$

$$f(x)=\sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \qquad ,c_i=\frac{2}{b^2 J^2{}_{n+1}(\alpha_i b)}\int_0^b xJ_n(\alpha_i x)f(x)dx$$

2) α_i are defined by $hJ_n(\alpha b)+\alpha b J_n'(\alpha b)=0$

$$f(x)=\sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \quad ,c_i=\frac{2\alpha_i^2}{(\alpha_i^2 b^2-n^2+h^2)J_n^2(\alpha_i b)}\int_0^b xJ_n(\alpha_i x)f(x)dx$$

3) α_i are defined by $J_n'(\alpha b)=0$

$$f(x)=c_1+\sum_{i=2}^{\infty} c_i J_n(\alpha_i x) \qquad ,c_1=\frac{2}{b^2}\int_0^b xf(x)dx$$

$$,c_i=\frac{2}{b^2 J_0^2(\alpha_i b)}\int_0^b xJ_n(\alpha_i x)f(x)dx$$

Fourier-Bessel Series

▪**Convergence of a Fourier-Bessel Series:**
Sufficient conditions for the convergence of a Fourier-Bessel series are not particularly restrictive

Theorem 12.4 **Conditions for Convergence**

If f and f' are piecewise continuous on the open interval $(0, b)$, then a Fourier-Bessel expansion of f converges to $f(x)$ at any point where f is continuous and to the average $[f(x+) + f(x-)]/2$ at a point where f is discontinuous.

Fourier-Bessel Series

☑ Example 1

Expansion in a Fourier-Bessel Series

Expand f(x)=x, 0<x<3, in a Fourier-Bessel series, using Bessel functions of order one that satisfy the boundary condition $J_1(3\alpha) = 0$

$b = 3,$

1) α_i are defined by $J_n(\alpha_i b) = 0$

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x)$$

$$, c_i = \frac{2}{b^2 J^2{}_{n+1}(\alpha_i b)} \int_0^b x J_n(\alpha_i x) f(x) dx$$

$$c_i = \frac{2}{3^2 J_2{}^2(3\alpha_i)} \int_0^3 x^2 J_1(\alpha_i x) dx$$

$t = \alpha_i x, \; dx = dt/\alpha_i, \; x^2 = t^2/\alpha_i{}^2,$

$$\frac{d}{dt}[t^2 J_2(t)] = t^2 J_1(t)$$

$$c_i = \frac{2}{3^2 J_2{}^2(3\alpha_i)} \int_0^{3\alpha_i} \frac{1}{\alpha_i^3} t^2 J_1(t) dt$$

recall $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

$$c_i = \frac{2}{9 J_2{}^2(3\alpha_i)} \frac{1}{\alpha_i^3} \int_0^{3\alpha_i} \frac{d}{dt}[t^2 J_2(t)] dt = \frac{2}{9 J_2{}^2(3\alpha_i)} \frac{1}{\alpha_i^3} 9\alpha_i^2 J_2(3\alpha_i) = \frac{2}{\alpha_i J_2(3\alpha_i)}$$

$$\therefore f(x) = 2\sum_{i=1}^{\infty} c_i J_1(\alpha_i x) = 2\sum_{i=1}^{\infty} \frac{1}{\alpha_i J_2(3\alpha_i)} J_1(\alpha_i x)$$

Fourier-Bessel Series

☑ Example 2

Expansion in a Fourier-Bessel Series

Expand f(x)=x, 0<x<3, in a Fourier-Bessel series, using Bessel functions of order one that satisfy the boundary condition $J_1(3\alpha)+\alpha J_1'(3\alpha)=0$

2) α_i are defined by $hJ_n(\alpha b)+\alpha bJ_n'(\alpha b)=0$

$$f(x)=\sum_{i=1}^{\infty}c_iJ_n(\alpha_i x)$$

$$,c_i=\frac{2\alpha_i^2}{(\alpha_i^2b^2-n^2+h^2)J_n^2(\alpha_i b)}\int_0^b xJ_n(\alpha_i x)f(x)dx$$

the only thing that changes in the expansion is the value of the square norm.

$$J_1(3\alpha)+\alpha J_1'(3\alpha)=0$$

Multiplying the boundary condition by 3 gives

$$3J_1(3\alpha)+3\alpha J_1'(3\alpha)=0$$ **→h=3, b=3, and n=1.**

$$c_i=\frac{2\alpha_i^2}{(\alpha_i^2b^2-n^2+h^2)J_n^2(\alpha_i b)}\int_0^b xJ_n(\alpha_i x)f(x)dx=\frac{2\alpha_i^2}{(\alpha_i^2 3^2-1^2+3^2)J_1^2(3\alpha_i)}\int_0^3 x^2J_1(\alpha_i x)dx$$

$$=\frac{2\alpha_i^2}{(\alpha_i^2 3^2-1^2+3^2)J_1^2(3\alpha_i)}\frac{1}{\alpha_i^3}9\alpha_i^2J_2(3\alpha_i)=\frac{18\alpha_iJ_2(3\alpha_i)}{(9\alpha_i^2+8)J_1^2(3\alpha_i)}$$

$$\therefore f(x)=18\sum_{i=1}^{\infty}\frac{\alpha_iJ_2(3\alpha_i)}{(9\alpha_i^2+8)J_1^2(3\alpha_i)}J_1(\alpha_i x)$$

$$c_i=\frac{18\alpha_iJ_2(3\alpha_i)}{(9\alpha_i^2+8)J_1^{\,2}(3\alpha_i)},$$

Fourier-Legendre Series

Definition 12.9 **Fourier-Legendre Series**

The Fourier-Legendre series of a function f defined on the interval $(-1,1)$ is given by

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$$

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) dx$$

▪**In an alternative form, let** $x = \cos\theta$

$$F(\theta) = \sum_{n=0}^{\infty} c_n P_n(\cos\theta)$$

$$c_n = \frac{2n+1}{2} \int_{0}^{\pi} F(\theta) P_n(\cos\theta) \sin\theta d\theta$$

$$F(\theta) = f(\cos\theta)$$

Fourier-Legendre Series

▪**Convergence of a Fourier-Legendre Series:**

Theorem 12.5 **Conditions for Convergence**

If f and f' are piecewise continuous on the open interval $(0, b)$, then a Fourier-Bessel expansion of f converges to $f(x)$ at any point where f is continuous and to the average $\left[f(x+) + f(x-)\right]/2$ at a point where f is discontinuous.

Fourier-Legendre Series

$$f(x)=\sum_{n=0}^{\infty}c_nP_n(x)$$

$$c_n=\frac{2n+1}{2}\int_{-1}^{1}f(x)P_n(x)dx$$

$$P_0(x)=1$$

$$P_1(x)=x$$

$$P_2(x)=\frac{1}{2}(3x^2-1)$$

$$P_3(x)=\frac{1}{2}(5x^3-3x)$$

$$P_4(x)=\frac{1}{8}(35x^4-30x^2+3)$$

☑ Example 3 Expansion in a Fourier-Bessel Series

Write out the first four nonzero terms in the Fourier-Legendre expansion of

$$f(x)=\begin{cases}0, & -1<x<0\\ 1, & 0\le x<1\end{cases}$$

$$c_0=\frac{1}{2}\int_{-1}^{1}f(x)P_0(x)dx=\frac{1}{2}\int_{0}^{1}1\cdot 1dx=\frac{1}{2}$$

$$c_1=\frac{3}{2}\int_{-1}^{1}f(x)P_1(x)dx=\frac{3}{2}\int_{0}^{1}1\cdot xdx=\frac{3}{4}$$

$$c_2=\frac{5}{2}\int_{-1}^{1}f(x)P_2(x)dx=\frac{5}{2}\int_{0}^{1}1\cdot\frac{1}{2}(3x^2-1)dx=0$$

$$c_3=\frac{7}{2}\int_{-1}^{1}f(x)P_3(x)dx=\frac{5}{2}\int_{0}^{1}1\cdot\frac{1}{2}(5x^3-3x)dx=-\frac{7}{16}$$

$$c_4=\frac{9}{2}\int_{-1}^{1}f(x)P_4(x)dx=\frac{9}{2}\int_{0}^{1}1\cdot\frac{1}{8}(35x^3-30x^2+3)dx=0$$

$$c_5=\frac{11}{2}\int_{-1}^{1}f(x)P_5(x)dx=\frac{11}{2}\int_{0}^{1}1\cdot\frac{1}{8}(63x^5-70x^3+15x)dx=\frac{11}{32}$$

$$\therefore f(x)=\frac{1}{2}P_0(x)+\frac{3}{4}P_1(x)-\frac{7}{16}P_3(x)+\frac{11}{32}P_5(x)+\cdots$$

6-3. Sturm-Liouville BVP /Eigen function Expansion

Regular Sturm-Liouville Problem

subject to $u(0,t)=0,\ \left.\dfrac{\partial u}{\partial x}\right|_{x=1} = -h\,u(1,t),\quad h>0,\ t>0$

$u(x,0)=1,\quad 0<x<1$

Separation of variables $u(x,t)=X(x)T(t)$

$X''+\lambda X=0\ ,\quad T'+k\lambda T=0$

$u(0,t)=0 \rightarrow X(0)=0$

$\left.\dfrac{\partial u}{\partial x}\right|_{x=1} = -h\,u(1,t)$:

$\left.\dfrac{\partial u}{\partial x}\right|_{x=1} = X'(1)\,T(t) \qquad \therefore\ X'(1) = -h\,X(1)$ ⇒ $X''+\lambda X=0,\ X(0)=0,\ X'(1)+hX(1)=0\ \cdots(1)^*$

$-u(1,t) = -h\,X(1)\,T(t)$

▪**Regular Sturm-Liouville Problem** **B.V.P**

Solve : $\dfrac{d}{dx}[r(x)y']+[q(x)+\lambda p(x)]y=0$

Subject to: $A_1y(a)+B_1y'(a)=0$

$A_2y(b)+B_2y'(b)=0$

"Regular Sturm-Liouville Problem"

$$X'' + \lambda X = 0,\ X(0) = 0,\ X'(1) + hX(1) = 0 \quad \cdots (1)^*$$

"Regular Sturm-Liouville Problem"

General solution of (1) is $X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x,\ \lambda_n = \alpha_n^2$

By $X(0) = 0,\ c_1 = 0$ *thus*, $X(x) = c_2 \sin \alpha x$

By $X'(1) + hX(1) = 0$

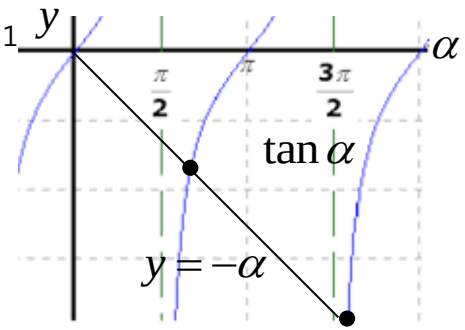
$$c_2 \left(\alpha \cos \alpha x + h \sin \alpha x \right)_{x=1} = c_2 \left(\alpha \cos \alpha + h \sin \alpha \right) = 0, \quad or \quad \tan \alpha = -\frac{\alpha}{h}$$

By boundary condition, $\tan \alpha = -\dfrac{\alpha}{h}$

then graph of $y = \tan\ \alpha$ *and* $y = -\dfrac{\alpha}{h},\ h > 0$

have an infinite number of intersection points, for a>0

Ex. h=1



If the consecutive **positive root** are denoted $\alpha_n,\ n = 1, 2, 3, \ldots.$

Eigenvalues of problem are $\lambda_n = \alpha_n^2$

and **eigenfunctions** are $\sin \alpha_n x,\ n = 1, 2, 3 \ldots \Rightarrow X(x) = c_2 \sin \alpha_n x$

$$X(x) = c_2 \sin \alpha_n x,\ n = 1, 2, 3\ldots$$

And the solution of first order ODE

$T' + k\lambda T = 0$ is $T(t) = c_2 e^{-k\alpha_n^2 t}$, $\lambda_n = \alpha_n^2$

$$u(0,t) = 0,\ \left.\frac{\partial u}{\partial x}\right|_{x=1} = -h\,u(1,t),\quad h > 0,\ t > 0$$

$$u(x,0) = 1,\quad 0 < x < 1$$

$$\therefore u_n = X(x)T(t) = A_n e^{-k\alpha_n^2 t} \sin \alpha_n x \quad \Rightarrow u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t} \sin \alpha_n x$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t} \sin \alpha_n x$$

By initial condition, $1 = \sum_{n=1}^{\infty} A_n \sin \alpha_n x$

This is not Fourier Sine Series but an expansion of u(x,0)=1 in terms of the orthogonal functions arising from the Sturm-Liouville problem

$$X'' + \lambda X = 0,$$
$$X(0) = 0,\ X'(1) + hX(1) = 0$$

$\therefore \{\sin \alpha_n\},\ n = 1, 2, 3, \cdots$

is an orthogonal set with respect to the weight function

$p(x) = 1$ **on the interval [0,1].**

Self-adjoint form of $X'' + \lambda X = 0$

$$\frac{d}{dx}[r(x)X'] + [q(x) + \lambda p(x)]X = 0$$

$$r(x) = 1, q(x) = 0, p(x) = 1$$

Definition 12.4 Orthogonal Set/Weight Function

A set of real-valued functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \cdots\}$ is said to be orthogonal with respect to a weight function $w(x)$ on an interval $[a,b]$ if

$$\int_a^b w(x)\phi_m(x)\phi_n(x)dx = 0, \quad m \neq n$$

$$f(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) + \cdots$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t} \sin \alpha_n x$$

By initial condition, $1 = \sum_{n=1}^{\infty} A_n \sin \alpha_n x$

$\therefore \{\sin \alpha_n\}, \quad n = 1, 2, 3, \cdots$

is an orthogonal set with respect to the weight function

$p(x) = 1$ **on the interval [0,1].**

$$\therefore A_n = \frac{\int_0^1 1 \cdot 1 \cdot \sin \alpha_n x dx}{\int_0^1 1 \cdot \sin^2 \alpha_n x dx} \longleftarrow \int_0^1 \sin^2 \alpha_n x dx = \frac{1}{2h}\left(h + \cos^2 \alpha_n\right)$$

$$c_n = \frac{\int_a^b f(x)w(x)\phi_n(x)dx}{\int_a^b w(x)\phi^2{}_n(x)dx}$$

$$\int_0^1 \sin^2 \alpha_n x dx = \frac{1}{2}\int_0^1 (1 - \cos 2\alpha_n x)dx$$

$$= \frac{1}{2}\left(1 - \frac{1}{2\alpha_n}\sin 2\alpha_n\right)$$

$$= \frac{1}{2}\left(\frac{2\alpha_n + 2\sin \alpha_n \cos \alpha_n}{2\alpha_n}\right)$$

$$= \frac{1}{2}\left(\frac{2\alpha_n + \frac{1}{h}2\alpha_n \cos \alpha_n{}^2}{2\alpha_n}\right)$$

$$= \frac{1}{2h}\left(h + \cos^2 \alpha_n\right)$$

Orthogonal Series Expansions

For a certain types of boundary conditions, the method of separation of variables and the superposition principle lead to an expansion of a function in an infinite series that is not a Fourier series

☑ Example 1 Using Orthogonal Series Expansions

Solve equation $k\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial u}{\partial t},\ 0<x<1,\ t>0$

subject to $u(0,t)=0,\ \dfrac{\partial u}{\partial x}\Big|_{x=1}=-hu(1,t),\ h>0, t>0$

$u(x,0)=1,\ 0<x<1$

$u(x,t)=\sum_{n=1}^{\infty}A_n e^{-k\alpha_n^2 t}\sin\alpha_n x$ By initial condition, $1=\sum_{n=1}^{\infty}A_n\sin\alpha_n x$

$\therefore \{\sin\alpha_n\},\ n=1,2,3,\cdots$

is an orthogonal set with respect to the weight function

$p(x)=1$ **on the interval [0,1].**

$$\int_0^1 \sin\alpha_n x dx=-\frac{1}{\alpha_n}\cos\alpha_n\Big|_0^1=\frac{1}{\alpha_n}(1-\cos\alpha_n)$$

$$\therefore A_n=\frac{\int_0^1 1\cdot 1\cdot\sin\alpha_n x dx}{\int_0^1 1\cdot\sin^2\alpha_n x dx}$$

$$\int_0^1\sin^2\alpha_n x dx=\frac{1}{2h}\left(h+\cos^2\alpha_n\right)$$

$$u(x,t)=\sum_{n=1}^{\infty}A_n e^{-k\alpha_n^2 t}\sin\alpha_n x$$ By initial condition, $$1=\sum_{n=1}^{\infty}A_n\sin\alpha_n x$$

$$\therefore \{\sin\alpha_n\},\ n=1,2,3,\cdots$$

is an orthogonal set with respect to the weight function

$p(x)=1$ **on the interval [0,1].**

$$\therefore A_n=\frac{\int_0^1 1\cdot 1\cdot\sin\alpha_n x dx}{\int_0^1 1\cdot\sin^2\alpha_n x dx}=\frac{\frac{1}{\alpha_n}(1-\cos\alpha_n)}{\frac{1}{2h}\left(h+\cos^2\alpha_n\right)}=\frac{2h(1-\cos\alpha_n)}{\alpha_n(h+\cos^2\alpha_n)}$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t} \sin \alpha_n x \quad , A_n = \frac{2h(1-\cos\alpha_n)}{\alpha_n (h + \cos^2 \alpha_n)}$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \frac{2h(1-\cos\alpha_n)}{\alpha_n (h + \cos^2 \alpha_n)} e^{-k\alpha_n^2 t} \sin \alpha_n x$$

Orthogonal Series Expansions

For a certain types of boundary conditions, the method of separation of variables and the superposition principle lead to an expansion of a function in an infinite series that is not a Fourier series

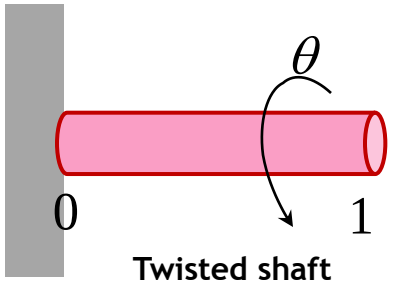
☑ Example 2 Using Orthogonal Series Expansions

The twist angle $\theta(x,t)$ of a torsionally vibrating shaft of unit length is determined from

Solve equation $a^2 \dfrac{\partial^2 \theta}{\partial x^2} = \dfrac{\partial^2 \theta}{\partial t^2}, \quad 0 < x < 1, \; t > 0$

subject to $\theta(0,t) = 0, \quad \left.\dfrac{\partial \theta}{\partial x}\right|_{x=1} = 0, \quad t > 0$

$\theta(x,0) = x, \quad \left.\dfrac{\partial \theta}{\partial t}\right|_{t=0} = 0, \quad 0 < x < 1$



Product Method

$\theta(x,t) = X(x)T(t)$

Then,

$X'' + \lambda X = 0$

$T'' + a^2 \lambda T = 0$

$X(0) = 0 \;\; and \;\; X'(1) = 0$

From $\theta(0,t) = X(0)T(t) = 0$

$T(t) \neq 0$ **otherwise θ(x,t) : trivial solution, So**

$X(0) = 0$

From $\theta'(0,t) = X'(0)T(t) = 0$

$T(t) \neq 0$ **otherwise θ(x,t) : trivial solution, So**

$X'(0) = 0$

$X'' + \lambda X = 0, \;\; X(0) = 0, X'(1) = 0$

Is a regular Sturm-Liouville problem.

is a regular Sturm-Liouville problem.

General solution is $X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$

By $X(0) = 0,\ c_1 = 0$

By $X'(1) = 0 \quad c_2 \cos \alpha = 0$

$$\therefore \alpha_n = (2n-1)\pi / 2 \quad , \lambda_n = \alpha_n^2 = (2n-1)^2 \pi^2 / 4$$

$$\therefore X(x) = c_2 \sin \alpha x = c_2 \sin\left(\frac{2n-1}{2}\right)\pi x,\ n = 1, 2, 3, \ldots.$$

$$T'' + a^2 \lambda T = 0, \quad \left.\frac{\partial \theta}{\partial t}\right|_{t=0} = 0$$

$$T(t) = c_3 \cos a\alpha_n t + c_4 \sin a\alpha_n t$$

$$T'(0) = 0 \quad \text{implies } c_4 = 0$$

$$\therefore T(t) = c_3 \cos a\alpha_n t = c_3 \cos a\left(\frac{2n-1}{2}\right)\pi t$$

$$X(x) = c_2 \sin\left(\frac{2n-1}{2}\right)\pi x, \; n = 1, 2, 3, \ldots.$$

$$, \lambda_n = \alpha_n^2 = (2n-1)^2 \pi^2 / 4$$

$$\theta(x,t) = X(x)T(t)$$

Then, $X'' + \lambda X = 0 \quad , T'' + a^2 \lambda T = 0$

$$X(x) = c_2 \sin\left(\frac{2n-1}{2}\right)\pi x, \; n = 1,2,3,....$$

$$T(t) = c_3 \cos a\left(\frac{2n-1}{2}\right)\pi t$$

$$, \lambda_n = \alpha_n^2 = (2n-1)^2 \pi^2 / 4$$

$$\therefore \theta_n(x,t) = X(x)T(t) = A_n \cos a\left(\frac{2n-1}{2}\right)\pi t \sin\left(\frac{2n-1}{2}\right)\pi x$$

$$\theta_n(x,t) = A_n \cos a\left(\frac{2n-1}{2}\right)\pi t \sin\left(\frac{2n-1}{2}\right)\pi x$$

$$\theta(x,t) = \sum_{n=1}^{\infty} A_n \cos a\left(\frac{2n-1}{2}\right)\pi t \sin\left(\frac{2n-1}{2}\right)\pi x$$

By initial condition,

$$\theta(x,0) = x = \sum_{n=1}^{\infty} A_n \sin\left(\frac{2n-1}{2}\right)\pi x \quad , where \quad A_n = \frac{\int_0^1 1\cdot x \sin\left(\frac{2n-1}{2}\right)\pi x dx}{\int_0^1 1\cdot \sin^2\left(\frac{2n-1}{2}\right)\pi x dx}$$

Self-adjoint form of $X'' + \lambda X = 0$

$$\frac{d}{dx}[r(x)X'] + [q(x) + \lambda p(x)]X = 0$$

$$r(x) = 1, q(x) = 0, p(x) = 1$$

$\{\sin \alpha_n\}, \; n = 1, 2, 3, \cdots$

is an orthogonal set with respect to the weight function

$p(x) = 1$ **on the interval [0,1].**

By initial condition,

$$\theta(x,0) = x = \sum_{n=1}^{\infty} A_n \sin\left(\frac{2n-1}{2}\right)\pi x$$

$$A_n = \frac{\int_0^1 1\cdot x \sin\left(\frac{2n-1}{2}\right)\pi x dx}{\int_0^1 1\cdot \sin^2\left(\frac{2n-1}{2}\right)\pi x dx}$$

$$\int_0^1 1\cdot x \sin\left(\frac{2n-1}{2}\right)\pi x dx$$

$$= \left[x\left(-\frac{2}{(2n-1)\pi}\cos\left(\frac{2n-1}{2}\right)\pi x \right)\right]_0^1 - \int_0^1 \left(-\frac{2}{(2n-1)\pi}\cos\left(\frac{2n-1}{2}\right)\pi x \right) dx$$

$$= \left(-\frac{2}{(2n-1)\pi}\cos\left(\frac{2n-1}{2}\right)\pi \right) + \left[\left(\frac{2}{(2n-1)\pi}\right)^2 \sin\left(\frac{2n-1}{2}\right)\pi x \right]_0^1$$

$$= (0) + \left(\left(\frac{2}{(2n-1)\pi}\right)^2 \sin\left(\frac{2n-1}{2}\right)\pi \right)$$

$$= \left(\frac{2}{(2n-1)\pi}\right)^2 (-1)^{n+1}$$

By initial condition,

$$\theta(x,0) = x = \sum_{n=1}^{\infty} A_n \sin\left(\frac{2n-1}{2}\right)\pi x$$

$$A_n = \frac{\int_0^1 1\cdot x\sin\left(\frac{2n-1}{2}\right)\pi x dx}{\int_0^1 1\cdot \sin^2\left(\frac{2n-1}{2}\right)\pi x dx}$$

$$\int_0^1 1\cdot x\sin\left(\frac{2n-1}{2}\right)\pi x dx = \left(\frac{2}{(2n-1)\pi}\right)^2 (-1)^{n+1}$$

$$\int_0^1 \sin^2\left(\frac{2n-1}{2}\right)\pi x dx = \frac{1}{2}\int_0^1 \left(1-\cos 2\left(\frac{2n-1}{2}\right)\pi x\right) dx$$

$$= \frac{1}{2}\left[x - \frac{1}{(2n-1)\pi}\sin(2n-1)\pi x\right]_0^1$$

$$= \frac{1}{2}$$

$$\theta(x,t) = \sum_{n=1}^{\infty} A_n \cos a\left(\frac{2n-1}{2}\right)\pi t \sin\left(\frac{2n-1}{2}\right)\pi x \ , A_n = \frac{8(-1)^{n+1}}{(2n-1)^2\pi^2}$$

$$\theta(x,t) = \sum_{n=1}^{\infty} \frac{8(-1)^{n+1}}{(2n-1)^2\pi^2} \cos a\left(\frac{2n-1}{2}\right)\pi t \sin\left(\frac{2n-1}{2}\right)\pi x$$

Eigenfunction Expansion*

$$u(x,0) = f(x)$$

$$\sum_{n=1}^{\infty} A_n \sin\sqrt{\lambda}x = f(x) \quad \Leftarrow \quad \therefore \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f(x)$$

Eigenfunction Expansion*

A real-valued functions $\phi(x)$ is said to be *square-integrable* with respect to a weight function $\rho(x) > 0$, if on an interval I,

$$\int_I \phi^2(x)\rho(x)dx < +\infty$$

Let $\{\phi_n(x)\}$ for a positive integer n, be an ***orthogonal*** set of square-integrable functions with a positive weight function $\rho(x)$ on an interval I,

Let $f(x)$ be a given function that can be represented by a uniformly convergent series of the form

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

Where the coefficients c_n are constants. Now multiplying both sides by $\phi_m(x)\rho(x)$ and integrating term by term over the interval I

$$\int_I f(x)\phi_m(x)\rho(x)dx = \sum_{n=1}^{\infty}\int_I c_n\phi_n(x)\phi_m(x)\rho(x)dx$$

and hence $$\int_I f(x)\phi_n(x)\rho(x)dx = c_n\int_I \phi_n{}^2(x)\rho(x)dx$$

Thus $$c_n = \frac{\int_I f\phi_n\rho dx}{\int_I \phi_n{}^2\rho dx}$$

*Tyn Myint-U, Partial Differential Equations of Mathematical Physics, Second Edition, Elsevier North Holland, 1980, p179~185

Eigenfunction Expansion*

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\sqrt{\lambda}x$$

Eigenfunction Expansion*

A real-valued functions $\phi(x)$ is said to be *square-integrable* with respect to a weight function $\rho(x) > 0$, if on an interval I,

$$\int_I \phi^2(x)\rho(x)dx < +\infty$$

Theorem 7.3.1 If f is represented by a uniformly convergent series

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

on an interval I, where $\phi_n(x)$ are square-integrable functions orthogonal with respect to a positive weight function $\rho(x)$

then, c_n are determined by

Thus $$c_n = \frac{\int_I f\phi_n\rho dx}{\int_I \phi_n{}^2\rho dx}$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\sqrt{\lambda}x$$

Eigenfunction Expansion*

When every continuous square-integrable function $f(x)$ can be expanded into an infinite series

$$f(x) = \sum_{k=1}^{\infty} c_k \phi_k(x)$$

The sequence of continuous square-integrable functions $\{\phi_k\}$ orthogonal with respect to the weight function ρ is said to be complete

Theorem 7.5.1

The eigenfunctions of any regular Sturm-Liouville system are complete in the space of functions that are piecewise continuous on the interval [a,b] with respect to the weight function $s(x)$.

Moreover, any piecewise smooth function on [a,b] that satisfies the end condition of the regular Sturm-Liouville system can be expanded in an absolutely and uniformly convergent series

$$f(x) = \sum_{k=1}^{\infty} c_k \phi_k(x)$$

Where c_k are given by $$c_k = \frac{\int_a^b f\phi_k s dx}{\int_a^b \phi_k{}^2 s dx}$$

*Tyn Myint-U, Partial Differential Equations of Mathematical Physics, Second Edition, Elsevier North Holland, 1980, p179~185

$$u(x,0) = f(x) \sim \sum_{n=1}^{\infty} A_n \sin\sqrt{\lambda}x$$

Eigenfunction Expansion*

If assume that f is a piecewise smooth function on [a,b], then by theorem 7.5.1 we can expand $f(x)$ in terms of eigenfunctions, and formally we write

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\sqrt{\lambda}x$$

With the coefficient A_n is given by $A_n = \dfrac{\int_0^L f(x)\sin\sqrt{\lambda}x\,dx}{\int_0^L \sin^2\sqrt{\lambda}x\,dx}$

CHAPTER 7. LINEAR ALGEBRA

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  - Buckling
- Diagonalization
- Singular Value Decomposition
- Quadratic Form
- Positive Definite Matrix & Optimization

7-1. LINEAR SYSTEMS AND MATRICES

Linear Systems Vs Matrices

$$x_1 + 2x_2 + x_3 = 1$$
$$3x_1 - x_2 - x_3 = 2$$
$$2x_1 + 3x_2 - x_3 = -3$$

row2 + row1x(−3)

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

row2 + row1x(−3)

$$3x_1 - x_2 - x_3 = 2$$
$$+) \quad -3x_1 - 6x_2 - 3x_3 = -3$$
$$-7x_2 - 4x_3 = -1$$

$$x_1 + 2x_2 + x_3 = 1$$
$$0 \cdot x_1 - 7x_2 - 4x_3 = -1$$
$$2x_1 + 3x_2 - x_3 = -3$$

$$\begin{bmatrix} 0 & -7 & -4 \\ 3 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$$

Linear Systems Vs Matrices

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 - x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 - x_3 &= -3 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 - 7x_2 - 4x_3 &= -1 \\ 2x_1 + 3x_2 - x_3 &= -3 \end{aligned}$$

row3 + row1x(−2)

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -4 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$$

row3 + row1x(−2)

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= -3 \\ +)\ -2x_1 - 4x_2 - 2x_3 &= -2 \\ \hline -x_2 - 3x_3 &= -5 \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 - 7x_2 - 4x_3 &= -1 \\ 0 \cdot x_1 - x_2 - 3x_3 &= -5 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -4 \\ 0 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -5 \end{bmatrix}$$

Linear Systems Vs Matrices

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 - x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 - x_3 &= -3 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 - 7x_2 - 4x_3 &= -1 \\ 0 \cdot x_1 - x_2 - 3x_3 &= -5 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -4 \\ 0 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -5 \end{bmatrix}$$

row 2 ↔ row 3

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 - x_2 - 3x_3 &= -5 \\ 0 \cdot x_1 - 7x_2 - 4x_3 &= -1 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & -7 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -1 \end{bmatrix}$$

row 2x(−1)

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 + x_2 + 3x_3 &= 5 \\ 0 \cdot x_1 - 7x_2 - 4x_3 &= -1 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -7 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$$

Linear Systems Vs Matrices

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 - x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 - x_3 &= -3 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 + x_2 + 3x_3 &= 5 \\ 0 \cdot x_1 - 7x_2 - 4x_3 &= -1 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -7 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$$

row 3+ row 2x7

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 + x_2 + 3x_3 &= 5 \\ 0 \cdot x_1 + 0 \cdot x_2 - 17x_3 &= 34 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 34 \end{bmatrix}$$

The last equations and matrix are equal to given equations.

Linear Independence

Definition 3.1 — **Linear Dependence / Independence**

A set of functions $f_1(x),\ f_2(x), \ldots,\ f_n(x)$ is said to be 'linearly dependent' on an interval I if there exist constant $c_1, c_2, \ldots c_n$, not all zero such that $c_1 f_1(x) + c_2 f_2(x) + \cdots c_n f_n(x) = 0$ for every x in the interval.

If the set of functions is not linearly dependent on the interval, it is said to be 'linearly independent'

In other words, a set of functions is 'linearly independent' if the only constants for

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

are $c_1 = c_2 = \cdots = c_n = 0$

"two functions are linearly independent when neither is a constant multiple of the other"

$$\begin{cases} f_1(x) = \sin 2x \\ f_2(x) = \sin x \cos x \end{cases} \quad on(-\infty, \infty)$$

$$f_2(x) = \sin x \cos x = \frac{1}{2}\sin 2x$$

$$\therefore\ f_2(x) = \frac{1}{2} f_1(x)$$

Linearly Dependent

$$\begin{cases} f_1(t) = e^t \\ f_2(t) = e^{2t} \end{cases} \quad on(-\infty, \infty)$$

$$c_1 e^t + c_2 e^{2t} = 0$$

Satisfied only when $c_1 = c_2 = 0$ on the interval

Linearly Independent

Linear Systems Vs Matrices

$$x_1 + 2x_2 + x_3 = 1$$
$$3x_1 - x_2 - x_3 = 2$$
$$2x_1 + 3x_2 - x_3 = -3$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$f_1 : x_1 + 2x_2 + x_3 - 1 = 0$$
$$f_2 : 0 \cdot x_1 + x_2 + 3x_3 - 5 = 0$$
$$f_3 : 0 \cdot x_1 + 0 \cdot x_2 + 17x_3 - 34 = 0$$

$$x_1 + 2x_2 + x_3 = 1$$
$$0 \cdot x_1 + x_2 + 3x_3 = 5$$
$$0 \cdot x_1 - + 0 \cdot x_2 + 17x_3 = 34$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 34 \end{bmatrix}$$

No. of equations which are linearly independent ?

$$c_1(f_1) + c_2(f_2) + c_3(f_3) = 0$$

$$c_1(x_1 + 2x_2 + x_3 - 1) + c_2(0 \cdot x_1 + x_2 + 3 \cdot x_3 - 5) + c_3(0 \cdot x_1 + 0 \cdot x_2 + 17x_3 - 34) = 0$$

$$(c_1 + 0 \cdot c_2 + 0 \cdot c_3)x_1 + (2c_1 + c_2 + 0 \cdot c_3)x_2 + (c_1 + 3c_2 + 17c_3)x_3 + (-c_1 - 5c_2 - 34c_3) = 0$$

$c_1 = 0$ $\quad c_2 = 0$ $\quad c_3 = 0$

$\therefore f_1,\ f_2, f_3$: **linearly independent.** $\longrightarrow$ **rank : 3**

Linear Systems Vs Matrices

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 - x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 - x_3 &= -3 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 + x_2 + 3x_3 &= 5 \\ 0 \cdot x_1 + 0 \cdot x_2 + 17x_3 &= 34 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 34 \end{bmatrix}$$

No. of equations which are linearly independent : 3

rank : 3

Unknown variables x_1, x_2, x_3 n=3

No. of equations which are linearly independent : 3 = **rank : 3** = **Unknown variables n=3** ⇨ **Unique Solution**

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 0 \cdot x_1 + x_2 + 3x_3 &= 5 \\ 0 \cdot x_1 + 2x_2 + 6x_3 &= 10 \end{aligned}$$

Linear Systems Vs Matrices

$$x_1 + 2x_2 + x_3 = 1$$
$$0 \cdot x_1 + x_2 + 3x_3 = 5$$
$$0 \cdot x_1 + 2x_2 + 6x_3 = 10$$

0.5*row 3

$$x_1 + 2x_2 + x_3 = 1$$
$$0 \cdot x_1 + x_2 + 3x_3 = 5$$
$$0 \cdot x_1 + x_2 + 3x_3 = 5$$

$$x_1 + 2x_2 + x_3 = 1$$
$$0 \cdot x_1 + x_2 + 3x_3 = 5$$
$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0$$

$$f_1 : x_1 + 2x_2 + x_3 - 1 = 0$$
$$f_2 : 0 \cdot x_1 + x_2 + 3x_3 - 5 = 0$$
$$f_3 : 0 \cdot x_1 + 2x_2 + 6x_3 - 10 = 0$$

No. of equations which are linearly independent ?

$$c_1(f_1) + c_2(f_2) + c_3(f_3) = 0$$

$$c_1(0) + c_2(0) + c_3(0) = 0$$

$$c_1(x_1 + 2x_2 + x_3 - 1) + c_2(0 \cdot x_1 + x_2 + 3 \cdot x_3 - 5) + c_3(0 \cdot x_1 + 2x_2 + 6x_3 - 10) = 0$$

$$(c_1 + 0 \cdot c_2 + 0 \cdot c_3)x_1 + (2c_1 + c_2 + 2c_3)x_2 + (c_1 + 3c_2 + 6c_3)x_3 + (-c_1 - 5c_2 - 10c_3) = 0$$

$$c_1 = 0$$

$$c_2 = -2c_3 \qquad c_3 = \textit{arbitrary number}$$

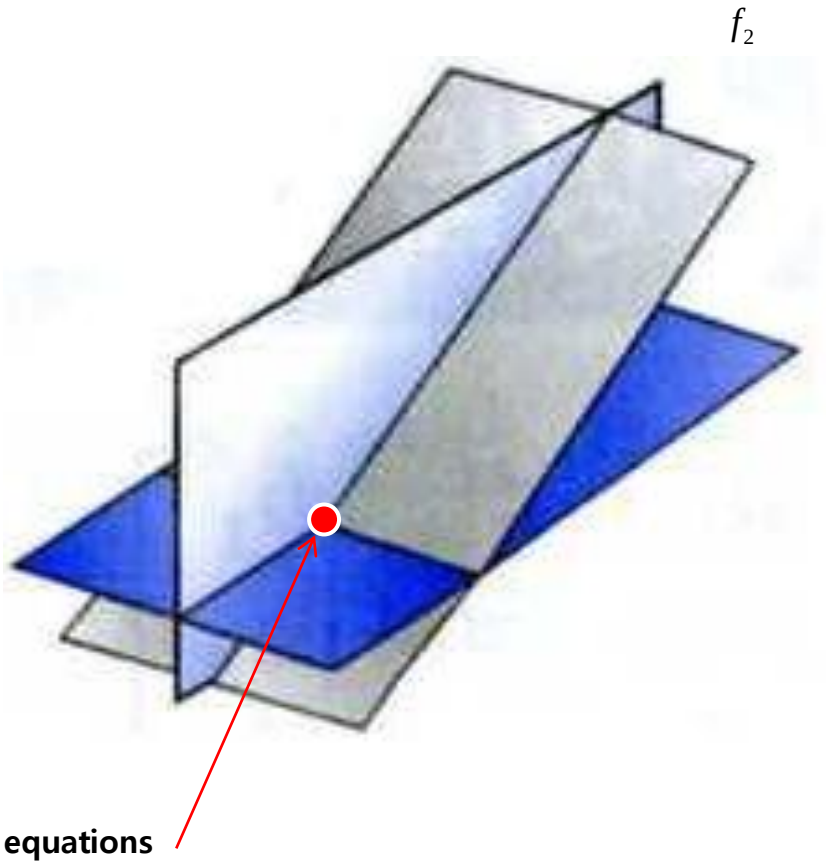
$\therefore f_1, f_2, f_3$: linearly ***dependent.***

Linear Systems Vs Matrices

$$\begin{cases} f_1 : x_1 + 2x_2 + x_3 - 1 = 0 \\ f_2 : 0 \cdot x_1 + x_2 + 3x_3 - 5 = 0 \\ f_3 : 0 \cdot x_1 + 2x_2 + 6x_3 - 10 = 0 \end{cases}$$

$$c_1(0) + c_2(0) + c_3(0) = 0$$

$$f_1 = 0 \text{ and } f_2 = 0 \text{ and } f_3 = 0$$



Only when (x_1, x_2, x_3) is the common solution for three equations

Linear Systems Vs Matrices

$$x_1 + 2x_2 + x_3 = 1$$
$$0 \cdot x_1 + x_2 + 3x_3 = 5$$
$$0 \cdot x_1 + 2x_2 + 6x_3 = 10$$

0.5*row 3

$$x_1 + 2x_2 + x_3 = 1$$
$$0 \cdot x_1 + x_2 + 3x_3 = 5$$
$$0 \cdot x_1 + x_2 + 3x_3 = 5$$

$$x_1 + 2x_2 + x_3 = 1$$
$$0 \cdot x_1 + x_2 + 3x_3 = 5$$
$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0$$

rank : 2

$$f_1 : x_1 + 2x_2 + x_3 - 1 = 0$$
$$f_2 : 0 \cdot x_1 + x_2 + 3x_3 - 5 = 0$$
$$f_3 : 0 \cdot x_1 + 2x_2 + 6x_3 - 10 = 0$$

No. of equations which are linearly independent ?

$$c_1(f_1) + c_2(f_2) = 0$$

$$c_1(x_1 + 2x_2 + x_3 - 1) + c_2(0 \cdot x_1 + x_2 + 3 \cdot x_3 - 5) = 0$$

$$(c_1 + 0 \cdot c_2)x_1 + (2c_1 + c_2)x_2 + (c_1 + 3c_2)x_3 + (-c_1 - 5c_2) = 0$$

$$c_1 = 0 \qquad c_2 = 0$$

$\therefore f_1, f_2$: linearly ***independent.***

Linear Systems Vs Matrices

$$x_1 + 2x_2 + x_3 = 1$$
$$0 \cdot x_1 + x_2 + 3x_3 = 5$$
$$0 \cdot x_1 + 2x_2 + 6x_3 = 10$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}$$

$$x_1 + 2x_2 + x_3 = 1$$
$$0 \cdot x_1 + x_2 + 3x_3 = 5$$
$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$$

No. of equations which are linearly independent : 2 = rank : 2 < Unknown variables n=3 ⇨ Infinite many solutions

Solution ?

$$\mathbf{Ax} = \lambda \mathbf{x} \longrightarrow \mathbf{x} = \mathbf{0}$$ Trivial Solution

Linear Systems Vs Matrices

$$\mathbf{A}=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$$

Rank : 3

Trivial *x*

Solution ?

$$\mathbf{Ax}=\lambda\mathbf{x}\longrightarrow \mathbf{x}=\mathbf{0}$$ Trivial Solution

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

x: Infinite many solutions

to have a solution x except x=0

$$rank(\mathbf{A}-\lambda\mathbf{I})<n \qquad \det(\mathbf{A}-\lambda\mathbf{I})=0$$

Zero row

λ: Eigenvalues, x: Eigenvectors
n: Unknown variables

x: Infinite many solutions → Objective function → Solution determined

Optimization Problem

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$$

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

when $\lambda=5$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-5\mathbf{I}=\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}$$

Row reduction

$$\begin{bmatrix}-7 & 2 & -3\\ 0 & -24/7 & -48/7\\ 0 & 0 & 0\end{bmatrix}$$

Rank : 2

infinite no. of *x*

Matrices

Matrix : a rectangular array of numbers (or functions) enclosed in brackets.

Entries
(or sometimes the elements)

Examples)

$$\begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix},$$

rows

$$\begin{bmatrix} e^{-x} & 2x^2 \\ e^{6x} & 4x \end{bmatrix}$$

square matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

columns

(no. of rows = no. of columns)

Matrices

We shall denote matrices by *capital boldface letters* A, B, C, ⋯ , or by writing the general entry in brackets ; thus A=[a_{jk}], and so on. By an *m* x *n* matrix (read *m* by *n* matrix) we mean a matrix with *m* rows and *n* columns – rows come always first!

***m* x *n* is called the size of the matrix. Thus an *m* x *n* matrix is of the form**

$$\mathbf{A} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \cdots (2).$$

Matrices

$$\mathbf{A} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

If $m = n$, we call A an n x n square matrix.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

main diagonal

If $m \neq n$, we call A a rectangular matrix.

Vectors

Matrix : a rectangular array of numbers (or functions) enclosed in brackets.

Vector : a matrix with only one row or column.

We shall denote vectors by *lowercase boldface letters* $\mathbf{a}, \mathbf{b}, \cdots$ or by its general component in brackets, $\mathbf{a}=[a_{ij}]$, and so on.

Examples)

row vector

$$\mathbf{a} = [a_1 \quad a_2 \quad a_3]$$

Components (entries)

column vector

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Matrices

Ex 1) Linear Systems, a Major Application of Matrices

In a system of linear equations, briefly called a linear system, such as

$$\begin{aligned} 4x_1 + 6x_2 + 9x_3 &= 6 \\ 6x_1 \qquad - 2x_3 &= 20 \\ 5x_1 - 8x_2 + \ x_3 &= 10 \end{aligned}$$

the coefficients of the unknowns x_1, x_2, x_3 are the entries of the coefficient matrix, call it *A*,

$$\mathbf{A} = \begin{bmatrix} 4 & 6 & 9 \\ 6 & 0 & -2 \\ 5 & -8 & 1 \end{bmatrix}.$$

The matrix

$$\tilde{\mathbf{A}} = \begin{bmatrix} 4 & 6 & 9 & 6 \\ 6 & 0 & -2 & 20 \\ 5 & -8 & 1 & 10 \end{bmatrix}$$

is called augmented matrix(첨가행렬) of the system.

We shall discuss this in great detail, beginning in Sec. 7.3

Matrices

Ex 2) Sales Figures in Matrix Form

Sales Figures for Three products I, II, III in a store on Monday (M), Tuesday (T), ⋯ may for each week be arrange in a matrix.

$$A = \begin{matrix} \\ \\ \\ \end{matrix} \begin{matrix} M & T & W & Th & F & S \end{matrix} \begin{bmatrix} 400 & 330 & 810 & 0 & 210 & 470 \\ 0 & 120 & 780 & 500 & 500 & 960 \\ 100 & 0 & 0 & 270 & 430 & 780 \end{bmatrix} \begin{matrix} \text{I} \\ \text{II} \\ \text{III} \end{matrix} .$$

If the company has ten stores, we can set up ten such matrices, on for each store. Then by adding corresponding entries of these matrices we can get a matrix showing the total sales of each product on each day.

Matrix Addition and Scalar Multiplication

Equality of Matrices

Two matrices $\mathbf{A}=[a_{jk}]$ and $\mathbf{B}=[b_{jk}]$ are equal, written $\mathbf{A}=\mathbf{B}$, if and only if they have the same size and the corresponding entries are equal, that is, $a_{11}=b_{11}$, $a_{12}=b_{12}$, and so on. Matrices that are not equal are called different. Thus, matrices of different sizes are always different.

Ex3) Equality of Matrices

$$\text{Let} \quad \mathbf{A}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{B}=\begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix}$$

$$\text{Then} \quad \mathbf{A}=\mathbf{B} \ \text{ if and only if } \quad \begin{aligned} a_{11}&=4, \quad a_{12}=0, \\ a_{21}&=3, \quad a_{22}=-1. \end{aligned}$$

Matrix Addition and Scalar Multiplication

Addition of Matrices

The sum of two matrices $\mathbf{A}=[a_{jk}]$ and $\mathbf{B}=[b_{jk}]$ of the same size is written $\mathbf{A}+\mathbf{B}$ and has the entries $a_{jk}+b_{jk}$ obtained by adding the corresponding entries of $\mathbf{A}$ and $\mathbf{B}$. Matrices of different sizes cannot be added.

Ex4) Addition of Matrices and Vectors

$$\text{If} \quad \mathbf{A}=\begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B}=\begin{bmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}, \quad \text{then} \quad \mathbf{A}+\mathbf{B}=\begin{bmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{bmatrix}.$$

$$\text{If} \quad \mathbf{a}=\begin{bmatrix} 5 & 7 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b}=\begin{bmatrix} -6 & 2 & 0 \end{bmatrix}, \quad \text{then} \quad \mathbf{a}+\mathbf{b}=\begin{bmatrix} -1 & 9 & 2 \end{bmatrix}.$$

Matrix Addition and Scalar Multiplication

Scalar Multiplication (Multiplication by a Number)

The product of any m x n matrix $\mathbf{A}=[a_{jk}]$ and any scalar c (number c) is written $c\mathbf{A}$ and is the m x n matrix $c\mathbf{A}=[ca_{jk}]$ obtained by multiplying each entry of $\mathbf{A}$ by c.

Ex5) Scalar Multiplication

$$\text{If} \quad \mathbf{A}=\begin{bmatrix} 2.7 & -1.8 \\ 0 & 0.9 \\ 9.0 & -4.5 \end{bmatrix}, \text{ then } \quad -\mathbf{A}=\begin{bmatrix} -2.7 & 1.8 \\ 0 & -0.9 \\ -9.0 & 4.5 \end{bmatrix}$$

$$\frac{10}{9}\mathbf{A}=\begin{bmatrix} 3 & -2 \\ 0 & 1 \\ 10 & -5 \end{bmatrix}, \quad 0\cdot\mathbf{A}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

zero matrix

Rules for Matrix Addition and Scalar Multiplication

$$(3)\quad \begin{cases} (a) & \mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A} \\ (b) & (\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C}) \quad (\text{written } \mathbf{A}+\mathbf{B}+\mathbf{C}) \\ (c) & \mathbf{A}+0=\mathbf{A} \\ (d) & \mathbf{A}+(-\mathbf{A})=0 \end{cases}$$

$$(4)\quad \begin{cases} (a) & c(\mathbf{A}+\mathbf{B})=c\mathbf{A}+c\mathbf{B} \\ (b) & (c+k)\mathbf{A}=c\mathbf{A}+k\mathbf{A} \\ (c) & c(k\,\mathbf{A})=(ck)\mathbf{A} \quad (\text{written } ck\mathbf{A}) \\ (d) & 1\mathbf{A}=\mathbf{A} \end{cases}$$

Multiplication of a Matrix by a Matrix

$$c_{jk} = \sum_{l=1}^{n} a_{jl} b_{lk} = a_{j1} b_{1k} + a_{j2} b_{2k} + \cdots + a_{jn} b_{nk} \cdots (1)$$

$$j = 1, \cdots, m$$
$$k = 1, \cdots, p.$$

$$m = 4 \left\{ \overbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}}^{n=3} \overbrace{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}}^{p=2} \right\} (r = 3) = \overbrace{\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \\ c_{41} & c_{42} \end{bmatrix}}^{p=2} \left\} \vphantom{\begin{bmatrix} a \\ a \\ a \\ a \end{bmatrix}} \right. m = 4$$

$$c_{21} = a_{21} b_{11} + a_{22} b_{21} + a_{23} b_{31}$$

$$\underset{[m\times n]}{\mathbf{A}}\ \underset{[n\times r]}{\mathbf{B}} = \underset{[m\times r]}{\mathbf{C}}$$

Multiplication of a Matrix by a Matrix

Ex 1) Matrix Multiplication

$$\mathbf{AB} = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & 14 & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}$$

$$c_{11} = 3\cdot 2 + 5\cdot 5 + (-1)\cdot 9 = 22$$

$$c_{23} = 4\cdot 3 + 0\cdot 7 + 2\cdot 1 = 14$$

The product BA is not defined.

Multiplication of a Matrix by a Matrix

$$\mathbf{A}\quad\mathbf{B}=\mathbf{C}$$

$$[m\times n]\ [n\times r]=[m\times r]$$

Ex 1) Matrix Multiplication

$$\begin{bmatrix}4 & 2\\ 1 & 8\end{bmatrix}\begin{bmatrix}3\\ 5\end{bmatrix}=\begin{bmatrix}4\cdot 3+2\cdot 5\\ 1\cdot 3+8\cdot 5\end{bmatrix}=\begin{bmatrix}22\\ 43\end{bmatrix}$$

$$\begin{bmatrix}3\\ 5\end{bmatrix}\begin{bmatrix}4 & 2\\ 1 & 8\end{bmatrix}$$ **is not defined.**

Ex3) Products of Row and Column Vectors

$$\begin{bmatrix}3 & 6 & 1\end{bmatrix}\begin{bmatrix}1\\ 2\\ 4\end{bmatrix}=\begin{bmatrix}19\end{bmatrix}$$

$$\begin{bmatrix}1\\ 2\\ 4\end{bmatrix}\begin{bmatrix}3 & 6 & 1\end{bmatrix}=\begin{bmatrix}3 & 6 & 1\\ 6 & 12 & 2\\ 12 & 24 & 4\end{bmatrix}.$$

Multiplication of a Matrix by a Matrix

$$\underset{[m\times n]}{\mathbf{A}}\ \underset{[n\times r]}{\mathbf{B}} = \underset{[m\times r]}{\mathbf{C}}$$

Ex 4) CAUTION! Matrix Multiplication is Not Commutative, AB≠BA in General

$$\mathbf{A}=\begin{bmatrix}1 & 1\\ 100 & 100\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}$$

$$\mathbf{AB}=\begin{bmatrix}1 & 1\\ 100 & 100\end{bmatrix}\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\mathbf{BA}=\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ 100 & 100\end{bmatrix}=\begin{bmatrix}99 & 99\\ -99 & -99\end{bmatrix}$$

$$\therefore \mathbf{AB}\neq\mathbf{BA}$$

It is interesting that this also shows that AB=0 does not necessarily imply BA=0 or A=0 or B=0.

Properties of Matrix Multiplication

$$(2)\begin{cases}(a) & (k\mathbf{A})\mathbf{B} = k(\mathbf{AB}) = \mathbf{A}(k\mathbf{B}) \\ (b) & \mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C} \\ (c) & (\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC} + BC \\ (d) & \mathbf{C}(\mathbf{A}+\mathbf{B}) = \mathbf{CA} + \mathbf{CB}\end{cases}$$

$(\text{written } k\mathbf{AB} \text{ or } \mathbf{A}k\mathbf{B})$

$(\text{written } \mathbf{ABC})$

(2*b*) is called the associative law (결합법칙)

(2*c*) and (2*d*) is called the distributive law (분배법칙)

$$\mathbf{A}\quad\mathbf{B}\quad=\quad\mathbf{C}$$
$$[m\times n]\ [n\times r]=[m\times r]$$

Properties of Matrix Multiplication

$$c_{jk}=\sum_{l=1}^{n}a_{jl}b_{lk}=a_{j1}b_{1k}+a_{j2}b_{2k}+\cdots+a_{jn}b_{nk}\cdots(1)$$

Since matrix multiplication is a multiplication of rows into columns, we can write the defining formula (1) more compactly as

$$\boxed{c_{jk}=\mathbf{a}_j\mathbf{b}_k\cdots(3)}\qquad j=1,\cdots,m;\quad k=1,\cdots,p.$$

where $\mathbf{a}_j$ is the *j*th row vector of A and $\mathbf{b}_k$ is the *k*th column vector of B, so that in agreement with (1),

$$\mathbf{a}_j\mathbf{b}_k=\begin{bmatrix}a_{j1} & a_{j2} & \cdots & a_{jn}\end{bmatrix}\begin{bmatrix}b_{1k}\\ \vdots\\ b_{nk}\end{bmatrix}=a_{j1}b_{1k}+a_{j2}b_{2k}+\cdots+a_{jn}b_{nk}$$

Properties of Matrix Multiplication

Ex 5) Product in Terms of Row and Column Vectors

If $\mathbf{A}=[a_{jk}]$ is of size 3x3 and $\mathbf{B}=[b_{jk}]$ is of size 3x4, then

$$\mathbf{AB}=\begin{bmatrix} \mathbf{a}_1\mathbf{b}_1 & \mathbf{a}_1\mathbf{b}_2 & \mathbf{a}_1\mathbf{b}_3 & \mathbf{a}_1\mathbf{b}_4 \\ \mathbf{a}_2\mathbf{b}_1 & \mathbf{a}_2\mathbf{b}_2 & \mathbf{a}_2\mathbf{b}_3 & \mathbf{a}_2\mathbf{b}_4 \\ \mathbf{a}_3\mathbf{b}_1 & \mathbf{a}_3\mathbf{b}_2 & \mathbf{a}_3\mathbf{b}_3 & \mathbf{a}_3\mathbf{b}_4 \end{bmatrix} \cdots (4).$$

$$\mathbf{A}=\begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix}, \quad \mathbf{B}=\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{bmatrix}$$

Parallel processing of products on the computer is facilitated by a variant of (3) for computing C=AB, which is used by standard algorithms (such as in Lapack). In this method, A is used as given, B is taken in terms of its column vectors, and the product is computed columnwise; thus,

$$\mathbf{AB}=\mathbf{A}\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix}=\begin{bmatrix} \mathbf{Ab}_1 & \mathbf{Ab}_2 & \cdots & \mathbf{Ab}_p \end{bmatrix} \cdots (5).$$

Properties of Matrix Multiplication

Ex 5) Product in Terms of Row and Column Vectors

$$\mathbf{AB} = \begin{bmatrix} \mathbf{a}_1\mathbf{b}_1 & \mathbf{a}_1\mathbf{b}_2 & \mathbf{a}_1\mathbf{b}_3 & \mathbf{a}_1\mathbf{b}_4 \\ \mathbf{a}_2\mathbf{b}_1 & \mathbf{a}_2\mathbf{b}_2 & \mathbf{a}_2\mathbf{b}_3 & \mathbf{a}_2\mathbf{b}_4 \\ \mathbf{a}_3\mathbf{b}_1 & \mathbf{a}_3\mathbf{b}_2 & \mathbf{a}_3\mathbf{b}_3 & \mathbf{a}_3\mathbf{b}_4 \end{bmatrix} \cdots (4).$$

$$\mathbf{AB} = \mathbf{A}\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} \mathbf{Ab}_1 & \mathbf{Ab}_2 & \cdots & \mathbf{Ab}_p \end{bmatrix} \cdots (5).$$

Columns of $\mathbf{B}$ are then assigned to different processors (individually or several to each processor), which simultaneously compute the columns of the product matrix $\mathbf{Ab}_1$, $\mathbf{Ab}_2$, etc.

Properties of Matrix Multiplication

Ex 6) Computing Products Columnwise by (5)

$$\mathbf{AB} = \mathbf{A}\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p\end{bmatrix} = \begin{bmatrix}\mathbf{Ab}_1 & \mathbf{Ab}_2 & \cdots & \mathbf{Ab}_p\end{bmatrix} \cdots (5).$$

$$\mathbf{AB} = \begin{bmatrix}4 & 1\\ -5 & 2\end{bmatrix}\begin{bmatrix}3 & 0 & 7\\ -1 & 4 & 6\end{bmatrix} = \begin{bmatrix}11 & 4 & 34\\ -17 & 8 & -23\end{bmatrix}$$

To obtain AB from (5), calculate the columns

$$\begin{bmatrix}4 & 1\\ -5 & 2\end{bmatrix}\begin{bmatrix}3\\ -1\end{bmatrix} = \begin{bmatrix}11\\ -17\end{bmatrix}, \quad \begin{bmatrix}4 & 1\\ -5 & 2\end{bmatrix}\begin{bmatrix}0\\ 4\end{bmatrix} = \begin{bmatrix}4\\ 8\end{bmatrix}, \quad \begin{bmatrix}4 & 1\\ -5 & 2\end{bmatrix}\begin{bmatrix}7\\ 6\end{bmatrix} = \begin{bmatrix}34\\ -23\end{bmatrix}$$

of AB and then write them as a single matrix, as shown in the fist formula on the right.

Applications of Matrix Multiplication

Ex 11) Computer Production. Matrix Times Matrix

Supercomp Ltd produces two computer models PC1086 and PC1186. The matrix A show the cost per computer (in thousands of dollars) and B the production figures for the year 2005 (in multiples of 1000 units). Find a matrix C that shows the shareholders the cost per quarter (in millions of dollars) for raw material, labor, and miscellaneous.

$$\mathbf{A} = \begin{matrix} \text{PC1086} & \text{PC1186} \\ \begin{bmatrix} 1.2 & 1.6 \\ 0.3 & 0.4 \\ 0.5 & 0.5 \end{bmatrix} & \begin{matrix} \text{Raw Components} \\ \text{Labor} \\ \text{Miscellaneous} \end{matrix} \end{matrix}$$

$$\mathbf{B} = \begin{matrix} \text{Quarter} \\ 1 \quad 2 \quad 3 \quad 4 \\ \begin{bmatrix} 3 & 8 & 6 & 9 \\ 6 & 2 & 4 & 3 \end{bmatrix} \end{matrix} \begin{matrix} \\ \text{PC1086} \\ \text{PC1186} \end{matrix}$$

Applications of Matrix Multiplication

Ex 11) Computer Production. Matrix Times Matrix

$$\mathbf{A} = \begin{matrix} \text{PC1086} & \text{PC1186} \\ \begin{bmatrix} 1.2 & 1.6 \\ 0.3 & 0.4 \\ 0.5 & 0.5 \end{bmatrix} \end{matrix} \begin{matrix} \\ \text{Raw Components} \\ \text{Labor} \\ \text{Miscellaneous} \end{matrix}$$

$$\mathbf{B} = \begin{matrix} & \text{Quarter} & & & \\ 1 & 2 & 3 & 4 & \\ \begin{bmatrix} 3 & 8 & 6 & 9 \\ 6 & 2 & 4 & 3 \end{bmatrix} & & & & \begin{matrix} \text{PC1086} \\ \text{PC1186} \end{matrix} \end{matrix}$$

$$\mathbf{C} = \mathbf{AB} = \begin{matrix} & \text{Quarter} & & & \\ 1 & 2 & 3 & 4 & \\ \begin{bmatrix} 13.2 & 12.8 & 13.6 & 15.6 \\ 3.3 & 3.2 & 3.4 & 3.9 \\ 5.1 & 5.2 & 5.4 & 6.3 \end{bmatrix} & & & & \begin{matrix} \text{Raw Components} \\ \text{Labor} \\ \text{Miscellaneous} \end{matrix} \end{matrix}$$

Applications of Matrix Multiplication

Ex 12) Weight Watching. Matrix Times Vector

Suppose that in a weight-watching program, a person of 185lb burns 350 cal/hr in walking (3 mph) , 500 in bicycling (13 mph) and 950 in jogging (5.5 mph) . Bill, weighing 185 lb, plans to exercise according to the matrix shown. Verify the calculations (W = Walking, B = Bicycling, J=Jogging).

$$\begin{array}{c} \\ \text{MON} \\ \text{WED} \\ \text{FRI} \\ \text{SAT} \end{array} \begin{array}{c} \begin{array}{ccc} \text{W} & \text{B} & \text{J} \end{array} \\ \begin{bmatrix} 1.0 & 0 & 0.5 \\ 1.0 & 1.0 & 0.5 \\ 1.5 & 0 & 0.5 \\ 2.0 & 1.5 & 1.0 \end{bmatrix} \end{array} \begin{bmatrix} 350 \\ 500 \\ 950 \end{bmatrix} = \begin{bmatrix} 825 \\ 1325 \\ 1000 \\ 2400 \end{bmatrix} \begin{array}{l} \\ \text{MON} \\ \text{WED} \\ \text{FRI} \\ \text{SAT} \end{array}$$

Applications of Matrix Multiplication

Ex 13) Markov Process. Powers of a Matrix. Stochastic Matrix

Suppose that the 2004 state of land use in a city of 60 mi^2 of built-up area is
C : Commercially Used 25 % ,I : Industrially Used 20 %
R : Residentially Used 55 %.
Find the states 2009, 2014, 2019, assuming that the transition probabilities for 5-year intervals are given by the matrix A and remain practically the same over the time considered.

$$\mathbf{A}: \quad \begin{matrix} \\ \text{To C} \\ \text{To I} \\ \text{To R} \end{matrix} \begin{matrix} \begin{matrix} \text{From C} & \text{From I} & \text{From R} \end{matrix} \\ \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix} \end{matrix}$$

Applications of Matrix Multiplication

Ex 13) Markov Process. Powers of a Matrix. Stochastic Matrix

A is a stochastic matrix(확률행렬), that is, a square matrix with all entries nonnegative and all column sums equal to 1. Our example concerns a Markov process, that is, a process for which the probability of entering a certain state depends only on the last state occupied (and the matrix A), not on any earlier state.

$$\mathbf{A}: \begin{array}{c} \\ \text{To C} \\ \text{To I} \\ \text{To R} \end{array} \begin{array}{c} \begin{array}{ccc} \text{From C} & \text{From I} & \text{From R} \end{array} \\ \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix} \end{array}$$

solution.

area

$$\begin{array}{c} \text{C} \\ \text{I} \\ \text{R} \end{array} \begin{bmatrix} 0.7\cdot 25+0.1\cdot 20+0\cdot 55 \\ 0.2\cdot 25+0.9\cdot 20+0.2\cdot 55 \\ 0.1\cdot 25+0\cdot 20+0.8\cdot 55 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix} \begin{bmatrix} 25 \\ 20 \\ 55 \end{bmatrix} = \begin{bmatrix} 19.5 \\ 34.0 \\ 46.5 \end{bmatrix}.$$

Applications of Matrix Multiplication

$$A: \begin{array}{c} \\ \text{To C} \\ \text{To I} \\ \text{To R} \end{array} \begin{array}{c} \begin{array}{ccc} \text{From C} & \text{From I} & \text{From R} \end{array} \\ \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix} \end{array}$$

Ex 13) Markov Process. Powers of a Matrix. Stochastic Matrix

area

$$\begin{array}{c} \\ \text{C} \\ \text{I} \\ \text{R} \end{array} \begin{bmatrix} 0.7\cdot 25+0.1\cdot 20+ & 0\cdot 55 \\ 0.2\cdot 25+0.9\cdot 20+0.2\cdot 55 \\ 0.1\cdot 25+ & 0\cdot 20+0.8\cdot 55 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix} \begin{bmatrix} 25 \\ 20 \\ 55 \end{bmatrix} = \begin{bmatrix} 19.5 \\ 34.0 \\ 46.5 \end{bmatrix}.$$

We see that the 2009 state vector (y) is the column vector

$$\mathbf{y} = \begin{bmatrix} 19.5 \\ 34.0 \\ 46.5 \end{bmatrix} = \mathbf{Ax} = \mathbf{A} \begin{bmatrix} 25 \\ 20 \\ 55 \end{bmatrix}$$

The vector (x) is the given 2004 state vector

The sum of the entries of y is 100%.

Similarly, you may verify that for 2014 and 2019 we get the state vectors.

$$\mathbf{z} = \mathbf{Ay} = \mathbf{A}(\mathbf{Ax}) = \mathbf{A}^2\mathbf{x} = \begin{bmatrix} 17.05 & 43.80 & 39.15 \end{bmatrix}^T$$

$$\mathbf{u} = \mathbf{Az} = \mathbf{A}(\mathbf{Ay}) = \mathbf{A}^3\mathbf{x} = \begin{bmatrix} 16.315 & 50.660 & 33.025 \end{bmatrix}^T$$

Transposition

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Transposition of Matrices and Vectors

The transposition of an m x n matrix $\mathbf{A}=[a_{jk}]$ is the n x m matrix $\mathbf{A}^T$ (read A transpose) that has the first row of A as its first column, the second row of A as its second column, and so on. Thus the transpose of A in (2) is $\mathbf{A}^T=[a_{kj}]$, written out

As a special case, transposition converts row vectors to column vectors and conversely

$$\mathbf{A}^{\mathrm{T}} = [a_{kj}] \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdot & \cdot & \cdots & \cdot \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \cdots (9)$$

Transposition

Ex 7) Transposition of Matrices and Vectors

$$\text{If } \mathbf{A} = \begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix}, \text{ then } \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}$$

Comparing this A little more compactly, we can write

$$\begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 3 & 8 \\ 0 & -1 \end{bmatrix},$$

$$\begin{bmatrix} 6 & 2 & 3 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}, \qquad \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 6 & 2 & 3 \end{bmatrix}$$

Rules for transposition

$$
\begin{cases}
(a) & (\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A} \\
(b) & (\mathbf{A}+\mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}} \\
(c) & (c\mathbf{A}^{\mathrm{T}}) = c\mathbf{A}^{\mathrm{T}} \\
(d) & (\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}.
\end{cases}
$$

Caution! **Note that in (d) the transposed matrices are in reversed order.**

Special Matrices

Symmetric matrix

$$\mathbf{A}^{\mathrm{T}} = \mathbf{A}, \ (\text{thus } a_{kj} = a_{jk})$$

Skew-Symmetric matrix

$$\mathbf{A}^{\mathrm{T}} = -\mathbf{A}, \ (\text{thus } a_{kj} = -a_{jk}, \text{hence } a_{jj} = 0).$$

Ex 8) Symmetric and Skew-Symmetric Matrices

$$\mathbf{A} = \begin{bmatrix} 20 & 120 & 200 \\ 120 & 10 & 150 \\ 200 & 150 & 30 \end{bmatrix}$$

Symmetric matrix

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$$

Skew-Symmetric matrix

Triangular Matrices

Ex 9) Upper and Lower Triangular Matrices

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix},$$

Upper Triangular Matrices

$$\begin{bmatrix} 2 & 0 & 0 \\ -8 & -1 & 0 \\ 7 & 6 & 8 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 9 & -3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 9 & 3 & 6 \end{bmatrix}.$$

Lower Triangular Matrices

Upper triangular matrices are square matrices that can have nonzero entries only on and above the main diagonal, whereas any entry below the diagonal must be zero. Similarly, lower triangular matrices can have nonzero entries only on and below the main diagonal. Any entry on the main diagonal of a triangular matrix may be zero or not.

Diagonal Matrices

Ex 10) Diagonal Matrix D. Scalar Matrix S. Unit Matrix I.

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Diagonal Matrices are square matrices that can have nonzero entries only on the main diagonal. Any entry above or below the main diagonal must be zero.

Diagonal Matrices

$$\mathbf{D}=\begin{bmatrix}2 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & 0\end{bmatrix},\qquad \mathbf{S}=\begin{bmatrix}c & 0 & 0\\ 0 & c & 0\\ 0 & 0 & c\end{bmatrix},\qquad \mathbf{I}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

We call S a scalar matrix because multiplication of any square matrix A of the same size by S has the same effect as the multiplication b`y a scalar, that is,

$$\mathbf{AS}=\mathbf{SA}=c\mathbf{A}\cdots(12).$$

In particular, a scalar matrix whose entries on the main diagonal are all 1 is called a unit matrix (or identity matrix) and is denoted by $\mathbf{I}_n$ or simply by I. for I, formula (12) becomes

$$\mathbf{AI}=\mathbf{IA}=\mathbf{A}\cdots(13).$$

Linear System, Coefficient Matrix, Augmented Matrix

A linear system of m equations in n unknowns $x_1, \cdots, x_n$ is a set of equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \quad \cdots (1)$$

given :

$a_{11}, \cdots a_{1n}$

$b_1, \cdots b_m$

coefficients

If $b_1 = \cdots = b_m = 0$, homogeneous system,
else nonhomogeneous system

The system is called linear because each variable x_j appears in the first power only, just as in the equation of a straight line.

Linear System, Coefficient Matrix, Augmented Matrix

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \quad \cdots(1)$$

A solution of (1) is a set of numbers $x_1, \cdots, x_n$ that satisfies all the m equations.

A solution vector of (1) is a vector $\mathbf{x}$ whose components form a solution of (1). If the system (1) is homogeneous, it has at least the trivial solution $x_1=0, \cdots, x_n=0$.

Matrix Form of the Linear system (1). From the definition of matrix multiplication we see that the m equations of (1) may be written as a single vector equation.

$$\mathbf{Ax} = \mathbf{b} \cdots (2)$$

Linear System, Coefficient Matrix, Augmented Matrix

$$\mathbf{Ax} = \mathbf{b} \cdots (2)$$

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{aligned}$$

where the coefficient matrix A=[a_{jk}] is the *m* x *n* matrix

$$\mathbf{A} = \overbrace{\begin{matrix} m \left\{ \right. \end{matrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}^{n}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix},$$

are column vectors. We assume that A is not a zero matrix.
Note that x has *n* components, whereas b has *m* components.

Linear System, Coefficient Matrix, Augmented Matrix

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{aligned}
\qquad \cdots(1)
$$

The matrix

$$
\tilde{\mathbf{A}} = \left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & \cdot \\
\cdot & \cdot & \cdots & \cdot & \cdot \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{array}\right]
$$

is called the augmented matrix(첨가행렬) of the system (1). The dashed vertical line could be omitted (as we shall do later); it is merely a reminder that the last column of $\tilde{\mathbf{A}}$ does not belong to A

The augmented matrix $\tilde{\mathbf{A}}$ determines the system (1) completely because it contains all the given numbers appearing in (1).

Linear System, Coefficient Matrix, Augmented Matrix

Ex 1) Geometric Interpretation. Existence and Uniqueness of Solutions

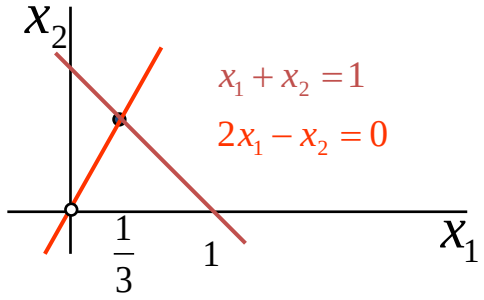
If $m = n = 2$, we have two equations in two unknowns x_1, x_2

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

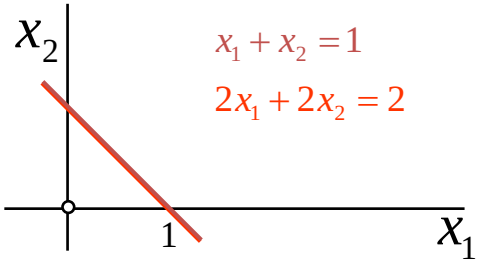
If we interpret x_1, x_2 as coordinate in the $x_1\,x_2$ -plane, then each of the two equations represents a straight line, and (x_1, x_2) is a solution if and only if the point P with coordinates x_1, x_2 lies on both lines. Hence there are three possible cases :

If the system is homogenous case (c) cannot happen

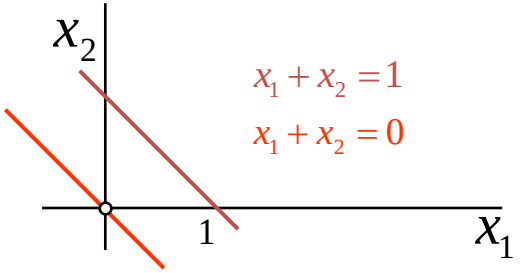
(a) precisely one solution if the lines intersect



(b) Infinitely many solutions if the lines coincide



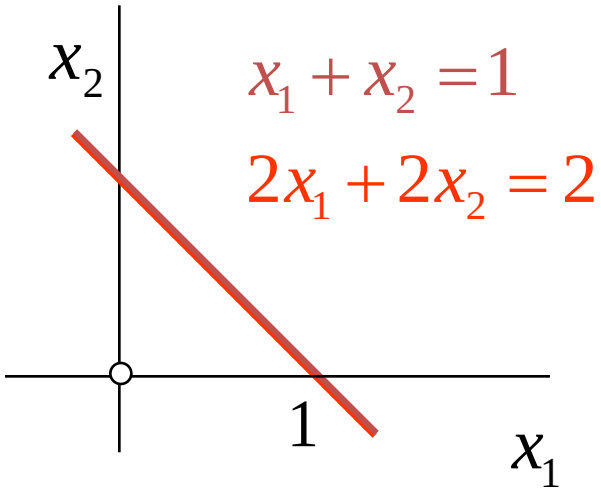
(c) No solution if the lines are parallel



Linear System, Coefficient Matrix, Augmented Matrix

Ex 1) Geometric Interpretation. Existence and Uniqueness of Solutions

(b) Infinitely many solutions if the lines coincide



Changing the form of the equation as

$$x_2 = 1 - x_1$$

if we choose a value of x_1, then the corresponding values of x_2 is uniquely determined.

x_1 : independent variable or design variable,
x_2 : dependent variable.

There would be many solutions, so we choose the best solution to obtain proper result.

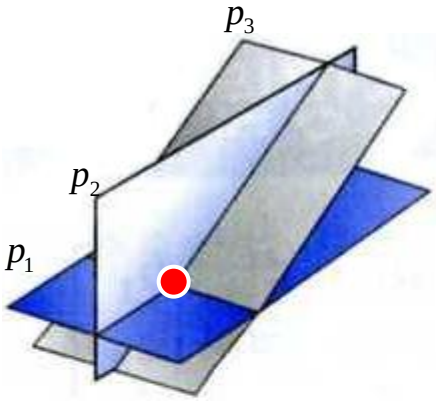
The problems like this is called optimization problem.

Linear System, Coefficient Matrix, Augmented Matrix

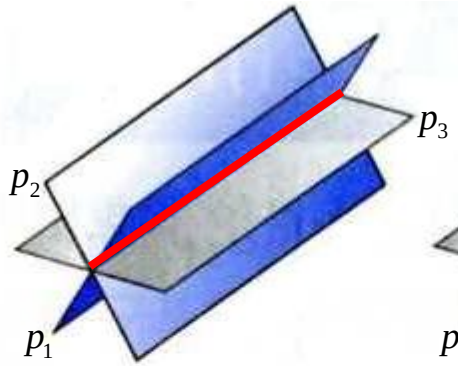
Ex 1) Geometric Interpretation. Existence and Uniqueness of Solutions

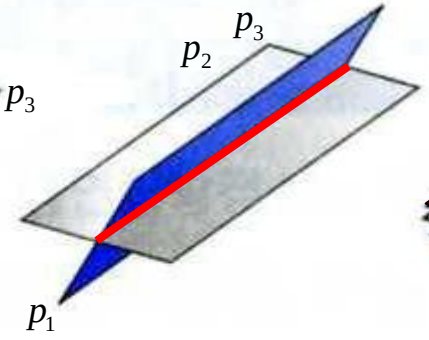
Fig. 156. Three equations in three unknowns interpreted as planes in space

$$p_1: a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$p_2: a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$p_3: a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

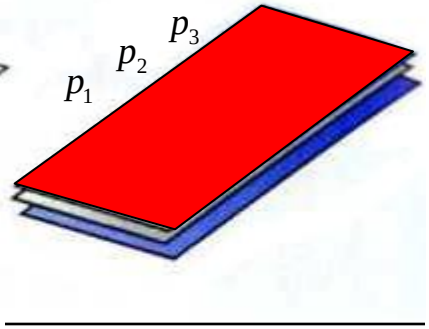


Unique solution

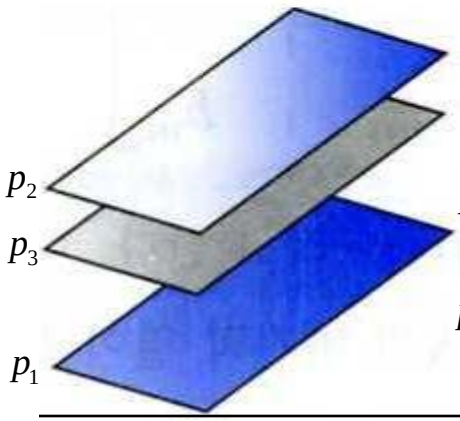


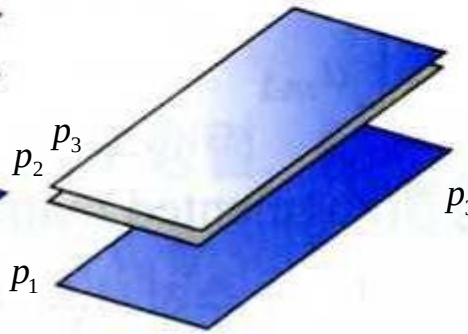


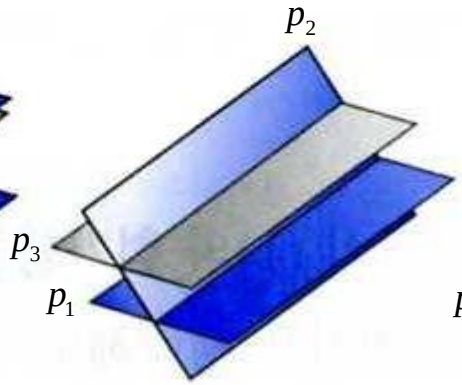
Infinitely many solutions

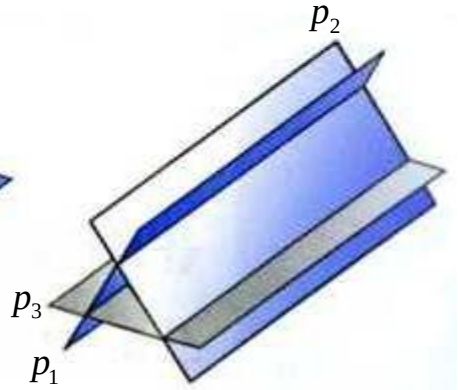


No solution









No solution

7-2. Column Picture and Linear Equations*

*Strang G., Introduction to Linear Algebra, Third edition, Wellesley-Cambridge Press, 2003, Ch.2.1, p21

Vectors and Linear Equations

The central problem of linear algebra is to solve a system of equations. Those equations are linear, which means that the unknowns are only multiplied by numbers - we never see x times y.

First example)

$$x - 2y = 1$$

$$3x + 2y = 11$$

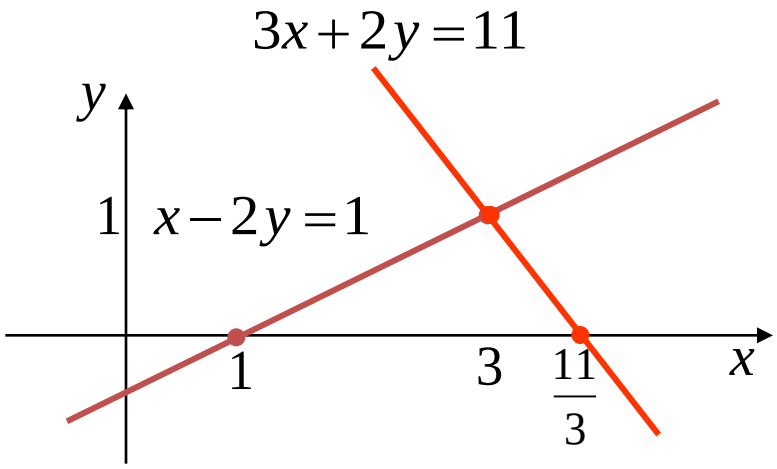


Figure 2.1 Row picture : The point (3, 1) where the lines meet is the solution

- solution of first equation
- solution of second equation

Slopes are important in calculus and this is linear algebra.

You can't miss the intersection point where the two lines meet. The point $x = 3, y = 1$ lies on both lines. That point solves both equations at once. This is the solution to our system of linear equation.

The row picture show two lines meeting at a single point.

Vectors and Linear Equations

I want to recognize the linear system as a "vector equation". Instead of numbers we need to see vectors. If you separate the original system into its columns instead of its rows, you get

$$x\begin{bmatrix}1\\3\end{bmatrix}+y\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}=\mathbf{b}$$

This has two column vectors on the left side. The problem is to find the combination of those vectors that equals the vector on the right.

We are multiplying the first column by x and the second column by y, and adding. With the right choices $x = 3, y = 1$, this produces 3 (column 1) + 1 (column 2) = b.

The column picture combines the column vectors on the left side to produce the vector b on the right side.

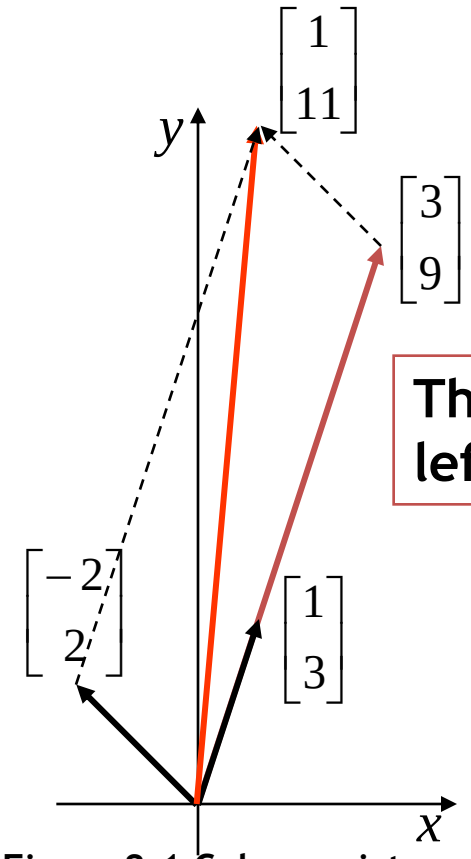
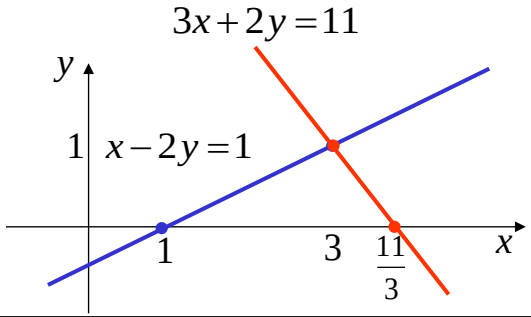


Figure 2.1 Column picture : A combination of columns produces the right side (1, 11).

비교



$$x-2y=1$$

$$3x+2y=11$$

Vectors and Linear Equations

$$x - 2y = 1$$
$$3x + 2y = 11$$

$$x\begin{bmatrix}1\\3\end{bmatrix} + y\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix} = \mathbf{b} \qquad \therefore x = 3,\ y = 1$$

The left side of the vector equation is a linear combination of the columns. The problem is to find the right coefficients $x = 3$ and $y = 1$. We are combining scalar multiplication and vector addition into one step. That step is crucially important, because it contains both of basic operations :

$$\textbf{Linear combination}\quad 3\begin{bmatrix}1\\3\end{bmatrix} + 1\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix}$$

Vectors and Linear Equations

The coefficient matrix on the left side of the equation is the 2 by 2 matrix $\mathbf{A}$:

$$\textbf{Coefficient matrix } \mathbf{A} = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$$

This is very typical of linear algebra, to look at a matrix by rows and by columns. Its rows give the row picture and its columns give the column picture. Same numbers, different pictures, same equations. We write those equations as a matrix problem $\mathbf{Ax} = \mathbf{b}$.

$$\textbf{Matrix equation } \mathbf{Ax} = \mathbf{b}: \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

The row picture deals with the two rows of $\mathbf{A}$. The column picture combines the columns. The numbers $x = 3$ and $y = 1$ go into the solution vector $\mathbf{x}$.

Vectors and Linear Equations
- Three Equations in Three Unknowns

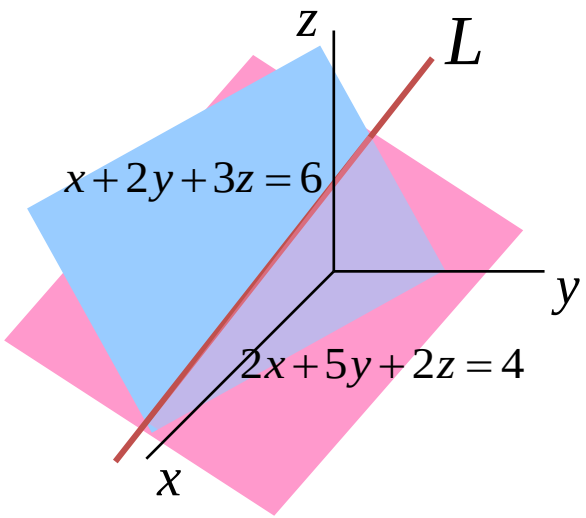
The three unknowns x, y, z. The linear equations $\mathbf{Ax} = \mathbf{b}$ are

$$x + 2y + 3z = 6$$

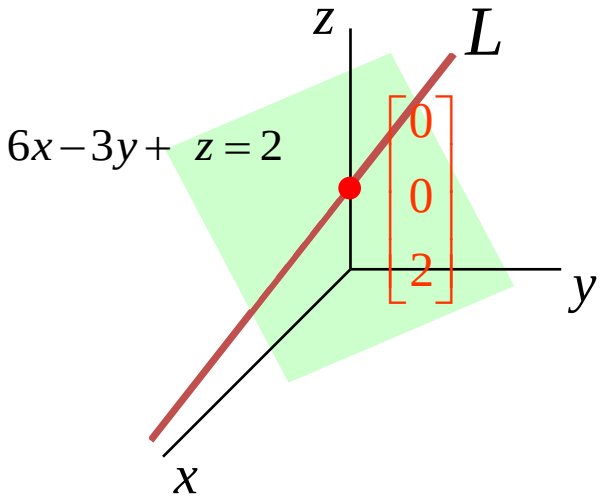
$$2x + 5y + 2z = 4$$

$$6x - 3y + z = 2$$

The row picture show three planes meeting at a single point.



The usual result of two equations in three unknowns is a intersect line L of solutions.



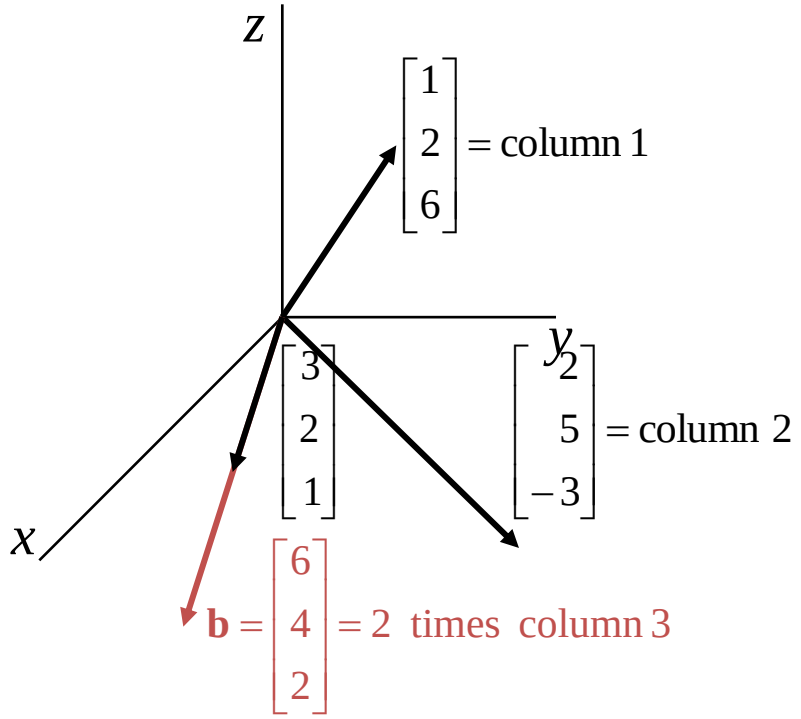
The third equation gives a third plane. It cuts the line L at a single point. That point lies on all three planes and it solves all three equations.

Vectors and Linear Equations - Three Equations in Three Unknowns

$$x+2y+3z=6$$
$$2x+5y+2z=4$$
$$6x-3y+z=2$$

The column picture starts with the vector form of the equations :

$$x\begin{bmatrix}1\\2\\6\end{bmatrix}+y\begin{bmatrix}2\\5\\-3\end{bmatrix}+z\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}6\\4\\2\end{bmatrix}$$



C The column picture combines three columns to produce the vector (6,4,2)

Figure 2.4 Column picture : $(x, y, z) = (0, 0, 2)$ because $2(3, 2, 1) = (6, 4, 2) = \mathbf{b}$.

The coefficient we need are $x = 0$, $y = 0$ and $z = 2$. This is also the intersection point of the three planes in the row picture.

Vectors and Linear Equations
- The Matrix Form of the Equations

$$\begin{aligned} x+2y+3z &= 6 \\ 2x+5y+2z &= 4 \\ 6x-3y+\ z &= 2 \end{aligned} \qquad (3)$$

Matrix equation :

$$\mathbf{Ax} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = \mathbf{b}$$

Coefficient matrix — unknown vector

We multiply the matrix $\mathbf{A}$ times the unknown vector $\mathbf{x}$ to get the right side $\mathbf{b}$.

Multiplication by rows : $\mathbf{Ax}$ comes from dot products, each row times the column $\mathbf{x}$:

$$\mathbf{Ax} = \begin{bmatrix} (\mathbf{row\ 1}) \bullet \mathbf{x} \\ (\mathbf{row\ 2}) \bullet \mathbf{x} \\ (\mathbf{row\ 3}) \bullet \mathbf{x} \end{bmatrix}.$$

Multiplication by columns : $\mathbf{Ax}$ is a combination of column vectors :

$$\mathbf{Ax} = x(\mathbf{column\ 1}) + y(\mathbf{column\ 2}) + z(\mathbf{column\ 3})$$

Vectors and Linear Equations
- The Matrix Form of the Equations

$$\mathbf{Ax} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = \mathbf{b}$$

$$\mathbf{Ax} = x(\mathbf{column\ 1}) + y(\mathbf{column\ 2}) + z(\mathbf{column\ 3})$$

When we substitute the solution $\mathbf{x} = (0, 0, 2)$, the multiplication $\mathbf{Ax}$ produces $\mathbf{b}$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = 2 \text{ times column } 3 = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}.$$

The first dot product in row multiplication is $(1, 2, 3) \bullet (0, 0, 2) = 6$. The other dot products are 4 and 2. Multiplication by columns is simply 2 times column 3.

$\mathbf{Ax}$ as a combination of the columns of $\mathbf{A}$.

7-3. Row-Equivalent Operation, Rank, Determinant

Elementary Row Operations. Row-Equivalent System

Elementary *Row* Operations for Matrices :

1) ***Interchange of two rows***

2) ***Addition of a constant multiple of one row to another row***

3) ***Multiplication of a row by a nonzero constant*** c

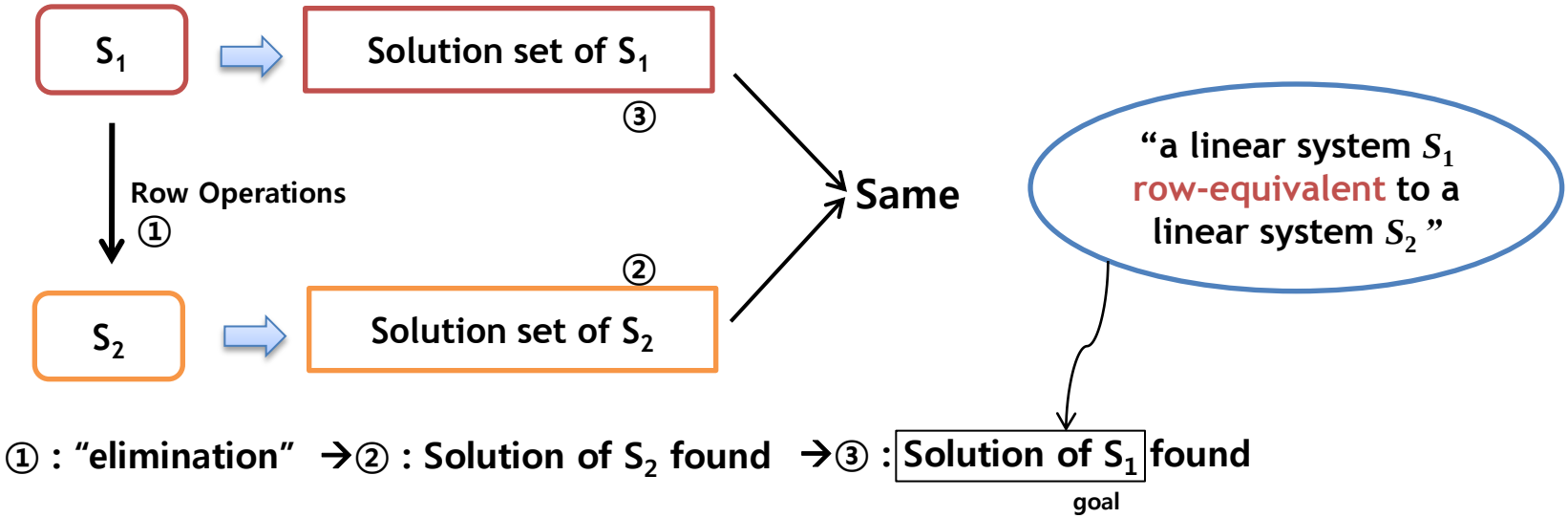
Elementary Operations for Equations :

1) ***Interchange of two equations***

2) ***Addition of a constant multiple of one equation to another equation***

3) ***Multiplication of an equation by a nonzero constant*** c

Elementary Row Operations. Row-Equivalent (행동치) System



Clearly, the interchange of two equations does not alter the solution set. Neither does that addition because we can undo it by a corresponding subtraction. Similarly for that multiplication, which we can undo by multiplying the new equation by $1/c$ (since $c \neq 0$), producing the original equation.

We now call a linear system S_1 row-equivalent to a linear system S_2 if S_1 can be obtained from S_2 by (finitely many!) row operations. Thus we have proved the following result, which also justifies the Gauss elimination.

Because of this theorem, systems having the same solution sets are often called equivalent systems. But note well that we are dealing with row operations.

No column operations on the augmented matrix are permitted in this context because they would generally alter the solution set.

GAUSS ELIMINATION

Gauss Elimination and Back Substitution

This is a standard elimination method for solving linear systems that proceeds systematically irrespective of particular features of the coefficients.

If a system is in "triangular form" we can solve it by "back substitution".

Triangular Matrices

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ -8 & -1 & 0 \\ 7 & 6 & 8 \end{bmatrix}$$

Gauss Elimination and Back Substitution

$$x_1 + 2x_2 + x_3 = 1$$
$$3x_1 - x_2 - x_3 = 2$$
$$2x_1 + 3x_2 - x_3 = -3$$

row2 + row1x(−3)

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

row2 + row1x(−3)

$$3x_1 - x_2 - x_3 = 2$$
$$+) \quad -3x_1 - 6x_2 - 3x_3 = -3$$
$$-7x_2 - 4x_3 = -1$$

$$x_1 + 2x_2 + x_3 = 1$$
$$-7x_2 - 4x_3 = -1$$
$$2x_1 + 3x_2 - x_3 = -3$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -4 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$$

Gauss Elimination and Back Substitution

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 - x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 - x_3 &= -3 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ -7x_2 - 4x_3 &= -1 \\ 2x_1 + 3x_2 - x_3 &= -3 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -4 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$$

row3 + row1x(−2)

row3 + row1x(−2)

$$\begin{aligned} & 2x_1 + 3x_2 - x_3 = -3 \\ +) \quad & -2x_1 - 4x_2 - 2x_3 = -2 \\ \hline & -x_2 - 3x_3 = -5 \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ -7x_2 - 4x_3 &= -1 \\ -x_2 - 3x_3 &= -5 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -4 \\ 0 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -5 \end{bmatrix}$$

Gauss Elimination and Back Substitution

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 - x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 - x_3 &= -3 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ -7x_2 - 4x_3 &= -1 \\ -x_2 - 3x_3 &= -5 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -4 \\ 0 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -5 \end{bmatrix}$$

row 2 ↔ row 3

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ -x_2 - 3x_3 &= -5 \\ -7x_2 - 4x_3 &= -1 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & -7 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -1 \end{bmatrix}$$

row 2x(−1)

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ x_2 + 3x_3 &= 5 \\ -7x_2 - 4x_3 &= -1 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -7 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$$

Gauss Elimination and Back Substitution

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 - x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 - x_3 &= -3 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ x_2 + 3x_3 &= 5 \\ -7x_2 - 4x_3 &= -1 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -7 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$$

row 3+ row 2x7

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ x_2 + 3x_3 &= 5 \\ 17x_3 &= 34 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 34 \end{bmatrix}$$

The last equations and matrix are equal to given equations.

Gauss Elimination and Back Substitution

$$x_1 + 2x_2 + x_3 = 1$$
$$x_2 + 3x_3 = 5$$
$$17x_3 = 34$$

x_1, x_2, x_3

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 34 \end{bmatrix}$$

$$x_3 = \frac{34}{17} = 2$$

$$x_2 + 3x_3 = x_2 + 3 \cdot 2 = 5$$

$$\therefore x_2 = -1$$

$$x_1 + 2x_2 + x_3 = x_1 + 2 \cdot (-1) + 2 = 1$$

$$\therefore x_1 = 1$$

We can solve this by "Back substitution", that is, solve the last equation for the variable, and then work backward, substituting the value of the variable into the above equation and solve it for another variable.

Gauss Elimination and Back Substitution

Since a linear system is completely determined by its augmented matrix, Gauss elimination can be done by merely considering the matrices.

augmented matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 3 & -1 & -1 & 2 \\ 2 & 3 & -1 & -3 \end{array}\right]$$

$$\vdots \qquad\qquad \vdots$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 34 \end{bmatrix} \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 17 & 34 \end{array}\right]$$

ROW-ECHELON FORM

Gauss Elimination : The Three Possible Cases of Systems

case 1 : Gauss Elimination if Infinitely Many Solutions Exist

three equations < four unknowns

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]$$

$$\begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 0.6x_1 + 1.5x_2 + 1.5x_3 - 5.4x_4 &= 2.7 \\ 1.2x_1 - 0.3x_2 + 0.3x_3 + 2.4x_4 &= 2.1 \end{aligned}$$

⇩

Row2-0.2*Row1

Row3-0.4*Row1

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{array}\right]$$

$$\begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 1.1x_2 + 1.1x_3 - 4.4x_4 &= 1.1 \\ -1.1x_2 - 1.1x_3 + 4.4x_4 &= -1.1 \end{aligned}$$

⇩

Row3+Row2

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 1.1x_2 + 1.1x_3 - 4.4x_4 &= 1.1 \\ 0 &= 0 \end{aligned}$$

Gauss Elimination : The Three Possible Cases of Systems

case 1 : Gauss Elimination if Infinitely Many Solutions Exist

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 = 8.0$$
$$1.1x_2 + 1.1x_3 - 4.4x_4 = 1.1$$
$$0 = 0$$

Back substitution.

$$\text{From the second equation} : x_2 = 1 - x_3 + 4x_4$$
$$\text{From the first equation} : x_1 = 1 - x_4$$

Since x_3 and x_4 remain arbitrary, we have infinitely many solutions.

If we choose a value of x_3 and a value of x_4, then the corresponding values of x_1 and x_2 are uniquely determined.

Gauss Elimination : The Three Possible Cases of Systems

case 2 : Gauss Elimination if no Solution Exists

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right] \qquad \begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ 2x_1 + x_2 + x_3 = 0 \\ 6x_1 + 2x_2 + 4x_3 = 6 \end{cases}$$

⇩

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & -2 & 2 & 0 \end{array}\right] \qquad \begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ -1/3x_2 + 1/3x_3 = -2 \\ -2x_2 + 4x_3 = 0 \end{cases}$$

Row2-2/3*Row1

Row3-2*Row1

⇩

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & 0 & 0 & 12 \end{array}\right] \qquad \begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ -1/3x_2 + 1/3x_3 = -2 \\ 0 = 12 \end{cases}$$

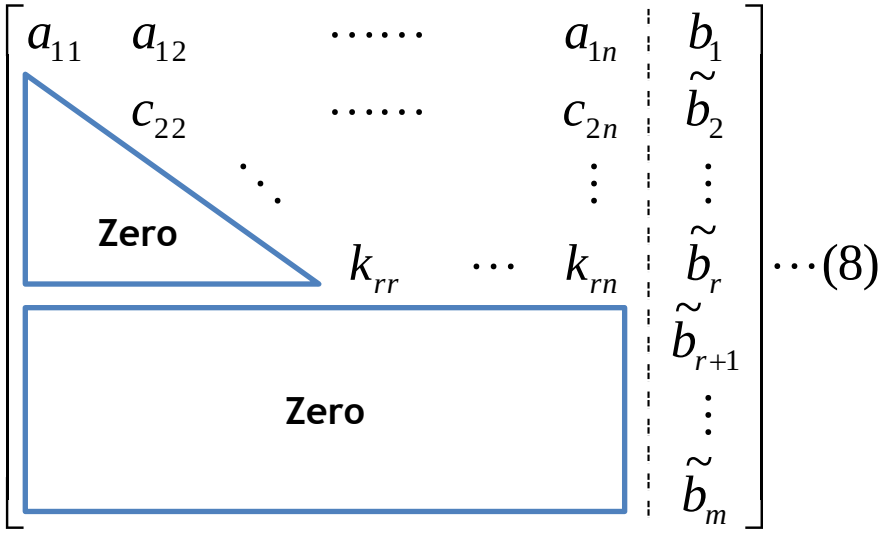
Row3-6*Row3

The false statement 0=12 show that the system has no solution.

Gauss Elimination : The Three Possible Cases of Systems

Row Echelon Form

At the end of the Gauss elimination (before the back substitution) the row-echelon form(행 사다리꼴) of the augmented matrix will be



$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots\cdots & a_{1n} & b_1 \\ & c_{22} & \cdots\cdots & c_{2n} & \tilde{b}_2 \\ & & \ddots & \vdots & \vdots \\ & & k_{rr} \cdots & k_{rn} & \tilde{b}_r \\ & & & & \tilde{b}_{r+1} \\ & & & & \vdots \\ & & & & \tilde{b}_m \end{array}\right] \cdots (8)$$

Here, $r \le m$ and $a_{11} \neq 0$, $c_{22} \neq 0$, ⋯, $k_{rr} \neq 0$, and all the entries in the blue triangle as well as in the blue rectangle are zero. From this we see that with respect to solutions of the system with augmented matrix (8) (and thus with respect to the originally given system) there are three possible cases:

(r : no. of equations, n : no. of unknowns, m : no. of rows)

Gauss Elimination : The Three Possible Cases of Systems

Row Echelon Form

(r : no. of equations, n : no. of unknowns, m : no. of rows)

(a) Exactly one solution (no. of equations 'r' = no. of unknowns 'n')

if $r = n$ and $\tilde{b}_{r+1}, \cdots \tilde{b}_m$, if present, are zero. To get the solution, solve the nth equation corresponding to (8) (which is $k_{nn}x_n = b_n$) for x_n, then the $(n\text{-}1)$st equation for x_{n-1}, and so on up the line.

(b) Infinitely many solutions (no. of equations < no. of unknowns)

if $r < n$ and $\tilde{b}_{r+1}, \cdots \tilde{b}_m$, if present, are zero. To obtain any of these solutions, choose values of x_{r-1}, ⋯, x_n arbitrary . Then solve the rth equation for x_r, then the $(r\text{-}1)$st equation for x_{r-1}, and so on up the line.

(c) No solution (no. of equations < no. of rows)

if $r < m$ and one of the entries $\tilde{b}_{r+1}, \cdots \tilde{b}_m$ is not zero.

$$n=3,\quad r=3,\quad m=4:\quad \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & (0) \end{array}\right]$$

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ -x_1 + 11x_2 + 24x_3 = 90 \\ 3x_1 - 3x_2 - 92x_3 = -190 \\ 2x_1 - 2x_2 + 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 - x_2 + x_3 = 0 \\ 0\cdot x_1 + 10x_2 + 25x_3 = 90 \\ 0\cdot x_1 + 0\cdot x_2 - 95x_3 = -190 \\ 0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 = 0 \end{cases}$$

$$n=4,\quad r=2,\quad m=3:\quad \left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & (0) \end{array}\right]$$

$$\begin{cases} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 = 8.0 \\ -x_1 - 0.3x_2 - 0.3x_3 + 0.2x_4 = -2.3 \\ 1.5x_1 + 1.0x_2 + 1.0x_3 - 2.5x_4 = 4.0 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 = 8.0 \\ 0\cdot x_1 + 1.1x_2 + 1.1x_3 - 4.4x_4 = 1.1 \\ 0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 + 0\cdot x_4 = 0.0 \end{cases}$$

$$n=3,\quad r=2,\quad m=3:\quad \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & 0 & 0 & 12 \end{array}\right]$$

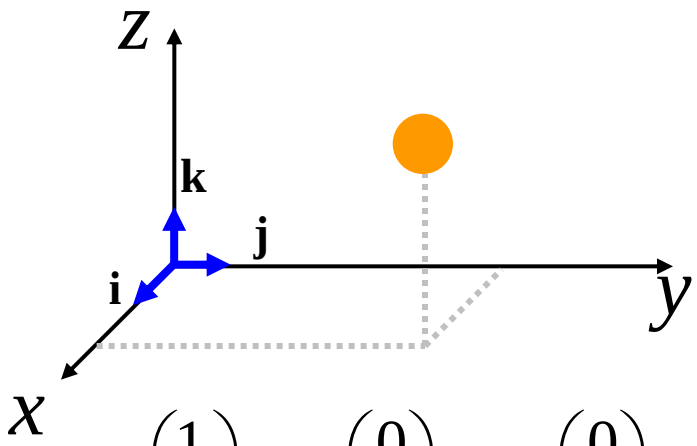
$$\begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ 9x_1 + 7x_2 + 2x_3 = 15 \\ 3x_1 + 2x_2 + x_3 = -9 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ -1/3x_2 + 1/3x_3 = -2 \\ 0 = 12 \end{cases}$$

Rank of a Matrix.
Linear Independence.

Linearly independent vectors



$$\mathbf{i}=\begin{pmatrix}1\\0\\0\end{pmatrix},\mathbf{j}=\begin{pmatrix}0\\1\\0\end{pmatrix},\mathbf{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

We can express the location of the point with i, j, k.

$$a\mathbf{i}+b\mathbf{j}+c\mathbf{k}=a\begin{pmatrix}1\\0\\0\end{pmatrix}+b\begin{pmatrix}0\\1\\0\end{pmatrix}+c\begin{pmatrix}0\\0\\1\end{pmatrix}$$

If the point is at the origin, the equation becomes

$$a\begin{bmatrix}1\\0\\0\end{bmatrix}+b\begin{bmatrix}0\\1\\0\end{bmatrix}+c\begin{bmatrix}0\\0\\1\end{bmatrix}=\mathbf{0}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

The equation above is satisfied if and only if $a=b=c=0$.

Then, i, j, k are linearly independent.

Linear Independence and Dependence of Vectors

Given any set of m vectors $\mathbf{a}_{(1)}, \cdots, \mathbf{a}_{(m)}$ (with the same number of components), a linear combination of these vectors is an expression of the form

$$c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)}$$

$c_1, c_2, \cdots, c_m$ are any scalars. Now consider the equation.

$$c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)} = \mathbf{0} \quad \cdots\cdots(1)$$

When $c_1 = c_2 = \cdots = c_m = \mathbf{0}$

Vector ⇕ 비교 Function

$\mathbf{a}_{(1)}, \mathbf{a}_{(2)}, \cdots, \mathbf{a}_{(m)}$ vectors **linearly independent set** or **linearly independent**.

Definition 3.1 Linear Dependence / Independence

A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is said to be 'linearly dependent' on an interval I if there exist constant $c_1, c_2, \ldots c_n$, not all zero such that $c_1 f_1(x) + c_2 f_2(x) + \cdots c_n f_n(x) = 0$ for every x in the interval.

If the set of functions is not linearly dependent on the interval, it is said to be 'linearly independent'

In other words, a set of functions is 'linearly independent' if the only constants for

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

are $c_1 = c_2 = \cdots = c_n = 0$

Linear Independence and Dependence of Vectors

$$c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)} = \mathbf{0} \quad \cdots\cdots(1)$$

If (1) also holds with scalars not all zero, we call these vectors linearly dependent, because then we can express (at least) one of them as a linear combination of the others. For instance, if (1) holds with, say, c_1=0, we can solve (1) for $\mathbf{a}_{(1)}$:

$$\mathbf{a}_{(1)} = k_2\mathbf{a}_{(2)} + \cdots + k_m\mathbf{a}_{(m)}, \quad (\text{where } k_j = -c_j / c_1)$$

(Some k_j's may be zero. Or even all of them, namely, if $\mathbf{a}_{(1)}$=0.)

Linear Independence and Dependence of Vectors

Ex 1) Linear Independence and Dependence

Vector

$$\mathbf{a}_{(1)} = [\,3,\ 0,\ 2,\ 2]$$
$$\mathbf{a}_{(2)} = [-6,\ 42,\ 24,\ 54]$$
$$\mathbf{a}_{(3)} = [21,\ -21,\ 0,\ -15]$$

$$6\mathbf{a}_{(1)} = [18,\ 0,\ 12,\ 12]$$
$$-\frac{1}{2}\mathbf{a}_{(2)} = [3,\ -21,\ -12,\ -27]$$
$$-\mathbf{a}_{(3)} = [-21,\ 21,\ 0,\ 15]$$

$$6\mathbf{a}_{(1)} - \frac{1}{2}\mathbf{a}_{(2)} - \mathbf{a}_{(3)} = [0,0,0,0]$$

The three vectors are linearly dependent

Linear Systems

$$\begin{cases} 3x_1 + 0\cdot x_2 + 2x_3 = 2 & ① \\ -6x_1 + 42x_2 + 24x_3 = 54 & ② \\ 21x_1 - 21x_2 + 0\cdot x_3 = -15 & ③ \end{cases}$$

⇩ ①x2+②

$$\begin{cases} 3x_1 + 0\cdot x_2 + 2x_3 = 2 \\ 0\cdot x_1 + 42x_2 + 28x_3 = 58 \\ 21x_1 - 21x_2 + 0\cdot x_3 = -15 \end{cases}$$

⇩ ①x(-7)+③

$$\begin{cases} 3x_1 + 0\cdot x_2 + 2x_3 = 2 \\ 0\cdot x_1 + 42x_2 + 28x_3 = 58 \\ 0\cdot x_1 - 21x_2 - 14x_3 = -29 \end{cases}$$

⇩ ②x(0.5)+③

$$\begin{cases} 3x_1 + 0\cdot x_2 + 2x_3 = 2 \\ 0\cdot x_1 + 42x_2 + 28x_3 = 58 \\ 0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 = 0 \end{cases}$$

The three equations are linearly dependent

Matrix

$$\left[\begin{array}{ccc|c} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{array}\right] \quad \left[\begin{array}{ccc} 3 & 0 & 2 \\ -6 & 42 & 24 \\ 21 & -21 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 21 & -21 & 0 & -15 \end{array}\right] \quad \left[\begin{array}{ccc} 3 & 0 & 2 \\ 0 & 42 & 28 \\ 21 & -21 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{array}\right] \quad \left[\begin{array}{ccc} 3 & 0 & 2 \\ 0 & 42 & 28 \\ 0 & -21 & -14 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{ccc} 3 & 0 & 2 \\ 0 & 42 & 28 \\ 0 & 0 & 0 \end{array}\right]$$

The three rows are linearly dependent

Rank of a Matrix

Rank of a Matrix

The rank of a matrix A

: "the maximum number of linearly independent row vectors" of A. rank A.

Ex 2) Rank

The matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix} \cdots (2)$$

has rank 2, because Example 1 shows that the first two row vectors are linearly independent, whereas all three row vectors are linearly dependent.

Note further that rank A=0 if and only if A=0 (zero matrix).

Rank of a Matrix

☑ Example 1

Rank of 3 x 4 Matrix

Consider the 3 x 4 matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 & 3 \\ 2 & -2 & 6 & 8 \\ 3 & 5 & -7 & 8 \end{pmatrix}.$$

With $\mathbf{u}_1$=(-1 1 -1 3), $\mathbf{u}_2$=(2 -2 6 8), and $\mathbf{u}_3$=(3 5 -7 8), we see that $4\mathbf{u}_1$-1/2$\mathbf{u}_2$+$\mathbf{u}_3$=0.
the set $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ is linearly dependent.

On the other hand, since neither $\mathbf{u}_1$ nor $\mathbf{u}_2$ is a constant multiple of the other set of row vectors $\mathbf{u}_1$, $\mathbf{u}_2$ is linearly independent. Hence by Definition, rank($\mathbf{A}$) = 2.

$$\begin{pmatrix} 1 & 1 & -1 & 3 \\ 2 & -2 & 6 & 8 \\ 3 & 5 & -7 & 8 \end{pmatrix} \qquad \begin{cases} x_1 + x_2 - x_3 = 3 \\ 2x_1 - 2x_2 + 6x_3 = 8 \\ 3x_1 + 5x_2 - 7x_3 = 8 \end{cases}$$

⇩ ①x(-2)+②

$$\begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -4 & 8 & 2 \\ 3 & 5 & -7 & 8 \end{pmatrix} \qquad \begin{cases} x_1 + x_2 - x_3 = 3 \\ 0 \cdot x_1 - 4x_2 + 8x_3 = 2 \\ 3x_1 + 5x_2 - 7x_3 = 8 \end{cases}$$

⇩ ①x(-3)+③

$$\begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -4 & 8 & 2 \\ 0 & 2 & -4 & -1 \end{pmatrix} \qquad \begin{cases} x_1 + x_2 - x_3 = 3 \\ 0 \cdot x_1 - 4x_2 + 8x_3 = 2 \\ 0 \cdot x_1 + 2x_2 - 4x_3 = -1 \end{cases}$$

⇩ ②x(0.5)+③

$$\begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -4 & 8 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{cases} x_1 + x_2 - x_3 = 3 \\ 0 \cdot x_1 - 4x_2 + 8x_3 = 2 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0 \end{cases}$$

rank($\mathbf{A}$) = 2

Rank and Row-Equivalent Matrices

Theorem : Row-Equivalent Matrices

Row-equivalent matrices have the same rank.

A_1 row-equivalent to a matrix A_2

→ rank is invariant under elementary row operations.

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 0 & 10 & 25 \\ 20 & 10 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 10 & 25 \\ 0 & 0 & -95 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 \\ 0.6 & 1.5 & 1.5 & -5.4 \\ 1.2 & -0.3 & -0.3 & 2.4 \end{bmatrix} \Rightarrow \begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 \\ 0 & 1.1 & 1.1 & -4.4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ 6 & 2 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 2 & 1 \\ 0 & -1/3 & 1/3 \\ 0 & 0 & 0 \end{bmatrix}$$

RANK AND LINEAR SYSTEM SOLUTIONS

Rank and Row-Equivalent Matrices

Theorem 8.4

Rank of a Matrix by Row Reduction

If a matrix A is row equivalent to a row-echelon form B, then
i) the row space of A = the row space of B
ii) the nonzero rows of B from a basis for the row space of A, and
iii) rank(A) = the number of nonzero rows in B

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{bmatrix}$$

$$\begin{matrix} ① \\ ② \\ ③ \\ \end{matrix}\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank : 3

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ -x_1 + 11x_2 + 24x_3 = 90 \\ 3x_1 - 3x_2 - 92x_3 = -190 \\ 2x_1 - 2x_2 + 2x_3 = 0 \end{cases}$$

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ 0 \cdot x_1 + 10x_2 + 25x_3 = 90 \\ 0 \cdot x_1 + 0 \cdot x_2 - 95x_3 = -190 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0 \end{cases}$$

$$\begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{bmatrix}$$

$$\begin{matrix} ① \\ ② \\ \end{matrix}\begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rank : 2

$$\begin{cases} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 = 8.0 \\ -x_1 - 0.3x_2 - 0.3x_3 + 0.2x_4 = -2.3 \\ 1.5x_1 + 1.0x_2 + 1.0x_3 - 2.5x_4 = 4.0 \end{cases}$$

$$\begin{cases} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 = 8.0 \\ 0 \cdot x_1 + 1.1x_2 + 1.1x_3 - 4.4x_4 = 1.1 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 0.0 \end{cases}$$

$$\begin{bmatrix} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{bmatrix}$$

$$\begin{matrix} ① \\ ② \\ ③ \\ \end{matrix}\begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

rank : 3

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ 9x_1 + 7x_2 + 2x_3 = 15 \\ 3x_1 + 2x_2 + x_3 = -9 \end{cases}$$

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ -1/3x_2 + 1/3x_3 = -2 \\ 0 = 12 \end{cases}$$

Rank and Row-Equivalent Matrices

☑ Example 3
Linear Independence /Dependence

Determine whether the set of vectors

$$\mathbf{u}_1 = <2,1,1>$$

$$\mathbf{u}_2 = <0,3,0>$$

$$\mathbf{u}_3 = <3,1,2>$$

in R^3 in linearly dependent or linearly independent.

Solution)

If we form a matrix $\mathbf{A}$ with the given vectors as rows, and if we row reduce $\mathbf{A}$ to a row-echelon form $\mathbf{B}$ with rank 3, then the set of vectors is linearly independent.
If $\text{rank}(\mathbf{A})<3$, then the set of vectors is linearly dependent.

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 3 & 1 & 2 \end{pmatrix} \xrightarrow{\text{row operations}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus $\text{rank}(\mathbf{A})=3$ and the set of vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ is linearly independent.

Rank and Linear Systems

$$\begin{aligned} x_1 + x_2 &= 1 \\ 4x_1 - x_2 &= -6 \\ 2x_1 - 3x_2 &= 8 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 4 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 8 \end{bmatrix} \quad \Rightarrow \quad \mathbf{Ax} = \mathbf{B}$$

$$[\mathbf{A}\,|\,\mathbf{B}] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 4 & -1 & -6 \\ 2 & -3 & 8 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 16 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

row operation　　row operation

rank(A|B)=3

different

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 4 & -1 \\ 2 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

row operation

rank(A)=2

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

No solution case

$$\begin{cases} x_1 + 0 \cdot x_2 = 1 \\ 0 \cdot x_1 + x_2 = 0 \\ 0 \cdot x_1 + 0 \cdot x_2 = 1 \end{cases}$$

rank(A)≠rank(A|B)　　False statement

Rank and Row-Equivalent Matrices

Theorem 8.5 — **Consistency of AX=B**

A linear system of equations AX=B is consistent if and only if the rank of the coefficient matrix A is the same as the rank of the augmented matrix of the system

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & & 0 \\ -1 & 1 & -1 & & 0 \\ 0 & 10 & 25 & & 90 \\ 20 & 10 & 0 & & 80 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & & 0 \\ 0 & 10 & 25 & & 90 \\ 0 & 0 & -95 & & -190 \\ 0 & 0 & 0 & & 0 \end{array}\right] \quad \begin{array}{l} \text{rank (A|B): 3} \\ \text{rank (A): 3} \end{array}$$

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ -x_1 + 11x_2 + 24x_3 = 90 \\ 3x_1 - 3x_2 - 92x_3 = -190 \\ 2x_1 - 2x_2 + 2x_3 = 0 \end{cases}$$

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ 0 \cdot x_1 + 10x_2 + 25x_3 = 90 \\ 0 \cdot x_1 + 0 \cdot x_2 - 95x_3 = -190 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0 \end{cases}$$

Solution
- One solution

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} \text{rank (A|B) : 2} \\ \text{rank (A): 2} \end{array}$$

$$\begin{cases} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 = 8.0 \\ -x_1 - 0.3x_2 - 0.3x_3 + 0.2x_4 = -2.3 \\ 1.5x_1 + 1.0x_2 + 1.0x_3 - 2.5x_4 = 4.0 \end{cases}$$

$$\begin{cases} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 = 8.0 \\ 0 \cdot x_1 + 1.1x_2 + 1.1x_3 - 4.4x_4 = 1.1 \\ 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 0.0 \end{cases}$$

Solution
- Many solutions

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & 0 & 0 & 12 \end{array}\right] \quad \begin{array}{l} \text{rank (A|B) : 3} \\ \text{rank (A): 2} \end{array}$$

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ 9x_1 + 7x_2 + 2x_3 = 15 \\ 3x_1 + 2x_2 + x_3 = -9 \end{cases}$$

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ -1/3x_2 + 1/3x_3 = -2 \\ 0 = 12 \end{cases}$$

No Solution

Rank in Terms of Column Vectors

Theorem : Rank in Terms of Column Vectors

The rank r of a matrix $\mathbf{A}$ equals the maximum number of linearly independent column vectors of $\mathbf{A}$. Hence $\mathbf{A}$ and its transpose $\mathbf{A}^T$ have the same rank.

Proof) Let $\mathbf{A}$ be an m x n matrix of rank($\mathbf{A}$) = r

Then by definition of rank, $\mathbf{A}$ has r linearly independent rows which we denote by $\mathbf{v}_{(1)}, \cdots, \mathbf{v}_{(r)}$ and all the rows $\mathbf{a}_{(1)}, \cdots, \mathbf{a}_{(m)}$ of $\mathbf{A}$ are linear combinations of those.

Rank in Terms of Column Vectors
- 3 by 3 matrix

let $\text{rank}\,\mathbf{A} = 3$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix},$$

3(= rank A) linearly independent rows (basis) :

$$\mathbf{v}_1 = \begin{bmatrix} v_{11} & v_{12} & v_{13} \end{bmatrix}$$
$$\mathbf{v}_2 = \begin{bmatrix} v_{21} & v_{22} & v_{23} \end{bmatrix}$$
$$\mathbf{v}_3 = \begin{bmatrix} v_{31} & v_{32} & v_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} c_{11}\mathbf{v}_1 + c_{12}\mathbf{v}_2 + c_{13}\mathbf{v}_3 \\ \hline c_{21}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + c_{23}\mathbf{v}_3 \\ \hline c_{31}\mathbf{v}_1 + c_{32}\mathbf{v}_2 + c_{33}\mathbf{v}_3 \end{bmatrix}$$

행벡터를 v1, v2, v3의 일차결합으로 표현함

Note

A의 rank가 3이므로 행벡터는 linearly independent 하다.

그러므로 A의 행벡터가 이루는 공간은 3개의 Basis를 갖는다. (ex : $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$)

즉 그 공간의 임의의 벡터(b)는 A의 행벡터로 표현될 수 있다. $\mathbf{b} = l\mathbf{a}_1 + m\mathbf{a}_2 + n\mathbf{a}_3 \quad (l, m, n : const)$

Ex) 3차원 공간상의 벡터 b를 basis $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ 의 linear combination으로 표현

$\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ 를 다른 basis로 표현한다면 3개의 basis가 필요하다.

$$\begin{aligned} \mathbf{b} &= l\mathbf{a}_1 + m\mathbf{a}_2 + n\mathbf{a}_3 \\ &= l(c_{11}\mathbf{v}_1 + c_{12}\mathbf{v}_2 + c_{13}\mathbf{v}_3) + m(c_{21}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + c_{23}\mathbf{v}_3) + n(c_{31}\mathbf{v}_1 + c_{32}\mathbf{v}_2 + c_{33}\mathbf{v}_3) \\ &= (lc_{11} + mc_{21} + nc_{31})\mathbf{v}_1 + (lc_{12} + mc_{22} + nc_{32})\mathbf{v}_2 + (lc_{13} + mc_{23} + nc_{33})\mathbf{v}_3 \\ &(l, m, n, c : const) \end{aligned}$$

Ex) 3차원 공간상의 벡터 b를 basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ 의 linear combination으로 표현

$\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ 를 다른 2개의 basis로 표현한다면

$$\begin{aligned} \mathbf{b} &= l\mathbf{a}_1 + m\mathbf{a}_2 + n\mathbf{a}_3 \\ &= l(c_{11}\mathbf{v}_1 + c_{12}\mathbf{v}_2) + m(c_{21}\mathbf{v}_1 + c_{22}\mathbf{v}_2) + n(c_{31}\mathbf{v}_1 + c_{32}\mathbf{v}_2) \\ &= (lc_{11} + mc_{21} + nc_{31})\mathbf{v}_1 + (lc_{12} + mc_{22} + nc_{32})\mathbf{v}_2 \\ &(l, m, n, c : const) \end{aligned}$$

Ex) $\mathbf{v}_1, \mathbf{v}_2$ 만으로는 3차원 공간상의 벡터 b를 표현할 수 없음

Rank in Terms of Column Vectors - 3 by 3 matrix

$$\mathbf{v}_1 = [v_{11} \quad v_{12} \quad v_{13}]$$
$$\mathbf{v}_2 = [v_{21} \quad v_{22} \quad v_{23}]$$
$$\mathbf{v}_3 = [v_{31} \quad v_{32} \quad v_{33}]$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} c_{11}\mathbf{v}_1 + c_{12}\mathbf{v}_2 + c_{13}\mathbf{v}_3 \\ c_{21}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + c_{23}\mathbf{v}_3 \\ c_{31}\mathbf{v}_1 + c_{32}\mathbf{v}_2 + c_{33}\mathbf{v}_3 \end{bmatrix}$$

행벡터 a의 성분을 새로운 basis v의 성분으로 표현하면

$$[\mathbf{a}_1] = [a_{11} \; a_{12} \; a_{13}] = c_{11}\mathbf{v}_1 + c_{12}\mathbf{v}_2 + c_{13}\mathbf{v}_3$$
$$= c_{11}[v_{11}, v_{12}, v_{13}] + c_{12}[v_{21}, v_{22}, v_{23}] + c_{13}[v_{31}, v_{32}, v_{33}]$$
$$= [(c_{11}v_{11} + c_{12}v_{21} + c_{13}v_{31}), (c_{11}v_{12} + c_{12}v_{22} + c_{13}v_{32}), (c_{11}v_{13} + c_{12}v_{22} + c_{13}v_{32})]$$

$$a_{11} = c_{11}v_{11} + c_{12}v_{21} + c_{13}v_{31}$$
$$a_{21} = c_{21}v_{11} + c_{22}v_{21} + c_{23}v_{31}$$
$$a_{31} = c_{31}v_{11} + c_{32}v_{21} + c_{33}v_{31}$$

$$a_{12} = c_{11}v_{12} + c_{12}v_{22} + c_{13}v_{32}$$
$$a_{22} = c_{21}v_{12} + c_{22}v_{22} + c_{23}v_{32}$$
$$a_{32} = c_{31}v_{12} + c_{32}v_{22} + c_{33}v_{32}$$

$$a_{13} = c_{11}v_{13} + c_{12}v_{23} + c_{13}v_{33}$$
$$a_{23} = c_{21}v_{13} + c_{22}v_{23} + c_{23}v_{33}$$
$$a_{33} = c_{31}v_{13} + c_{32}v_{23} + c_{33}v_{33}$$

Column Vector of A

linearly independent basis

$$\begin{bmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{bmatrix} = v_{1k} \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} + v_{2k} \begin{bmatrix} c_{12} \\ c_{22} \\ c_{32} \end{bmatrix} + v_{3k} \begin{bmatrix} c_{13} \\ c_{23} \\ c_{33} \end{bmatrix}$$

열벡터도 3개의 basis (c의 성분의 수는 v의 수와 동일함) 존재. 따라서 rank A^T = rank A

Rank in Terms of Column Vectors - 3 by 3 matrix

$$\mathbf{v}_1 = [v_{11} \quad v_{12} \quad v_{13}]$$
$$\mathbf{v}_2 = [v_{21} \quad v_{22} \quad v_{23}]$$
$$\mathbf{v}_3 = [v_{31} \quad v_{32} \quad v_{33}]$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} c_{11}\mathbf{v}_1 + c_{12}\mathbf{v}_2 + c_{13}\mathbf{v}_3 \\ c_{21}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + c_{23}\mathbf{v}_3 \\ c_{31}\mathbf{v}_1 + c_{32}\mathbf{v}_2 + c_{33}\mathbf{v}_3 \end{bmatrix}$$

Column Vector of A

$$\begin{bmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{bmatrix} = v_{1k} \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} + v_{2k} \begin{bmatrix} c_{12} \\ c_{22} \\ c_{32} \end{bmatrix} + v_{3k} \begin{bmatrix} c_{13} \\ c_{23} \\ c_{33} \end{bmatrix}$$

linearly independent basis

열벡터도 3개의 basis (c의 성분의 수는 v의 수와 동일함) 존재.
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Rank in Terms of Column Vectors - 3 by 3 matrix

$$\mathbf{v}_1 = [v_{11} \quad v_{12} \quad v_{13}]$$
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$$\mathbf{v}_3 = [v_{31} \quad v_{32} \quad v_{33}]$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} c_{11}\mathbf{v}_1 + c_{12}\mathbf{v}_2 + c_{13}\mathbf{v}_3 \\ c_{21}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + c_{23}\mathbf{v}_3 \\ c_{31}\mathbf{v}_1 + c_{32}\mathbf{v}_2 + c_{33}\mathbf{v}_3 \end{bmatrix}$$

Column Vector of A

linearly independent basis

$$\begin{bmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{bmatrix} = v_{1k} \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} + v_{2k} \begin{bmatrix} c_{12} \\ c_{22} \\ c_{32} \end{bmatrix} + v_{3k} \begin{bmatrix} c_{13} \\ c_{23} \\ c_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{bmatrix} = v_{1k} \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} + v_{2k} \begin{bmatrix} c_{12} \\ c_{22} \\ c_{32} \end{bmatrix} + v_{3k} u \begin{bmatrix} c_{12} \\ c_{22} \\ c_{23} \end{bmatrix}$$

$$\begin{bmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{bmatrix} = v_{1k} \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} + (v_{2k} + v_{3k}u) \begin{bmatrix} c_{12} \\ c_{22} \\ c_{32} \end{bmatrix}$$

⇩ A의 행백터 관점에서 basis의 개수가 줄어들게 되어 모순이 됨 $\mathbf{v}_1, \mathbf{v}_{new}$

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} c_{11}\mathbf{v}_1 + c_{12}\mathbf{v}_2 + uc_{12}\mathbf{v}_3 \\ c_{21}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + uc_{22}\mathbf{v}_3 \\ c_{31}\mathbf{v}_1 + c_{32}\mathbf{v}_2 + uc_{33}\mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} c_{11}\mathbf{v}_1 + c_{12}(1+u)(\mathbf{v}_2 + \mathbf{v}_3) \\ c_{21}\mathbf{v}_1 + c_{22}(1+u)(\mathbf{v}_2 + \mathbf{v}_3) \\ c_{31}\mathbf{v}_1 + c_{32}(1+u)(\mathbf{v}_2 + \mathbf{v}_3) \end{bmatrix} = \begin{bmatrix} c_{11}\mathbf{v}_1 + c_{12}(1+u)\mathbf{v}_{new} \\ c_{21}\mathbf{v}_1 + c_{22}(1+u)\mathbf{v}_{new} \\ c_{31}\mathbf{v}_1 + c_{32}(1+u)\mathbf{v}_{new} \end{bmatrix}$$

Rank in Terms of Column Vectors - 3 by 3 matrix

$$\mathbf{A}=\begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}=\begin{bmatrix}c_{11}\mathbf{v}_1+c_{12}\mathbf{v}_2+c_{13}\mathbf{v}_3\\ c_{21}\mathbf{v}_1+c_{22}\mathbf{v}_2+c_{23}\mathbf{v}_3\\ c_{31}\mathbf{v}_1+c_{32}\mathbf{v}_2+c_{33}\mathbf{v}_3\end{bmatrix}$$

이 열벡터들이 linearly dependent 하다면?

$$\begin{bmatrix}a_{1k}\\ a_{2k}\\ a_{3k}\end{bmatrix}=v_{1k}\begin{bmatrix}c_{11}\\ c_{21}\\ c_{31}\end{bmatrix}+v_{2k}\begin{bmatrix}c_{12}\\ c_{22}\\ c_{32}\end{bmatrix}+v_{3k}u\begin{bmatrix}c_{12}\\ c_{22}\\ c_{23}\end{bmatrix}$$

$$\begin{bmatrix}a_{1k}\\ a_{2k}\\ a_{3k}\end{bmatrix}=v_{1k}\begin{bmatrix}c_{11}\\ c_{21}\\ c_{31}\end{bmatrix}+(v_{2k}+v_{3k}u)\begin{bmatrix}c_{12}\\ c_{22}\\ c_{32}\end{bmatrix}$$

A의 열벡터

$$\begin{bmatrix}a_{1k}\\ a_{2k}\\ a_{3k}\end{bmatrix}=v_{1k}\begin{bmatrix}c_{11}\\ c_{21}\\ c_{31}\end{bmatrix}+v_{2k}\begin{bmatrix}c_{12}\\ c_{22}\\ c_{32}\end{bmatrix}+v_{3k}\begin{bmatrix}c_{13}\\ c_{23}\\ c_{33}\end{bmatrix}$$

∴ 이 열벡터들은 linearly independent 한 basis

열벡터도 3개의 basis (c의 성분의 수는 v의 수와 동일함) 존재.
따라서 rank A^T = rank A

A의 행백터 관점에서 basis의 개수가 줄어들게 되어 모순이 됨 $\mathbf{v}_1, \mathbf{v}_{new}$

$$\mathbf{A}=\begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}=\begin{bmatrix}c_{11}\mathbf{v}_1+c_{12}\mathbf{v}_2+uc_{12}\mathbf{v}_3\\ c_{21}\mathbf{v}_1+c_{22}\mathbf{v}_2+uc_{22}\mathbf{v}_3\\ c_{31}\mathbf{v}_1+c_{32}\mathbf{v}_2+uc_{33}\mathbf{v}_3\end{bmatrix}=\begin{bmatrix}c_{11}\mathbf{v}_1+c_{12}(1+u)(\mathbf{v}_2+\mathbf{v}_3)\\ c_{21}\mathbf{v}_1+c_{22}(1+u)(\mathbf{v}_2+\mathbf{v}_3)\\ c_{31}\mathbf{v}_1+c_{32}(1+u)(\mathbf{v}_2+\mathbf{v}_3)\end{bmatrix}=\begin{bmatrix}c_{11}\mathbf{v}_1+c_{12}(1+u)\mathbf{v}_{new}\\ c_{21}\mathbf{v}_1+c_{22}(1+u)\mathbf{v}_{new}\\ c_{31}\mathbf{v}_1+c_{32}(1+u)\mathbf{v}_{new}\end{bmatrix}$$

SOLUTIONS OF HOMOGENEOUS LINEAR SYSTEMS

Homogeneous Linear System with Fewer Equations Than Unknowns (2)

$$\mathbf{Ax} = \lambda\mathbf{x}$$

$$\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{0}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

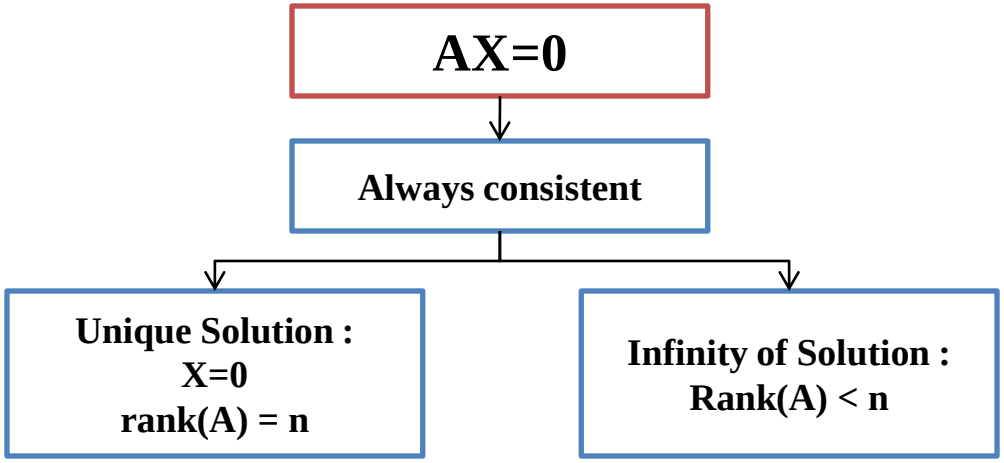
$$\begin{cases} \mathbf{A} : \text{matrix} \\ \lambda : \text{scalar} \\ \mathbf{x} : \text{vector} \end{cases}$$

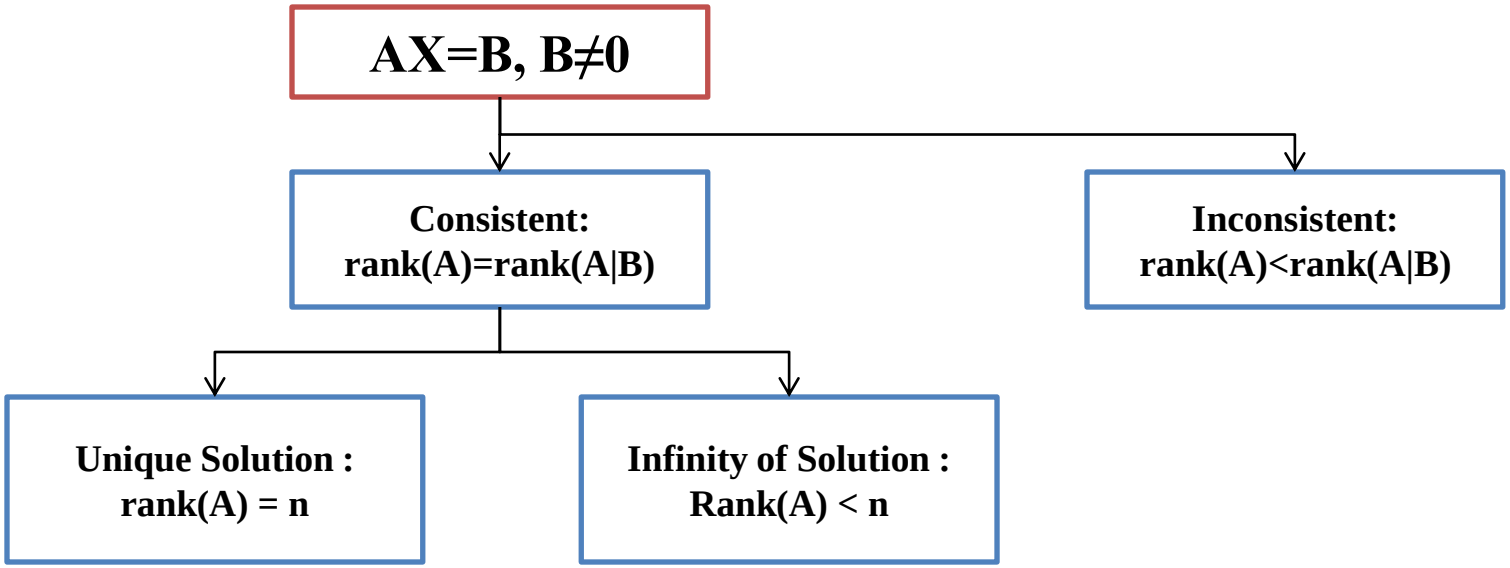
- If the rank $(\mathbf{A} - \lambda\mathbf{I})$ is equal to n, the number of component of $\mathbf{x}$, (the determinant of $(\mathbf{A} - \lambda\mathbf{I})$ is nonzero), we have a trivial solution ($\mathbf{x} = 0$).

- If the rank $(\mathbf{A} - \lambda\mathbf{I})$ is less than n, the number of component of $\mathbf{x}$, (the determinant of $(\mathbf{A} - \lambda\mathbf{I})$ is zero), we have Infinitely many solutions ($\mathbf{x} \neq 0$).

- A scalar λ such that the equation holds for some vector $\mathbf{x} \neq 0$ is called an eigenvalue of $\mathbf{A}$.
  - At that time, vector $\mathbf{x}$ is called eigenvector of $\mathbf{A}$.

Connection between the concept of rank of a matrix and the solution of linear system





SECOND- AND THIRD-ORDER DETERMINANTS

Determinant of second- and third order

Determinant of second order

$$D = \det A = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Determinant of third order

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= +a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

Determinant of Order n

Terms

In D we have n^2 entries a_{jk}, also n rows and n columns, and a main diagonal on which $a_{11}, a_{12}, \ldots, a_{nn}$ stand.

M_{jk} is called the minor of a_{jk} in D, and C_{jk} the cofactor of a_{jk} in D

For later use we note that D may also be written in terms of minors

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad (j = 1, 2, \cdots, n)$$

$$D = \sum_{j=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad (k = 1, 2, \cdots, n)$$

$$C_{jk} = (-1)^{j+k} M_{jk}$$

Determinant of Order *n*

A determinant of order *n* is a scalar associated with an *n* x *n* matrix A=[a_{jk}], which is written

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

and is defined for *n*=1 by

$$D = a_{11}$$

Determinant of Order n

For $n \geq 2$ by

$$D = a_{j1}C_{j1} + a_{j2}C_{j2} + \cdots + a_{jn}C_{jn} \quad (j = 1, 2, \cdots, n)$$

or

$$D = a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk} \quad (k = 1, 2, \cdots, n)$$

Here,

$$C_{jk} = (-1)^{j+k} M_{jk}$$

M_{jk} is a determinant of order n-1, namely, the determinant of the submatrix of A obtained A by omitting the row and column of the entry a_{jk}, that is, the jth row and the kth column.

Determinant of Order n

1) n=1

$$\mathbf{A} = [a_{11}] \qquad \therefore \det \mathbf{A} = a_{11}$$

2) n=2

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det \mathbf{A} = a_{11} \begin{vmatrix} \blacksquare & \blacksquare \\ \blacksquare & a_{22} \end{vmatrix} - a_{12} \begin{vmatrix} \blacksquare & \blacksquare \\ a_{21} & \blacksquare \end{vmatrix}$$

$$= a_{11}a_{22} - a_{12}a_{21}$$

Determinant of Order *n*

3) $n=3$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det \mathbf{A} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Determinant :
(Minors and Cofactors of a Third-Order Determinant)

$$C_{jk} = (-1)^{j+k} M_{jk}$$

Find minors and cofactors.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{12} = (-1)^{1+2} M_{12} = -M_{12}$$

1) 1st row

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{11} = (-1)^{1+1} M_{11} = M_{11}$$

$$C_{13} = (-1)^{1+3} M_{13} = M_{13}$$

Determinant : (Minors and Cofactors of a Third-Order Determinant)

$$C_{jk} = (-1)^{j+k} M_{jk}$$

Find minors and cofactors.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{22} = (-1)^{2+2} M_{22} = M_{22}$$

2) 2nd row

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{21} = (-1)^{2+1} M_{21} = -M_{21}$$

$$C_{23} = (-1)^{2+3} M_{23} = -M_{23}$$

Determinant : (Minors and Cofactors of a Third-Order Determinant)

$$C_{jk} = (-1)^{j+k} M_{jk}$$

Find minors and cofactors.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$C_{32} = (-1)^{3+2} M_{32} = -M_{32}$$

3) 3rd row

$$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$C_{31} = (-1)^{3+1} M_{31} = M_{31}$$

$$M_{13} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$C_{33} = (-1)^{3+3} M_{33} = M_{33}$$

Determinant : (Minors and Cofactors of a Third-Order Determinant)

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad (j = 1, 2, \cdots, n)$$

(Expansions of a Third-Order Determinant)

Find determinant

$$\det \mathbf{A} = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix}$$

1) 1st rows

$$= 1\begin{vmatrix} 6 & 4 \\ 0 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix}$$

$$= 1(12 - 0) - 3(4 + 4) + 0(0 + 6) = -12$$

Determinant : (Minors and Cofactors of a Third-Order Determinant)

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad (j = 1, 2, \cdots, n)$$

(Expansions of a Third-Order Determinant)

Find determinant.

$$\det \mathbf{A} = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix}$$

2) 2nd rows

$$= -2\begin{vmatrix} \blacksquare & 3 & 0 \\ \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & 0 & 2 \end{vmatrix} + 6\begin{vmatrix} 1 & \blacksquare & 0 \\ \blacksquare & \blacksquare & \blacksquare \\ -1 & \blacksquare & 2 \end{vmatrix} - 4\begin{vmatrix} 1 & 3 & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \\ -1 & 0 & \blacksquare \end{vmatrix}$$

$$= -2(6-0) + 6(2+0) - 4(0+3) = -12$$

Determinant :
(Minors and Cofactors of a Third-Order Determinant)

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad (j = 1, 2, \cdots, n)$$

(Expansions of a Third-Order Determinant)

Find determinant.

$$\det \mathbf{A} = \begin{vmatrix} -3 & 0 & 0 \\ 6 & 4 & 0 \\ -1 & 2 & 5 \end{vmatrix}$$

$$= -3 \begin{vmatrix} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & 4 & 0 \\ \blacksquare & 2 & 5 \end{vmatrix} - 0 \begin{vmatrix} \blacksquare & \blacksquare & \blacksquare \\ 6 & \blacksquare & 0 \\ -1 & \blacksquare & 5 \end{vmatrix} + 0 \begin{vmatrix} \blacksquare & \blacksquare & \blacksquare \\ 6 & 4 & \blacksquare \\ -1 & 2 & \blacksquare \end{vmatrix}$$

$$= -3 \begin{vmatrix} 4 & 0 \\ 2 & 5 \end{vmatrix} = -3 \cdot 4 \cdot 5 = -60$$

Solving linear systems of two equations

Solve the linear systems of two equations

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad \cdots ①$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \quad \cdots ②$$

1. General Solution

$① \times a_{22} - ② \times a_{12}$:

$$(a_{11}a_{22} - a_{12}a_{21})x_1 = b_1a_{22} - a_{12}b_2$$

$$\therefore x_1 = \frac{b_1a_{22} - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}} \quad (a_{11}a_{22} - a_{12}a_{21} \neq 0)$$

$① \times (-a_{21}) + ② \times a_{11}$:

$$(a_{11}a_{22} - a_{12}a_{21})x_1 = a_{11}b_2 - b_1a_{21}$$

$$\therefore x_2 = \frac{a_{11}b_2 - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \quad (a_{11}a_{22} - a_{12}a_{21} \neq 0)$$

Solving linear systems of two equations

Solve the linear systems of two equations

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad \cdots ①$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \quad \cdots ②$$

2. Use Cramer's rule

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D} = \frac{b_1 a_{22} - a_{12} b_2}{D} \qquad (D \neq 0)$$

$(D = 0)$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{D} = \frac{a_{11} b_2 - b_1 a_{21}}{D} \qquad (D \neq 0)$$

$$x_1 = \frac{b_1 a_{22} - a_{12} b_2}{a_{11} a_{22} - a_{12} a_{21}}$$

$$x_2 = \frac{a_{11} b_2 - b_1 a_{21}}{a_{11} a_{22} - a_{12} a_{21}}$$

$$D = \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} a_{22} - a_{12} a_{21}$$

Solving linear systems of three equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$x_1 = \frac{D_1}{D}, \qquad x_2 = \frac{D_2}{D}; \qquad x_3 = \frac{D_3}{D}$$

$$D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Solving linear systems of three equations

Note that D_1, D_2, D_3 are obtained by replacing Columns 1, 2, 3.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \quad \Rightarrow \quad \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\mathbf{b}}$$

$$\mathbf{A} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \Rightarrow \quad D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

Solving linear systems of three equations

Note that D_1, D_2, D_3 are obtained by replacing Columns 1, 2, 3.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \quad \Rightarrow \quad \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\mathbf{b}}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \Rightarrow \quad D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

Solving linear systems of three equations

Note that D_1, D_2, D_3 are obtained by replacing Columns 1, 2, 3.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \quad \Rightarrow \quad \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\mathbf{b}}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \Rightarrow \quad D_3 = \begin{vmatrix} a_{11} & a_{21} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{23} & b_3 \end{vmatrix}$$

Cramer's Rule

Cramer's Theorem (Solution of Linear Systems by Determinants)
(a) If a linear system of n equations in the same number of unknowns $x_1, \dots, x_n$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

has a nonzero coefficient determinant D=det(A), the system has precisely one solution. This solution is given by the formulas

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad , \quad \cdots \quad , \quad x_n = \frac{D_n}{D}$$

Where D_k is the determinant obtained from D by replacing in D the kth column by the column with the entries $b_1, \dots, b_n$.

Cramer's Rule

Cramer's Theorem (Solution of Linear Systems by Determinants)

$$D_k = \begin{vmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & \cdots & b_2 & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix} \quad \begin{matrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_n \end{matrix}$$

replace

$$D_k = b_1 C_{1k} + b_2 C_{2k} + \cdots + b_n C_{nk}$$

(b) Hence if the system is homogeneous and D≠0, it has only the trivial solution x_1=0, … , x_n=0. If D=0, the homogeneous system also has nontrivial solutions.

(참고) 3차 연립방정식의 해

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = p \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = q \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = r \end{cases}$$

$$a_{21}a_{11}x_1 + a_{21}a_{12}x_2 + a_{21}a_{13}x_3 = a_{21}p$$

$$- \left| \; a_{11}a_{21}x_1 + a_{11}a_{22}x_2 + a_{11}a_{23}x_3 = a_{11}q \right.$$

$$(a_{21}a_{12} - a_{11}a_{22})x_2 + (a_{21}a_{13} - a_{11}a_{23})x_3 = a_{21}p - a_{11}q$$

$$a_{31}a_{11}x_1 + a_{31}a_{12}x_2 + a_{31}a_{13}x_3 = a_{31}p$$

$$- \left| \; a_{11}a_{31}x_1 + a_{11}a_{32}x_2 + a_{11}a_{33}x_3 = a_{11}r \right.$$

$$(a_{31}a_{12} - a_{11}a_{32})x_2 + (a_{31}a_{13} - a_{11}a_{33})x_3 = a_{31}p - a_{11}r$$

Solving linear systems of two equations

Solve the linear systems of two equations

$$(D=0)\quad D=\det A=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}=0$$

$$\begin{cases} a_{11}x_1+a_{12}x_2=b_1 & \text{①} \\ a_{21}x_1+a_{22}x_2=b_2 & \text{②} \end{cases}$$

$$a_{11}=\frac{a_{12}a_{21}}{a_{22}}$$

$$\begin{cases} a_{21}x_1+a_{22}x_2=\dfrac{a_{22}}{a_{12}}b_1 \\ a_{21}x_1+a_{22}x_2=b_2 \end{cases} \qquad \frac{a_{12}a_{21}}{a_{22}}x_1+a_{12}x_2=b_1$$

$$\begin{cases} a_{21}x_1+a_{22}x_2=\dfrac{a_{22}}{a_{12}}b_1 \\ 0\cdot x_1+0\cdot x_2=\dfrac{a_{22}}{a_{12}}b_1-b_2 \end{cases}$$

when $\dfrac{a_{22}}{a_{12}}b_1-b_2=0$

$$\begin{cases} a_{21}x_1+a_{22}x_2=\dfrac{a_{22}}{a_{12}}b_1 \\ 0\cdot x_1+0\cdot x_2=0 \end{cases}$$

Linearly independent equation : 1
Variables : 2

$$\left[\begin{array}{cc|c} a_{21} & a_{22} & \dfrac{a_{22}}{a_{12}}b_1 \\ 0 & 0 & 0 \end{array}\right]$$

rank(A)=1=rank(A|B)
rank(A)=1<2 unknowns

$$\frac{a_{22}}{a_{12}}b_1-b_2\neq 0=b'$$

$$\begin{cases} a_{21}x_1+a_{22}x_2=\dfrac{a_{22}}{a_{12}}b_1 \\ 0\cdot x_1+0\cdot x_2=b' \end{cases}$$

False statement

$$\left[\begin{array}{cc|c} a_{21} & a_{22} & \dfrac{a_{22}}{a_{12}}b_1 \\ 0 & 0 & b' \end{array}\right]$$

rank(A)=1≠2=rank(A|B)

Homogeneous linear systems $\begin{cases} a_{11}x_1+a_{12}x_2=0 \\ a_{21}x_1+a_{22}x_2=0 \end{cases}$

$$A\mathbf{x}=\mathbf{0} \qquad A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det A=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=0$$

$$A\mathbf{x}=\mathbf{0} \qquad A=\begin{bmatrix} a_{11} & a_{12} \\ 0 & 0 \end{bmatrix}$$

Trivial Solution $\mathbf{x}=\mathbf{0}$

Nontrivial many solutions

Behavior of an nth-Order Determinant under Elementary Row Operations

Theorem 1. Behavior of an nth-Order Determinant under Elementary Row Operations

(a) Interchange of two rows multiplies the value of the determinant by -1.

(b) Addition of a multiple of a row to another row does not alter the value of the determinant.

(c) Multiplication of a row by a nonzero constant c multiplies the value of the determinant by c.

Behavior of an nth-Order Determinant under Elementary Row Operations

Proof. ***(a) Interchange of two rows multiplies the value of the determinant by -1*** **by induction.**

The statement holds for n=2 because

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \qquad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad$$

(a) holds for determinants of order n-1(≥2) and show that it then holds determinants of order n.
Let D be of order n. Let E be one of those interchanged. Expand D and E by a row that is not one of those interchanged

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk}, \qquad E = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} N_{jk}$$

Behavior of an nth-Order Determinant under Elementary Row Operations

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk}, \qquad E = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} N_{jk}$$

N_{jk} is obtained from the minor M_{jk} of a_{jk} in D by interchange of those two rows which have been interchanged in D. Now these minors are of order n-1.

➔ **The induction hypothesis applies**

$$M_{jk} = -N_{jk}$$

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} \left(-N_{jk}\right) = -E$$

Behavior of an nth-Order Determinant under Elementary Row Operations

Proof. ***(b) Addition of a multiple of a row to another row does not alter the value of the determinant.***

Add c times Row i to Row j.

Let $\tilde{D}$ be the new determinant. Its entries in Row j are $a_{jk}+ca_{ik}$.

$$D=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \Rightarrow \tilde{D}=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1}+ca_{i1} & a_{j2}+ca_{i2} & \cdots & a_{jn}+ca_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Behavior of an *n*th-Order Determinant under Elementary Row Operations

We can write $\tilde{D}$ by the *j*th row.

$$\tilde{D} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1}+ca_{i1} & a_{j2}+ca_{i2} & \cdots & a_{jn}+ca_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{k=1}^{n} (-1)^{j+k} \left(a_{jk} + ca_{ik} \right) M_{jk}$$

$$= \underbrace{\sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk}}_{D_1} + c \underbrace{\sum_{k=1}^{n} (-1)^{j+k} a_{ik} M_{jk}}_{D_2} = D_1 + cD_2$$

Behavior of an *n*th-Order Determinant under Elementary Row Operations

$$D_1 = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = D$$

Behavior of an nth-Order Determinant under Elementary Row Operations (Previous)

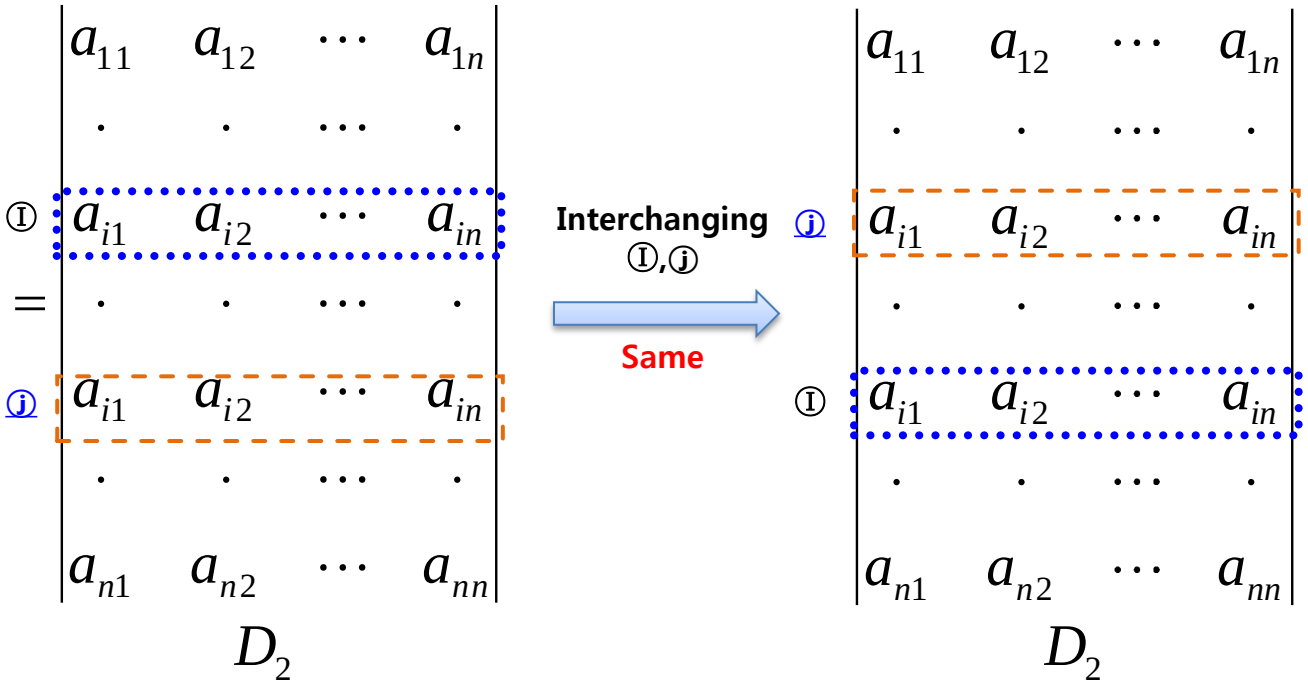
$$D_2 = \sum_{k=1}^{n} (-1)^{j+k} a_{ik} M_{jk} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ \textcircled{i}\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ \textcircled{j}\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad \textcircled{i}\text{x(-1)+}\textcircled{j} \quad = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

It has a_{ik} in both Row i and Row j.
Interchanging these two rows gives D_2 back, but on the other hand it gives $-D_2$ by (a). (D_2=-D_2=0)

Behavior of an nth-Order Determinant under Elementary Row Operations (modified)

It has a_{ik} in both Row i and Row j.
Interchanging these two rows gives D_2 back,

$$D_2 = \sum_{k=1}^{n} (-1)^{j+k} a_{ik} M_{jk} =$$



D_2 $\qquad$ D_2

but on the other hand interchanging these two rows gives $-D_2$ by Theorem(a).

D_2 $\xrightarrow{\text{Interchanging ①,ⓙ}}$ $-D_2$

Together $D_2 = -D_2 = 0$.

Behavior of an *n*th-Order Determinant under Elementary Row Operations

$D_1 = D$

$D_2 = 0$

$$\tilde{D} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1}+ca_{i1} & a_{j2}+ca_{i2} & \cdots & a_{jn}+ca_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + c \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= D_1 + cD_2$$

$$= D + c \cdot 0$$

$$= D$$

Behavior of an *n*th-Order Determinant under Elementary Row Operations

Proof. (c) Multiplication of a row by a nonzero constant c multiplies the value of the determinant by c

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \Rightarrow \tilde{D} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ ca_{j1} & ca_{j2} & \cdots & ca_{jn} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Expand the determinant by the *j*th row.

$$\tilde{D} = \sum_{k=1}^{n} (-1)^{j+k} ca_{jk} M_{jk} = c \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} = cD$$

Further Properties of nth-Order Determinants

Theorem 2. Further Properties of nth-Order Determinants

(d) Transposition leaves the value of a determinant unaltered.

(e) A zero row or column renders the value of a determinant zero.

(f) Proportional rows or columns render the value of a determinant zero. In particular, a determinant with two identical rows or columns has the value zero.

Further Properties of *n*th-Order Determinants

Proof.
(d) Transposition leaves the value of a determinant unaltered.

Transposition is defined as for matrices, that is, the *j*th row becomes the *j*th column of the transpose.

Proof.
(e) A zero row or column renders the value of a determinant zero.

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ . & . & \cdots & . \\ 0 & 0 & \cdots & 0 \\ . & . & \cdots & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk}$$

$$= \sum_{k=1}^{n} (-1)^{j+k} 0 \cdot M_{jk} = 0$$

Further Properties of nth-Order Determinants

Proof.

(f) Proportional rows or columns render the value of a determinant zero. In particular, a determinant with two identical rows or columns has the value zero.

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ ca_{i1} & ca_{i2} & \cdots & ca_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = c \times \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad \begin{matrix} \text{①} \\ \\ \\ \\ \text{ⓙ} \\ \\ \\ \end{matrix}$$

①x(-1)+ⓙ

$$= c \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = c \times 0 = 0$$

Theorem (1.c) Multiplication of a row by a nonzero constant c multiplies the value of the determinant by c

Theorem. ***(1.b) Addition of a multiple of a row to another row does not alter the value of the determinant.***

Determinant of a Triangular Matrix

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \Rightarrow M_{11}$$

➔ It is also a determinant of a triangular matrix.

$$= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

$$= a_{11}C_{11} \qquad \left(\because a_{12} = a_{13} = \cdots = a_{1n} = 0\right)$$

$$= a_{11}M_{11}$$

$$= a_{11} \times a_{22} \times \cdots \times a_{nn}$$

(참고) 3차 연립방정식의 해

$$(a_{21}a_{12}-a_{11}a_{22})x_2+(a_{21}a_{13}-a_{11}a_{23})x_3=a_{21}p-a_{11}q$$
$$(a_{31}a_{12}-a_{11}a_{32})x_2+(a_{31}a_{13}-a_{11}a_{33})x_3=a_{31}p-a_{11}r$$

$$x_3=\frac{(a_{22}a_{31}-a_{21}a_{32})p-(a_{31}a_{12}-a_{11}a_{32})q+(a_{21}a_{12}-a_{11}a_{22})r}{a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}-a_{13}a_{22}a_{31}}$$

$$\det(\mathbf{A})=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}-a_{13}a_{22}a_{31}$$

$$\begin{vmatrix} a_{11} & a_{12} & p \\ a_{21} & a_{22} & q \\ a_{31} & a_{32} & r \end{vmatrix} = (a_{22}a_{31}-a_{21}a_{32})p-(a_{31}a_{12}-a_{11}a_{32})q+(a_{21}a_{12}-a_{11}a_{22})r$$

$$\therefore x_3=\frac{D_3}{D}$$

Rank in Terms of Determinants

Theorem 3. Rank in Terms of Determinants

An m x n matrix $\mathbf{A}=[a_{jk}]$ has rank $r \geq 1$ if and only if $\mathbf{A}$ has an r x r submatrix with nonzero determinant, whereas every square submatrix with more than r rows than $\mathbf{A}$ has (or does not have!) has determinant equal to zero.

In particular, if $\mathbf{A}$ is square, n x n, it has rank n if and only if

$$\det D \neq 0$$

INVERSE OF A MATRIX. GAUSS-JORDAN ELIMINATION

Notation of inverse matrix

In this inverse section, only *square matrices* are considered exclusively.

Notation of inverse of an n x n matrix A=[a_{jk}] : A^{-1}

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$, where I is the n x n unit matrix.

Nonsingular matrix : A matrix that has an inverse.

(If a matrix has an inverse, the inverse is unique)

Singular matrix : A matrix that has no inverse.

Proof of uniqueness of inverse matrix

If B and C are inverses of A $(\mathbf{AB} = \mathbf{I} \ \& \ \mathbf{CA} = \mathbf{I})$,

We obtain $$\mathbf{B} = \mathbf{IB} = (\mathbf{CA})\mathbf{B} = \mathbf{C}(\mathbf{AB}) = \mathbf{CI} = \mathbf{C}$$

(the uniqueness of inverse)

Theorem1. Existence of the Inverse

The inverse A^{-1} of an n x n matrix A exists if and only if rank A = n, thus if and only if $\det A \neq 0$.

Hence A is nonsingular if rank $A = n$,
and is singular if rank $A < n$.

Inverse by the Gauss-Jordan Method

For Practical determination of the inverse $\mathbf{A}^{-1}$ of a nonsingular $n \times n$ matrix $\mathbf{A}$, Gauss elimination can be used.
: This method is called Gauss-Jordan elimination

Step 1. Make augmented matrix. $\tilde{\mathbf{A}} = [\mathbf{A}\ \mathbf{I}]$

Step 2. Make Multiplication of $\mathbf{AX}=\mathbf{I}$ by $\mathbf{A}^{-1}$
(by applying Gauss elimination to $\tilde{\mathbf{A}} = [\mathbf{A}\ \mathbf{I}]$**)**

⟶ This gives a matrix of the form $[\mathbf{U}\ \mathbf{H}]$

Step 3. Reduce $\mathbf{U}$ by further elementary row operations to diagonal form.
(Eliminate the entries of $\mathbf{U}$ above the main diagonal and making the diagonal entries all 1 by multiplication. See the example next page.)

Inverse of a Matrix. Gauss-Jordan elimination.

Determine the inverse $\mathbf{A}^{-1}$ of

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

Step 1. Make augmented matrix.

$$[\mathbf{A}\ \mathbf{I}] = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

Step 2. Make Multiplication of $\mathbf{AX}=\mathbf{I}$ by $\mathbf{A}^{-1}$ by applying Gauss elimination to

Inverse of a Matrix. Gauss-Jordan elimination

$$\mathbf{AX}=\mathbf{I} \Rightarrow \mathbf{A}^{-1}\mathbf{AX}=\mathbf{A}^{-1}\mathbf{I} \Rightarrow \mathbf{IX}=\mathbf{A}^{-1}$$

$$[\mathbf{A}\,|\,\mathbf{I}] \Rightarrow \text{Gauss-Jordan Elimination} \Rightarrow [\mathbf{I}\,|\,\mathbf{A}^{-1}]$$

$$[\mathbf{A}\ \mathbf{I}]=\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

Row2 + 3Row1
Row3 − 3Row1

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right]$$

Row3 − Row2

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right]$$

Row1 +0.4Row3

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 0 & -0.6 & -0.4 & 0.4 \\ 0 & 2 & 0 & -2.6 & -0.4 & 1.4 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right]$$

Row2 +1.4Row3

$$\left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 0.7 & -0.2 & -0.3 \\ 0 & 2 & 0 & -2.6 & -0.4 & 1.4 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right]$$

diagonal matrix

Row1 -0.5Row2

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

-Row1
0.5Row2
-0.2Row3

Inverse of a Matrix. Gauss-Jordan elimination

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

$\mathbf{A}^{-1}$

Check the result.

Let $\mathbf{AA^{-1} = B}$

$$= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

$$b_{11} = (-1)\times(-0.7)+1\times(-1.3)+2\times 0.8 = 1$$
$$b_{12} = (-1)\times(0.2)+1\times(-0.2)+2\times 0.2 = 0$$
$$b_{13} = (-1)\times(0.3)+1\times(0.7)+2\times(-0.2) = 0$$
$$b_{21} = (3)\times(-0.7)+(-1)\times(-1.3)+1\times(0.8) = 0$$
$$b_{22} = (3)\times(0.2)+(-1)\times(-0.2)+1\times(0.2) = 1$$
$$b_{23} = (3)\times(0.3)+(-1)\times(0.7)+1\times(-0.2) = 0$$
$$b_{31} = (-1)\times(-0.7)+(3)\times(-1.3)+4\times(0.8) = 0$$
$$b_{32} = (-1)\times(0.2)+(3)\times(-0.2)+4\times(0.2) = 0$$
$$b_{33} = (-1)\times(0.3)+(3)\times(0.7)+4\times(-0.2) = 1$$

$$\therefore \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Useful formulas for Inverses
Theorem 2. Inverse of a Matrix

cofactor: $C_{jk} = (-1)^{j+k} M_{jk}$

M_{jk} : a determinant of order $n-1$

The inverse of a nonsingular $n \times 2n$ matrix $A=[a_{jk}]$ is given by

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}[\mathbf{C}_{jk}]^{\mathsf{T}} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ . & . & . & . \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \cdots (4)$$

Where C_{jk} is *the cofactor of* a_{jk} in det A

CAUTION! Note well that in A^{-1}, the cofactor C_{jk} occupies the same place as a_{kj} (not a_{jk}) does in A.)

Proof)

Let $\mathbf{B} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ . & . & . & . \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$ **and show that BA=I**

Useful formulas for Inverses
Theorem 2. Inverse of a Matrix

$(k = l),\ D = a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk} \cdots (9)$

$(k \neq l),\ a_{1l}C_{1k} + a_{2l}C_{2k} + \cdots + a_{nl}C_{nk} = 0 \cdots (10)$

Let

$$\mathbf{BA} = \mathbf{G} = [g_{kl}] \ldots (5)$$

and then show that G = I.

By definition of matrix multiplication and because of the form of B in (4)

$$g_{kl} = \sum_{s=1}^{n} \frac{C_{sk}}{\det \mathbf{A}} a_{sl}$$

$$= \frac{1}{\det \mathbf{A}} \left(a_{1l}C_{1k} + \cdots + a_{nl}C_{nk} \right) \cdots (6)$$

In Sec 7.7 (9) and (10),

If $l = k$, $a_{1l}C_{1k} + \cdots + a_{nl}C_{nk} = \det \mathbf{A}$

$l \neq k$, $a_{1l}C_{1k} + \cdots + a_{nl}C_{nk} = 0$

Hence,

$$g_{kk} = \frac{1}{\det \mathbf{A}} \det \mathbf{A} = 1$$

$$g_{lk} = 0$$

***g_{kk}* are the entries of main diagonal of matrix G. and that means only entries of main diagonal is 1.**

$\therefore$ G = I

$\therefore$ B = $\mathbf{A}^{-1}$

$$\mathbf{BA} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{s1} \\ C_{12} & C_{22} & \cdots & C_{s2} \\ . & . & . & . \\ C_{1k} & C_{2k} & \cdots & C_{sk} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{s1} \\ a_{12} & a_{22} & \cdots & a_{s2} \\ . & . & . & . \\ a_{1l} & a_{2l} & \cdots & a_{sl} \end{bmatrix}$$

Inverse of 2 x 2 Matrix

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \cdot & \cdot & \cdot & \cdot \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Ex 2)

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

$$\det \mathbf{A} = 3 \cdot 4 - 1 \cdot 2 = 10$$

$$C_{11} = 4, \qquad C_{21} = -1,$$

$$C_{12} = -2, \qquad C_{22} = 3,$$

$$\therefore \mathbf{A}^{-1} = \frac{1}{10} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.3 \end{bmatrix}$$

Inverse of 3 x 3 Matrix

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \cdot & \cdot & \cdot & \cdot \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}, \quad \mathbf{A}^{-1} \ ?$$

$$\det \mathbf{A} = -1 \cdot \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} - 1 \begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} + 2 \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix}$$

$$= 7 - 13 + 16 = 10$$

$$C_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7, \qquad C_{21} = -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2,$$

$$C_{12} = -\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = -13, \qquad C_{22} = \begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} = -2,$$

$$C_{13} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8, \qquad C_{23} = -\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} = 2,$$

$$C_{31} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3,$$

$$C_{32} = -\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = 7,$$

$$C_{33} = \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} = -2,$$

$$\therefore \mathbf{A}^{-1} = \frac{1}{10} \begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

Inverse of Diagonal matrices

$\mathbf{A}=[a_{jk}]$, $a_{jk}=0$ when $j\neq k$, have an inverse if and only if all $a_{jj}\neq 0$. Then $\mathbf{A}^{-1}$ is diagonal, too, with entries $1/a_{11},\cdots, 1/a_{nn}$.

Proof) For a diagonal matrix we have in (4)

$$\frac{C_{11}}{D}=\frac{a_{22}\cdots a_{nn}}{a_{11}a_{22}\cdots a_{nn}}=\frac{1}{a_{11}},\quad etc.$$

Ex 4) Inverse of Diagonal Matrix

$$\mathbf{A}=\begin{bmatrix}-0.5 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 1\end{bmatrix},\qquad \therefore \mathbf{A}^{-1}=\begin{bmatrix}-\frac{1}{0.5} & 0 & 0\\ 0 & \frac{1}{4} & 0\\ 0 & 0 & \frac{1}{1}\end{bmatrix}=\begin{bmatrix}-2 & 0 & 0\\ 0 & 0.25 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Inverse of Products

Products can be inverted by taking the inverse of each factor and multiplying these inverses in reverse order,

Hence for more than two factors, $(\mathbf{AC})^{-1} = \mathbf{C}^{-1}\mathbf{A}^{-1}$

$$(\mathbf{AC}\cdots\mathbf{PQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1}\cdots\mathbf{C}^{-1}\mathbf{A}^{-1}$$

$$(\mathbf{AC})(\mathbf{C}^{-1}\mathbf{A}^{-1}) = \mathbf{ACC}^{-1}\mathbf{A}^{-1} = \mathbf{AIA}^{-1} = \mathbf{AA}^{-1} = \mathbf{I}$$

$$(\mathbf{C}^{-1}\mathbf{A}^{-1})(\mathbf{AC}) = \mathbf{C}^{-1}\mathbf{A}^{-1}\mathbf{AC} = \mathbf{C}^{-1}\mathbf{IC} = \mathbf{C}^{-1}\mathbf{C} = \mathbf{I}$$

Determinants of Matrix Products

For any *n* x *n* matrices A and B,

$$\det(\mathbf{AB}) = \det(\mathbf{BA}) = \det \mathbf{A} \cdot \det \mathbf{B}$$

If A or B is singular, so are AB and BA

Now A and B be nonsingular. Then we can reduce A to a diagonal matrix $\hat{A} = [a_{jk}]$ by Gauss-Jordan steps.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \hat{\mathbf{A}} = \begin{bmatrix} \hat{a}_{11} & 0 & \cdots & 0 \\ 0 & \hat{a}_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \hat{a}_{nn} \end{bmatrix}.$$

A and $\hat{A}$ are row-equivalent matrices.

Determinants of Matrix Products

A and $\hat{\mathrm{A}}$ are row-equivalent matrices.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \hat{\mathbf{A}} = \begin{bmatrix} \hat{a}_{11} & 0 & \cdots & 0 \\ 0 & \hat{a}_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \hat{a}_{nn} \end{bmatrix}.$$

$$\mathbf{Ax} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ b_n \end{bmatrix} = \mathbf{b}$$

$$\hat{\mathbf{A}}\mathbf{x} = \begin{bmatrix} \hat{a}_{11} & 0 & \cdots & 0 \\ 0 & \hat{a}_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \hat{a}_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ . \\ \hat{b}_n \end{bmatrix} = \hat{\mathbf{b}}$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \hat{\mathbf{A}}^{-1}\hat{\mathbf{b}}$$

Determinants of Matrix Products

$\mathbf{A}$ and $\hat{\mathbf{A}}$ are row-equivalent matrices.

Addition of a multiple of a row to another row does not alter the value of the determinant.

$$\therefore \det(\mathbf{AB}) = \det(\hat{\mathbf{A}}\mathbf{B})$$

$$\hat{\mathbf{A}}\mathbf{B} = \begin{bmatrix} \hat{a}_{11} & 0 & \cdots & 0 \\ 0 & \hat{a}_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \hat{a}_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & & \vdots & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} \hat{a}_{11}b_{11} & \hat{a}_{11}b_{12} & \cdots & \hat{a}_{11}b_{1n} \\ \hat{a}_{22}b_{21} & \hat{a}_{22}b_{22} & \cdots & \hat{a}_{22}b_{2n} \\ & & \vdots & \\ \hat{a}_{nn}b_{n1} & \hat{a}_{nn}b_{n2} & \cdots & \hat{a}_{nn}b_{nn} \end{bmatrix}$$

$$\det(\hat{\mathbf{A}}\mathbf{B}) = \begin{vmatrix} \hat{a}_{11}b_{11} & \hat{a}_{11}b_{12} & \cdots & \hat{a}_{11}b_{1n} \\ \hat{a}_{22}b_{21} & \hat{a}_{22}b_{22} & \cdots & \hat{a}_{22}b_{2n} \\ & & \vdots & \\ \hat{a}_{nn}b_{n1} & \hat{a}_{nn}b_{n2} & \cdots & \hat{a}_{nn}b_{nn} \end{vmatrix} = \hat{a}_{11}\hat{a}_{22}\cdots\hat{a}_{nn} \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & & \vdots & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix}$$

Determinants of Matrix Products

A and $\hat{\mathrm{A}}$ are row-equivalent matrices.

$$\therefore \det(\mathbf{AB}) = \det(\hat{\mathbf{A}}\mathbf{B})$$

$$\det(\hat{\mathbf{A}}\mathbf{B}) = \begin{vmatrix} \hat{a}_{11}b_{11} & \hat{a}_{11}b_{12} & \cdots & \hat{a}_{11}b_{1n} \\ \hat{a}_{22}b_{21} & \hat{a}_{22}b_{22} & \cdots & \hat{a}_{22}b_{2n} \\ & & \vdots & \\ \hat{a}_{nn}b_{n1} & \hat{a}_{nn}b_{n2} & \cdots & \hat{a}_{nn}b_{nn} \end{vmatrix} = \hat{a}_{11}\hat{a}_{22}\cdots\hat{a}_{nn} \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & & \vdots & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix}$$

$$= \hat{a}_{11}\hat{a}_{22}\cdots\hat{a}_{nn} \cdot \det(\mathbf{B})$$

$$= \det(\mathbf{A}) \cdot \det(\mathbf{B})$$

$$\therefore \det(\mathbf{AB}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$$

7-4. EIGENVALUES, EIGENVECTORS

EIGENVALUES, EIGENVECTORS

Eigenvalues, Eigenvectors

Let $\mathbf{A}=[a_{jk}]$ be a given matrix n x n matrix and consider the vector equation

$$\mathbf{Ax} = \lambda\mathbf{x}$$

Our task is to determine $\mathbf{x}$'s and λ's that satisfy this equation. Clearly, the zero vector $\mathbf{x} = 0$ is a solution of this equation for any value λ, because $\mathbf{A0}=\mathbf{0}$. This is of no interest.

(a) Eigenvalue: A value of λ for which this equation has a solution $\mathbf{x}\neq\mathbf{0}$ (= characteristic value, latent root)

(b) Eigenvector: The corresponding solutions $\mathbf{x}\neq\mathbf{0}$ of this equation

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

Determine eigenvalues and eigenvectors

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

$$-5x_1 + 2x_2 = \lambda x_1$$

$$2x_1 - 2x_2 = \lambda x_2$$

$$(-5-\lambda)x_1 + 2x_2 = 0$$

$$2x_1 + (-2-\lambda)x_2 = 0$$

(a) Eigenvalues

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

This can be written in matrix notation.

$$\mathbf{Ax} - \lambda \mathbf{x} = 0$$

$$\mathbf{Ax} - \lambda \mathbf{Ix} = 0$$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$$

Eigenvalues, Eigenvectors

$D(\lambda)$: characteristic determinant

$D(\lambda)=0$: characteristic equation

Ex) Determination of Eigenvalues and Eigenvectors)

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

We see that this is a homogeneous linear system. By Cramer's rule it has a nontrivial solution $\mathbf{x}\neq 0$ if and only if its coefficient determinant is zero.

$$D(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}=0$$

$$(-5-\lambda)(-2-\lambda)-4=0$$

$$\lambda^2+7\lambda+6=0$$

$$\lambda_1=-1$$

$$\lambda_2=-6$$

$$(-5-\lambda)x_1+2x_2=0\cdots(a)$$

$$2x_1+(-2-\lambda)x_2=0\cdots(b)$$

$x_1=0, x_2=0$: **Trivial solution**

To have non-trivial solution

No. of Linearly independent equation < Unknowns

$(a)\times n=(b)$ **Linearly dependent, n≠0**

$$(-5-\lambda)n=2\cdots(c)$$

$$2n=(-2-\lambda)\cdots(d)$$

$(c)/(d)$

$$\frac{(-5-\lambda)}{2}=\frac{2}{(-2-\lambda)}$$

$$\therefore(-5-\lambda)(-2-\lambda)-4=0$$

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

$$(-5-\lambda)x_1 + 2x_2 = 0 \qquad \lambda_1 = -1$$

$$2x_1 + (-2-\lambda)x_2 = 0 \qquad \lambda_2 = -6$$

-Eigenvector of A corresponding to λ_1

$$\lambda_1 = -1$$

The equation is

$$-4x_1 + 2x_2 = 0$$

$$2x_1 - x_2 = 0$$

$$\therefore x_2 = 2x_1$$

From this, we determine eigenvector corresponding to λ_1 up to a scalar multiple.
If we choose x_1=1, we obtain

$$\mathbf{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

What happen if choose x_1=something else?

$$1:\mathbf{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$2:\mathbf{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$3:\mathbf{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\vdots$$

'basis'

Eigenvalues, Eigenvectors

$$D(\lambda)=\begin{vmatrix}-5-\lambda & 2\\ 2 & -2-\lambda\end{vmatrix}=0$$

Ex) Determination of Eigenvalues and Eigenvectors)

$$(-5-\lambda)x_1+2x_2=0 \qquad \lambda_1=-1$$

$$2x_1+(-2-\lambda)x_2=0 \qquad \lambda_2=-6$$

-Eigenvector of A corresponding to λ_2

$$\lambda_2=-6$$

The equation is

$$x_1+2x_2=0$$

$$2x_1+4x_2=0$$

$$\therefore x_2=-x_1/2$$

From this, we determine eigenvector corresponding to λ_2 up to a scalar multiple. If we choose x_1=2, we obtain

$$\mathbf{x}_2=\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}x_1\\ -x_1/2\end{bmatrix}=\begin{bmatrix}2\\ -1\end{bmatrix}$$

Eigenvalues, Eigenvectors

Ex) Determination of Eigenvalues and Eigenvectors)

$$(-5-\lambda)x_1 + 2x_2 = 0 \qquad \lambda_1 = -1$$

$$2x_1 + (-2-\lambda)x_2 = 0 \qquad \lambda_2 = -6$$

-Eigenvector of A corresponding to λ_1

$$\lambda_2 = -6$$

The equation is

$$x_1 + 2x_2 = 0$$

$$2x_1 + 4x_2 = 0$$

$$\therefore x_2 = -x_1/2$$

From this, we determine eigenvector corresponding to λ_2 up to a scalar multiple.
If we choose x_1=2, we obtain

$$\mathbf{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1/2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

What happen if choose x_2=something else?

$$1:\mathbf{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1/2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$2:\mathbf{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1/2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$3:\mathbf{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1/2 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\vdots$$

'basis'

General case of Eigenvectors, Eigenvalues

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= \lambda x_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= \lambda x_2 \\ &\vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= \lambda x_n \end{aligned}$$

$$\begin{aligned} (a_{11} - \lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn} - \lambda)x_n &= 0 \end{aligned}$$

In matrix notation,

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$$

General case of Eigenvectors, Eigenvalues

$$D(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

(a) Characteristic matrix: $\mathbf{A}$- $\lambda\mathbf{I}$

(b) Characteristic determinant: $D(\lambda)$

(c) Characteristic polynomial: a polynomial of nth degree in λ by developing $D(\lambda)$

Eigenvalues, Eigenvectors

Theorem 8.1 Eigenvalues

The eigenvalues of a square matrix A are the roots of the characteristic equation of A.

$$D(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

Hence an n x n matrix has at least one eigenvalue and at most n numerically different eigenvalues.

Eigenvalues, Eigenvectors

Ex.)Multiple Eigenvalues

Find the eigenvalues and eigenvector of

$$\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Sol) For our matrix, the characteristic determinant gives the characteristic equation

$$-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$$

The roots (eigenvalues of $\mathbf{A}$) are

$$\lambda_1 = 5,\ \lambda_2 = \lambda_3 = -3$$

Eigenvalues, Eigenvectors

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

Ex.)Multiple Eigenvalues

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

(1) $\lambda=5$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-5\mathbf{I}=\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}$$

row reduction

$$\begin{bmatrix}-7 & 2 & -3\\ 0 & -24/7 & -48/7\\ 0 & 0 & 0\end{bmatrix}$$

Choosing $x_3=-1$

we have

$$-\frac{24}{7}x_2-\frac{48}{7}x_3=0$$

$$\therefore x_2=2$$

$$-7x_1+2x_2-3x_3=0$$

$$\therefore x_1=1$$

$$\mathbf{x}_1=\begin{bmatrix}1 & 2 & -1\end{bmatrix}^{\mathbf{T}}$$

if choose $x_3=1$

we have

$$-\frac{24}{7}x_2-\frac{48}{7}x_3=0$$

$$\therefore x_2=-2$$

$$-7x_1+2x_2-3x_3=0$$

$$\therefore x_1=-1$$

$$\mathbf{x}_1=\begin{bmatrix}-1 & -2 & 1\end{bmatrix}^{\mathbf{T}}$$

$$\mathbf{x}_1=(-1)\times\begin{bmatrix}1 & 2 & -1\end{bmatrix}^{\mathbf{T}}$$

'basis'

Eigenvalues, Eigenvectors

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{vmatrix}$$

$$=(5-\lambda)(\lambda+3)^2=\mathbf{0}$$

Ex.)Multiple Eigenvalues

(2) $\lambda=-3$

$$\mathbf{A}-\lambda\mathbf{I}=\mathbf{A}-3\mathbf{I}$$

$$=\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}$$

row reduction:

$$\begin{bmatrix}1 & 2 & -3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$x_1+2x_2-3x_3=0$$

$$x_1=-2x_2+3x_3$$

Choosing $x_2=1,\ x_3=0$

we have $x_1=-2$

$$\mathbf{x}_2=\begin{bmatrix}-2 & 1 & 0\end{bmatrix}^T$$

Choosing $x_2=0,\ x_3=1$

we have $x_1=3$

$$\mathbf{x}_3=\begin{bmatrix}3 & 0 & 1\end{bmatrix}^T$$

$$\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}\Rightarrow\begin{cases}x_1+2x_2-3x_3=0\\ 2x_1+4x_2-6x_3=0\\ -_1-2x_2+3x_3=0\end{cases}$$

$$\Downarrow$$

$$x_1+2x_2-3x_3=0$$

we have only one equation

and three variables

two free variables (x_2, x_3)

Eigenvalues, Eigenvectors

$$\mathbf{Ax} = \lambda\mathbf{x}$$

Ex.) Real Matrices with Complex Eigenvalues and Eigenvectors

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Characteristic Equation:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\therefore \lambda = \pm i$$

$$(1)\ \lambda = i$$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$$

$$-ix_1 + x_2 = 0$$

Choosing $x_1 = 1$

we have $x_2 = i$

$$\mathbf{x}_1 = \begin{bmatrix} 1 & i \end{bmatrix}^{\mathrm{T}}$$

Eigenvalues, Eigenvectors

$$\mathbf{Ax} = \lambda\mathbf{x}$$

Ex.) Real Matrices with Complex Eigenvalues and Eigenvectors

$(2)\ \lambda = -i$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}$$

$$ix_1 + x_2 = 0$$

Choosing $x_1 = 1$

we have $x_2 = -i$

$$\mathbf{x}_2 = \begin{bmatrix} 1 & -i \end{bmatrix}^{\mathrm{T}}$$

Eigenvalues of the Transpose

Theorem 8.3 Eigenvalues of the Transpose

The transpose A^T of a square matrix A has the same eigenvalues as A

Proof

Transposition does not change the value of the characteristic determinant

Theorem 2. Further Properties of *n*th-Order Determinants

(d) Transposition leaves the value of a determinant unaltered.

(e) A zero row or column renders the value of a determinant zero.

(f) Proportional rows or columns render the value of a determinant zero. In particular, a determinant with two identical rows or columns has the value zero.

Sum and Multiplication of Eigenvalues

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$D(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix}$$

$$= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12} \cdot a_{21}$$

$$= \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12} \cdot a_{21}$$

$$= \lambda^2 - (a_{11} + a_{22})\lambda + \det \mathbf{A}$$

$$= (\lambda - \lambda_1)(\lambda - \lambda_2)$$

$$= \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0$$

$$\therefore a_{11} + a_{12} = \lambda_1 + \lambda_2$$

$$\therefore \det \mathbf{A} = \lambda_1 \cdot \lambda_2$$

The sum of these *n* eigenvalues equals the sum of the entries on the main diagonal of A, called trace of A;

$$\therefore \text{trace } \mathbf{A} = \sum_{j=1}^{n} a_{jj} = \sum_{k=1}^{n} \lambda_k$$

The product of the eigenvalues equals the determinant of A,

$$\det \mathbf{A} = \lambda_1 \lambda_2 \cdots \lambda_n$$

Example of Eigenvalue Problems
-Markov Matrix
-Stretching of an Elastic Membrane

Markov matrix

Linear equations Ax = b come from steady state problems. Eigenvalues have their greatest importance in dynamic problems. The solution of *d*u/*dt* = Au is changing with time - Growing or decaying or oscillating. We can't find it by elimination. This chapter enters a new part of linear algebra. All matrices in this chapter are square.

A good model comes from the powers $A, A^2, A^3, \ldots$ of a matrix.

Suppose you need the hundredth power A^{100}.

$$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \quad \begin{bmatrix} 0.70 & 0.45 \\ 0.30 & 0.55 \end{bmatrix} \quad \begin{bmatrix} 0.650 & 0.525 \\ 0.350 & 0.475 \end{bmatrix} \quad \cdots \quad \begin{bmatrix} 0.6000 & 0.6000 \\ 0.4000 & 0.4000 \end{bmatrix}$$

$$\mathbf{A} \qquad \mathbf{A}^2 \qquad \mathbf{A}^3 \qquad\qquad \mathbf{A}^{100}$$

A^{100} was found by using the eigenvalues of A, not by multiplying 100 matrices.

Those eigenvalues are a new way to see into the heart of a matrix.

To explain eigenvalues, we first explain eigenvectors.

need the hundredth power A^{100}

Markov matrix

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

Almost all vectors change direction, when they are multiplied by **A**.

Certain exceptional vectors **x** are in the same direction as **Ax**. Those are the "eigenvectors". Multiply an eigenvector by **A**, and the vector **Ax** is a number λ times the original **x**. The basic equation is **Ax** = λ**x**.

$$\mathbf{x}_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} \quad \text{and} \quad \mathbf{A}\mathbf{x}_1 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \mathbf{x}_1 \qquad \therefore \lambda_1 = 1$$

A vector — Multiply by A — Same direction — Eigenvector — Eigenvalue

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{A}\mathbf{x}_2 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} = \frac{1}{2}\mathbf{x}_2 \qquad \therefore \lambda_2 = \frac{1}{2}$$

A vector — Multiply by A — Same direction — Eigenvector — Eigenvalue

need the hundredth power A^{100}

Markov matrix

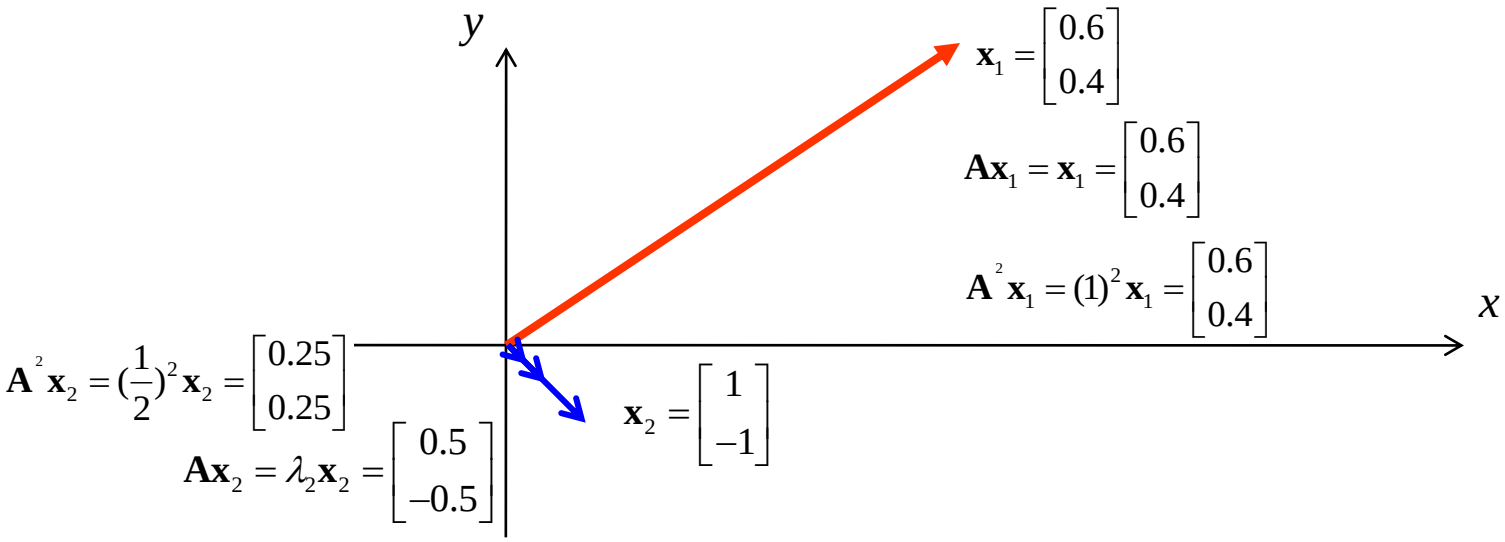
$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

If we again multiply x_1 by A, we still get x_1. Every power of A will give $A^n x_1 = x_1$. Multiplying x_2 by A gave $\frac{1}{2} x_2$, and if we multiply again we get $(\frac{1}{2})^2 x_2$. When A is squared, the eigenvectors x_1 and x_1 stay the same. The λ's are now 1^2 and $(\frac{1}{2})^2$. The eigenvalues are squared.

$$\mathbf{A}\mathbf{x}_1 = \mathbf{x}_1, \qquad \mathbf{A}\mathbf{x}_2 = \frac{1}{2}\mathbf{x}_2$$

$$\mathbf{A}^2\mathbf{x}_1 = \mathbf{A}(\mathbf{A}\mathbf{x}_1) = \mathbf{A}\mathbf{x}_1 = \mathbf{x}_1, \qquad \mathbf{A}^2\mathbf{x}_2 = \mathbf{A}(\mathbf{A}\mathbf{x}_2) = \mathbf{A}\left(\frac{1}{2}\mathbf{x}_2\right) = \frac{1}{2}\mathbf{A}\mathbf{x}_2 = \frac{1}{2}\frac{1}{2}\mathbf{x}_2 = \frac{1}{2}^2\mathbf{x}_2$$

The eigenvalues of A^{100} are $1^{100} = 1$ and $\frac{1}{2}^{100}$ = very small number.



need the hundredth power A^{100}

Markov matrix

$$\mathbf{A}=\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix},\quad \mathbf{x}_1=\begin{bmatrix}0.6\\ 0.4\end{bmatrix},\ \mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}^n\mathbf{x}_1=\mathbf{x}_1,\quad \mathbf{A}^n\mathbf{x}_2=\left(\tfrac{1}{2}\right)^n\mathbf{x}_2$$

Other vectors do change direction. But all other vectors are combinations of the two eigenvectors. The first column of A is the combination $\mathbf{x}_1+0.2\mathbf{x}_2$:

$$\begin{bmatrix}0.8\\ 0.2\end{bmatrix}=\mathbf{x}_1+0.2\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}0.2\\ -0.2\end{bmatrix}$$

Multiplying by A gives the first column of A^2 . Do it separately for $\mathbf{x}_1$ and $0.2\mathbf{x}_2$.

$$\mathbf{A}\begin{bmatrix}0.8\\ 0.2\end{bmatrix}=\mathbf{A}(\mathbf{x}_1+0.2\mathbf{x}_2)=\mathbf{A}\mathbf{x}_1+0.2\mathbf{A}\mathbf{x}_2=\mathbf{x}_1+0.2\cdot\tfrac{1}{2}\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}0.1\\ -0.1\end{bmatrix}$$

Each eigenvector is multiplied by its eigenvalue, when we multiply by A. We didn't need thse eigenvectors to find A^2. But it is the good way to do 99 multiplications.

$$\mathbf{A}^{99}\begin{bmatrix}0.8\\ 0.2\end{bmatrix}=\mathbf{x}_1+0.2\cdot\left(\tfrac{1}{2}\right)^{99}\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}very\\ small\\ vector\end{bmatrix}$$

need the hundredth power A^{100}

Markov matrix

$$\mathbf{A}=\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix},\quad \mathbf{x}_1=\begin{bmatrix}0.6\\ 0.4\end{bmatrix},\ \mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}^n\mathbf{x}_1=\mathbf{x}_1,\quad \mathbf{A}^n\mathbf{x}_2=\left(\tfrac{1}{2}\right)^n\mathbf{x}_2$$

Other vectors do change direction. But all other vectors are combinations of the two eigenvectors. The first column of A is the combination $\mathbf{x}_1 + 0.2\mathbf{x}_2$:

$$\begin{bmatrix}0.3\\ 0.7\end{bmatrix}=\mathbf{x}_1-0.3\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}0.3\\ -0.3\end{bmatrix}$$

Multiplying by A gives the first column of A^2 . Do it separately for $\mathbf{x}_1$ and $0.2\mathbf{x}_2$.

$$\mathbf{A}\begin{bmatrix}0.3\\ 0.7\end{bmatrix}=\mathbf{A}(\mathbf{x}_1-0.3\mathbf{x}_2)=\mathbf{A}\mathbf{x}_1-0.3\mathbf{A}\mathbf{x}_2=\mathbf{x}_1-0.3\cdot\tfrac{1}{2}\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}0.15\\ -0.15\end{bmatrix}$$

Each eigenvector is multiplied by its eigenvalue, when we multiply by A. We didn't need thse eigenvectors to find A^2. But it is the good way to do 99 multiplications.

$$\mathbf{A}^{99}\begin{bmatrix}0.3\\ 0.7\end{bmatrix}=\mathbf{x}_1-0.3\cdot\left(\tfrac{1}{2}\right)^{99}\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}very\\ small\\ vector\end{bmatrix}$$

need the hundredth power A^{100}

Markov matrix

$$\mathbf{A}=\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix},\quad \mathbf{x}_1=\begin{bmatrix}0.6\\ 0.4\end{bmatrix},\ \mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}^n\mathbf{x}_1=\mathbf{x}_1,\quad \mathbf{A}^n\mathbf{x}_2=\left(\tfrac{1}{2}\right)^n\mathbf{x}_2$$

$$\mathbf{A}^{99}\begin{bmatrix}0.8\\ 0.2\end{bmatrix}=\mathbf{x}_1+0.2\cdot\left(\tfrac{1}{2}\right)^{99}\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+\begin{bmatrix}very\\ small\\ vector\end{bmatrix}$$

$$\mathbf{A}^{99}\begin{bmatrix}0.3\\ 0.7\end{bmatrix}=\mathbf{x}_1-0.3\cdot\left(\tfrac{1}{2}\right)^{99}\mathbf{x}_2=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}-\begin{bmatrix}very\\ small\\ vector\end{bmatrix}$$

$$\Rightarrow\quad \therefore \mathbf{A}^{99}\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix}=\begin{bmatrix}0.6 & 0.6\\ 0.4 & 0.4\end{bmatrix}$$

$$\therefore \mathbf{A}^{100}=\mathbf{A}^{99}\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix}=\begin{bmatrix}0.6 & 0.6\\ 0.4 & 0.4\end{bmatrix}$$

The eigenvector x_1 is a "steady state" that doesn't change (because $\lambda_1 = 1$).

The eigenvector x_2 is a "decaying mode" that virtually disappears (because $\lambda_2 = 0.5$).

The higher power of A, the closer its columns approach the steady state.

We mention that this particular A is a Markov matrix. Its entries are positive and every column adds to 1. Those facts guarantee that the largest eigenvalue is $\lambda = 1$. Its eigenvector $x_1 = (0.6, 0.4)$ is the steady state - which all columns of A^k will approach.

Markov matrix

$$\mathbf{A}=\begin{bmatrix}0.8 & 0.3\\ 0.2 & 0.7\end{bmatrix},\qquad \mathbf{x}_1=\begin{bmatrix}0.6\\ 0.4\end{bmatrix},\ \mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}^n\mathbf{x}_1=\mathbf{x}_1,\qquad \mathbf{A}^n\mathbf{x}_2=\left(\tfrac{1}{2}\right)^n\mathbf{x}_2$$

find : $\mathbf{A}^{100}\begin{bmatrix}3.8\\ -0.8\end{bmatrix}$

$$\begin{bmatrix}3.8\\ -0.8\end{bmatrix}=3\mathbf{x}_1+2\mathbf{x}_2=3\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+2\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}\begin{bmatrix}3.8\\ -0.8\end{bmatrix}=3\mathbf{A}\mathbf{x}_1+2\mathbf{A}\mathbf{x}_2=3\mathbf{x}_1+2\frac{1}{2}\mathbf{x}_2$$

$$=3\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+2\frac{1}{2}\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{A}^2\begin{bmatrix}3.8\\ -0.8\end{bmatrix}=3\mathbf{A}^2\mathbf{x}_1+2\mathbf{A}^2\mathbf{x}_2=3\mathbf{x}_1+2\left(\frac{1}{2}\right)^2\mathbf{x}_2$$

$$=3\begin{bmatrix}0.6\\ 0.4\end{bmatrix}+2\left(\frac{1}{2}\right)^2\begin{bmatrix}1\\ -1\end{bmatrix}$$

Markov Matrix

always $\lambda_1 = 1,\ |\lambda_2| < 1$

Given : Markov matrix A

$$\mathbf{A} = \begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix}$$

$$0 \le a,\ b < 1$$

Find : The eigenvalues of A

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a-\lambda & b \\ 1-a & 1-b-\lambda \end{vmatrix}$$

$$= (a-\lambda)(1-b-\lambda) - b(1-a)$$

$$= \lambda^2 - (1+a-b)\lambda + a - ab - b + ab$$

$$= \lambda^2 - (1+a-b)\lambda + a - b$$

$$= \lambda^2 - (1+a-b)\lambda + 1\cdot(a-b)$$

$$= (\lambda - 1)(\lambda - (a-b))$$

$$(\lambda - 1)(\lambda - (a-b)) = 0$$

$$\therefore \lambda_1 = 1,\ \lambda_2 = a - b$$

$$0 < a < 1$$

$$0 < b < 1$$

$$0 - 1 = -1 < \lambda_2 = a - b < 1 = 1 - 0$$

$$\therefore \lambda_1 = 1,\ |\lambda_2| < 1$$

Matrix as a transformation

Solving linear systems

$$x_1 + 2x_2 + x_3 = 1$$
$$0 \cdot x_1 + x_2 + 3x_3 = 5$$
$$0 \cdot x_1 + 2x_2 + 6x_3 = 10$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix} \quad \mathbf{Ax} = \mathbf{B}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 2 & 6 & 10 \end{array}\right] \quad [\mathbf{A} \mid \mathbf{B}]$$

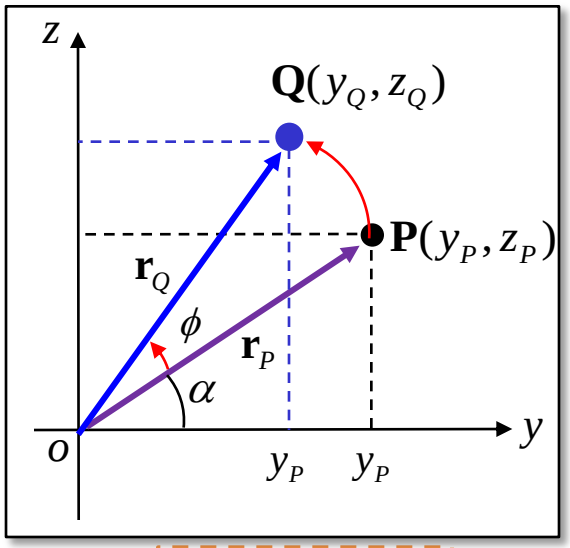
$$\mathbf{y} = \mathbf{Az}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

z : input

y : output

A : Transformation



$$\begin{bmatrix} y_Q \\ z_Q \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} y_P \\ z_P \end{bmatrix}$$

점의 회전 변환

$$\mathbf{Ax} = \lambda\mathbf{x}$$

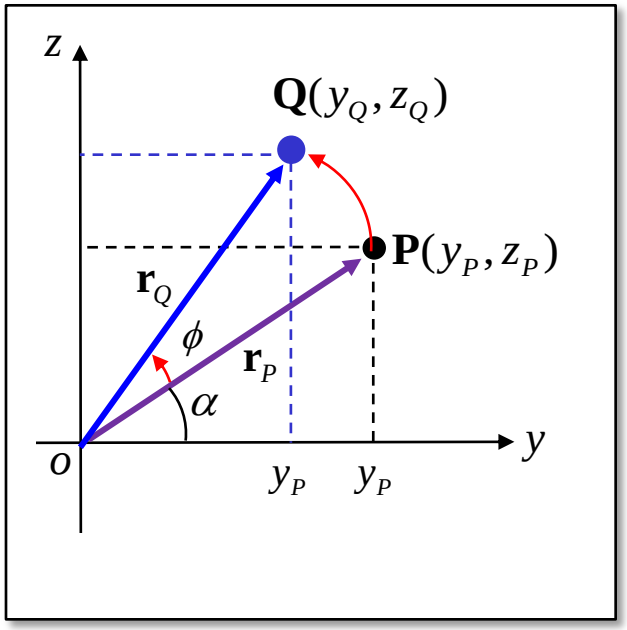
To see transformation properties

Point Transformation and Coordinate System Transformation

✓ 고정된 좌표계에서 물체의 회전

Given: oyz 에서 정의된 점 P의 좌표값

Find : 점 P 을 oyz 에 대해 ϕ 만큼 회전시킨 점 Q구하기



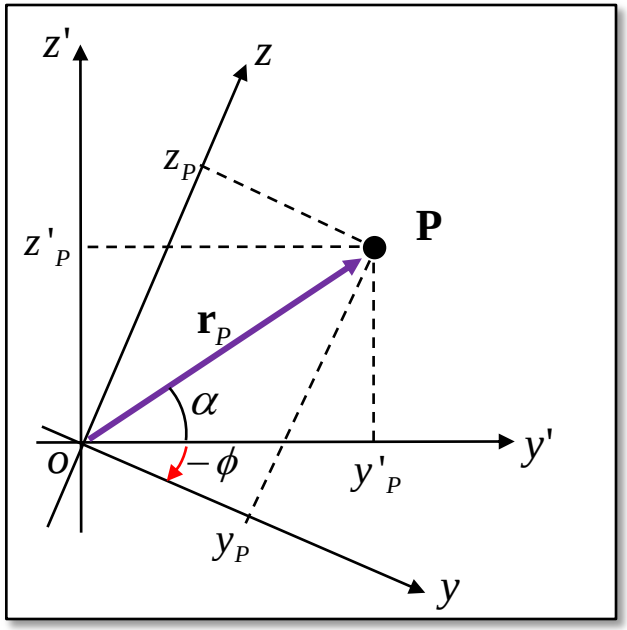
$$\begin{bmatrix} y_Q \\ z_Q \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} y_P \\ z_P \end{bmatrix}$$

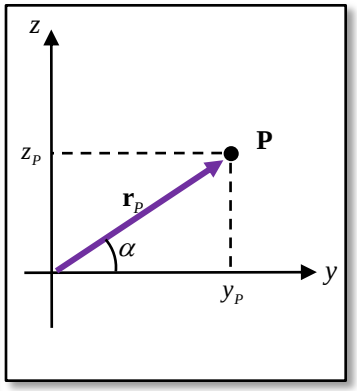
점의 회전 변환

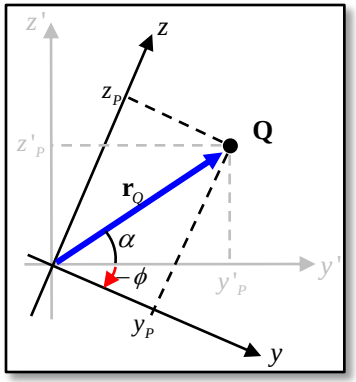
✓ 좌표계 회전

Given: $oy'z'$ 에서 정의된 점 P의 좌표값

Find: $oy'z'$ 에 대해 $-\phi$ 만큼 회전한 새로운 좌표계 oyz 에서의 P의 좌표값







$$\begin{bmatrix} y_P \\ z_P \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} y'_P \\ z'_P \end{bmatrix}$$

좌표계 회전 변환

$oy'z'$: *Body fixed coordinate*

oyz : *Global coordinate*

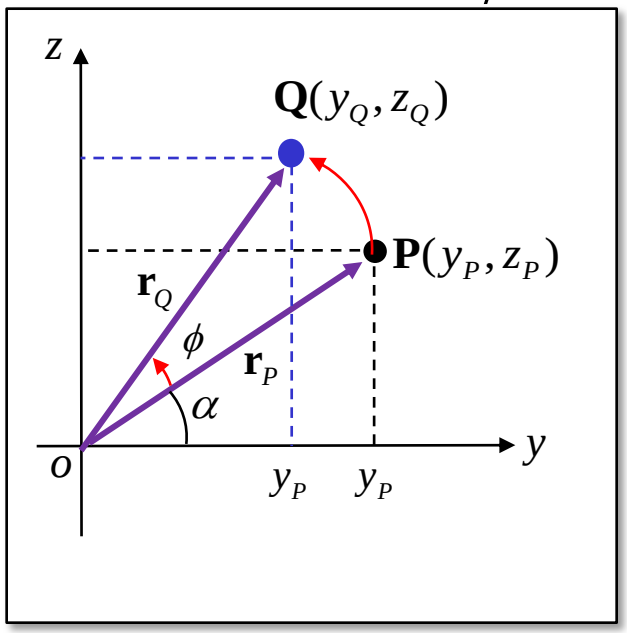
점을 ϕ 만큼 회전시키는 변환 행렬과 좌표계를 $-\phi$ 만큼 회전시키는 변환 행렬이 동일함

$$\mathbf{r}_p = \mathbf{r}_Q$$

Point Transformation and Coordinate System Transformation

Given: oyz 에서 정의된 점 P의 좌표값

Find: 점 P 을 oyz 에 대해 ϕ 만큼 회전시킨 점 Q구하기



① 점 P,Q의 좌표를 각으로 표현하면,

$$y_P = |\mathbf{r}_P|\cos\alpha \qquad y_Q = |\mathbf{r}_Q|\cos(\alpha+\phi)$$

$$z_P = |\mathbf{r}_P|\sin\alpha \qquad z_Q = |\mathbf{r}_Q|\sin(\alpha+\phi)$$

② 삼각함수 합공식

$$\sin(\alpha+\phi) = \sin\alpha\cos\phi + \cos\alpha\sin\phi$$

$$\cos(\alpha+\phi) = \cos\alpha\cos\phi - \sin\alpha\sin\phi$$

③ 점 Q의 좌표를 삼각함수의 차공식으로 전개하면,

$$\begin{aligned} y_Q &= |\mathbf{r}_Q|\cos(\alpha+\phi) \\ &= |\mathbf{r}_Q|\cos\alpha\cos\phi - |\mathbf{r}_Q|\sin\alpha\sin\phi \\ &= (|\mathbf{r}_P|\cos\alpha)\cos\phi - (|\mathbf{r}_P|\sin\alpha)\sin\phi \quad ,(|\mathbf{r}_P| = |\mathbf{r}_Q|) \\ &= y_P\cos\phi - z_P\sin\phi \end{aligned}$$

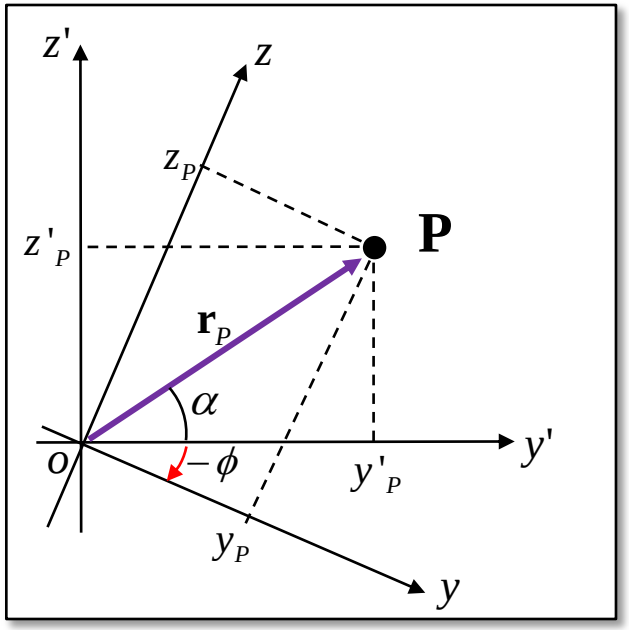
$$\begin{aligned} z_Q &= |\mathbf{r}_Q|\sin(\alpha+\phi) \\ &= |\mathbf{r}_Q|\sin\alpha\cos\phi + |\mathbf{r}_Q|\cos\alpha\sin\phi \\ &= (|\mathbf{r}_P|\sin\alpha)\cos\phi + (|\mathbf{r}_P|\cos\alpha)\sin\phi \quad ,(|\mathbf{r}_P| = |\mathbf{r}_Q|) \\ &= z_P\cos\phi + y_P\sin\phi \end{aligned}$$

④ 행렬로 표현하면,

$$\begin{bmatrix} y_Q \\ z_Q \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} y_P \\ z_P \end{bmatrix}$$

Point Transformation and Coordinate System Transformation

Given: $oy'z'$ 에서 정의된 점 P의 좌표값
Find: $oy'z'$ 에 대해 $-\phi$ 만큼 회전한 새로운 좌표계 oyz 에서의 P의 좌표값



① 점 P의 좌표를 각으로 표현하면,

$$y'_P = |\mathbf{r}_P|\cos\alpha \qquad y_P = |\mathbf{r}_P|\cos(\alpha+\phi)$$
$$z'_P = |\mathbf{r}_P|\sin\alpha \qquad z_P = |\mathbf{r}_P|\sin(\alpha+\phi)$$

② 삼각함수 합공식

$$\sin(\alpha+\phi) = \sin\alpha\cos\phi + \cos\alpha\sin\phi$$
$$\cos(\alpha+\phi) = \cos\alpha\cos\phi - \sin\alpha\sin\phi$$

③ 점 P의 좌표를 삼각함수의 차공식으로 전개하면,

$$\begin{aligned} y_P &= |\mathbf{r}_P|\cos(\alpha+\phi) \\ &= |\mathbf{r}_P|\cos\alpha\cos\phi - |\mathbf{r}_P|\sin\alpha\sin\phi \\ &= (|\mathbf{r}_P|\cos\alpha)\cos\phi - (|\mathbf{r}_P|\sin\alpha)\sin\phi \\ &= y'_P\cos\phi - z'_P\sin\phi \end{aligned}$$

$$\begin{aligned} z_P &= |\mathbf{r}_P|\sin(\alpha+\varphi) \\ &= |\mathbf{r}_P|\sin\alpha\cos\varphi + |\mathbf{r}_P|\cos\alpha\sin\varphi \\ &= (|\mathbf{r}_P|\sin\alpha)\cos\varphi + (|\mathbf{r}_P|\cos\alpha)\sin\varphi \\ &= z'_P\cos\varphi + y'_P\sin\varphi \end{aligned}$$

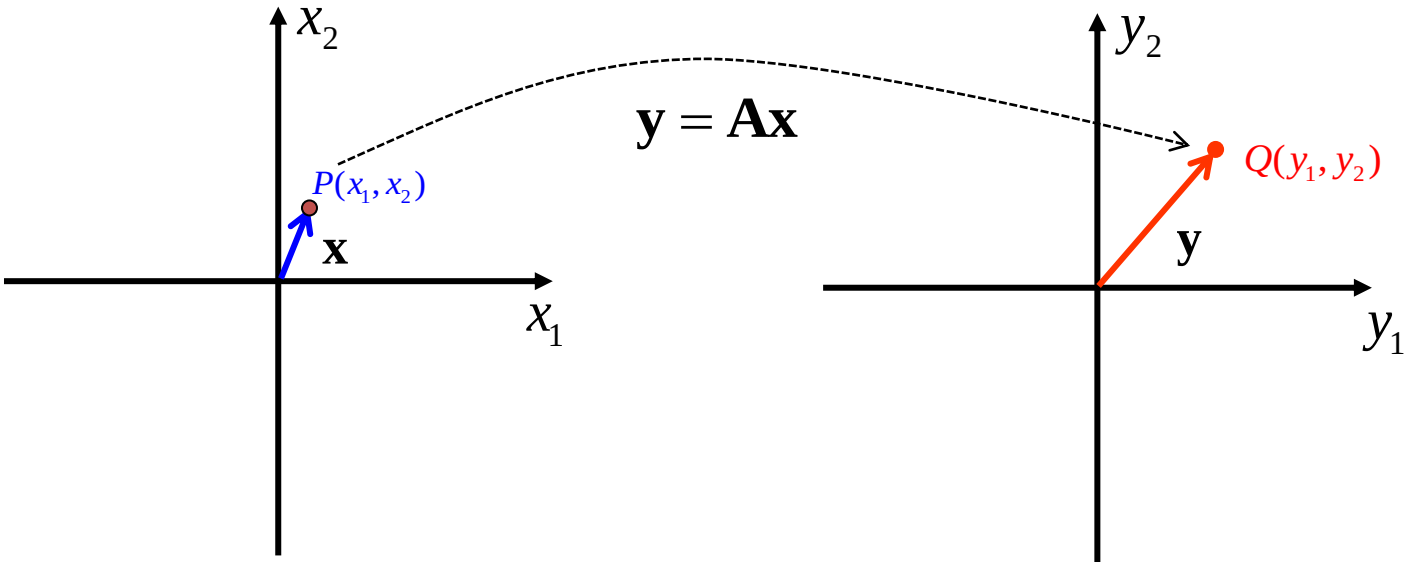
④ 행렬로 표현하면,

$$\begin{bmatrix} y_P \\ z_P \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} y'_P \\ z'_P \end{bmatrix}$$

Stretching of an Elastic Membrane

An elastic membrane on the z_1z_2-plane with boundary circle $x_1^2+x_2^2=1$ is stretched so that a point $P(x_1, x_2)$ goes over into the point $Q(y_1, y_2)$ given by

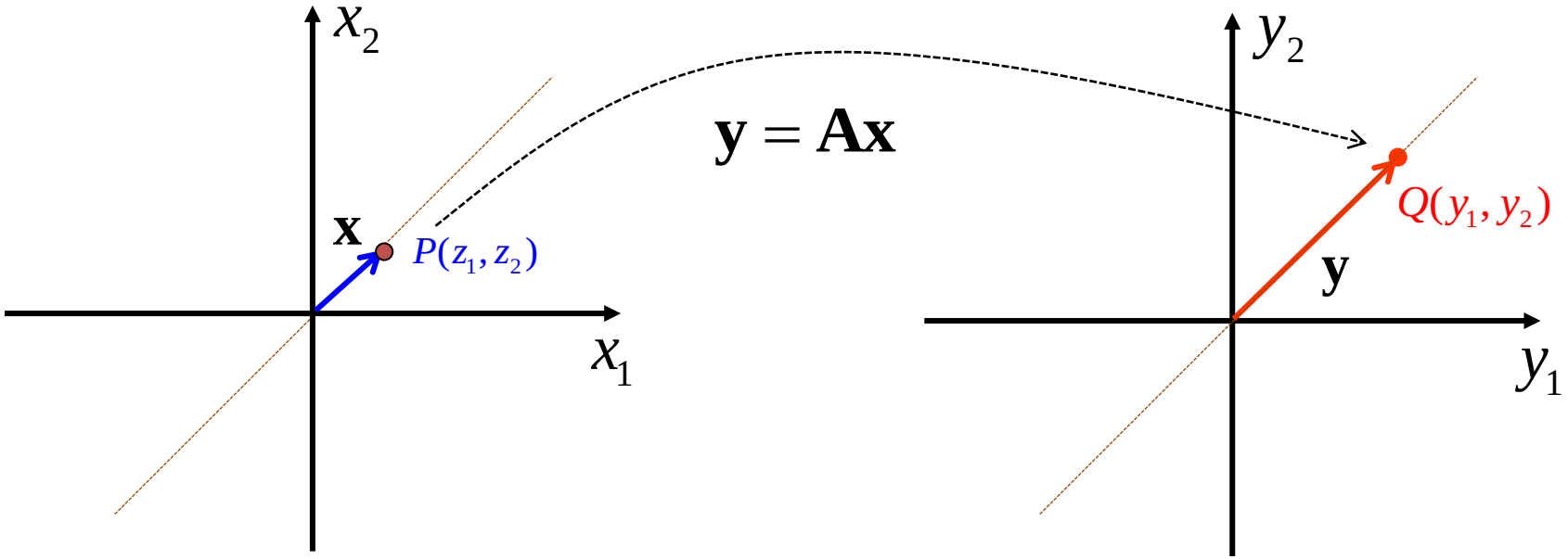
$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{A}\mathbf{x} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



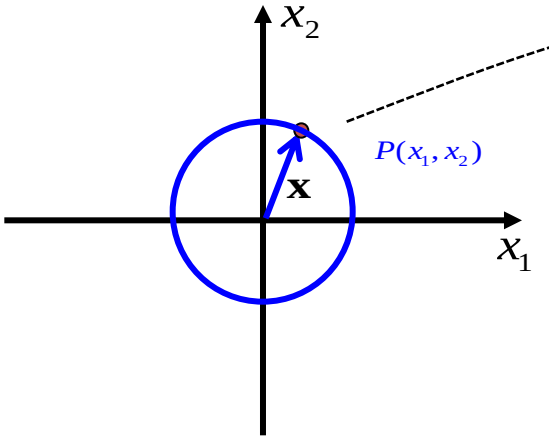
continues...

Stretching of an Elastic Membrane

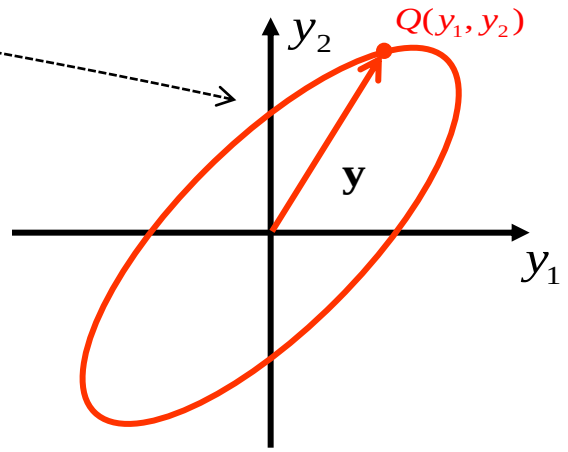
Find the principle directions, that is, the directions of the position vector $\mathbf{x}$ of P for which the directions of the position vector $\mathbf{y}$ of Q is the same or exactly opposite. What shape does the boundary circle take under this deformation?



Stretching of an Elastic Membrane



$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{A}\mathbf{x} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Given:

Transformed :

$$x_1^2 + x_2^2 = 1$$

$$\Rightarrow \quad \begin{cases} y_1 = 5x_1 + 3x_2 \\ y_2 = 3x_1 + 5x_2 \end{cases}$$

$$\Rightarrow \quad \begin{cases} x_1 = \dfrac{1}{16}(5y_1 - 3y_2) \\ x_2 = \dfrac{1}{16}(-3y_1 + 5y_2) \end{cases}$$

$$\frac{1}{16^2}(5y_1 - 3y_2)^2 + \frac{1}{16^2}(-3y_1 + 5y_2)^2 = 1$$

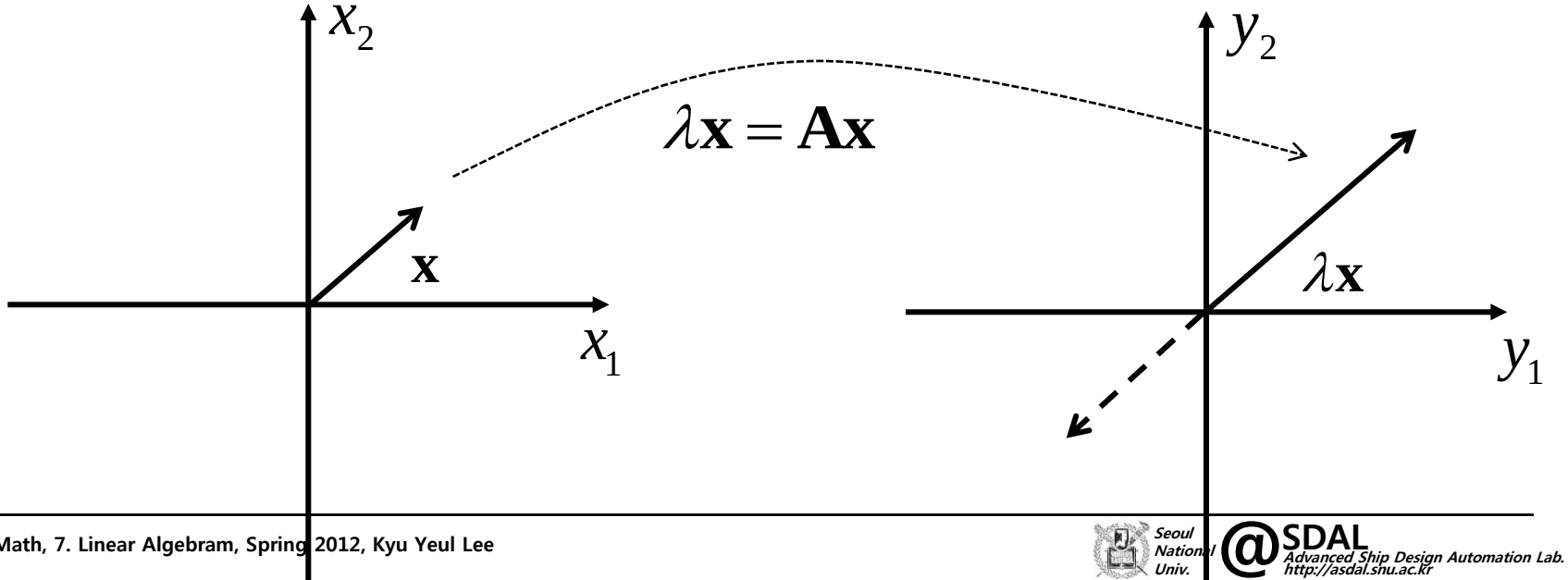
$$34y_1^2 - 60y_1y_2 + 34y_2^2 = 256$$

Stretching of an Elastic Membrane

An elastic membrane on the z_1z_2-plane with boundary circle $x_1^2+x_2^2=1$ is stretched so that a point $P(x_1, x_2)$ goes over into the point $Q(y_1, y_2)$ given by

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{A}\mathbf{x} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find the principle directions, that is, the directions of the position vector $\mathbf{x}$ of P for which the directions of the position vector $\mathbf{y}$ of Q is the same or exactly opposite. What shape does the boundary circle take under this deformation?



Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x after the stretch is the same direction before the stretch.

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix}5-\lambda & 3\\ 3 & 5-\lambda\end{vmatrix}=0$$

$$\lambda^2-10\lambda+25-9=\lambda^2-10\lambda+16$$

$$=(\lambda-8)(\lambda-2)=0$$

$$\therefore \lambda_1=8,\ \lambda_2=2$$

(1) $\lambda=\lambda_1=8$

$$-3x_1+3x_2=0$$

$$3x_1-3x_2=0$$

$$\therefore x_1=x_2$$

For instance, $x_1=x_2=1$

$$\mathbf{x}_1=\begin{bmatrix}1 & 1\end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{Ax}_1=\begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}8\\ 8\end{bmatrix}$$

$$=8\begin{bmatrix}1\\ 1\end{bmatrix}=\lambda_1\mathbf{x}_1$$

Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x after the stretch is the same direction before the stretch.

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

Characteristic Equation:

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$$\begin{vmatrix}5-\lambda & 3\\ 3 & 5-\lambda\end{vmatrix}=0$$

$$\lambda^2-10\lambda+25-9=\lambda^2-10\lambda+16$$

$$=(\lambda-8)(\lambda-2)=0$$

$$\therefore \lambda_1=8,\ \lambda_2=2$$

(2) $\lambda=\lambda_2=2$

$$3x_1+3x_2=0$$

$$3x_1+3x_2=0$$

$$\therefore x_1=-x_2$$

For instance, $x_1=1,\ x_2=-1$

$$\mathbf{x}_2=\begin{bmatrix}1 & -1\end{bmatrix}^{\mathrm{T}}$$

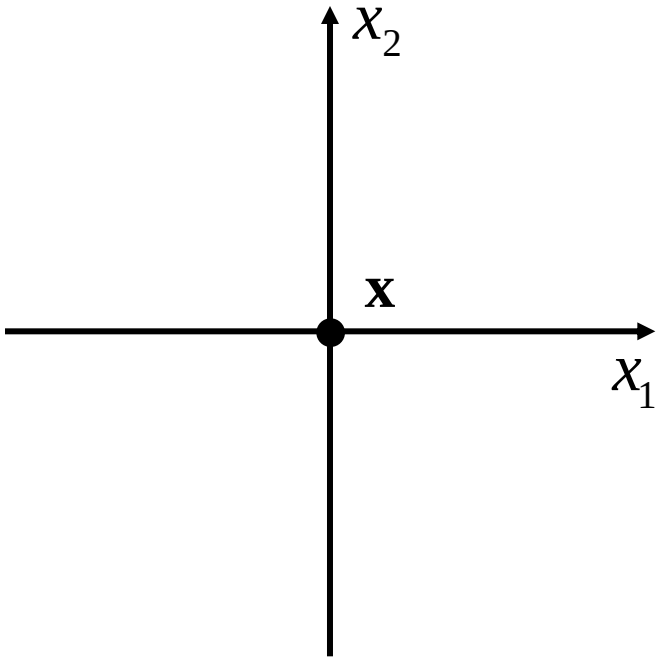
$$\mathbf{Ax}_2=\begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix}=\begin{bmatrix}2\\ -2\end{bmatrix}$$

$$=2\begin{bmatrix}1\\ -1\end{bmatrix}=\lambda_2\mathbf{x}_2$$

Transformation and trivial solution

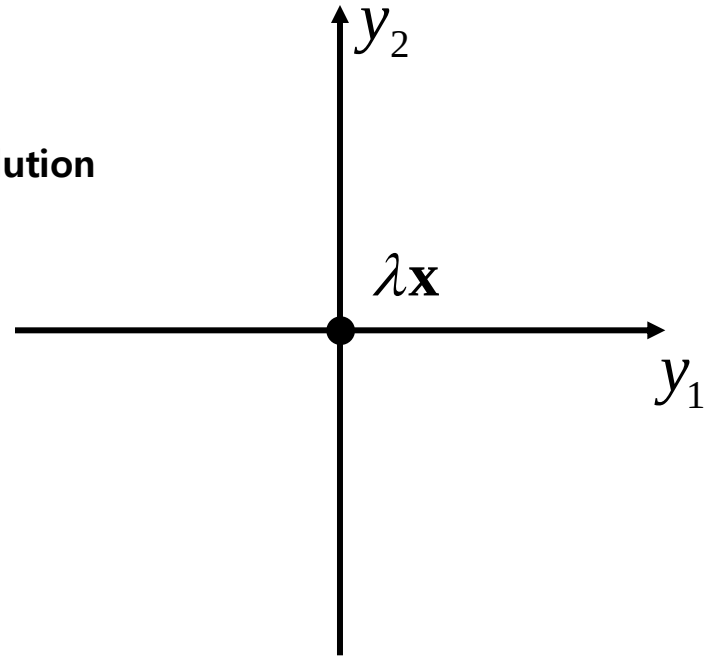
$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{A}\mathbf{x} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find the principle directions, that is, the directions of the position vector $\mathbf{x}$ of P for which the directions of the position vector $\mathbf{y}$ of Q is the same or exactly opposite. What shape does the boundary circle take under this deformation?



$$\lambda\mathbf{x} = \mathbf{A}\mathbf{x}$$

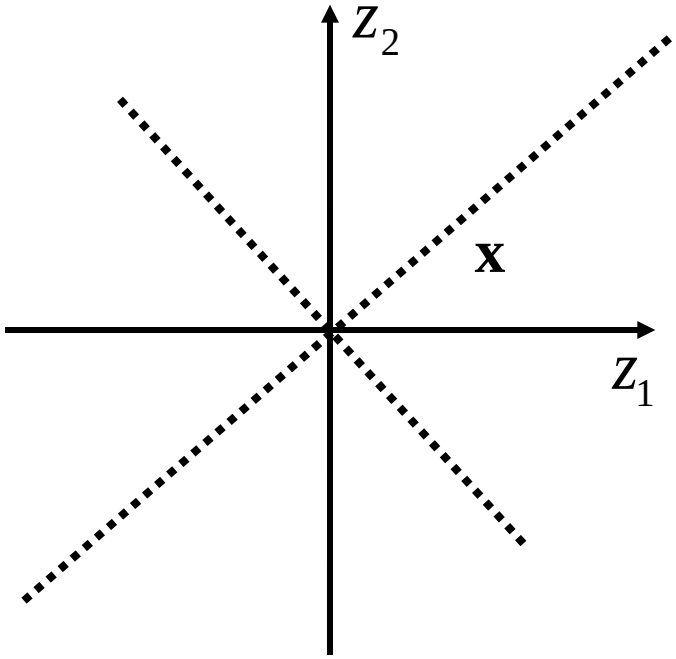
$\mathbf{x} = \mathbf{0}$:Trivial solution



Transformation and trivial solution

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{A}\mathbf{x} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find the principle directions, that is, the directions of the position vector $\mathbf{x}$ of P for which the directions of the position vector $\mathbf{y}$ of Q is the same or exactly opposite. What shape does the boundary circle take under this deformation?

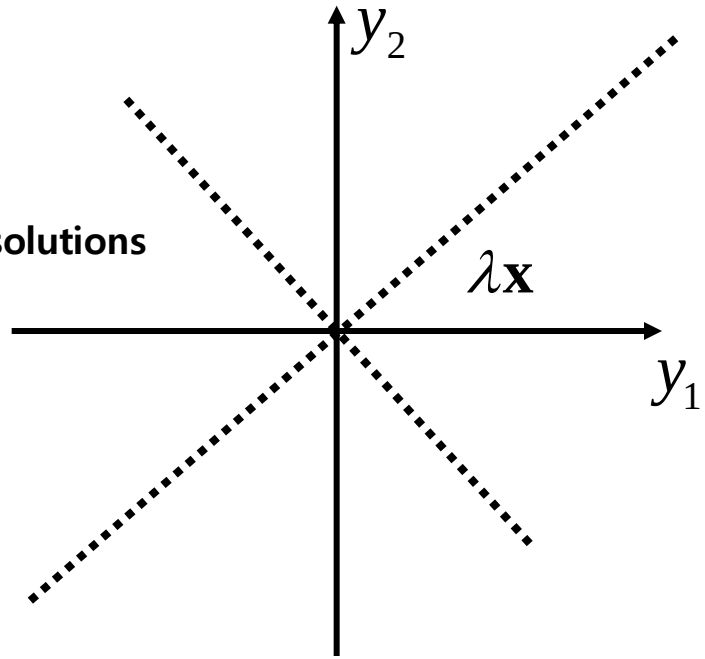


$$\lambda\mathbf{x} = \mathbf{A}\mathbf{x}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

$\mathbf{x}$ **:Nontrivial many solutions**



Stretching of an Elastic Membrane

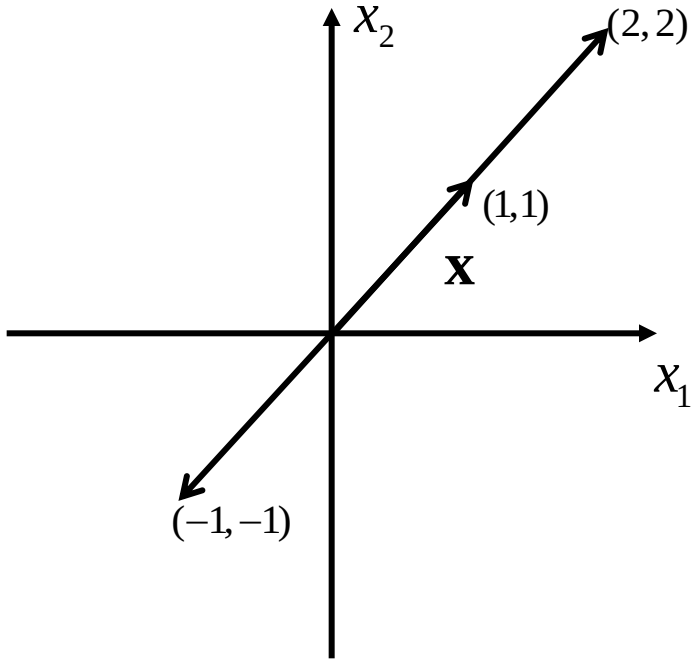
Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x after the stretch is the same direction before the stretch.

$$1)\ \lambda_1 = 8$$

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

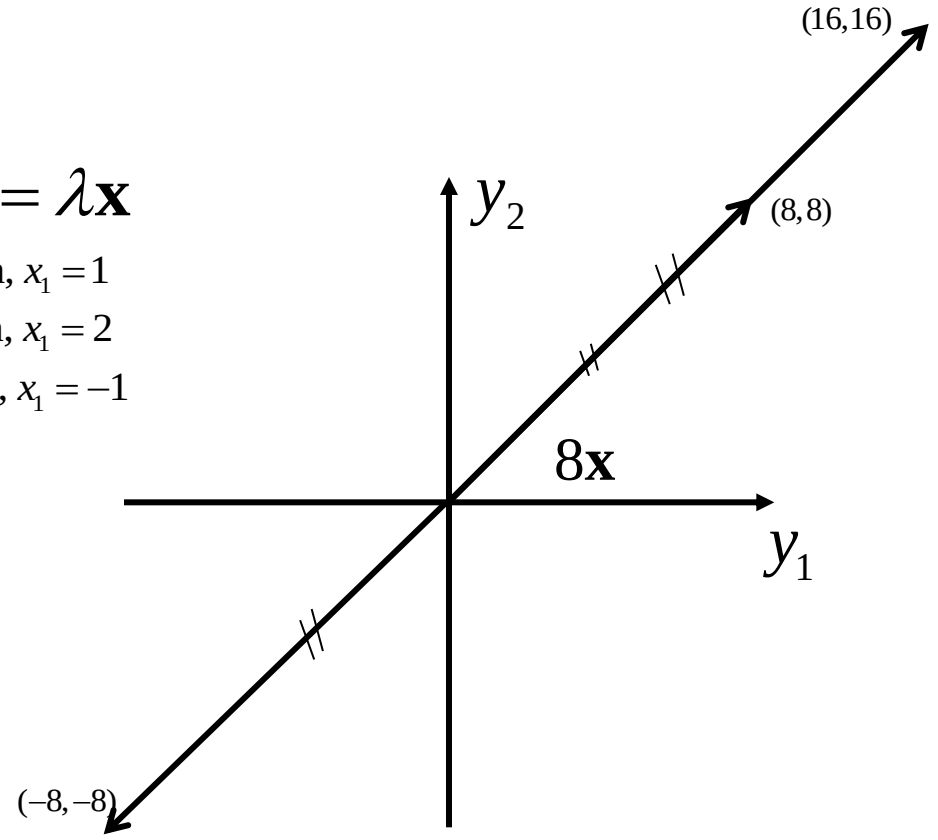


$$\mathbf{Ax} = \lambda\mathbf{x}$$

when, $x_1 = 1$

when, $x_1 = 2$

when, $x_1 = -1$



Stretching of an Elastic Membrane

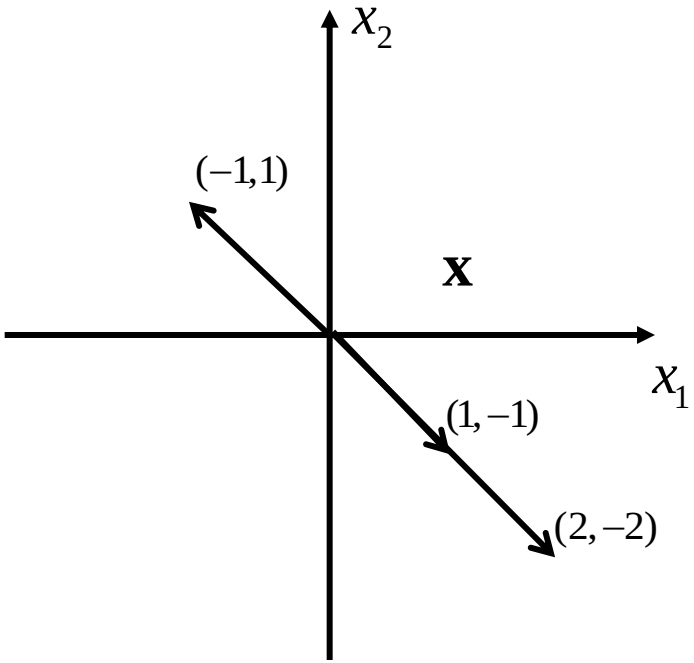
Use Eigenvalue, Eigenvector

When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of x after the stretch is the same direction before the stretch.

1) $\lambda_1 = 2$

$$\mathbf{Ax}_2 = \lambda_2 \mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

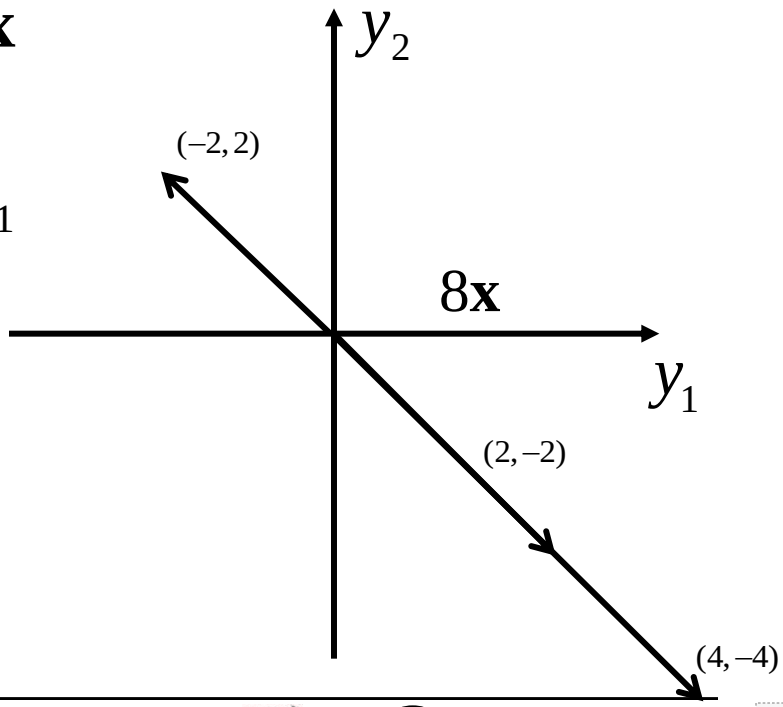


$$\mathbf{Ax} = \lambda\mathbf{x}$$

when, $x_1 = 1$

when, $x_1 = 2$

when, $x_1 = -1$



Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

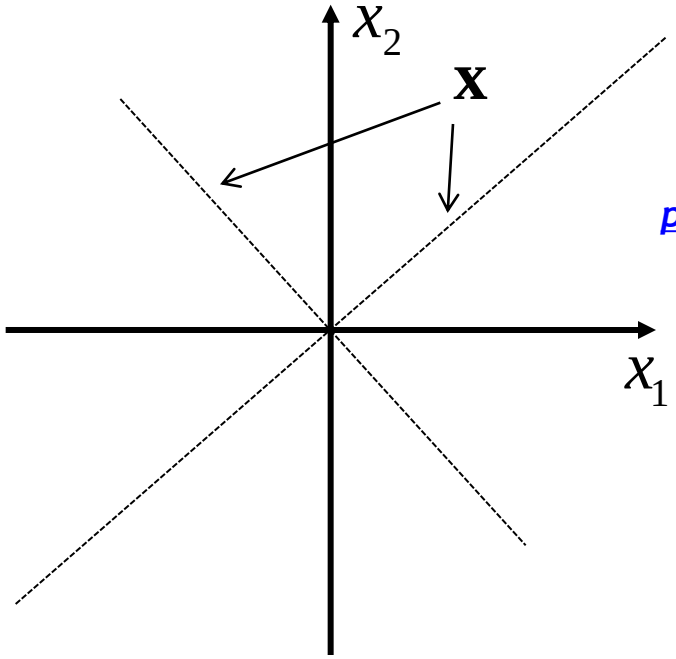
When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

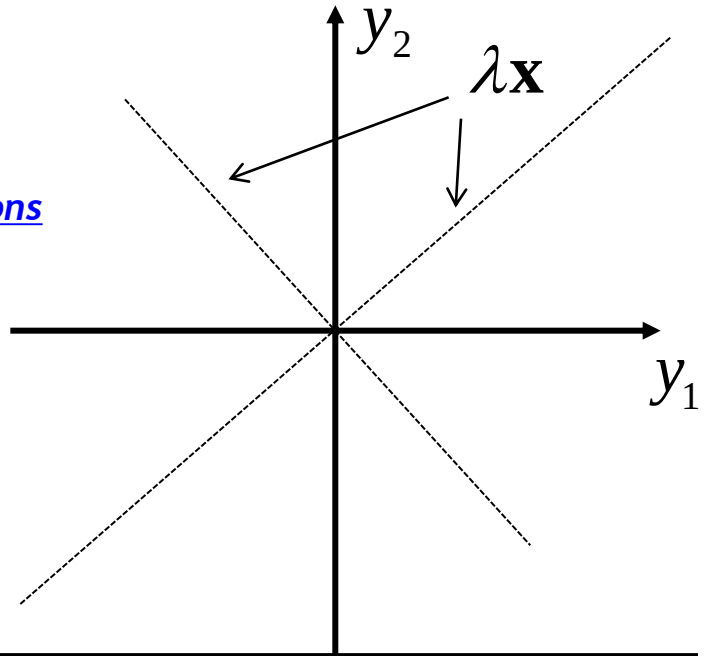
$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

$$\lambda\mathbf{x} = \mathbf{Ax}$$



principle directions



Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

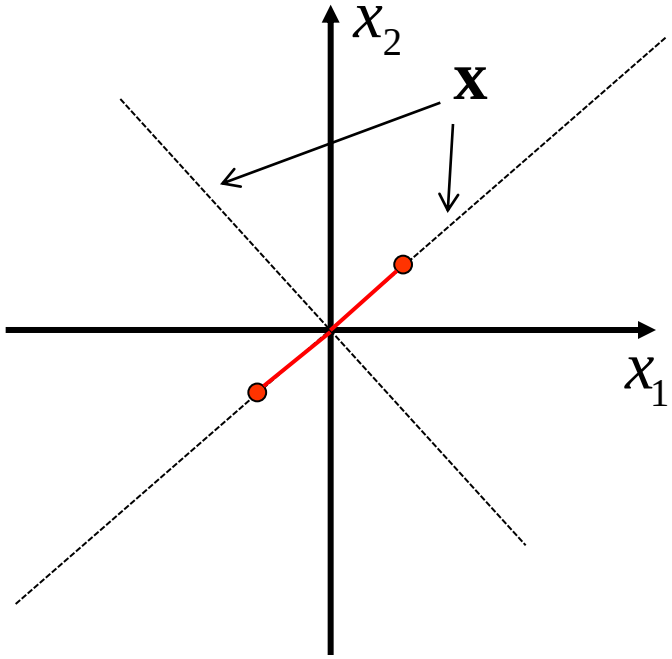
When $\mathbf{Ax}=\lambda\mathbf{x}$.

It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

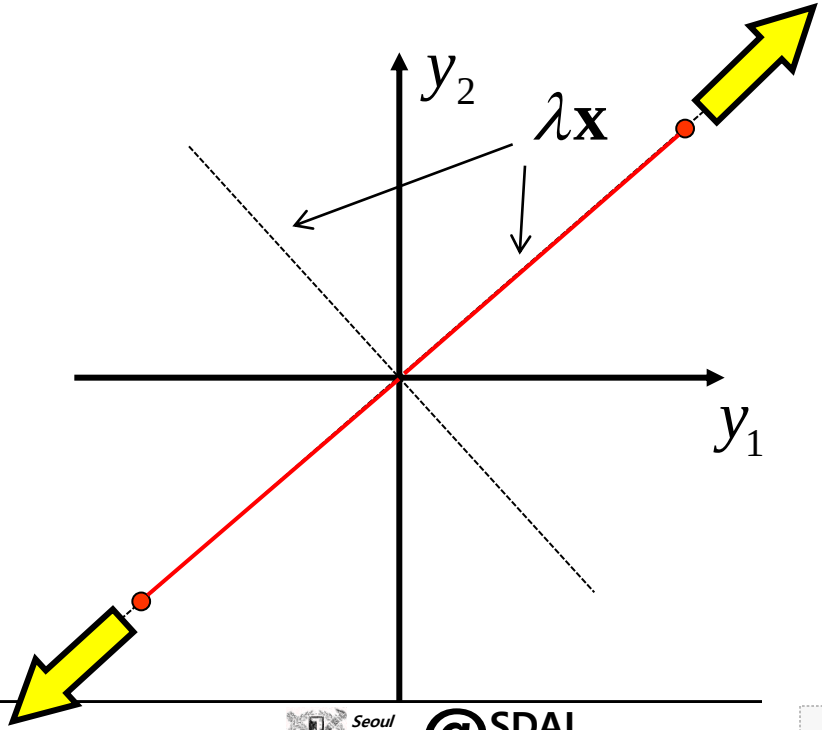
$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

$$\lambda\mathbf{x} = \mathbf{Ax}$$



Stretching by 8 times



Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

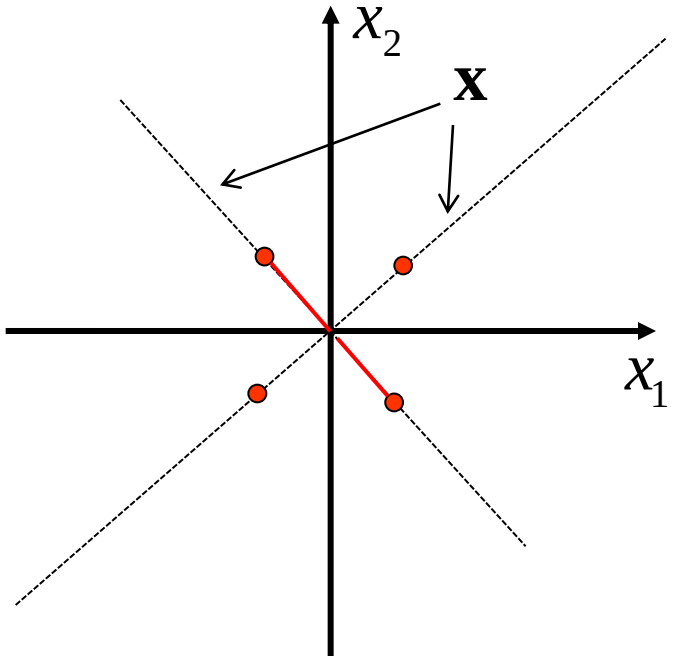
When $\mathbf{Ax}=\lambda\mathbf{x}$.

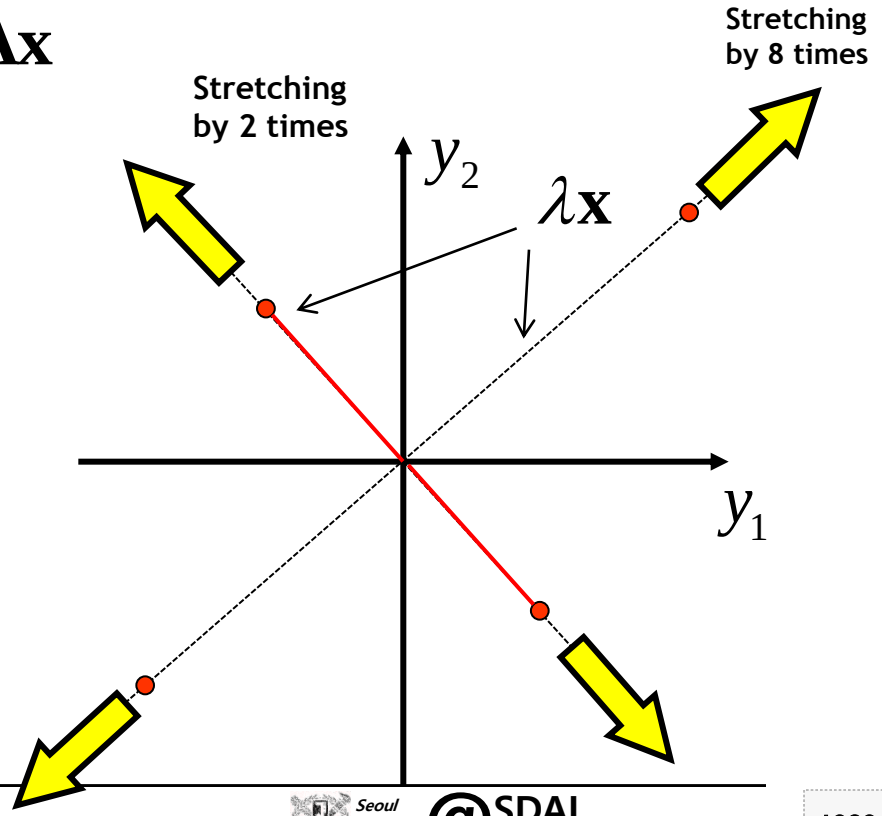
It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

$$\lambda\mathbf{x} = \mathbf{Ax}$$





Stretching of an Elastic Membrane

Use Eigenvalue, Eigenvector

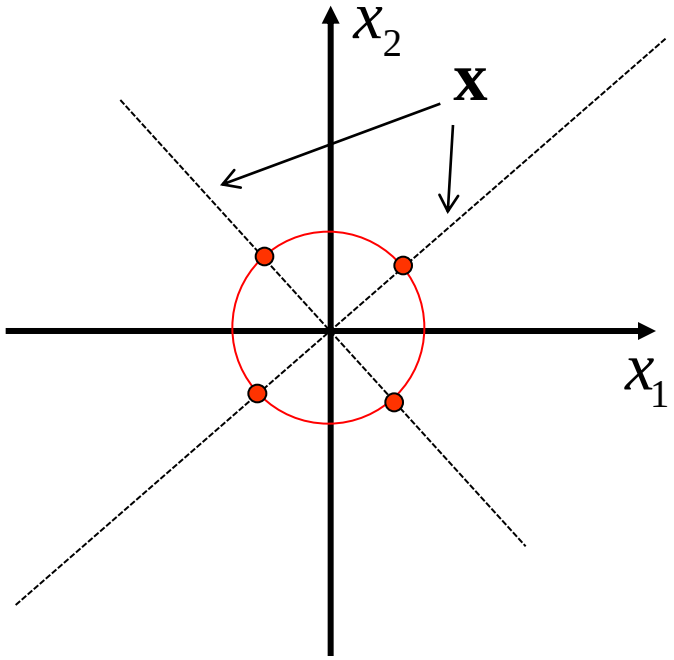
When $\mathbf{Ax}=\lambda\mathbf{x}$.

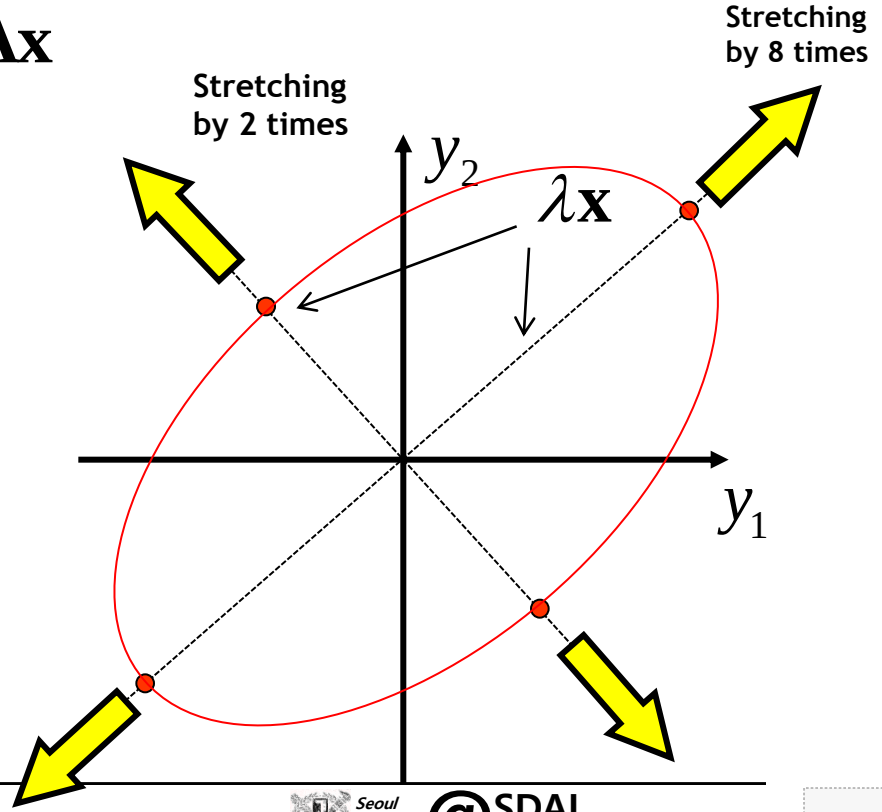
It means that the direction of $\mathbf{x}$ *after the stretch is the same direction before the stretch.*

$$\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1 = 8\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

$$\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2 = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

$$\lambda\mathbf{x} = \mathbf{Ax}$$





Stretching of an Elastic Membrane

So $\lambda_1 = 8, \ \mathbf{x}_1 = [1 \quad 1]^T$

➔ stretching by 8(= λ_1) *times* to the direction of $\mathbf{x}_1=[1 \ \ 1]^T$.

$\lambda_2 = 2, \ \mathbf{x}_2 = [1 \quad -1]^T$

➔ stretching by 2(= λ_2) *times* to the direction of $\mathbf{x}_2=[1 \ \ -1]^T$.

This is *principal directions*. This vector make 45° and 135° angles with the positive x_1-direction.

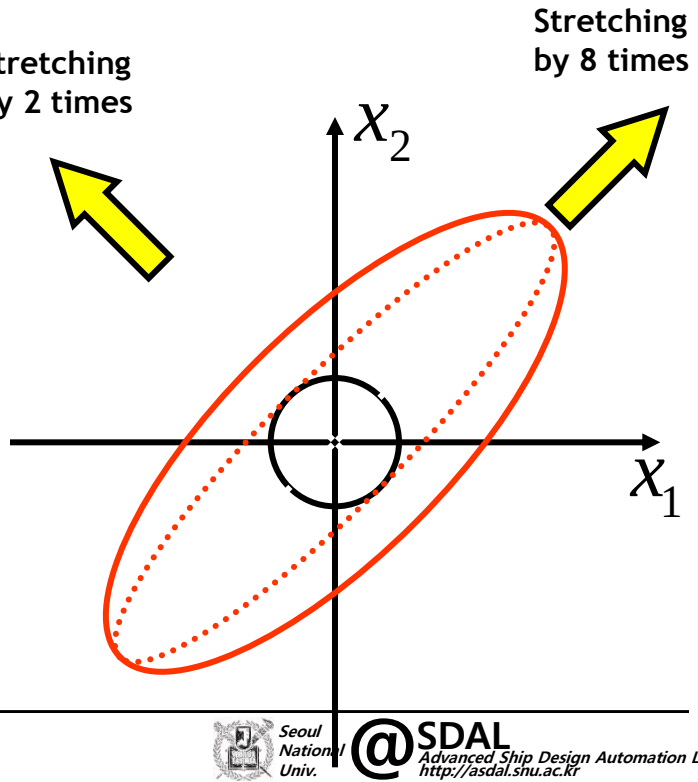
$$\mathbf{x}_1^T \mathbf{x}_2 = [1 \quad 1]\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

that eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal

A real square matrix is orthogonal

if and only if column vectors $a_1, \dots, a_n$ form an orthonormal system,

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j^T \mathbf{a}_k = \begin{cases} 0 & if \quad j \neq k \\ 1 & if \quad j = k \end{cases}$$



Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

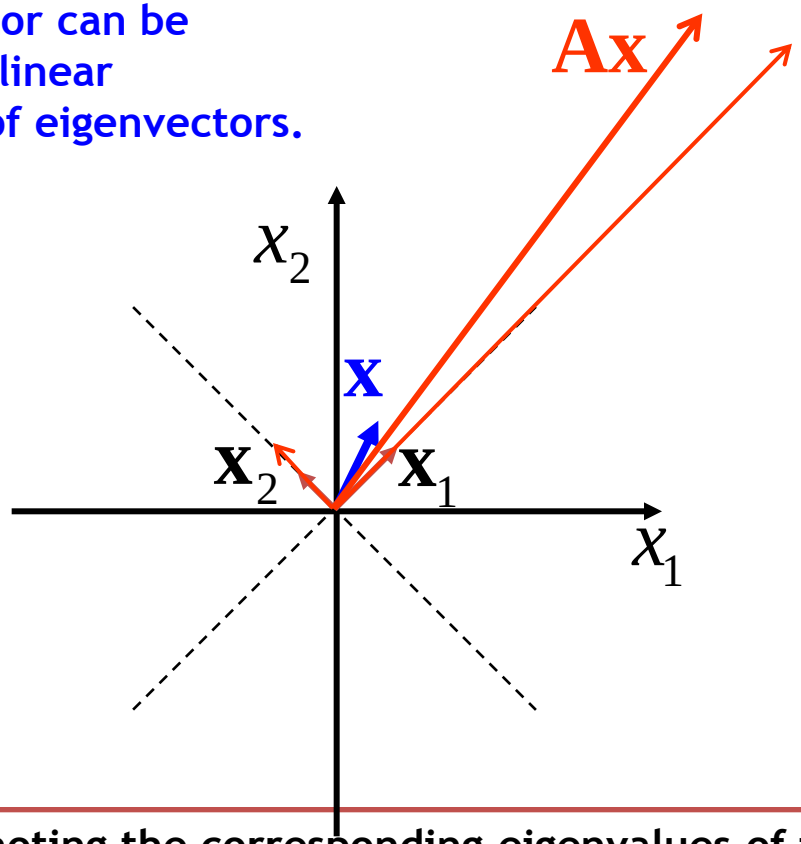
$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Ax} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$$

$$\mathbf{x} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

Arbitrary vector can be expressed by linear combination of eigenvectors.



Denoting the corresponding eigenvalues of the matrix A by $\lambda_1, \ldots, \lambda_n$, we have $\mathbf{Ax}_j = \lambda_j \mathbf{x}_j$, so that we simply obtain

$$\begin{aligned} \mathbf{y} = \mathbf{Ax} &= \mathbf{A}\left(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n\right) \\ &= c_1\mathbf{Ax}_1 + c_2\mathbf{Ax}_2 + \cdots + c_n\mathbf{Ax}_n \\ &= c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \cdots + c_n\lambda_n\mathbf{x}_n \end{aligned}$$

Stretching of an Elastic Membrane

$\mathbf{Ax} = \lambda \mathbf{x}$

$$\mathbf{x} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

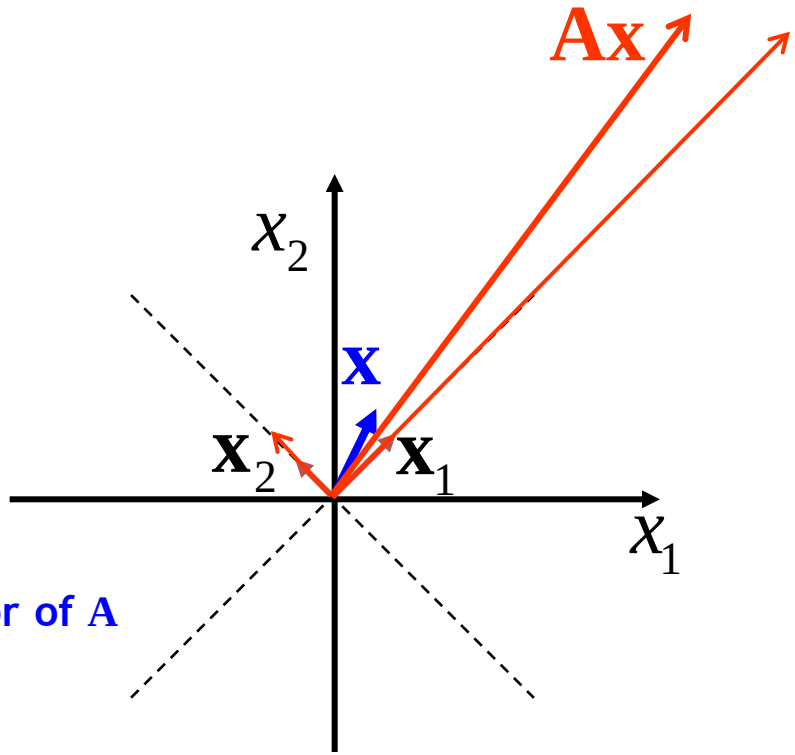
$$\mathbf{Ax} = \mathbf{A}(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2)$$

$$= \alpha \mathbf{A} \mathbf{x}_1 + \beta \mathbf{A} \mathbf{x}_2$$

$$= \alpha \lambda_1 \mathbf{x}_1 + \beta \lambda_2 \mathbf{x}_2$$

$$= 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$

$\mathbf{x}_1$, $\mathbf{x}_2$ is eigenvector of A



$$\therefore \mathbf{Ax} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$

Ax

complicated action of A on an arbitrary vector x

decompose

$8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$

Sum of simple actions (multiplication by scalars) on the eigenvectors of A

Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

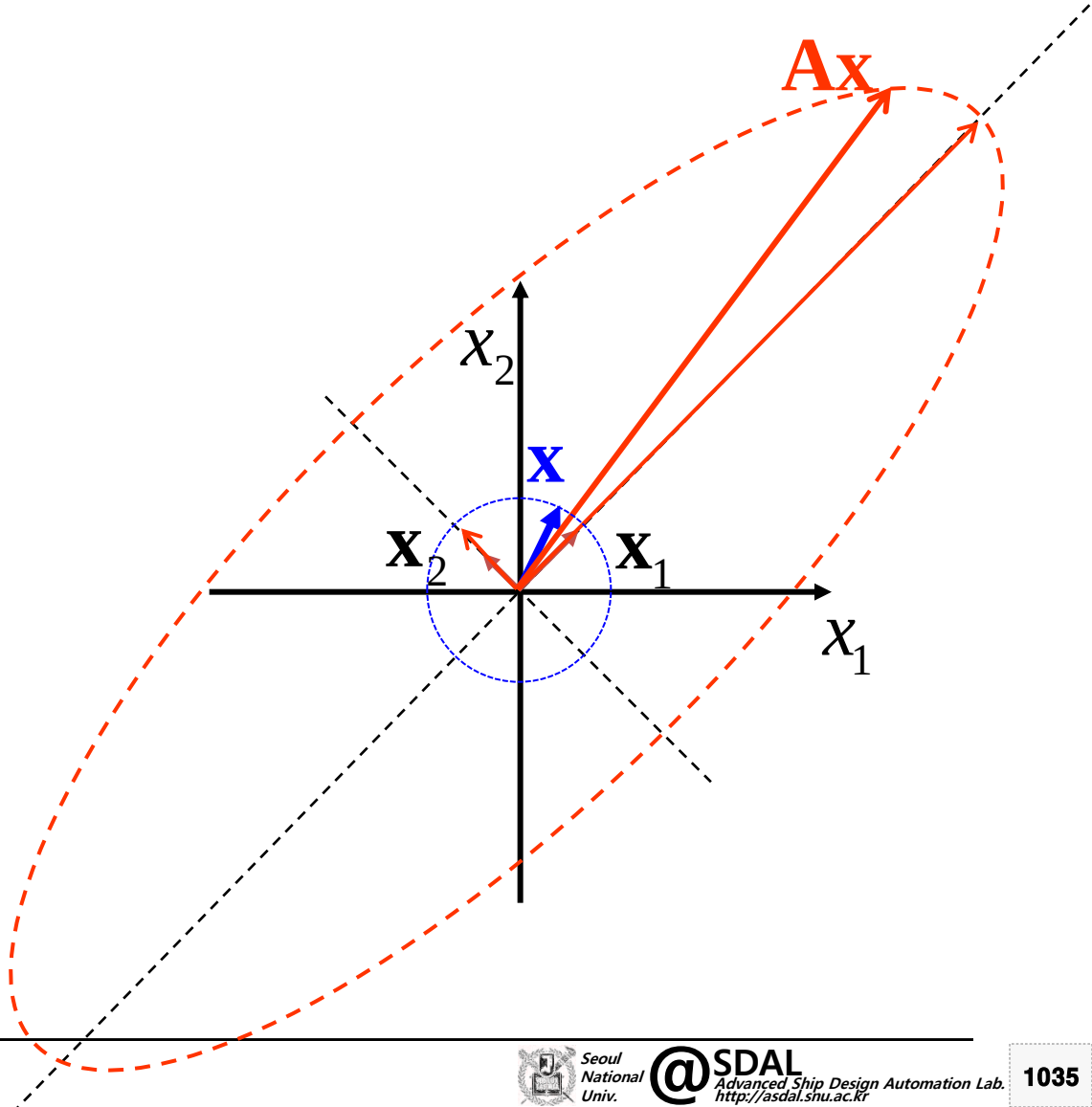
$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Ax} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{x} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\therefore \mathbf{Ax} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

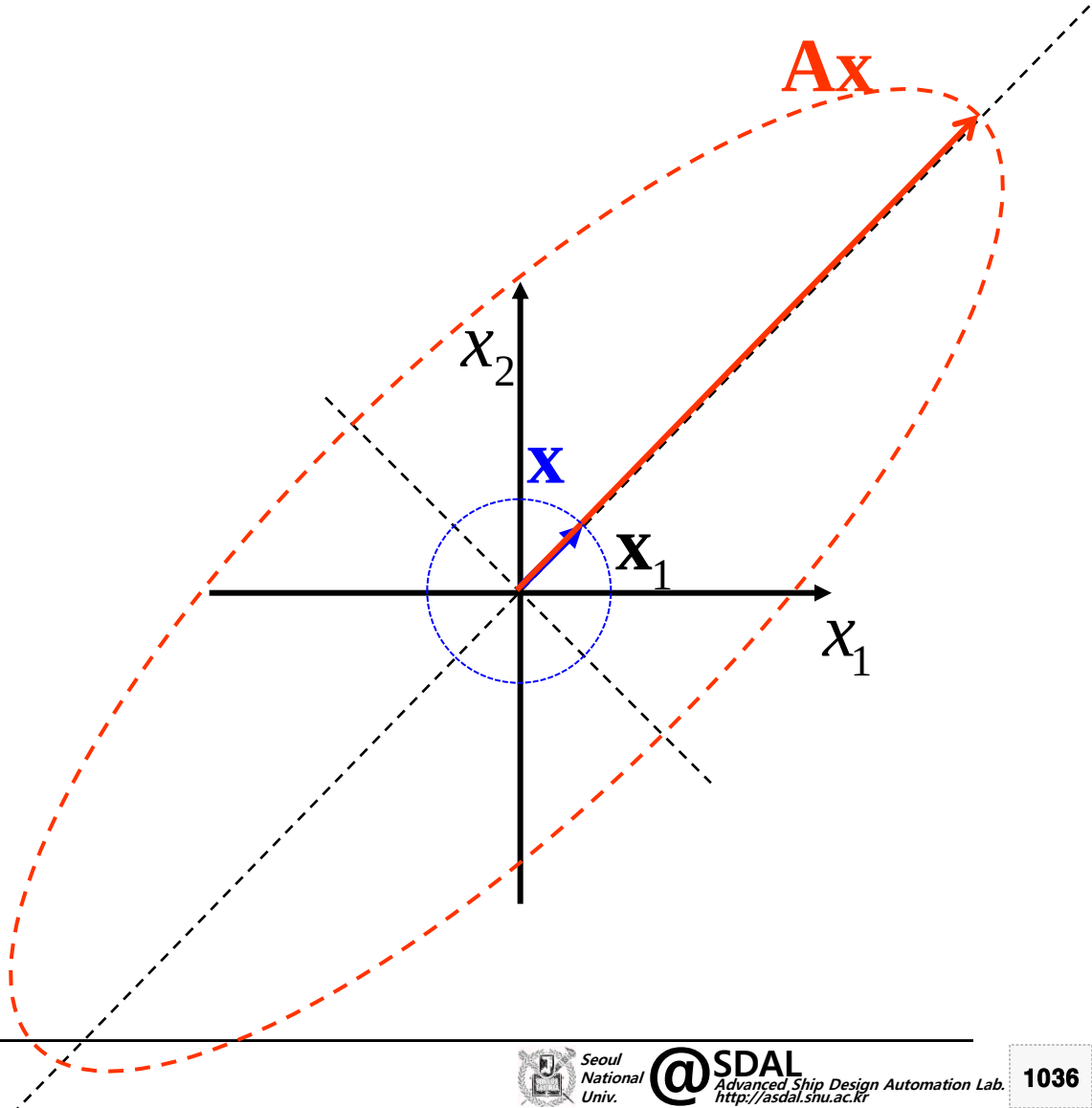
$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Ax} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{x} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\therefore \mathbf{Ax} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

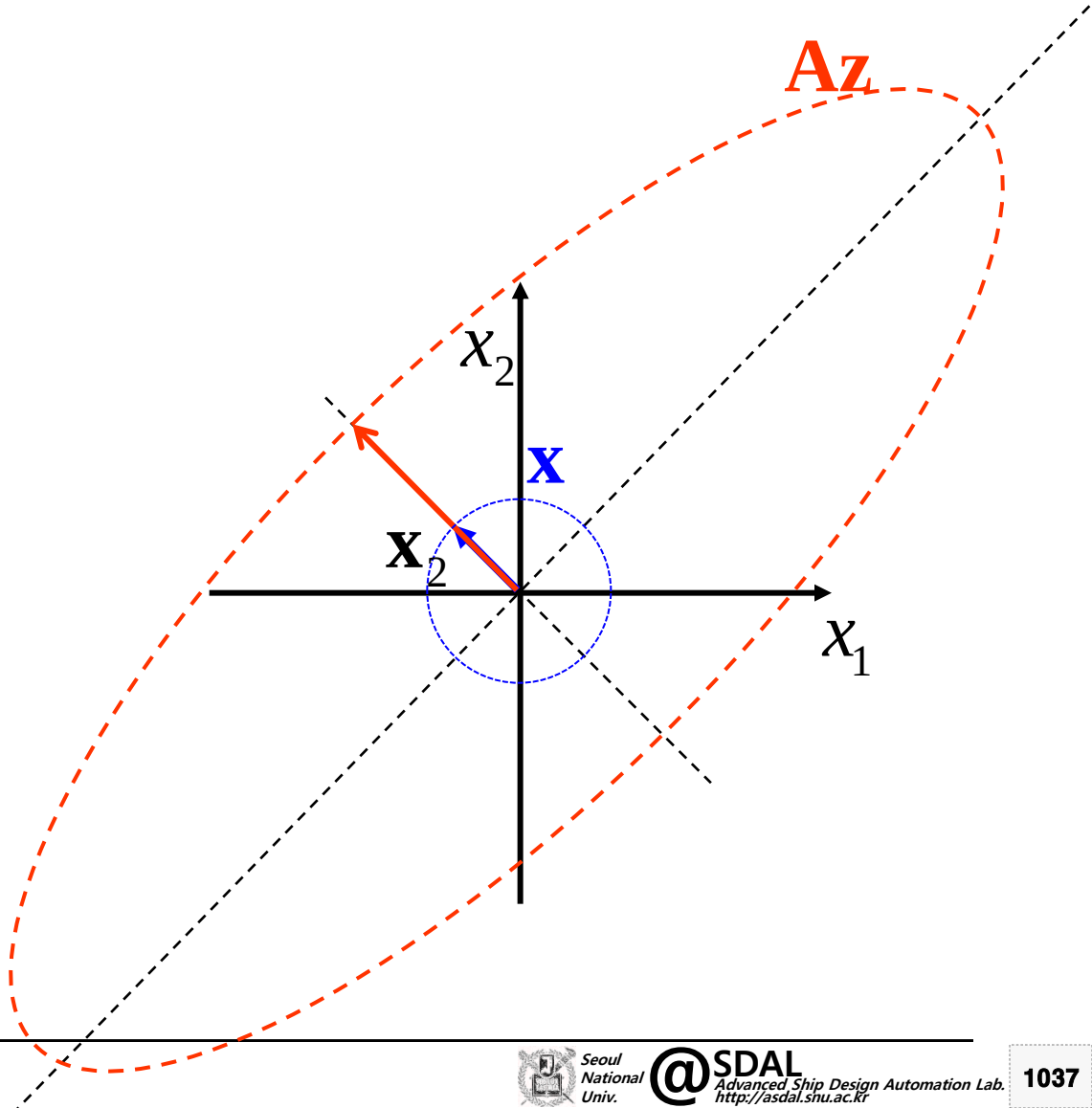
$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Ax} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{x} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\therefore \mathbf{Ax} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda\mathbf{x}$$

We transformed membrane with boundary circle(Conic) by symmetric transform matrix(case I). (eigenvalues are not orthogonal)

What happen in other cases ?

Possible Cases

Object / Transform Matrix	Conic (Circle, Ellipse, Hyperbola, Parabola)	Arbitrary shape (Rectangle in this example)
Symmetric matrix (eigenvalues are orthogonal)	**Case I**	**Case II**
Non-symmetric matrix (eigenvalues are not orthogonal)	**Case III**	**Case IV**

Conic Section(Conic) : curve that results by cutting a double-napped cone with a plane[1)]

[1)] Anton, Busby, Contemporary Linear Algebra, Wiley, 2003, Ch.8.4, p.502

Stretching of an Elastic Membrane

$$\mathbf{Ax} = \lambda \mathbf{x}$$

Case II) Transformation of **Rectangle shape** by **Symmetric Transform Matrix**,
(eigenvalues are orthogonal)

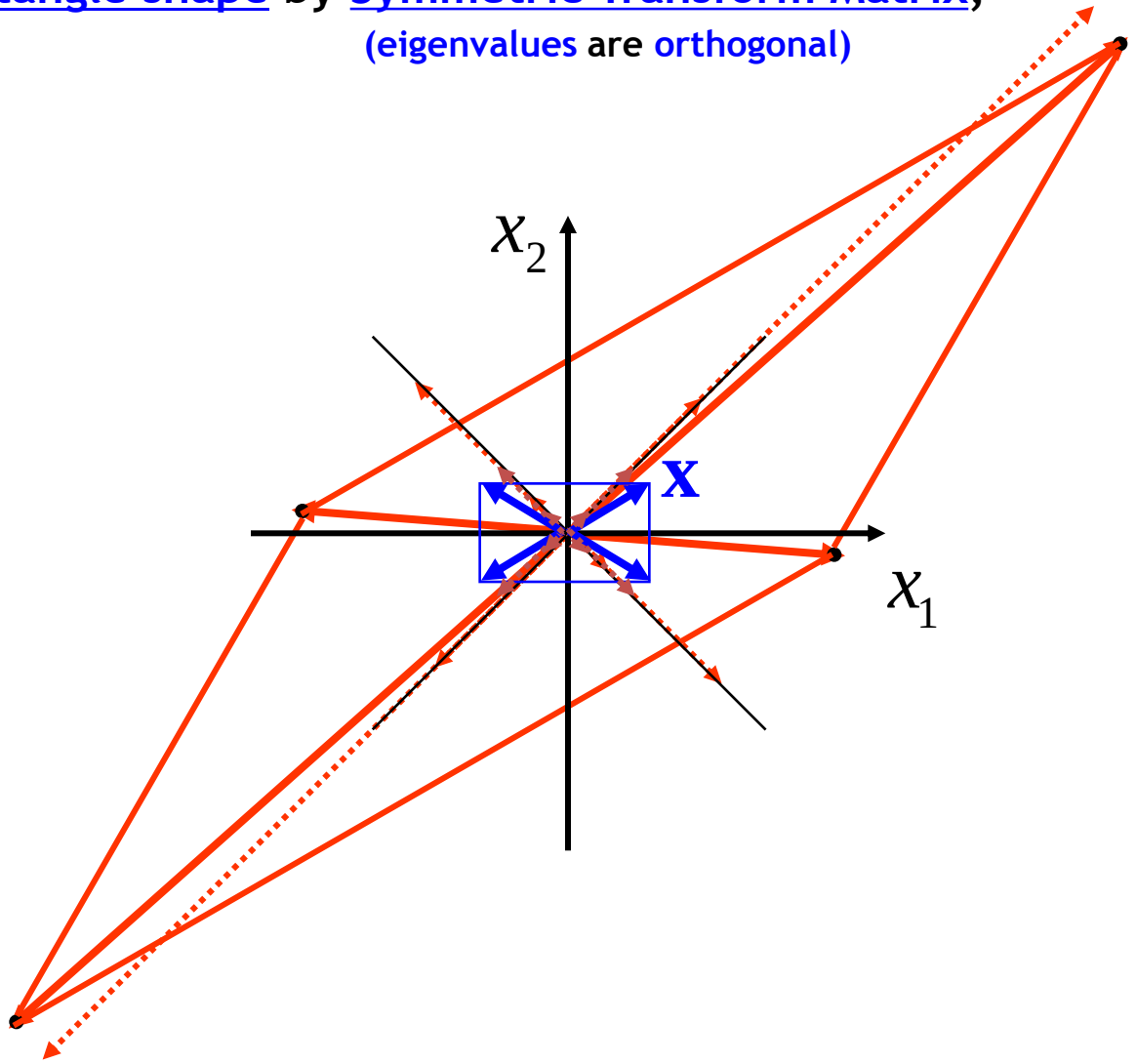
$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Ax} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$$

$$\mathbf{x} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$$

$$\therefore \mathbf{Ax} = 8\alpha \mathbf{x}_1 + 2\beta \mathbf{x}_2$$



Stretching of an Elastic Membrane

Case III) Transformation of Conic shape by Non-symmetric Transform Matrix,

(eigenvalues are not orthogonal)

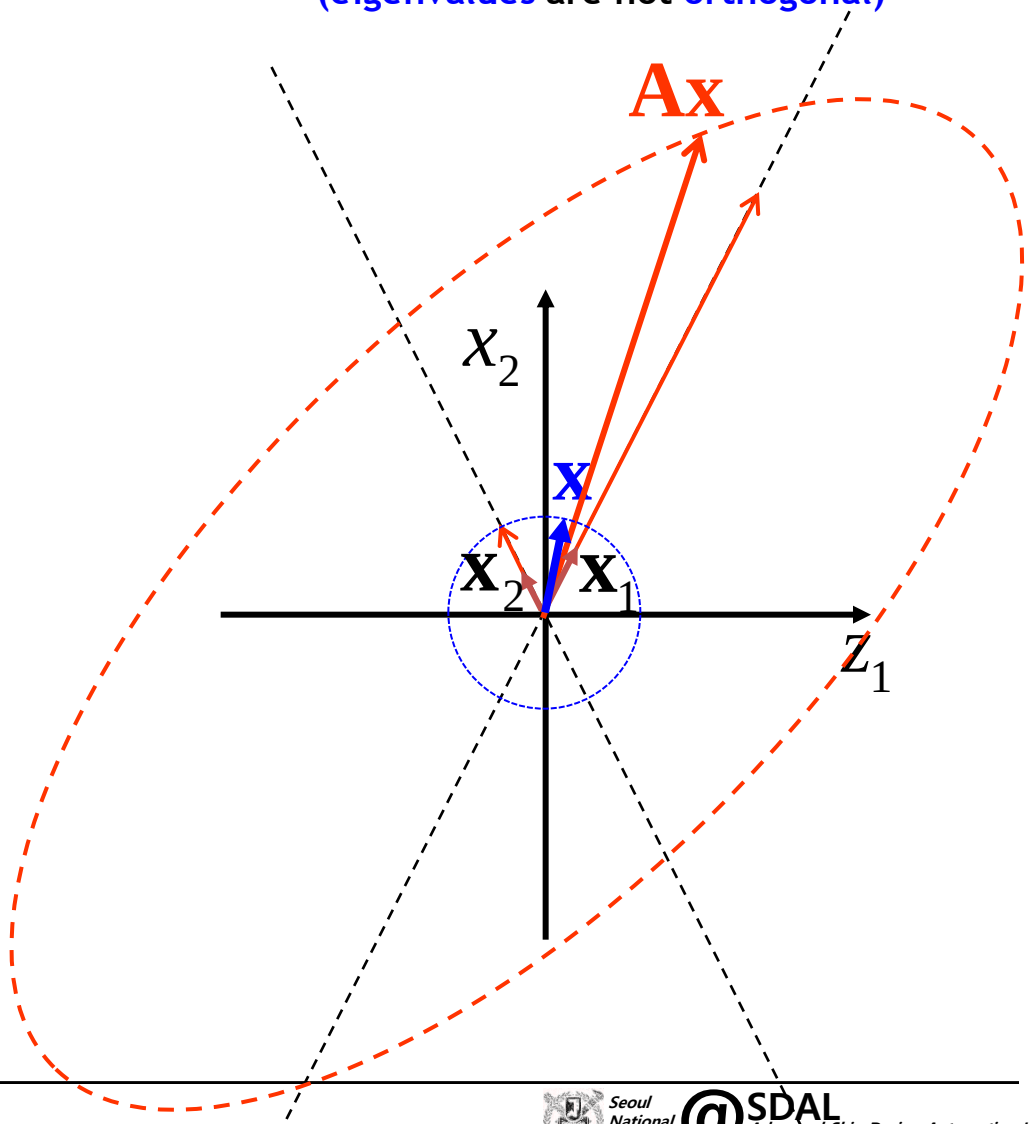
$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Ax} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$$

$$\mathbf{x} = \alpha\,\mathbf{x}_1 + \beta\,\mathbf{x}_2$$

$$\therefore \mathbf{Ax} = 8\alpha\,\mathbf{x}_1 + 2\beta\,\mathbf{x}_2$$



Stretching of an Elastic Membrane

Case II) Transformation of Rectangle shape by Non-symmetric Transform Matrix,
(eigenvalues are not orthogonal)

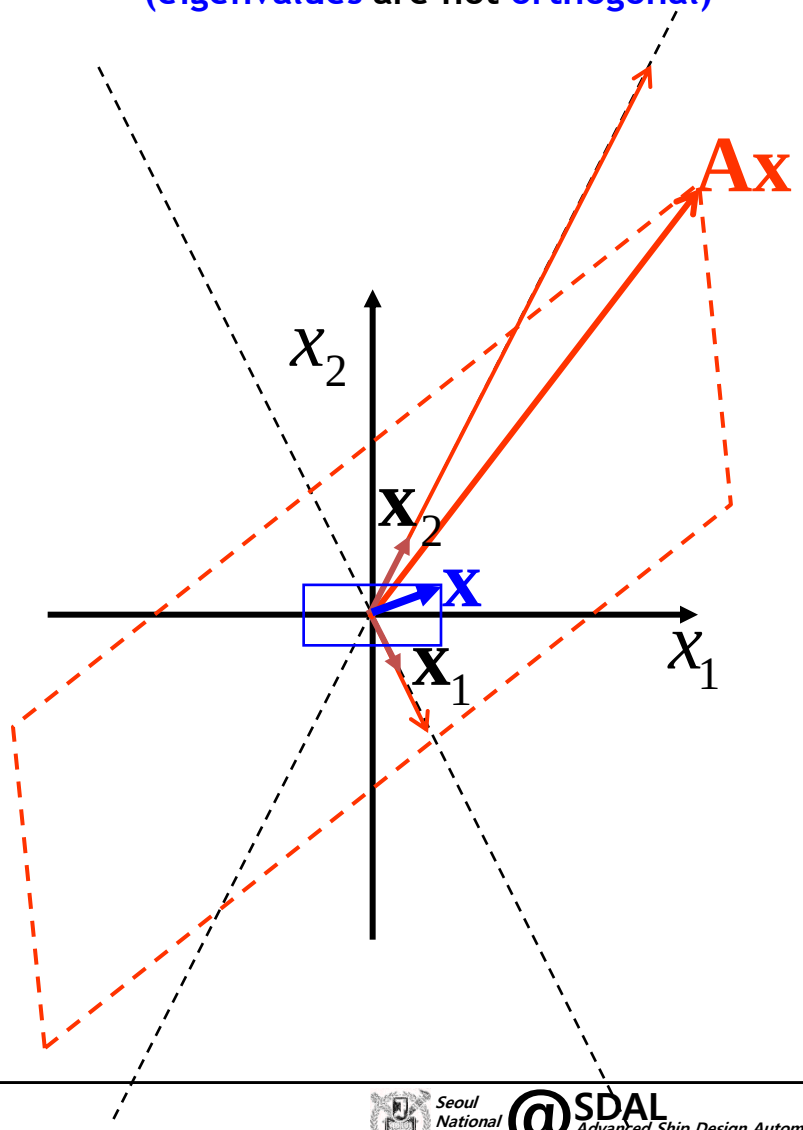
$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Ax} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\lambda_1 = 8,\ \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$$

$$\lambda_2 = 2,\ \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$$

$$\mathbf{x} = \alpha\,\mathbf{x}_1 + \beta\,\mathbf{x}_2$$

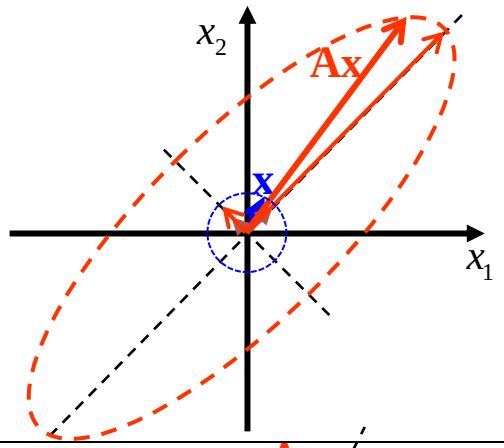
$$\therefore \mathbf{Ax} = 8\alpha\,\mathbf{x}_1 + 2\beta\,\mathbf{x}_2$$

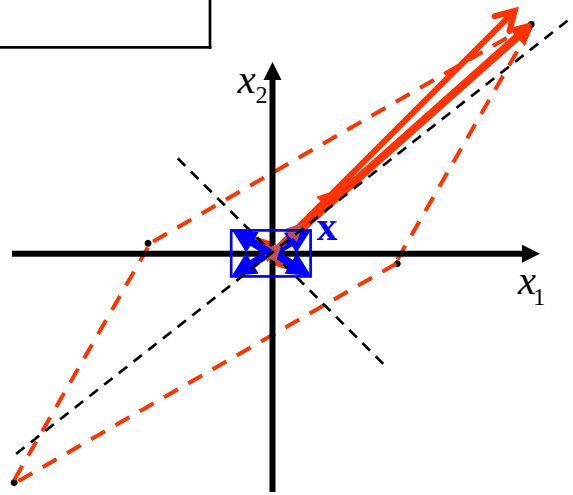


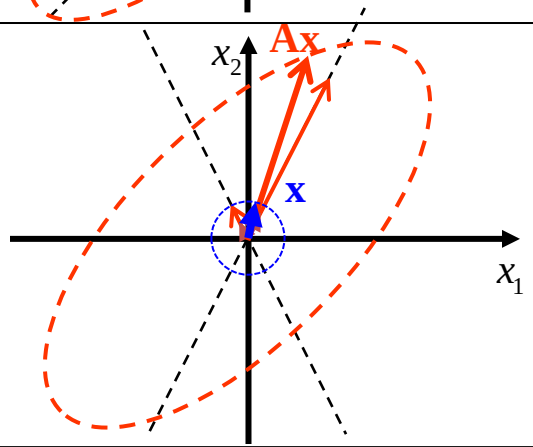
Stretching of an Elastic Membrane

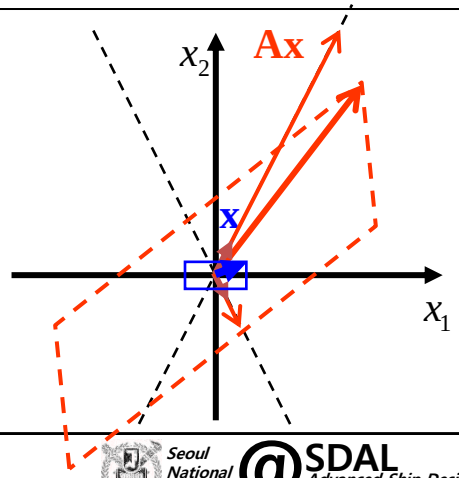
$$\mathbf{Ax} = \lambda \mathbf{x}$$

Transform Matrix \ Object	Conic (Circle, Ellipse, Hyperbola, Parabola)	Arbitrary shape (Rectangle in this example)
Symmetric matrix (eigenvalues are orthogonal)	**Case I**	**Case II**
Non-symmetric matrix (eigenvalues are not orthogonal)	**Case III**	**Case IV**









EIGENBASES.

Eigenbasis

$\mathbf{y} = \mathbf{A}\mathbf{z}$: a transformation

If we are interested in a transformation $\mathbf{y} = \mathbf{A}\mathbf{z}$, such an "*eigenbasis*" (basis of eigenvectors) - if it exists - is of great advantage because then we can represent any $\mathbf{z}$ in R^n uniquely as *a linear combination of the eigenvectors* $\mathbf{x}_1, \cdots, \mathbf{x}_n$, say,

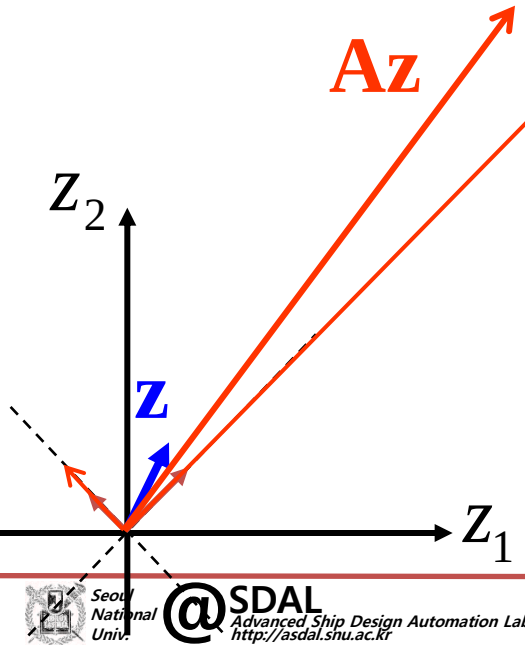
$$\mathbf{z} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$$

And denoting the corresponding eigenvalues of the matrix $\mathbf{A}$ by $\lambda_1, \cdots, \lambda_n$, we have $\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j$, so that we simply obtain

$$\begin{aligned}\mathbf{y} = \mathbf{A}\mathbf{z} &= \mathbf{A}\left(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n\right) \\ &= c_1\mathbf{A}\mathbf{x}_1 + c_2\mathbf{A}\mathbf{x}_2 + \cdots + c_n\mathbf{A}\mathbf{x}_n \\ &= c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \cdots + c_n\lambda_n\mathbf{x}_n\end{aligned}$$

Ex.)

$$\mathbf{y} = \mathbf{A}\mathbf{z} = \begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} \quad \lambda_1 = 8,\ \mathbf{x}_1 = \begin{bmatrix}1 & 1\end{bmatrix}^T \quad \lambda_2 = 2,\ \mathbf{x}_2 = \begin{bmatrix}1 & -1\end{bmatrix}^T$$



Basis of Eigenvectors

Theorem 8.4.1 Basis of Eigenvectors

If an n x n matrix A has n distinct eigenvalues, then

A has a basis of eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ for R^n.

Proof

All we have to show is that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent.

Suppose they are not.

⇒ 1 to r : independent
r is the largest integer that is a linearly independent

⇒ $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$: independent

$\{\mathbf{x}_1, \cdots, \mathbf{x}_{r+1}\}$: dependent

⇒ $c_1\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{x}_{r+1} = 0$

Basis of Eigenvectors (Proof)

$$c_1\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{x}_{r+1} = 0 \quad \rightarrow ①$$

Multiply both sides by $\mathbf{A}$

$$c_1\mathbf{A}\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{A}\mathbf{x}_{r+1} = 0$$

Use $\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j$

$$c_1\lambda_1\mathbf{x}_1 + \cdots + c_{r+1}\lambda_{r+1}\mathbf{x}_{r+1} = 0 \rightarrow ②$$

② - λ_{r+1} x ①:

$$c_1(\lambda_1 - \lambda_{r+1})\mathbf{x}_1 + \cdots + c_r(\lambda_r - \lambda_{r+1})\mathbf{x}_r = 0$$

$\mathbf{x}_1, \ldots, \mathbf{x}_r$ is linearly independent.

$$c_1(\lambda_1 - \lambda_{r+1}) = \cdots = c_r(\lambda_r - \lambda_{r+1}) = 0$$

All the eigenvalues are distinct.

$$\lambda_1 - \lambda_{r+1} \neq 0$$
$$\vdots$$
$$\lambda_r - \lambda_{r+1} \neq 0$$
$$\therefore c_1 = \cdots = c_r = 0$$

With this, ① reduces to

$$c_{r+1}\mathbf{x}_{r+1} = 0$$
$$\therefore c_{r+1} = 0$$

This contradicts the fact that not all scalars in ① are zero.

Basis of Eigenvectors

Example (Eigenbasis. Nondistinct Eigenvalues. Nonexistence)

$$\mathbf{A}=\begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix}$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}5-\lambda & 3\\ 3 & 5-\lambda\end{vmatrix}=(5-\lambda)^2-9=0$$

$$\lambda_1=8,\ \lambda_2=2$$

(1) $\lambda=\lambda_1=8$

$$-3x_1+3x_2=0$$
$$3x_1-3x_2=0$$
$$\therefore x_1=x_2$$
$$x_1=x_2=1$$
$$\mathbf{x}_1=\begin{bmatrix}1 & 1\end{bmatrix}^{\mathrm{T}}$$

(2) $\lambda=\lambda_2=2$

$$3x_1+3x_2=0$$
$$3x_1+3x_2=0$$
$$\therefore x_1=-x_2$$
$$x_1=1,\ x_2=-1$$
$$\mathbf{x}_2=\begin{bmatrix}1 & -1\end{bmatrix}^{\mathrm{T}}$$

➔ **eigenbasis for R^n**

$$\mathbf{A}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-\lambda & 1\\ 0 & -\lambda\end{vmatrix}=(-\lambda)^2=0$$

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0\quad,\lambda=0$$

$$\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$0\cdot x_1+1\cdot x_2=0\qquad\therefore\ \mathbf{x}=\begin{bmatrix}k\\ 0\end{bmatrix}$$

Matrix A may *not have enough linearly independent eigenvectors* to make up a basis.

7-5. Symmetric, Skew-Symmetric, and Orthogonal Matrices

Symmetric, Skew-Symmetric, and Orthogonal Matrices

A *real* square matrix $\mathbf{A} = [a_{jk}]$ is called

symmetric if transposition leaves it unchanged

$$\mathbf{A}^{\mathrm{T}} = \mathbf{A}, \qquad \text{thus} \quad a_{kj} = a_{jk}$$

skew-symmetric if transposition gives the negative of $\mathbf{A}$

$$\mathbf{A}^{\mathrm{T}} = -\mathbf{A}, \qquad \text{thus} \quad a_{kj} = -a_{jk}$$

orthogonal if transposition gives the inverse of $\mathbf{A}$

$$\mathbf{A}^{\mathrm{T}} = \mathbf{A}^{-1}$$

Example 8.3-1 (1)
(Symmetric, Skew-Symmetric, and Orthogonal Matrices)

The matrices

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}, \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

are symmetric, skew-symmetric, and orthogonal, respectively, as you should verify. Every skew-symmetric matrix has all main diagonal entries zero. (Can you prove this?)

Example 8.3-1 (2)

(Symmetric, Skew-Symmetric, and Orthogonal Matrices)

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}^T = \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

➔ **symmetric**

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & -9 & 12 \\ 9 & 0 & -20 \\ -12 & 20 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

➔ **skew-symmetric**

Example 8.3-1 (3)
(Symmetric, Skew-Symmetric, and Orthogonal Matrices)

Symmetric: $\mathbf{A}^T=\mathbf{A}$
Skew-symmetric: $\mathbf{A}^T=-\mathbf{A}$
Orthogonal: $\mathbf{A}^T=\mathbf{A}^{-1}$

$$\mathbf{A}=\begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix},\quad \mathbf{A}^{\mathbf{T}}=\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}^{\mathbf{T}}=\begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\mathbf{A}\cdot\mathbf{A}^{\mathbf{T}}=\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}\begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \mathbf{A}^{\mathbf{T}}=\mathbf{A}^{-1}$$ ➔ **orthogonal**

Example 8.3-1 (4)
(Symmetric, Skew-Symmetric, and Orthogonal Matrices)

Symmetric: $\mathbf{A}^T=\mathbf{A}$
Skew-symmetric: $\mathbf{A}^T=-\mathbf{A}$
Orthogonal: $\mathbf{A}^T=\mathbf{A}^{-1}$

From definitions skew-symmetric matrix is

$$\mathbf{A}^{\mathrm{T}} = -\mathbf{A}, \qquad \text{thus} \quad a_{kj} = -a_{jk}$$

If $j = k$, then

$$a_{jj} = -a_{jj}$$

$$\therefore a_{jj} = 0$$

So every skew-symmetric matrix has all main diagonal entries zero.

Example 8.3-2 (1)

Any real square matrix $\mathbf{A}$ may be written as the sum of a symmetric matrix $\mathbf{R}$ and a skew-symmetric matrix $\mathbf{S}$, where

$$\mathbf{R} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^{\mathbf{T}}\right), \qquad \mathbf{S} = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^{\mathbf{T}}\right)$$

$$\mathbf{A} = \begin{bmatrix} 9 & 5 & 2 \\ 2 & 3 & -8 \\ 5 & 4 & 3 \end{bmatrix} \qquad \mathbf{R} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^{\mathbf{T}}\right) = \begin{bmatrix} 9.0 & 3.5 & 3.5 \\ 3.5 & 3.0 & -2.0 \\ 3.5 & -2.0 & 3.0 \end{bmatrix}$$

$$\mathbf{A}^{\mathbf{T}} = \begin{bmatrix} 9 & 2 & 5 \\ 5 & 3 & 4 \\ 2 & -8 & 3 \end{bmatrix} \qquad \mathbf{S} = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^{\mathbf{T}}\right) = \begin{bmatrix} 0 & 1.5 & -1.5 \\ -1.5 & 0 & -6.0 \\ 1.5 & 6.0 & 0 \end{bmatrix}$$

Example 8.3-2 (2)

$$\mathbf{R}=\frac{1}{2}\left(\mathbf{A}+\mathbf{A}^{\mathrm{T}}\right), \qquad \mathbf{S}=\frac{1}{2}\left(\mathbf{A}-\mathbf{A}^{\mathrm{T}}\right)$$

$$\mathbf{A}=\begin{bmatrix}9 & 5 & 2\\ 2 & 3 & -8\\ 5 & 4 & 3\end{bmatrix}=\mathbf{R}+\mathbf{S}$$

$$=\begin{bmatrix}9.0 & 3.5 & 3.5\\ 3.5 & 3.0 & -2.0\\ 3.5 & -2.0 & 3.0\end{bmatrix}+\begin{bmatrix}0 & 1.5 & -1.5\\ -1.5 & 0 & -6.0\\ 1.5 & 6.0 & 0\end{bmatrix}$$

Eigenvalues of Symmetric and Skew-Symmetric Matrices

Theorem 8.1 Eigenvalues of Symmetric and Skew-Symmetric Matrices

(a) The eigenvalues of a symmetric matrix are *real*

(b) The eigenvalues of a skew-symmetric matrix are *pure imaginary or zero.*

Example 8.3-3
(Eigenvalues of Symmetric and Skew-Symmetric Matrices)

From example 8.2-1

$$\mathbf{A} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

➔ **symmetric**

Characteristic Equation:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)^2 - 3 = 0$$

$$\therefore \lambda = 2,\ 8 \quad \text{➔ real}$$

From example 8.3-1

$$\mathbf{A} = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 9 & -12 \\ -9 & -\lambda & 20 \\ 12 & -20 & -\lambda \end{vmatrix} = 0$$

$$-\lambda(\lambda^2 + 400) - 9(9\lambda - 240)$$
$$-12(180 + 12\lambda) = 0$$

$$\lambda^3 + 625\lambda = 0$$

$$\therefore \lambda = 0,\ \pm 25i \quad \text{➔ imaginary}$$

Orthogonal Transformations and Orthogonal Matrices

Orthogonal Transformations are transformations

$$\mathbf{y} = \mathbf{A}\mathbf{x} \quad \text{where } \mathbf{A} \text{ is an orthogonal matrix}$$

With each vector $\mathbf{x}$ in R^n such a transformation assigns a vector $\mathbf{y}$ in R^n.
For instance, the plane rotation through an angle θ.

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is an orthogonal transformation. It can be shown that any orthogonal transformation in the plane or in three-dimensional space is a *rotation*.

Invariance of Inner Product

Theorem 8.3.2 Invariance of Inner Product
An orthogonal transformation preserves the value of the inner product of vectors $\mathbf{a}$ and $\mathbf{b}$ in R^n, defined by

$$\mathbf{a}\cdot\mathbf{b}=\mathbf{a}^{\mathrm{T}}\mathbf{b}=\begin{bmatrix}a_1 & \cdots & a_n\end{bmatrix}\begin{bmatrix}b_1\\ \vdots\\ b_n\end{bmatrix}$$

That is, for any $\mathbf{a}$ and $\mathbf{b}$ in R^n, orthogonal n x n matrix $\mathbf{A}$, and $\mathbf{u} = \mathbf{Aa}$, $\mathbf{v} = \mathbf{Ab}$ we have $\mathbf{u}\cdot\mathbf{v}=\mathbf{a}\cdot\mathbf{b}$.
Hence the transformation also preserves the *length* or *norm* of any vector $\mathbf{a}$ in R^n given by

$$\|\mathbf{a}\|=\sqrt{\mathbf{a}\cdot\mathbf{a}}=\sqrt{\mathbf{a}^{\mathrm{T}}\mathbf{a}}$$

Invariance of Inner Product (Proof)

Let $\mathbf{A}$ be orthogonal. Let $\mathbf{u} = \mathbf{Aa}$ and $\mathbf{v} = \mathbf{Ab}$. We must show that $\mathbf{u} \cdot \mathbf{v} = \mathbf{a} \cdot \mathbf{b}$.

$$(\mathbf{Aa})^{\mathrm{T}} = \mathbf{a}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$$ ➔ by (10d) in Sec. 7.2

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$ ➔ $\mathbf{A}$ is orthogonal.

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^{\mathrm{T}}\mathbf{v} = (\mathbf{Aa})^{\mathrm{T}}\mathbf{Ab} &= \mathbf{a}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{Ab} \\ &= \mathbf{a}^{\mathrm{T}}\mathbf{Ib} \\ &= \mathbf{a}^{\mathrm{T}}\mathbf{b} \\ &= \mathbf{a} \cdot \mathbf{b}\end{aligned}$$

Orthonormality of Column and Row Vectors

Theorem 8.3.3

A real square matrix is orthogonal if and only if column vectors $a_1, \dots, a_n$ (and also its row vectors) form an orthonormal system, that is,

orthogonal (orthonomal) vectors

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j{}^{\mathrm{T}}\mathbf{a}_k = \begin{cases} 0 & if \quad j \neq k \\ 1 & if \quad j = k \end{cases}$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{x}_1{}^{\mathrm{T}}\mathbf{x}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

Let $\mathbf{A}$ be orthogonal. Then $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{I}$, in terms of column vector $\mathbf{a}_1, \dots, \mathbf{a}_n$,

orthogonal matrix

$$\mathbf{I} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} \mathbf{a}_1{}^{\mathrm{T}} \\ \vdots \\ \mathbf{a}_n{}^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1{}^{\mathrm{T}}\mathbf{a}_1 & \mathbf{a}_1{}^{\mathrm{T}}\mathbf{a}_2 & \cdots & \mathbf{a}_1{}^{\mathrm{T}}\mathbf{a}_n \\ \cdot & \cdot & \cdots & \cdot \\ \mathbf{a}_n{}^{\mathrm{T}}\mathbf{a}_1 & \mathbf{a}_n{}^{\mathrm{T}}\mathbf{a}_2 & \cdots & \mathbf{a}_n{}^{\mathrm{T}}\mathbf{a}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Orthonormality of Column and Row Vectors (Proof)

(a) Let $\mathbf{A}$ be orthogonal. Then $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}^{T}\mathbf{A} = \mathbf{I}$, in terms of column vector $\mathbf{a}_1, \dots, \mathbf{a}_n$,

$$\mathbf{I} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{A}^{T}\mathbf{A} = \begin{bmatrix} \mathbf{a}_1^{T} \\ \vdots \\ \mathbf{a}_n^{T} \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^{T}\mathbf{a}_1 & \mathbf{a}_1^{T}\mathbf{a}_2 & \cdots & \mathbf{a}_1^{T}\mathbf{a}_n \\ . & . & \cdots & . \\ \mathbf{a}_n^{T}\mathbf{a}_1 & \mathbf{a}_n^{T}\mathbf{a}_2 & \cdots & \mathbf{a}_n^{T}\mathbf{a}_n \end{bmatrix}$$

c.f.) Expression of a matrix-transpose in terms of column vectors

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$$

$$\mathbf{A}^{T} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^{T} \\ \mathbf{a}_2^{T} \\ \mathbf{a}_3^{T} \end{bmatrix}$$

여기서 행렬 A는 열벡터(column vector)의 곱이 아닌 배열이므로 행렬을 transpose할 때 그 순서가 바뀌지 않음

Determinant of an Orthogonal Matrix

Theorem 8.3.4 Determinant of an Orthogonal Matrix
The determinant of an orthogonal matrix has the value +1 or -1.

Proof

From $\det \mathbf{AB} = \det \mathbf{A} \cdot \det \mathbf{B}$ (Sec. 7.8, Theorem 4) and $\det \mathbf{A}^T = \det \mathbf{A}$ (Sec. 7.7, Theorem 2d), we get for an orthogonal matrix

$$
\begin{aligned}
1 = \det \mathbf{I} &= \det\left(\mathbf{A}\mathbf{A}^{-1}\right) = \det\left(\mathbf{A}\mathbf{A}^{\mathbf{T}}\right) \\
&= \det \mathbf{A} \cdot \det \mathbf{A}^{\mathbf{T}} \\
&= \det \mathbf{A} \cdot \det \mathbf{A} \\
&= \left(\det \mathbf{A}\right)^2
\end{aligned}
$$

Example 8.3-4

From example 1

$$\mathbf{A} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

➔ **orthogonal**

$$\det \mathbf{A} = \frac{2}{3}\left(-\frac{4}{9}-\frac{2}{9}\right)- \frac{1}{3}\left(\frac{4}{9}-\frac{1}{9}\right)+\frac{2}{3}\left(-\frac{4}{9}-\frac{2}{9}\right)$$

$$= -1$$

From example 3

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

➔ **orthogonal**

$$\det \mathbf{A} = \cos^2\theta + \sin^2\theta$$

$$= 1$$

Eigenvalues of an Orthogonal Matrix

Theorem 8.3.5 Eigenvalues of an Orthogonal Matrix
The eigenvalues of an orthogonal matrix A are *real or complex* conjugated in pairs and *have absolute value 1*.

Proof

The first part of the statement holds for any real matrix A because *its characteristic polynomial has real coefficients*.
$|\lambda|=1$ ➔ proved in Sec. 8.5.

Example 8.3.5 (1)

From example 1

$$\mathbf{A}=\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \qquad \det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix} \frac{2}{3}-\lambda & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3}-\lambda & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3}-\lambda \end{vmatrix}=0$$

➔ **orthogonal**

$$\left(\frac{2}{3}-\lambda\right)\left[\left(\frac{2}{3}-\lambda\right)\left(-\frac{2}{3}-\lambda\right)-\frac{1}{3}\cdot\frac{2}{3}\right]-\frac{1}{3}\left[-\frac{2}{3}\left(-\frac{2}{3}-\lambda\right)-\frac{1}{3}\cdot\frac{1}{3}\right]$$

$$+\frac{2}{3}\left[-\frac{2}{3}\cdot\frac{2}{3}-\left(\frac{2}{3}-\lambda\right)\frac{1}{3}\right]=0$$

$$-\lambda^3+\frac{2}{3}\lambda^2+\frac{2}{3}\lambda-1=0$$

Example 8.3.5 (2)

$$-\lambda^3+\frac{2}{3}\lambda^2+\frac{2}{3}\lambda-1=0$$

$$\left(\lambda+1\right)\left(\lambda^2-\frac{5}{3}\lambda+1\right)=0$$

$$\therefore \lambda=-1,\ \frac{5\pm i\sqrt{11}}{6}$$

Problem Set 8.3-3 (Eigenvalues) (1)

Are the eigenvalues of $\mathbf{A}+\mathbf{B}$ of the form $\lambda_j + \mu_j$ where λ_j and μ_j are the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$, respectively?

example 8.2-1

$$\mathbf{A}=\begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix}, \qquad \lambda = 8,\ 2$$

example 8.2-4

$$\mathbf{B}=\begin{bmatrix}-5 & 2\\ 2 & -2\end{bmatrix}, \qquad \lambda = -1,\ -6$$

$$\mathbf{A}+\mathbf{B}=\begin{bmatrix}0 & 5\\ 5 & 3\end{bmatrix}$$

Problem Set 8.3-3 (Eigenvalues) (2)

$$\mathbf{A}=\begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix}, \qquad \lambda=8,\ 2$$

$$\mathbf{B}=\begin{bmatrix}-5 & 2\\ 2 & -2\end{bmatrix}, \quad \lambda=-1,\ -6$$

$$\mathbf{A}+\mathbf{B}=\begin{bmatrix}0 & 5\\ 5 & 3\end{bmatrix}$$

$$\det(\mathbf{A}+\mathbf{B}-\lambda\mathbf{I})=\begin{bmatrix}-\lambda & 5\\ 5 & 3-\lambda\end{bmatrix}$$

$$=(-\lambda)(3-\lambda)-25$$

$$=\lambda^2-3\lambda-25$$

$$\lambda=\frac{3\pm\sqrt{3^2-4\cdot(-25)}}{2}$$

$$=\frac{3\pm\sqrt{109}}{2}$$

The eigenvalues of $\mathbf{A}+\mathbf{B}$ are not of the form $\lambda_j+\mu_j$

Problem Set 8.3-4 (Orthogonality)

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

Prove that eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal. Give an example.

Let $\mathbf{Ax} = \lambda\mathbf{x},\ \mathbf{Ay} = \mu\mathbf{y}$

where $\mathbf{x} \neq \mathbf{0},\ \mathbf{y} \neq \mathbf{0}$

$$\text{Thus } \lambda\mathbf{x}^{\mathrm{T}} = (\mathbf{Ax})^{\mathrm{T}} = \mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = \mathbf{x}^{\mathrm{T}}\mathbf{A}$$

$$\lambda\mathbf{x}^{\mathrm{T}}\mathbf{y} = \mathbf{x}^{\mathrm{T}}\mathbf{Ay} = \mathbf{x}^{\mathrm{T}}\mu\mathbf{y} = \mu\mathbf{x}^{\mathrm{T}}\mathbf{y}.$$

$$\mathbf{x}^{\mathrm{T}}\mathbf{y} = 0,\ (\because \lambda \neq \mu)$$

It proves orthogonality

Problem Set 8.3-4 (Orthogonality)

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

Prove that eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal. Give an example.

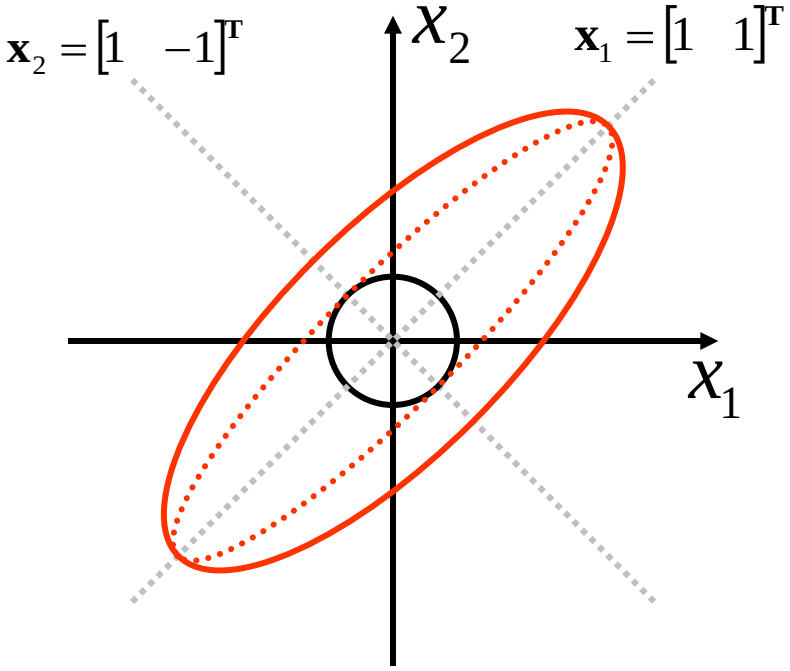
example 8.2-1

$$\mathbf{A}=\begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix},$$

$$\lambda_1=8,\ \mathbf{x}_1=\begin{bmatrix}1 & 1\end{bmatrix}^{\mathbf{T}}$$

$$\lambda_2=2,\ \mathbf{x}_2=\begin{bmatrix}1 & -1\end{bmatrix}^{\mathbf{T}}$$

$$\mathbf{x}_1\cdot\mathbf{x}_2=\mathbf{x}_1{}^{\mathbf{T}}\mathbf{x}_2=\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix}=0$$



Problem Set 8.3-5 (Skew-Symmetric matrix)

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

Show that the inverse of a skew-symmetric matrix is skew-symmetric matrix.

Let A is a skew-symmetric matrix, and $B = A^{-1}$ then,

$$B^T = B^T I = B^T(AB) = (B^T A)B = (A^T B)^T B$$

$$= (-AB)^T B, \quad (\because A \text{ is a skew-symmetric matrix.})$$

$$= (-I)^T B = -IB = -B$$

So, B is also a skew-symmetric matrix.

Problem Set 8.3-6 (Skew-Symmetric matrix with odd n)

Symmetric: $\mathbf{A}^T=\mathbf{A}$
Skew-symmetric: $\mathbf{A}^T=-\mathbf{A}$
Orthogonal: $\mathbf{A}^T=\mathbf{A}^{-1}$

Do there exist nonsingular skew-symmetric n x n matrices with odd n?

No.

Let's consider a n x n matrix $\mathbf{A}$

$$\det \mathbf{A} = \det \mathbf{A}^{\mathbf{T}} = \det(-\mathbf{A}) = (-1)^n \det \mathbf{A}$$

If n is odd, then

$$\det \mathbf{A} = -\det \mathbf{A} = 0$$

So, there is no nonsingular skew-symmetric n x n matrix with odd n

Problem Set 8.3-7 (Orthogonal matrix)

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

Do there exist skew-symmetric orthogonal 3 x 3 matrices ?

A *real* square matrix $A = [a_{jk}]$ is called *orthogonal* if transposition gives the inverse of A.

But, there is no nonsingular skew-symmetric n x n matrix with odd n.

So, there isn't existent skew-symmetric orthogonal 3 x 3 matrices

Problem Set 8.3-8 (Symmetric matrix)

Symmetric: $A^T=A$
Skew-symmetric: $A^T=-A$
Orthogonal: $A^T=A^{-1}$

Do there exist nondiagonal symmetric 3 x 3 matrices that are orthogonal ?

Yes, for example

$$\mathbf{A} = \begin{matrix} x \\ \\ \\ \end{matrix} \begin{matrix} y \\ \\ \\ \end{matrix} \begin{matrix} z \\ \\ \\ \end{matrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

two vectors which are orthogonal in the xy-plane.

a vector which is on the z-axis.

Problem Set 8.3-20 (Orthogonal Matrices) (1)

(a) Products. Inverse. Prove that the product of two orthogonal matrices is orthogonal, and so is the inverse of an orthogonal matrix. What does this mean in terms of rotation?

$$\text{Let } \mathbf{A}^{\mathrm{T}} = \mathbf{A}^{-1},\ \mathbf{B}^{\mathrm{T}} = \mathbf{B}^{-1}$$ **(Orthogonal matrices)**

$$\text{then, } (\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = \mathbf{B}^{-1}\mathbf{A}^{-1} = (\mathbf{AB})^{-1}$$

In terms of rotations it means that the composite of rotations and the inverse of a rotation are rotations.

Problem Set 8.3-20 (Orthogonal Matrices) (2)

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdots (6)$$

(b) Rotation. Show that (6) is an orthogonal transformation. Verify that it satisfies Theorem 3. Find the inverse transformation.

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{A}\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \mathbf{A}^{\mathrm{T}} = \mathbf{A}^{-1}$$

So, (6) is an orthogonal transformation

Problem Set 8.3-20 (Orthogonal Matrices) (3)

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdots (6)$$

Theorem 8.3.3 orthonormal system

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j{}^{\mathrm{T}} \mathbf{a}_k = \begin{cases} 0 & if \quad j \neq k \\ 1 & if \quad j = k \end{cases}$$

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$$

$$\mathbf{a}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$\mathbf{a}_1{}^{\mathrm{T}} \mathbf{a}_1 = \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = 1,$$

$$\mathbf{a}_2{}^{\mathrm{T}} \mathbf{a}_2 = \begin{bmatrix} -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = 1,$$

Problem Set 8.3-20 (Orthogonal Matrices) (4)

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdots (6)$$

Theorem 8.3.3 orthonormal system

$$\mathbf{a}_j \cdot \mathbf{a}_k = \mathbf{a}_j{}^{\mathrm{T}} \mathbf{a}_k = \begin{cases} 0 & if \quad j \neq k \\ 1 & if \quad j = k \end{cases}$$

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$$

$$\mathbf{a}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}, \quad \mathbf{a}_1{}^{\mathrm{T}} \mathbf{a}_1 = \mathbf{a}_2{}^{\mathrm{T}} \mathbf{a}_2 = 1,$$

$$\mathbf{a}_2{}^{\mathrm{T}} \mathbf{a}_2 = \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = 0,$$

∴ (6) satisfies Theorem 3

Problem Set 8.3-20 (Orthogonal Matrices) (5)

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdots (6)$$

(c) Powers. Write a program for computing powers $\mathbf{A}^m$ (m = 1,2…) of a 2 x 2 matrix A and their spectra. Apply it to in (call it A). To what rotation does A correspond? Do the eigenvalues of $\mathbf{A}^m$ have a limit as $m \to \infty$?

Problem Set 8.3-18

$\lambda = 1,\ \cos\theta \pm i\sin\theta$

$$\mathbf{A} = \begin{bmatrix} 0.96 & -0.28 \\ 0.28 & 0.96 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\tan\theta = \frac{0.28}{0.96}, \quad \theta = \tan^{-1}\left(\frac{0.28}{0.96}\right) = 16.26^\circ$$

Because eigenvalues of $\mathbf{A}^m$ is $(\cos 16.26^\circ \pm i \sin 16.26^\circ)^m$, $\mathbf{A}^m$ is divergent.

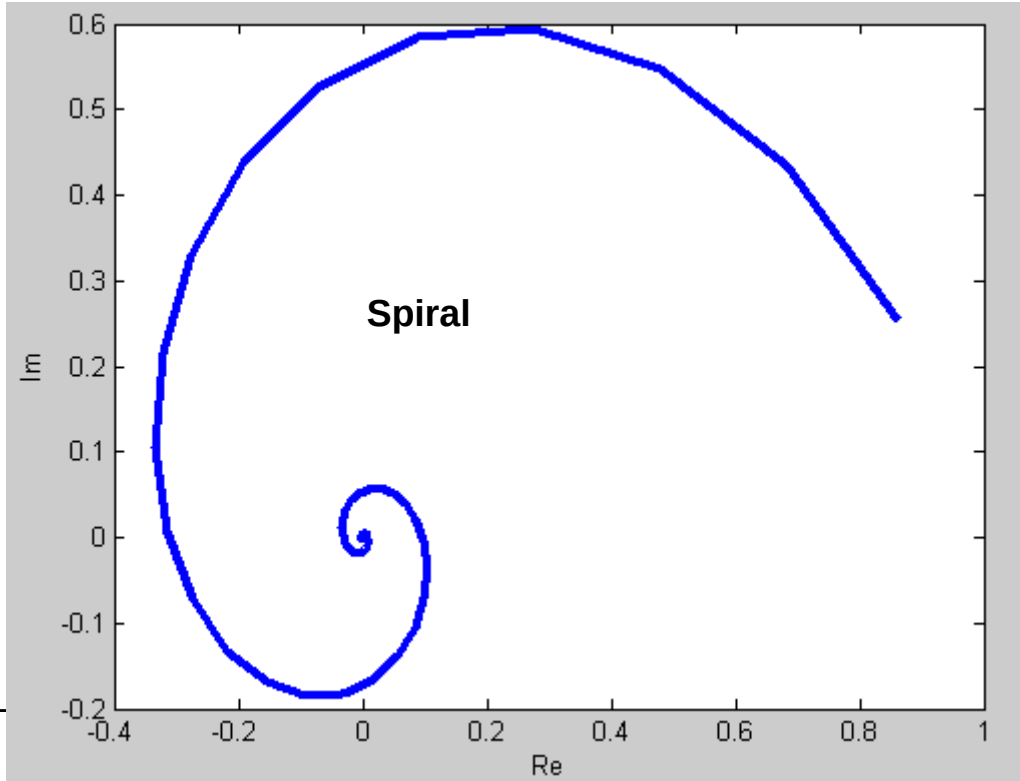
Problem Set 8.3-20 (Orthogonal Matrices) (6)

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdots (6)$$

(d) Compute the eigenvalues of $(0.9A)^m$, where A is the matrix in Prob. 9. Plot them as points. What is their limit? Along what kind of curve do these points approach the limit?

Eigenvalues of $(0.9A)^m$ is $0.9^m(\cos 16.26^\circ \pm i \sin 16.26^\circ)^m$, A^m is convergent to zero.

```
>> for m=1:100
x(m)=0.9^m.*cos(16.26*pi/180*m);
y(m)=0.9^m.*sin(16.26*pi/180*m);
end
>> plot(x,y); xlabel('Re'); ylabel('Im');
```



Problem Set 8.3-20 (Orthogonal Matrices) (7)

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdots (6)$$

(e) Find A such that y = Ax is a counterclockwise rotation through 30° in the plane?

$$\mathbf{A} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

7-6. EIGENBASES

Eigenbasis

If we are interested in a transformation $\mathbf{y} = \mathbf{A}\mathbf{x}$, such an "*eigenbasis*" (basis of eigenvectors) - if it exists - is of great advantage because then we can represent any $\mathbf{x}$ in R^n uniquely as *a linear combination of the eigenvectors* $\mathbf{x}_1, \dots, \mathbf{x}_n$, say,

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$$

And denoting the corresponding eigenvalues of the matrix $\mathbf{A}$ by $\lambda_1, \dots, \lambda_n$, we have $\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j$, so that we simply obtain

$$\mathbf{y} = \mathbf{A}\mathbf{x} = \mathbf{A}\left(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n\right)$$

$$= c_1\mathbf{A}\mathbf{x}_1 + c_2\mathbf{A}\mathbf{x}_2 + \cdots + c_n\mathbf{A}\mathbf{x}_n$$

$$= c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \cdots + c_n\lambda_n\mathbf{x}_n$$

Basis of Eigenvectors

Theorem 8.4.1 Basis of Eigenvectors
If an n x n matrix A has n distinct eigenvalues, then A has a basis of eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ for R^n.

Proof

All we have to show is that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent. *Suppose they are not*.

⇨ 1 to r : independent
r is the largest integer that is a linearly independent

⇨ $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$: independent
$\{\mathbf{x}_1, \cdots, \mathbf{x}_{r+1}\}$: dependent

⇨ $c_1\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{x}_{r+1} = 0$

Basis of Eigenvectors (Proof)

$$c_1\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{x}_{r+1} = 0 \quad \rightarrow ①$$

Multiply both sides by A

$$c_1\mathbf{A}\mathbf{x}_1 + \cdots + c_{r+1}\mathbf{A}\mathbf{x}_{r+1} = 0$$

Use $\mathbf{A}\mathbf{x}_j = \lambda_j\mathbf{x}_j$

$$c_1\lambda_1\mathbf{x}_1 + \cdots + c_{r+1}\lambda_{r+1}\mathbf{x}_{r+1} = 0 \rightarrow ②$$

② - λ_{j+1} x ①:

$$c_1(\lambda_1 - \lambda_{r+1})\mathbf{x}_1 + \cdots + c_r(\lambda_r - \lambda_{r+1})\mathbf{x}_r = 0$$

$\mathbf{x}_1$, … , $\mathbf{x}_r$ is linearly independent.

$$c_1(\lambda_1 - \lambda_{r+1}) = \cdots = c_r(\lambda_r - \lambda_{r+1}) = 0$$

All the eigenvalues are distinct.

$$\lambda_1 - \lambda_{r+1} \neq 0$$
$$\vdots$$
$$\lambda_r - \lambda_{r+1} \neq 0$$
$$\therefore c_1 = \cdots = c_r = 0$$

With this, ① reduces to

$$c_{r+1}\mathbf{x}_{r+1} = 0$$
$$\therefore c_{r+1} = 0$$

This contradicts the fact that not all scalars in ① are zero.

Example 8.4-1

(Eigenbasis. Nondistinct Eigenvalues. Nonexistence)

$$\mathbf{A}=\begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix}$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}5-\lambda & 3\\ 3 & 5-\lambda\end{vmatrix}=0$$

$$(5-\lambda)^2-9=0$$

$$\lambda_1=8,\quad \mathbf{x}_1=\begin{bmatrix}1 & 1\end{bmatrix}^{\mathbf{T}}$$

$$\lambda_2=2,\quad \mathbf{x}_2=\begin{bmatrix}1 & -1\end{bmatrix}^{\mathbf{T}}$$

➔**an eigenbasis for R^n (See example 2 in Sec. 8.1.)**

$$\mathbf{A}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-\lambda & 1\\ 0 & -\lambda\end{vmatrix}=0$$

$$(-\lambda)^2=0$$

$$\lambda=0,\quad \mathbf{x}=\begin{bmatrix}k\\ 0\end{bmatrix}$$

Matrix A may *not have enough linearly independent eigenvectors* to make up a basis.

7-7. SYMMETRIC MATRICES

Symmetric Matrices

Theorem 8.4.2 Symmetric Matrices
A symmetric matrix has an orthonormal basis of eigenvectors for R^n.

From example 8.4.1

$$\mathbf{A} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$$

normalize $\rightarrow \mathbf{x}_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$$

normalize $\rightarrow \mathbf{x}_2 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^T$

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_1^T \mathbf{x}_2$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$= 0$$

So $\mathbf{x}_1$, $\mathbf{x}_2$ is an orthonormal basis of eigenvectors.

(참고) Taylor series expansion과 극소점 (1)

$$\text{Given}: f(x), \frac{df}{dx}, \frac{d^2 f}{dx^2}, \cdots \text{ at } x$$

$$\text{Find}: f(x+\Delta x)$$

Taylor series expansion

Higher Order Terms

$$f(x+\Delta x) = f(x) + \frac{df}{dx}\Delta x + \frac{1}{2}\frac{d^2 f}{dx^2}\Delta x^2 + H.O.T$$

$f(x)$가 극소값을 가질 조건은?

$f(x)$가 주위의 함수값 $f(x+\Delta x)$보다 항상 작아야 함

$$f(x+\Delta x) - f(x) = \frac{df}{dx}\Delta x + \frac{1}{2}\frac{d^2 f}{dx^2}\Delta x^2 + H.O.T > 0$$

(참고) Taylor series expansion과 극소점 (2)

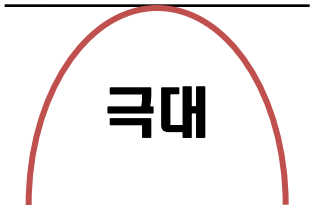
$$f(x+\Delta x)-f(x)=\frac{df}{dx}\Delta x+\frac{1}{2}\frac{d^2f}{dx^2}\Delta x^2+H.O.T>0$$

$$if\ \ \Delta x<0, \qquad \square \qquad >0$$

$$if\ \ \Delta x>0, \qquad >0 \qquad >0$$

$x+\Delta x$ 가 x보다 크거나 작은 것과 관계 없이 항상 $f(x+\Delta x)$가 $f(x)$ 보다 커야 하므로,

$$\therefore \left.\frac{df}{dx}\right|_{x=x^*}=0$$



$f(x+\Delta x)-f(x)$ 의 다른 항 중에 가장 값이 큰 항이 두 번째 항(2계 미분계수)이므로,

$$\therefore \left.\frac{d^2f}{dx^2}\right|_{x=x^*}>0$$

(참고) Taylor series expansion과 극소점 (3)

$$\text{Given}: f(x_1, x_2), \frac{\partial f}{\partial x_1}, \frac{\partial^2 f}{\partial x_1 \partial x_2}, \frac{\partial^2 f}{\partial {x_1}^2}, \frac{\partial^2 f}{\partial {x_2}^2} \cdots \text{ at } (x_1, x_2)$$

$$\text{Find}: \quad f(x_1 + \Delta x_1, x_2 + \Delta x_2)$$

Taylor series expansion (단, 3차 이상의 고차항은 무시할 경우)

$$f(x_1 + \Delta x_1, x_2 + \Delta x_2) = f(x_1, x_2) + \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2$$

$$+ \frac{1}{2}\left(\frac{\partial^2 f}{\partial x_1^2}{\Delta x_1}^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\Delta x_1 \cdot \Delta x_2 + \frac{\partial^2 f}{\partial x_2^2}{\Delta x_2}^2 \right)$$

$f(x_1, x_2)$가 극소값을 가질 조건은?

$f(x_1, x_2)$가 주위의 함수값 $f(x_1 + \Delta x_1, x_2 + \Delta x_2)$보다 항상 작아야 함

(참고) Taylor series expansion과 극소점 (4)

$$f(x_1+\Delta x_1, x_2+\Delta x_2) - f(x_1, x_2)$$

$$= \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x_1^2}\Delta x_1{}^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\Delta x_1 \cdot \Delta x_2 + \frac{\partial^2 f}{\partial x_2^2}\Delta x_2{}^2 \right) > 0$$

(x_1, x_2)가 $(x_1+\Delta x_1, x_2+\Delta x_2)$보다 크거나 작은 것과 관계 없이 항상 $f(x_1, x_2)$가 $f(x_1+\Delta x_1, x_2+\Delta x_2)$ 보다 커야 하므로,

$$\therefore \left.\frac{\partial f}{\partial x_1}\right|_{x_1,\, x_2} = \left.\frac{\partial f}{\partial x_2}\right|_{x_1,\, x_2} = 0.$$

$f(x_1+\Delta x_1, x_2+\Delta x_2) - f(x_1, x_2)$의 다른 항 중에 가장 값이 큰 항이 2계 미분계수와 관련된 항이다.

$$\therefore \frac{\partial^2 f}{\partial x_1^2}\Delta x_1{}^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\Delta x_1 \cdot \Delta x_2 + \frac{\partial^2 f}{\partial x_2^2}\Delta x_2{}^2 > 0$$

(참고) Taylor series expansion과 극소점 (5)

$f(x_1^*, x_2^*)$가 (x_1^*, x_2^*)주위의 함수값보다 항상 작아야 함

$$\frac{\partial^2 f}{\partial x_1^2}\Delta x_1{}^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\Delta x_1 \cdot \Delta x_2 + \frac{\partial^2 f}{\partial x_2^2}\Delta x_2{}^2 > 0$$

(x_1, x_2)가 (x_1^*, x_2^*)보다 크거나 작은 것과 관계 없이 항상 위 식이 성립하게 하는 $\frac{\partial^2 f}{\partial x_1^2}, \frac{\partial^2 f}{\partial x_1 \partial x_2}, \frac{\partial^2 f}{\partial x_2^2}$ 에 대한 조건을 알아야 함.

위 식을 행렬을 이용해서 표현하면 다음과 같다.

$$\frac{\partial^2 f}{\partial x_1^2}\Delta x_1{}^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\Delta x_1 \cdot \Delta x_2 + \frac{\partial^2 f}{\partial x_2^2}\Delta x_2{}^2$$

$$= \begin{bmatrix}\Delta x_1 & \Delta x_2\end{bmatrix}\begin{bmatrix}\frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2}\end{bmatrix}\begin{bmatrix}\Delta x_1 \\ x_2 - x_2^*\end{bmatrix}$$

(참고) Taylor series expansion과 극소점 (6)

$$\frac{\partial^2 f}{\partial x_1^2}\Delta x_1{}^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}\Delta x_1 \cdot \Delta x_2 + \frac{\partial^2 f}{\partial x_2^2}\Delta x_2{}^2$$

$$= \underbrace{\begin{bmatrix}\Delta x_1 & \Delta x_2\end{bmatrix}}_{\mathbf{x}^T}\underbrace{\begin{bmatrix}\frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2}\end{bmatrix}}_{\mathbf{H}}\underbrace{\begin{bmatrix}\Delta x_1 \\ \Delta x_2\end{bmatrix}}_{\mathbf{x}} = \mathbf{x}^T\mathbf{H}\mathbf{x} > 0$$

따라서 x에 무관하게 행렬 x^THx가 항상 양수가 되는 H에 대한조건을 알면, 언제 $f(x_1, x_2)$가 극소값을 갖는지 알 수 있다.

H의 모든 고유치가 양수이면, x^THx가 항상 양수이다. (2차 형식 부분에서 설명) ▶

7-8. SIMILAR MATRICES

Similar Matrices. Similarity Transformation

Definition. Similar Matrices. Similarity Transformation
An n x n matrix $\hat{\mathbf{A}}$ is called similar to an n x n matrix $\mathbf{A}$ if

for some (nonsingular!) n x n matrix $\mathbf{P}$. This transformation, which gives $\hat{\mathbf{A}}$ from $\mathbf{A}$, is called *a similarity transformation.*

$$\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

Eigenvalues and Eigenvectors of Similar Matrices

$\mathbf{Ax} = \lambda\mathbf{x}$

Theorem 8.4.3 Eigenvalues and Eigenvectors of Similar Matrices

If $\hat{\mathbf{A}}$ is similar to $\mathbf{A}$, then $\hat{\mathbf{A}}$ has the same eigenvalues as $\mathbf{A}$. Furthermore, if $\mathbf{x}$ is an eigenvector of $\mathbf{A}$, then $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ is an eigenvector of $\hat{\mathbf{A}}$ corresponding to the same eigenvalue.

Proof

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}(\lambda\mathbf{x}) = \lambda\mathbf{P}^{-1}\mathbf{x} \quad \rightarrow \text{①}$$

Use $\mathbf{I} = \mathbf{PP}^{-1}$

$$\mathbf{P}^{-1}\mathbf{Ax} = \mathbf{P}^{-1}\mathbf{AIx} = \underbrace{\mathbf{P}^{-1}\mathbf{A}(\mathbf{PP}^{-1})}_{\hat{\mathbf{A}}}\mathbf{x} = \hat{\mathbf{A}}\mathbf{P}^{-1}\mathbf{x} \quad \rightarrow \text{②}$$

From ①, ②

$$\hat{\mathbf{A}}\mathbf{P}^{-1}\mathbf{x} = \lambda\mathbf{P}^{-1}\mathbf{x}$$

If $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$, then

$$\hat{\mathbf{A}}\mathbf{y} = \lambda\mathbf{y}$$

→ $\mathbf{y}$ is an eigenvector of $\hat{\mathbf{A}}$

Indeed, $\mathbf{P}^{-1}\mathbf{x} = 0$ would give

$$\mathbf{x} = \mathbf{Ix} = \mathbf{PP}^{-1}\mathbf{x} = \mathbf{P}\cdot 0 = 0$$

→ contradicting $\mathbf{x} \neq 0$

$$\therefore \mathbf{y} = \mathbf{P}^{-1}\mathbf{x} \neq 0$$

Example 8.4-3 (1)
(Eigenvalues and Vectors of Similar Matrices)

$$\mathbf{A}=\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix},\quad \mathbf{P}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$\mathbf{P}^{-1}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}^{-1}=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}$$

Get similar matrix $\hat{\mathrm{A}}$.

$$\hat{\mathbf{A}}=\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

$$=\begin{bmatrix}4 & -3\\ -1 & 4\end{bmatrix}\begin{bmatrix}6 & -3\\ 4 & -1\end{bmatrix}\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$$

$$=\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}6-\lambda & -3\\ 4 & -1-\lambda\end{vmatrix}=0$$

$$(6-\lambda)(-1-\lambda)+12=0$$

$$\lambda^2-5\lambda+6=0 \quad\Rightarrow\quad \lambda_1=3,\ \lambda_2=2$$

$$\det(\hat{\mathbf{A}}-\lambda\mathbf{I})=\begin{vmatrix}3-\lambda & 0\\ 0 & 2-\lambda\end{vmatrix}=0$$

$$(3-\lambda)(2-\lambda)=0 \quad\Rightarrow\quad \lambda_1=3,\ \lambda_2=2$$

So A and $\hat{\mathrm{A}}$ has the same eigenvalue.

Example 8.4-3 (2)
(Eigenvalues and Vectors of Similar Matrices)

$$\mathbf{P}^{-1} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$$

$$\hat{\mathbf{A}} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad \lambda_1 = 3, \quad \lambda_2 = 2$$

(1) $\lambda = \lambda_1 = 3$

$$\mathbf{A} - \lambda_1 \mathbf{I} = \begin{bmatrix} 3 & -3 \\ 4 & -4 \end{bmatrix}$$

$$3x_1 - 3x_2 = 0$$

Let $x_1 = 1$, $\therefore \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$

(2) $\lambda = \lambda_2 = 2$

$$\mathbf{A} - \lambda_2 \mathbf{I} = \begin{bmatrix} 4 & -3 \\ 4 & -3 \end{bmatrix}$$

$$4x_1 - 3x_2 = 0$$

Let $x_1 = 3$, $\therefore \mathbf{x}_2 = \begin{bmatrix} 3 & 4 \end{bmatrix}^{\mathrm{T}}$

Eigenvectors of $\hat{\mathbf{A}}$ is

$$\mathbf{y}_1 = \mathbf{P}^{-1}\mathbf{x}_1 = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{y}_2 = \mathbf{P}^{-1}\mathbf{x}_2 = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

7-9. DIAGONALIZATION

Diagonalization of a Matrix

Theorem 8.4.4 Diagonalization of Matrix

If an n x n matrix $\mathbf{A}$ has a basis of eigenvectors, then

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$

is *diagonal*, with the eigenvalues of $\mathbf{A}$ as the entries on the main diagonal. Here $\mathbf{X}$ is the matrix with these eigenvectors as column vectors. Also $\mathbf{D}^m = \mathbf{X}^{-1}\mathbf{A}^m\mathbf{X}$

Diagonalization of a Matrix (Proof) (1)

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ constitute a basis of eigenvectors of $\mathbf{A}$ for R^n. Let the corresponding eigenvalues of A be $\lambda_1, \ldots, \lambda_n$.

$$\mathbf{A}\mathbf{x}_1 = \lambda_1 \mathbf{x}_1$$
$$\vdots$$
$$\mathbf{A}\mathbf{x}_n = \lambda_n \mathbf{x}_n$$

Then $\mathbf{X} = [\, \mathbf{x}_1 \ \ldots \ \mathbf{x}_n \,]$ has rank n, by Theorem 3 in Sec. 7.4.

Hence $\mathbf{X}^{-1}$ exists.

$$\mathbf{AX} = \mathbf{A}\begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{A}\mathbf{x}_1 & \cdots & \mathbf{A}\mathbf{x}_n \end{bmatrix}$$
$$= \begin{bmatrix} \lambda_1 \mathbf{x}_1 & \cdots & \lambda_n \mathbf{x}_n \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}}_{\mathbf{D}}$$
$$= \mathbf{XD}$$

$$\mathbf{AX} = \mathbf{XD}$$

$$\therefore \mathbf{D} = \mathbf{X}^{-1}\mathbf{AX}$$

$$\mathbf{D}^2 = \mathbf{DD} = \mathbf{X}^{-1}\mathbf{AXX}^{-1}\mathbf{AX}$$
$$= \mathbf{X}^{-1}\mathbf{AIAX} = \mathbf{X}^{-1}\mathbf{A}^2\mathbf{X}$$

Diagonalization of a Matrix (Proof) (2)

$$\mathbf{AX} = \mathbf{A}\begin{bmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_n\end{bmatrix} = \begin{bmatrix}\mathbf{Ax}_1 & \cdots & \mathbf{Ax}_n\end{bmatrix}$$

Let $n = 2$.

$$\mathbf{x}_1 = \begin{bmatrix}x_{11} & x_{12}\end{bmatrix}^{\mathrm{T}},\ \mathbf{x}_2 = \begin{bmatrix}x_{21} & x_{22}\end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{AX} = \mathbf{A}\begin{bmatrix}\mathbf{x}_1 & \mathbf{x}_2\end{bmatrix}$$

$$= \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}x_{11} & x_{21}\\ x_{12} & x_{22}\end{bmatrix}$$

$$= \begin{bmatrix}a_{11}x_{11} + a_{12}x_{21} & a_{11}x_{12} + a_{12}x_{22}\\ a_{21}x_{11} + a_{22}x_{21} & a_{21}x_{12} + a_{22}x_{22}\end{bmatrix}$$

$$= \begin{bmatrix}\mathbf{Ax}_1 & \mathbf{Ax}_2\end{bmatrix}$$

Diagonalization of a Matrix (Proof) (3)

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix}$$

$$\begin{aligned} \mathbf{A}\mathbf{x}_1 &= \lambda_1 \mathbf{x}_1 \\ \mathbf{A}\mathbf{x}_2 &= \lambda_2 \mathbf{x}_2 \\ \mathbf{A}\mathbf{x}_3 &= \lambda_3 \mathbf{x}_3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{A}\mathbf{x}_1 &= \mathbf{x}_1 \cdot \lambda_1 + \mathbf{x}_2 \cdot 0 + \mathbf{x}_3 \cdot 0 \\ \mathbf{A}\mathbf{x}_2 &= \mathbf{x}_1 \cdot 0 + \mathbf{x}_2 \cdot \lambda_2 + \mathbf{x}_3 \cdot 0 \\ \mathbf{A}\mathbf{x}_3 &= \mathbf{x}_1 \cdot 0 + \mathbf{x}_2 \cdot 0 + \mathbf{x}_3 \cdot \lambda_3 \end{aligned}$$

$$\mathbf{A}\begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix}$$

$$= \left[\begin{array}{lll|lll|lll} \mathbf{x}_1 \cdot \lambda_1 + \mathbf{x}_2 \cdot 0 & + \mathbf{x}_3 \cdot 0 & & \mathbf{x}_1 \cdot 0 & + \mathbf{x}_2 \cdot \lambda_2 + \mathbf{x}_3 \cdot 0 & & \mathbf{x}_1 \cdot 0 & + \mathbf{x}_2 \cdot 0 & + \mathbf{x}_3 \cdot \lambda_3 \end{array}\right]$$

$$= \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}}_{\mathbf{D}} = \mathbf{X}\mathbf{D}$$

$$\therefore \mathbf{A}\mathbf{X} = \mathbf{X}\mathbf{D}$$

$$\therefore \mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$

$$\mathbf{A} = \mathbf{X}\mathbf{D}\mathbf{X}^{-1}$$

Example 8.4-4 (1)
(Diagonalization)

Diagonalize

$$\mathbf{A}=\begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1.0 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix}$$

Characteristic Equation:

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

$$\begin{vmatrix} 7.3-\lambda & 0.2 & -3.7 \\ -11.5 & 1.0-\lambda & 5.5 \\ 17.7 & 1.8 & -9.3-\lambda \end{vmatrix}=0$$

$$(7.3-\lambda)[(1.0-\lambda)(-9.3-\lambda)-5.5\cdot 1.8]$$
$$-0.2[-11.5(-9.3-\lambda)-5.5\cdot 17.7]$$
$$-3.7[-11.5\cdot 1.8-(1.0-\lambda)17.7]=0$$

$$-\lambda^3-\lambda^2+12\lambda=0$$

$$\lambda_1=3 \quad \mathbf{x}_1=\begin{bmatrix}-1 & 3 & -1\end{bmatrix}^{\mathrm{T}}$$
$$\lambda_2=-4 \quad \mathbf{x}_2=\begin{bmatrix}1 & -1 & 3\end{bmatrix}^{\mathrm{T}}$$
$$\lambda_3=0 \quad \mathbf{x}_3=\begin{bmatrix}2 & 1 & 4\end{bmatrix}^{\mathrm{T}}$$

$$\therefore \mathbf{X}=\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

$\mathbf{x}_1$ $\mathbf{x}_2$ $\mathbf{x}_3$

$$\mathbf{X}^{-1}=\begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

Example 8.4-4 (2)
(Diagonalization)

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$

$$= \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix} \begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1.0 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

7-10. Quadratic Forms

Quadratic Forms

Definition: a quadratic form Q in the components $x_1, \dots, x_n$ of a vector $\mathbf{x}$ is a sum of n^2 terms, namely,

$$Q = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}x_j x_k$$

$$= a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n$$
$$+ a_{21}x_2x_1 + a_{22}x_2^2 + \cdots + a_{2n}x_2x_n$$
$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots +$$
$$+ a_{n1}x_nx_1 + a_{n2}x_nx_2 + \cdots + a_{nn}x_n^2$$

$\mathbf{A} = [a_{jk}]$ is called of the ***coefficient matrix*** of the form. We may assume that $\mathbf{A}$ is ***symmetric***, because we can take off-diagonal terms together in pairs and write the result as a sum of two equal terms.

Example 8.4-5

(Quadratic Form. Symmetric Coefficient Matrix)

다음 2차형식을 $\mathbf{x}^T\mathbf{C}\mathbf{x}$의 형태로 나타내시오.

$$Q = 3x_1^2 + 4x_1x_2 + 6x_2x_1 + 2x_2^2$$

$$= x_1(3x_1 + 4x_2) + x_2(6x_1 + 2x_2)$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3x_1 + 4x_2 \\ 6x_1 + 2x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}^T\mathbf{C}\mathbf{x}$$

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}^T\mathbf{C}\mathbf{x}$$

Symmetric matrix

$$Q = 3x_1^2 + 5x_1x_2 + 5x_2x_1 + 2x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Principal Axes Theorem

$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x} \longleftarrow \mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$
$$= \mathbf{x}^T\mathbf{X}\mathbf{D}\mathbf{X}^T\mathbf{x} \longleftarrow \mathbf{y} = \mathbf{X}^{-1}\mathbf{x} = \mathbf{X}^T\mathbf{x}$$
$$= \mathbf{y}^T\mathbf{D}\mathbf{y} \qquad \mathbf{y}^T = (\mathbf{X}^T\mathbf{x})^T = \mathbf{x}\mathbf{X}^T$$

Theorem 8.4.5 Principal Axes Theorem

The substitution $\mathbf{x} = \mathbf{X}\mathbf{y}$ transforms a quadratic form

$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}x_j x_k \qquad (a_{kj} = a_{jk})$$

to the *principal axes form* or *canonical form*

$$Q = \mathbf{y}^T\mathbf{D}\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

where $\lambda_1, \ldots, \lambda_n$ are the (not necessarily distinct) eigenvalues of the *symmetric* matrix $\mathbf{A}$, and $\mathbf{X}$ is an *orthogonal* matrix with corresponding eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$, respectively, as column vectors.

Principal Axes Theorem

Theorem 8.4.2 Symmetric Matrices

A symmetric matrix has an orthonormal basis of eigenvectors for R^n.

(Quadratic Forms) $$Q = \mathbf{x}^T \mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk} x_j x_k$$

By the Theorem 8.4.2 the *symmetric* coefficient matrix $\mathbf{A}$ has an *orthonormal basis of eigenvectors* $\mathbf{x}_1, \dots, \mathbf{x}_n$. Let $\mathbf{X}$ be

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix}$$

$\mathbf{X}$ is *orthogonal*, so that $\mathbf{X}^{-1}=\mathbf{X}^T$, we obtain

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$

$$\therefore \mathbf{A} = \mathbf{X}\mathbf{D}\mathbf{X}^{-1} = \mathbf{X}\mathbf{D}\mathbf{X}^T$$

$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{x}^T(\mathbf{X}\mathbf{D}\mathbf{X}^T)\mathbf{x}$$

Symmetric: $\mathbf{A}^T=\mathbf{A}$
Skew-symmetric: $\mathbf{A}^T=-\mathbf{A}$
Orthogonal: $\mathbf{A}^T=\mathbf{A}^{-1}$

Principal Axes Theorem

$$Q = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \mathbf{x}^{\mathrm{T}}\mathbf{X}\mathbf{D}\mathbf{X}^{\mathrm{T}}\mathbf{x}$$

If we set $\mathbf{X}^{\mathrm{T}}\mathbf{x} = \mathbf{y}$, then, since $\mathbf{X}^{\mathrm{T}} = \mathbf{X}^{-1}$, we get

$$\mathbf{x} = \left(\mathbf{X}^{\mathrm{T}}\right)^{-1}\mathbf{y} = \mathbf{X}\mathbf{y}$$

Furthermore, we have

$$\mathbf{x}^{\mathrm{T}}\mathbf{X} = \left(\mathbf{X}^{\mathrm{T}}\mathbf{x}\right)^{\mathrm{T}} = \mathbf{y}^{\mathrm{T}}$$

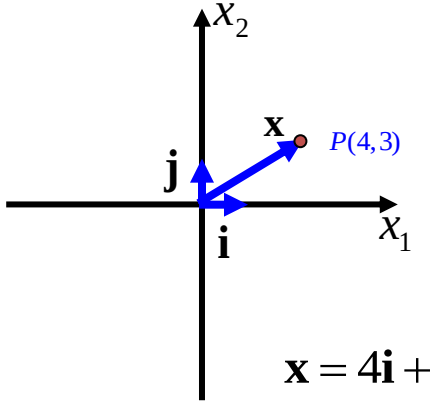
So Q becomes simply

$$Q = \mathbf{x}^{\mathrm{T}}\mathbf{X}\mathbf{D}\mathbf{X}^{\mathrm{T}}\mathbf{x} = \mathbf{y}^{\mathrm{T}}\mathbf{D}\mathbf{y}$$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

Principal Axes Theorem

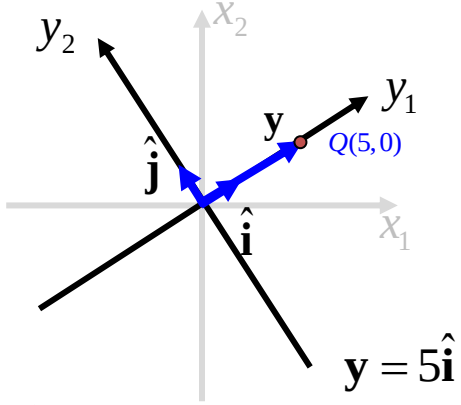
Transformation of Axis



$$\mathbf{x} = 4\mathbf{i} + 3\mathbf{j}$$

Choose proper axis

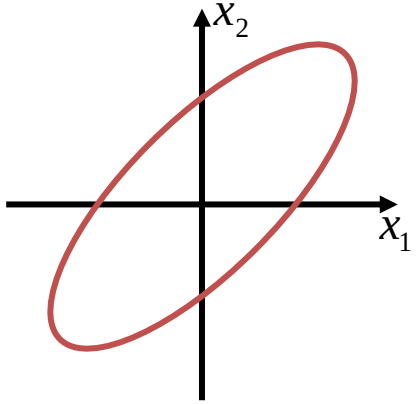
(Coordinate Transformation)



$$\mathbf{y} = 5\hat{\mathbf{i}}$$

Simple expression.
Easily recognition of magnitude of vector

Principal Axes Theorem

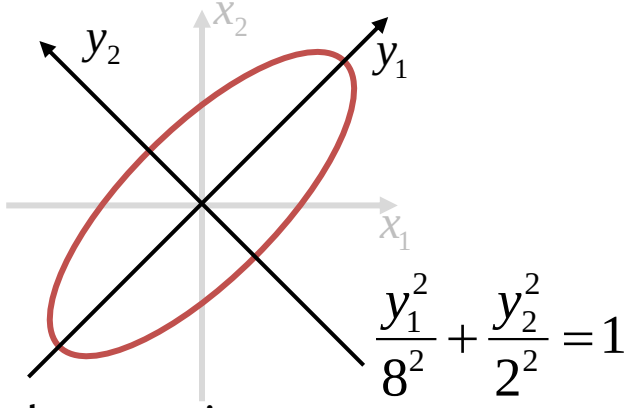


$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

Choose proper axis

(Coordinate Transformation)

$$\mathbf{x} = \mathbf{X}\mathbf{y}$$



$$\frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1$$

Simple expression.
Easily recognition of magnitude of principal axis of ellipse.

Principal Axes Theorem

Ex) Transformation to Principal Axes. Conic Sections

Find out what type of conic section the following quadratic form represents and transform it to principal axes. (Ex 8.2-1)

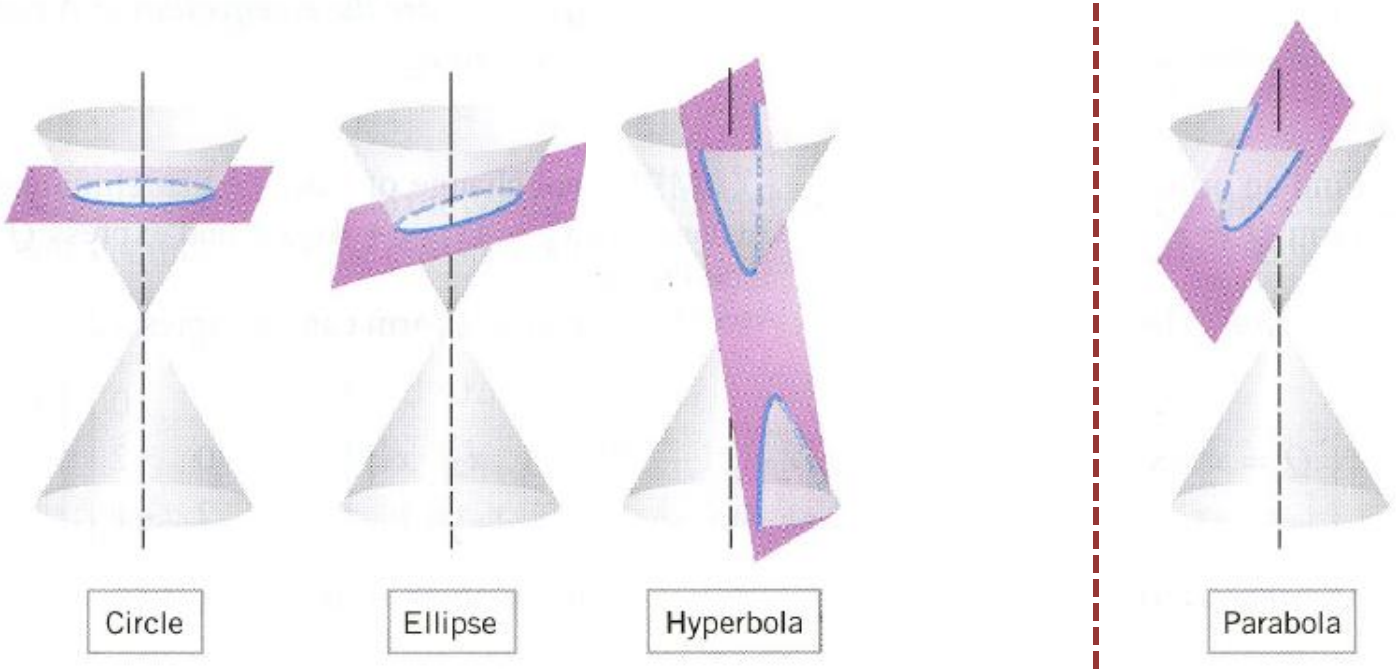
$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

Conic Section ?

Principal Axes Theorem

(Reference : Conic section)

Conic Section(Conic) : curve that results by cutting a double-napped cone with a plane[1)]



Standard form : $ax_1^2 + cx_2^2 + f = 0$,($b=d=e=0$)

Central conic : $ax_1^2 + 2bx_1x_2 + cx_2^2 + f = 0$,($d=e=0$)

: rotated conic in standard position about the origin

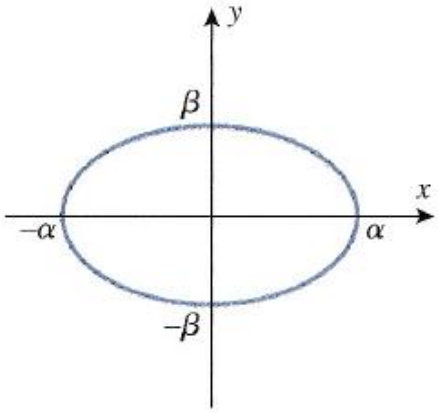
Conic section : $ax_1^2 + 2bx_1x_2 + cx_2^2 + dx_1 + ex_2 + f = 0$,($a \sim f$: constants)

[1)] Anton, Busby, Contemporary Linear Algebra, Wiley, 2003, Ch.8.4, p.502

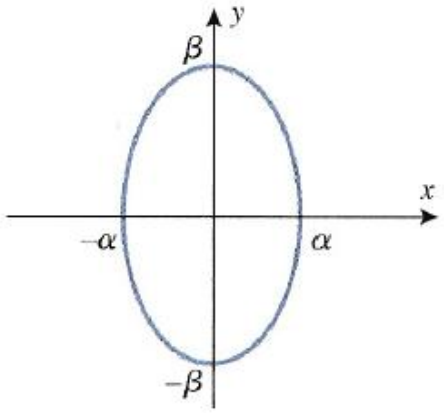
Principal Axes Theorem
(Reference : Conic section)

Standard forms of the central conics (represent a conic in standard position) [1)]

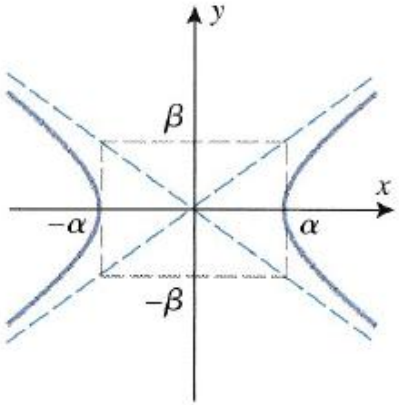
$$ax_1^2 + cx_2^2 + f = 0$$



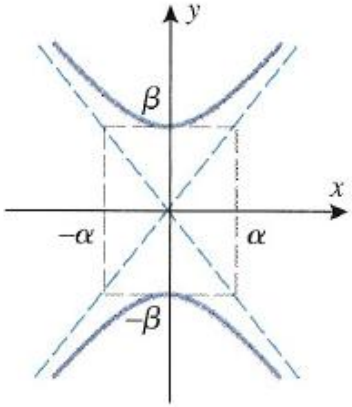
$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1 \quad (\alpha \geq \beta > 0)$$



$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1 \quad (\beta \geq \alpha > 0)$$



$$\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1 \quad (\alpha > 0, \beta > 0)$$



$$\frac{y^2}{\beta^2} - \frac{x^2}{\alpha^2} = 1 \quad (\alpha > 0, \beta > 0)$$

[1)] Anton, Busby, Contemporary Linear Algebra, Wiley, 2003, Ch.8.4, p.502

Principal Axes Theorem

$$\mathbf{y} = \mathbf{X}^{-1}\mathbf{x} = \mathbf{X}^{\mathrm{T}}\mathbf{x}$$

Ex) Transformation to Principal Axes. Conic Sections

Find out what type of conic section the following quadratic form represents and transform it to principal axes. (Ex 8.2-1)

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

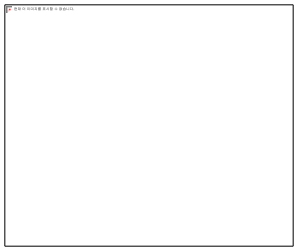
$$\mathbf{A} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$$

Characteristic Equation:

$$\begin{vmatrix} 17-\lambda & -15 \\ -15 & 17-\lambda \end{vmatrix} = 0$$

$$(17-\lambda)^2 - 15^2 = 0 \qquad \therefore \lambda_1 = 2,\ \lambda_2 = 32$$

Eigenvectors



$$Q = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \mathbf{x}^{\mathrm{T}}\mathbf{X}\mathbf{D}\mathbf{X}^{\mathrm{T}}\mathbf{x} = \mathbf{y}^{\mathrm{T}}\mathbf{D}\mathbf{y}$$

$$= [y_1, y_2] \begin{bmatrix} 2 & 0 \\ 0 & 32 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= 2y_1^2 + 32y_2^2 = 128$$

$$\therefore \frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1$$

Principal Axes Theorem

Ex) Transformation to Principal Axes. Conic Sections

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x}$$

$$\lambda_1 = 2,\ \lambda_2 = 32$$

Principal Axes

$$\mathbf{A} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$$

1) $\lambda = \lambda_1 = 2$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}$$

$$15x_1 - 15x_2 = 0$$

From this we get normalized eigenvector $\mathbf{x}_1$.

$$\mathbf{x}_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$$

2) $\lambda = \lambda_2 = 32$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix}$$

$$-15x_1 - 15x_2 = 0$$

From this we get normalized eigenvector $\mathbf{x}_2$.

$$\mathbf{x}_2 = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$$

Principal Axes Theorem

Ex) Transformation to Principal Axes. Conic Sections

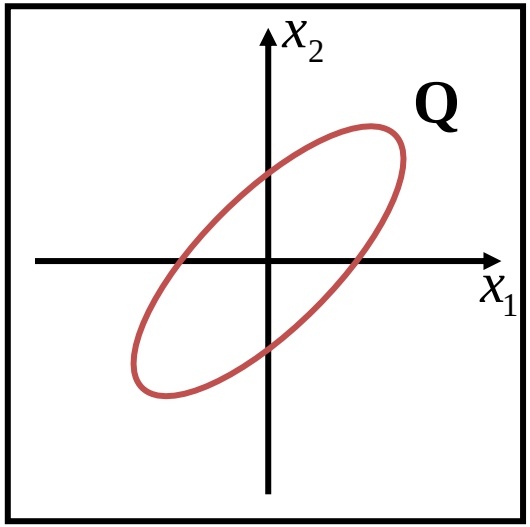
$$\mathbf{x} = \mathbf{X}\mathbf{y}$$

$$\mathbf{X} = [\mathbf{x}_1 \quad \mathbf{x}_2]$$

$$\mathbf{x}_1 = \left[1/\sqrt{2} \quad 1/\sqrt{2}\right]^{\mathbf{T}}$$

$$\mathbf{x}_2 = \left[-1/\sqrt{2} \quad 1/\sqrt{2}\right]^{\mathbf{T}}$$

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

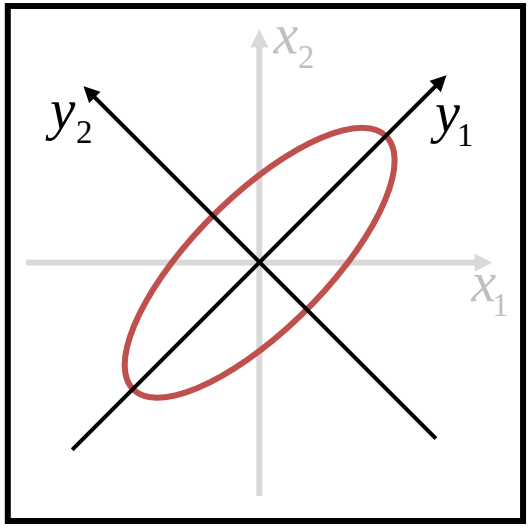


$$\therefore \mathbf{x} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\mathbf{x} = \mathbf{X}\mathbf{y}$$

$$= \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1$$



➔This means a 45° rotation (of principal axes)

(Ref.) Rotational Transformation

$$\frac{z_1'^2}{8^2}+\frac{z_2'^2}{2^2}=1 \quad \longleftarrow \quad z_1'=\frac{1}{\sqrt{2}}(z_1+z_2), \quad z_2'=\frac{1}{\sqrt{2}}(-z_1+z_2)$$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} z_1' \\ z_2' \end{bmatrix}$$

$$\begin{bmatrix} z_1' \\ z_2' \end{bmatrix}=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

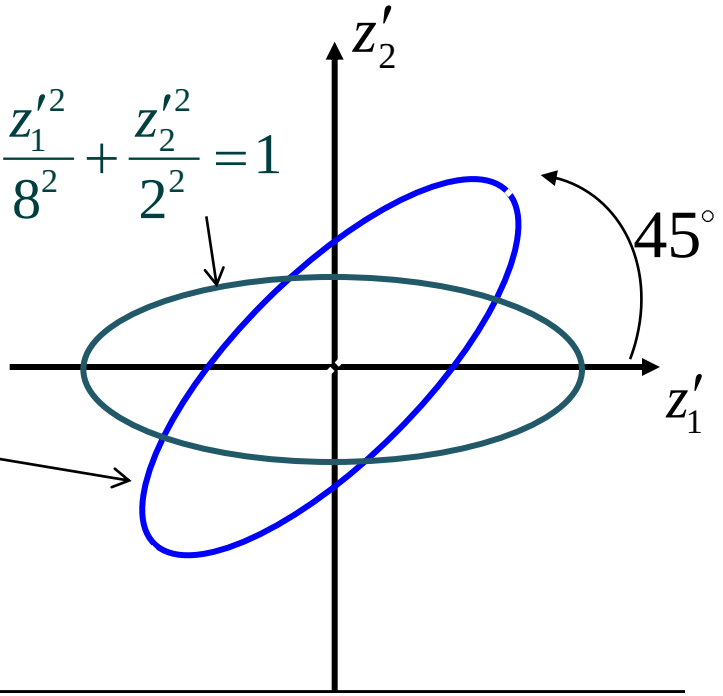
$$\begin{bmatrix} z_1' \\ z_2' \end{bmatrix}=\begin{bmatrix} \cos 45^\circ & \sin 45^\circ \\ -\sin 45^\circ & \cos 45^\circ \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\frac{(z_1+z_2)^2}{8^2}+\frac{(-z_1+z_2)^2}{2^2}=2$$

$$4(z_1^2+2z_1z_2+z_2^2)+64(z_1^2-2z_1z_2+z_2^2)=2$$

$$68z_1^2-120z_1z_2+68z_2^2=512$$

$$17z_1^2-30z_1z_2+17z_2^2=128$$



Principal Axes Theorem

Ex) Transformation to Principal Axes. Conic Sections

$$\mathbf{x} = \mathbf{X}\mathbf{y}$$

$$\mathbf{X} = [\mathbf{x}_1 \quad \mathbf{x}_2]$$

$$\mathbf{x}_1 = \left[1/\sqrt{2} \quad 1/\sqrt{2}\right]^{\mathbf{T}}$$

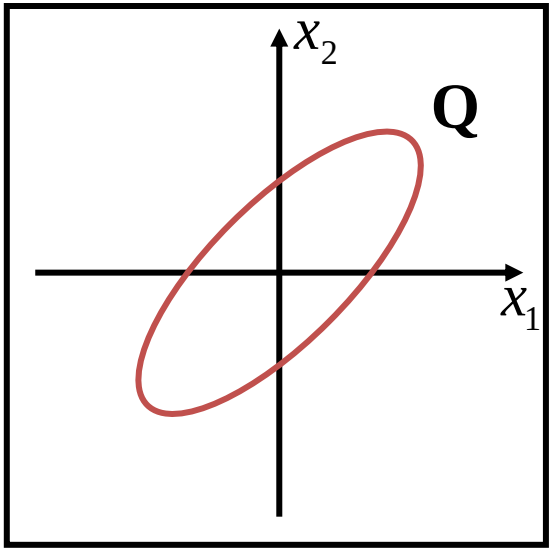
$$\mathbf{x}_2 = \left[-1/\sqrt{2} \quad 1/\sqrt{2}\right]^{\mathbf{T}}$$

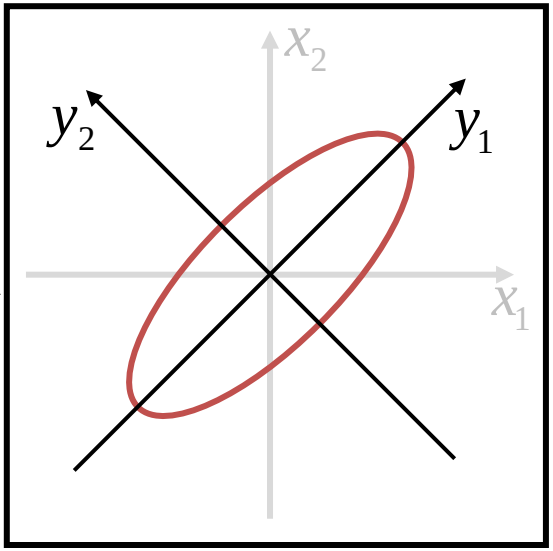
$$\therefore \mathbf{x} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

$$\mathbf{x} = \mathbf{X}\mathbf{y}$$

$$\frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1$$





(Review) Stretching of Elastic Membrain

Transform Matrix \ Object	Conic (Circle, Ellipse, Hyperbola, Parabola)	Arbitrary shape (Rectangle in this example)
Symmetric matrix (eigenvalues are orthogonal)	**Case I**	**Case II**
Non-symmetric matrix (eigenvalues are not orthogonal)	**Case III**	**Case IV**

➔This example of transformation of principal axes corresponds to Case I, and magnitude of transformation matrix is 1.

Quadratic form(Definiteness)

A quadratic form $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ and its (symmetric!) matrix A are called

(a) positive definite if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$,

(b) negative definite if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq 0$,

(c) indefinite if Q(x) takes both positive and negative values.

Quadratic form(Definiteness)

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(b) negative definite if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq 0$,

(c) indefinite if Q(x) takes both positive and negative values.

A necessary and sufficient condition for positive definiteness is that all the "principal minors" are positive, that is,

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \cdots, \quad \det \mathbf{A} > 0$$

Show that the form in Prob. 23 is positive definite, whereas that in Prob. 19 is indefinite.

Quadratic form(Definiteness)

A necessary and sufficient condition for positive definiteness is that all the "principal minors" are positive, that is,

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \cdots, \quad \det \mathbf{A} > 0$$

$$\mathbf{A} = \begin{bmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix}$$

$$a_{11} = 4 > 0$$

$$\begin{vmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{vmatrix} = 4 \cdot 2 - \left(\sqrt{3}\right)^2 = 5 > 0$$

➔ **positive definite**

$$\mathbf{A} = \begin{bmatrix} 1 & 12 \\ 12 & -6 \end{bmatrix}$$

$$a_{11} = 1 > 0$$

$$\begin{vmatrix} 1 & 12 \\ 12 & -6 \end{vmatrix} = -6 - 12^2 = -150 < 0$$

➔ **indefinite**

Quadratic form(Definiteness)

the eigenvalues of $\mathbf{A}$ are

(a) positive definite: all positive

(b) negative definite: all negative

(c) indefinite: both positive and negative

$$Q = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}, \quad \mathbf{x} = \mathbf{X}\mathbf{y}$$

$$= \mathbf{y}^{\mathrm{T}}\mathbf{A}\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \quad \textbf{(1)}$$

Because $y = X^{-1}x$, if $x \neq 0$, then $y \neq 0$.

From equation (1),

If all eigenvalues are positive, $Q(x)$ is positive.
If all eigenvalues are negative, $Q(x)$ is negative.

Application
In Optimization

Quadratic Forms. Transformation to Principal Axes

By definition, a quadratic form Q in the components x_1, … , x_n of a vector $\mathbf{x}$ is a sum of n^2 terms, namely,

$$Q = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}x_j x_k$$

$$= a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n$$

$$+ a_{21}x_2x_1 + a_{22}x_2^2 + \cdots + a_{2n}x_2x_n$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ a_{n1}x_nx_1 + a_{n2}x_nx_2 + \cdots + a_{nn}x_n^2$$

$\mathbf{A} = [a_{jk}]$ is called of the *coefficient matrix* of the form. We may assume that A is *symmetric*, because we can take off-diagonal terms together in pairs and write the result as a sum of two equal terms.

7-11. Jordan form

죠르단형식(Jordan form) (1)

대칭행렬이 아닌 임의의 행렬 A의 고유치 중에서 만약 중복되는 것이 존재할 경우에는 대각변환(Diagonalization)을 할 수 없다. 이런 경우에는 죠르단 형식(Jordan form)을 이용하여 유사 대각화를 할 수 있다.

죠르단 블록

$$J_p(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$$

p x p의 정방행렬 $J_p(\lambda)$를 고유치 λ 에 대한 p차의 죠르단 블록(Jordan block)이라 함.

죠르단 형식

몇 개의 죠르단블록을 대각선상의 부행렬(sub-matrix)로서 나열한 것을 죠르단형식(Jordan form)이라고 한다.

예)

$$J(\lambda) = \begin{bmatrix} J_1(\lambda_1) & 0 & 0 \\ 0 & J_2(\lambda_1) & 0 \\ 0 & 0 & J_2(\lambda_2) \end{bmatrix}$$

죠르단형식(Jordan form) (2)

죠르단블록

$$J_p(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}_{(p \times p)}$$

죠르단형식의 예)

$$J(\lambda) = \begin{bmatrix} J_1(\lambda_1) & 0 & 0 \\ 0 & J_2(\lambda_1) & 0 \\ 0 & 0 & J_2(\lambda_2) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}$$

$J(\lambda)$의 고유치 : λ_1(3중), λ_2(2중)

$J_1(\lambda_1)$: λ_1에 대한 1차 죠르단 블록

$J_2(\lambda_1)$: λ_1에 대한 2차 죠르단 블록

$J_2(\lambda_2)$: λ_2에 대한 2차 죠르단 블록

죠르단형식 (Jordan form) (3)

$$J(\lambda) = \begin{bmatrix} J_1(\lambda_1) & & 0 \\ & J_2(\lambda_1) & \\ 0 & & J_2(\lambda_2) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}$$

임의의 정방 행렬 A가 주어졌을 때, 다음과 같은 상사변환을 통하여 A의 죠르단형식을 구할 수 있다.

$$\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$$

변환행렬

따라서 J는 A와 고유치가 같고, 고유벡터는 다른 행렬이다.

A의 고유벡터를 x라 하면 J의 고유벡터는 $\mathbf{T}^{-1}\mathbf{x}$임.

(공학수학 8장 참고 $\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$

$\hat{\mathbf{A}}$ 의 고유치는 A의 고유치와 같다. $\hat{\mathbf{A}}$ 의 고유벡터는 $\mathbf{P}^{-1}\mathbf{x}$이다.)

Review: Diagonalization

Theorem 8.4.4 Diagonalization of Matrix

If an n x n matrix $\mathbf{A}$ has a basis of eigenvectors, then

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$

is *diagonal*, with the eigenvalues of $\mathbf{A}$ as the entries on the main diagonal. Here $\mathbf{X}$ is the matrix with these eigenvectors as column vectors.

Also $\mathbf{D}^m = \mathbf{X}^{-1}\mathbf{A}^m\mathbf{X}$

Example of Jordan Form

Example: Find the eigenvalues and eigenvectors of A

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

$$|\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda - 1 & -1 & -2 \\ 0 & \lambda - 1 & -3 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 1)^2(\lambda - 2) = 0$$

Eigenvalues : $\lambda_1 = 1, \quad \lambda_2 = 2$

Repeatness : $r_1 = 2, \quad r_2 = 1$

$$\mathbf{A}\mathbf{v}^1 = \lambda\mathbf{v}^1$$

고유벡터 고유치

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

고유치 : $\lambda_1 = 1, \quad \lambda_2 = 2$

중복도 : $r_1 = 2, \quad r_2 = 1$

① λ_1에 대한 고유벡터 구하기

$$\lambda_1\mathbf{I} - \mathbf{A} = \mathbf{I} - \mathbf{A} = \begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & -3 \\ 0 & 0 & -1 \end{bmatrix}$$

$$(\lambda_1\mathbf{I} - \mathbf{A})\mathbf{v}_1^1 = \begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & -3 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbf{0}$$

$$-v_2 - 2v_3 = 0$$

$$-3v_3 = 0$$

$$-v_3 = 0$$

$$\therefore \mathbf{v}_1^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{A}\mathbf{v}^1 = \lambda\mathbf{v}^1$$

고유벡터 고유치

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \mathbf{v}_1^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

고유치 : $\lambda_1 = 1, \quad \lambda_2 = 2$

중복도 : $r_1 = 2, \quad r_2 = 1$

② λ_2에 대한 고유벡터 구하기

$$(\lambda_2\mathbf{I} - \mathbf{A})\mathbf{v}_2^1 = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbf{0}$$

$$v_1 - v_2 - 2v_3 = 0$$

$$v_2 - 3v_3 = 0$$

We assume one unknowns.
미지수는 3개인데 식은 2개이므로,

$$\text{choosing} \quad v_3 = 1,$$

$$v_2 = 3, \quad v_1 = 5$$

$$\therefore \mathbf{v}_2^1 = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

$$\mathbf{Av}_1 = \mathbf{v}_1 \cdot \lambda_1 + \mathbf{v}_2 \cdot 0 + \mathbf{v}_3 \cdot 0$$
$$\mathbf{Av}_2 = \mathbf{v}_1 \cdot 0 + \mathbf{v}_2 \cdot \lambda_2 + \mathbf{v}_3 \cdot 0$$
$$\mathbf{Av}_3 = \mathbf{v}_1 \cdot 0 + \mathbf{v}_2 \cdot 0 + \mathbf{v}_3 \cdot \lambda_3$$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{v}_1^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2^1 = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

고유치 : $\lambda_1 = 1, \quad \lambda_2 = 2$

중복도 : $r_1 = 2, \quad r_2 = 1$

A는 3 x 3 행렬인데, 고유벡터는 2개(2 < 3)이다. (고유벡터 1개 부족)

$$\mathbf{Av}_1^1 = \lambda_1 \mathbf{v}_1^1$$
$$\mathbf{Av}_2^1 = \lambda_2 \mathbf{v}_2^1$$

⇒

$$\mathbf{Av}_1^1 = \mathbf{v}_1^1 \cdot \lambda_1 + \mathbf{v}_2^1 \cdot 0$$
$$\mathbf{Av}_2^1 = \mathbf{v}_1^1 \cdot 0 + \mathbf{v}_2^1 \cdot \lambda_2$$

⇒ **Matrix form**

$$\mathbf{A}\begin{bmatrix} \mathbf{v}_1^1 & \mathbf{v}_2^1 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1^1 & \mathbf{v}_2^1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$[3\times 3][3\times 2] \quad [3\times 2] \quad [2\times 2]$$

$\begin{bmatrix} \mathbf{v}_1^1 & \mathbf{v}_2^1 \end{bmatrix}$의 **역행렬은 존재하지 않음.**

따라서 대각화 ($D = X^{-1}AX$)는 불가능 함.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{v}_1^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2^1 = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

고유벡터는 아니나 두 개의 고유벡터와 일차 독립인 벡터를 이용하여 최대한 대각행렬과 비슷한 행렬을 만들고자 한다.

1) 중복도가 2인 고유치 λ_1과 고유벡터 를 이용하여 고유벡터 $\mathbf{v}_1^1$ 와 일차독립인 벡터 를 $\mathbf{v}_1^1$ 구한다.
2) 그리고 를 열벡터로 하는 행렬 T를 만든다.
3) 그 다음 행렬 T를 변환행렬로 사용해서 대각행렬과 최대한 비슷한 행렬(조르단형식)을 만든다.

$$\mathbf{v}_1^1, \quad \mathbf{v}_1^2, \quad \mathbf{v}_2^1$$

$$\mathbf{A}=\begin{bmatrix}1&1&2\\0&1&3\\0&0&2\end{bmatrix},\ \mathbf{v}_1^1=\begin{bmatrix}1\\0\\0\end{bmatrix},\ \mathbf{v}_2^1=\begin{bmatrix}5\\3\\1\end{bmatrix}$$

λ_1과 $\mathbf{v}_1^1$ 을 이용하여 새로운 벡터 구하기

대각행렬과 유사한 형태(조르단형식)를 만들기 위하여 다음과 같은 벡터 $\mathbf{v}_1^2$ 를 가정한다:

$$\mathbf{A}\mathbf{v}_1^2=\mathbf{v}_1^1+\lambda_1\mathbf{v}_1^2$$

$$(\lambda_1\mathbf{I}-\mathbf{A})\mathbf{v}_1^2=-\mathbf{v}_1^1$$

양변에 $(\lambda_1 I - A)$ 곱하기

$$(\lambda_1\mathbf{I}-\mathbf{A})^2\mathbf{v}_1^2=-(\lambda_1\mathbf{I}-\mathbf{A})\mathbf{v}_1^1=\mathbf{0}$$

$$(\lambda_1\mathbf{I}-\mathbf{A})^2\mathbf{v}_1^2=\mathbf{0}$$

$$(\lambda_1\mathbf{I}-\mathbf{A})^2\mathbf{v}_1^2$$

$$=\begin{bmatrix}0&-1&-2\\0&0&-3\\0&0&-1\end{bmatrix}\begin{bmatrix}0&-1&-2\\0&0&-3\\0&0&-1\end{bmatrix}$$

$$=\begin{bmatrix}0&0&5\\0&0&3\\0&0&1\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\mathbf{0}$$

$$\Rightarrow\ 5v_3=0,\quad 3v_3=0,\quad v_3=0$$

따라서

$$\mathbf{v}_1^2=\begin{bmatrix}1\\0\\0\end{bmatrix}\ or\ \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$\mathbf{v}_1^1$과 선형독립이어야 하므로

Jordan Form

고유치 : $\lambda_1 = 1,\ \lambda_2 = 2$

중복도 : $r_1 = 2,\ r_2 = 1$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}, \mathbf{v}_1^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_1^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2^1 = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

고유벡터와 일차독립인 벡터 (가정)

$$\mathbf{A}\mathbf{v}_1^1 = \lambda_1 \mathbf{v}_1^1$$

$$\mathbf{A}\mathbf{v}_1^2 = \mathbf{v}_1^1 + \lambda_1 \mathbf{v}_1^2$$

$$\mathbf{A}\mathbf{v}_2^1 = \lambda_2 \mathbf{v}_2^1$$

고유벡터

$$\mathbf{A}\mathbf{v}_1^1 = \mathbf{v}_1^1 \cdot \lambda_1 + \mathbf{v}_1^2 \cdot 0 + \mathbf{v}_2^1 \cdot 0$$

$$\mathbf{A}\mathbf{v}_1^2 = \mathbf{v}_1^1 \cdot 1 + \mathbf{v}_1^2 \cdot \lambda_1 + \mathbf{v}_2^1 \cdot 0$$

$$\mathbf{A}\mathbf{v}_2^1 = \mathbf{v}_1^1 \cdot 0 + \mathbf{v}_1^2 \cdot 0 + \mathbf{v}_2^1 \cdot \lambda_2$$

$$\mathbf{A}\begin{bmatrix} \mathbf{v}_1^1 & \mathbf{v}_1^2 & \mathbf{v}_2^1 \end{bmatrix}$$

$$= \left[\mathbf{v}_1^1 \cdot \lambda_1 + \mathbf{v}_1^2 \cdot 0 + \mathbf{v}_2^1 \cdot 0 \;\middle|\; \mathbf{v}_1^1 \cdot 1 + \mathbf{v}_1^2 \cdot \lambda_1 + \mathbf{v}_2^1 \cdot 0 \;\middle|\; \mathbf{v}_1^1 \cdot 0 + \mathbf{v}_1^2 \cdot 0 + \mathbf{v}_2^1 \cdot \lambda_2 \right]$$

$$= \begin{bmatrix} \mathbf{v}_1^1 & \mathbf{v}_1^2 & \mathbf{v}_2^1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1^1 & \mathbf{v}_1^2 & \mathbf{v}_2^1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \mathbf{TJ}$$

$$\therefore \mathbf{AT} = \mathbf{TJ} \Rightarrow \mathbf{T}^{-1}\mathbf{AT} = \mathbf{J}$$ <- Jordan Form

여기서, T는 $\mathbf{v}_1^1,\ \mathbf{v}_1^2,\ \mathbf{v}_2^1$ 를 기저로 하는 변환 행렬

Jordan Form: J

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{J}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

T를 변환행렬로서 사용하여 Jordan Form 구하기:

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \mathbf{J} = \begin{bmatrix} \mathbf{J}_2(\lambda_1) & 0 \\ 0 & \mathbf{J}_1(\lambda_2) \end{bmatrix}$$

Eigen Values of Jordan form

$$\mathbf{A}=\begin{bmatrix}1&1&2\\0&1&3\\0&0&2\end{bmatrix},\ \mathbf{v}_1^1=\begin{bmatrix}1\\0\\0\end{bmatrix},\ \mathbf{v}_2^1=\begin{bmatrix}5\\3\\1\end{bmatrix}$$

고유치 : $\lambda_1=1,\ \lambda_2=2$
중복도 : $r_1=2,\ r_2=1$

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T}=\mathbf{J}$$

$$\mathbf{J}=\begin{bmatrix}1&1&0\\0&1&0\\0&0&2\end{bmatrix}$$

$$\mathbf{J}\mathbf{x}=\lambda\mathbf{x}$$

$$(\mathbf{J}-\lambda\mathbf{I})\mathbf{x}=0$$

$$|\mathbf{J}-\lambda\mathbf{I}|=\begin{vmatrix}1-\lambda&1&0\\0&1-\lambda&0\\0&0&2-\lambda\end{vmatrix}=(1-\lambda)^2(2-\lambda)=0$$

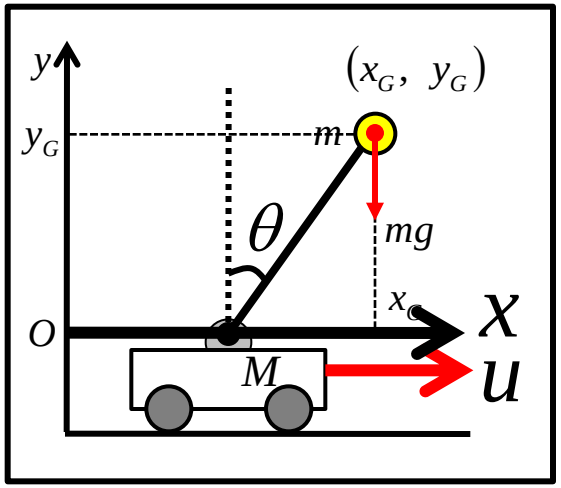
고유치 : $\lambda_1=1,\ \lambda_2=2$
중복도 : $r_1=2,\ r_2=1$

J는 A의 상사행렬이며,
상사행렬은 동일한 고유치를 가진다는 것을 확인할 수 있음.

Theorem 8.4.3 Eigenvalues and Eigenvectors of Similar Matrices
If $\hat{A}$ is similar to A, then $\hat{A}$ has the same eigenvalues as A.
Furthermore, if x is an eigenvector of A, then $y = P^{-1}x$ is an eigenvector of $\hat{A}$ corresponding to the same eigenvalue.

Jordan form의 응용

Controllability of an Inverted Pendulum



u : x방향 힘

x_C: 수레의 x방향 변위

θ : Inverted Pendulum의 각도

문제: 수레에 x방향 힘 u 를 가하여 아래의 상태 변수를 컨트롤할 수 있는가?

$\dot{x}_C$: 수레의 x방향 속도

x_C : 수레의 x방향 변위

$\dot{\theta}$: Inverted Pendulum의 각속도

θ : Inverted Pendulum의 각도

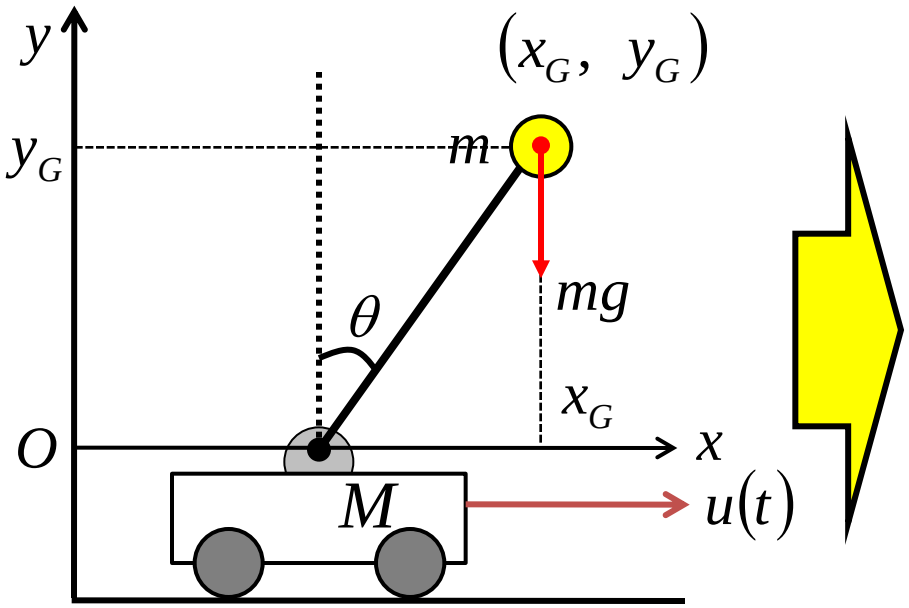
$x_1 = x_C,\ x_2 = \dot{x}_C,\ x_3 = \theta,\ x_4 = \dot{\theta}$ 라고 했을 때,

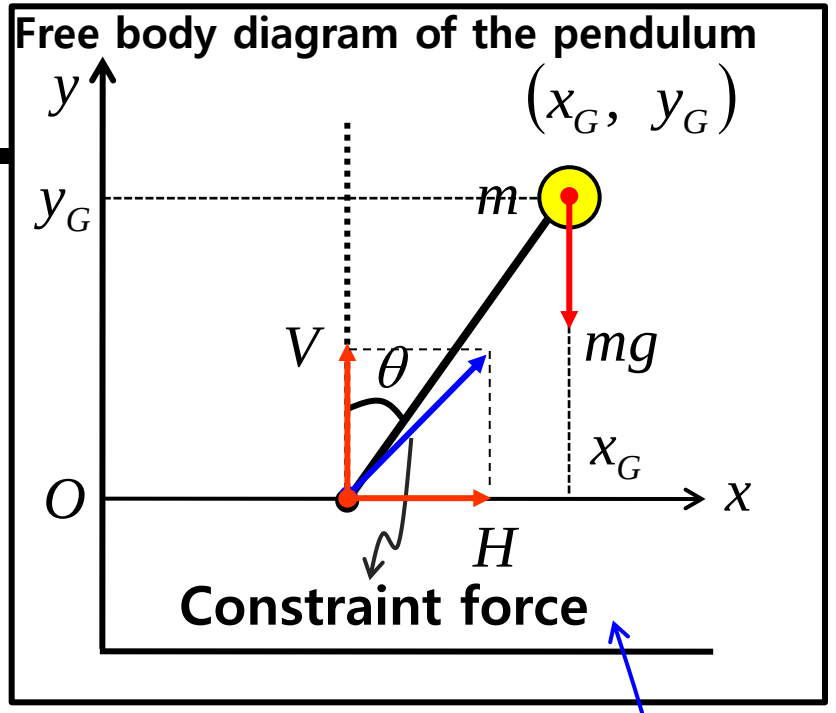
각 상태변수들의 변화율 $\dot{x}_1,\ \dot{x}_2,\ \dot{x}_3, \dot{x}_4$ 를 변경할 수 있다면

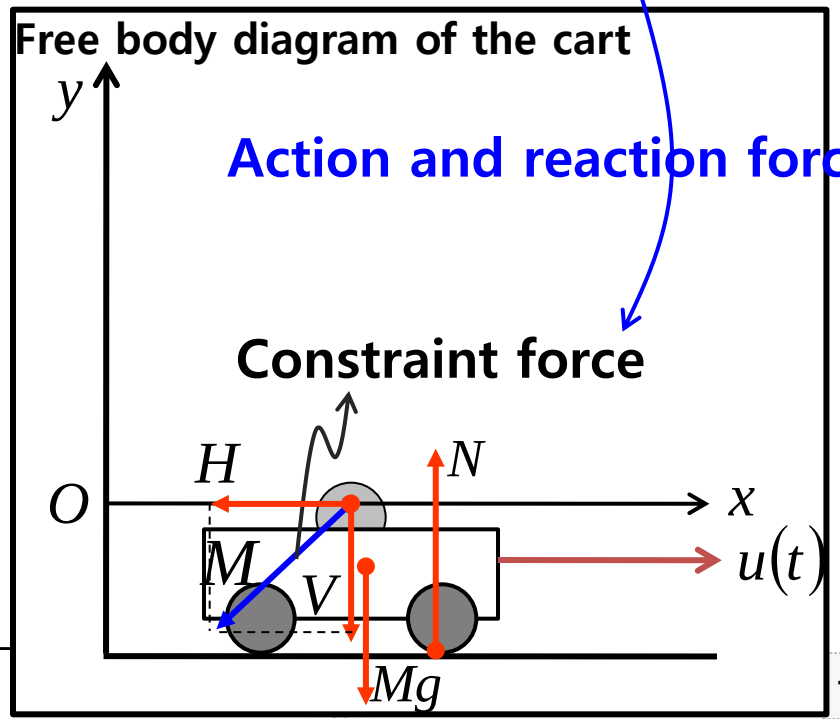
제어 가능하다고 할 수 있다.

Inverted Pendulum

- 수학적 모델링





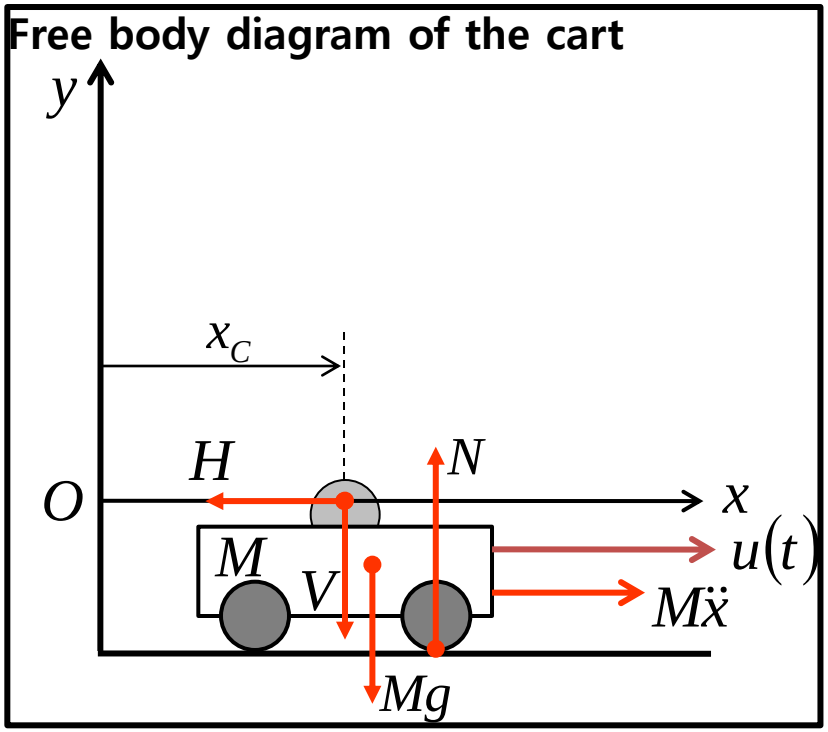


N: 수레가 바닥으로 부터 받는 반력

Inverted Pendulum

- 수학적 모델링

$$\sum F = m\frac{d^2x}{dt^2}$$



1) 수레의 수평 운동

L.H.S : $\sum F_x = u(t) - H$

R.H.S : $M\ddot{x}_C$

$$u(t) - H = M\ddot{x}_C$$

2) 수레의 수직 운동

L.H.S : $\sum F = -V - Mg + N = 0$

R.H.S : $m\ddot{y}_C$

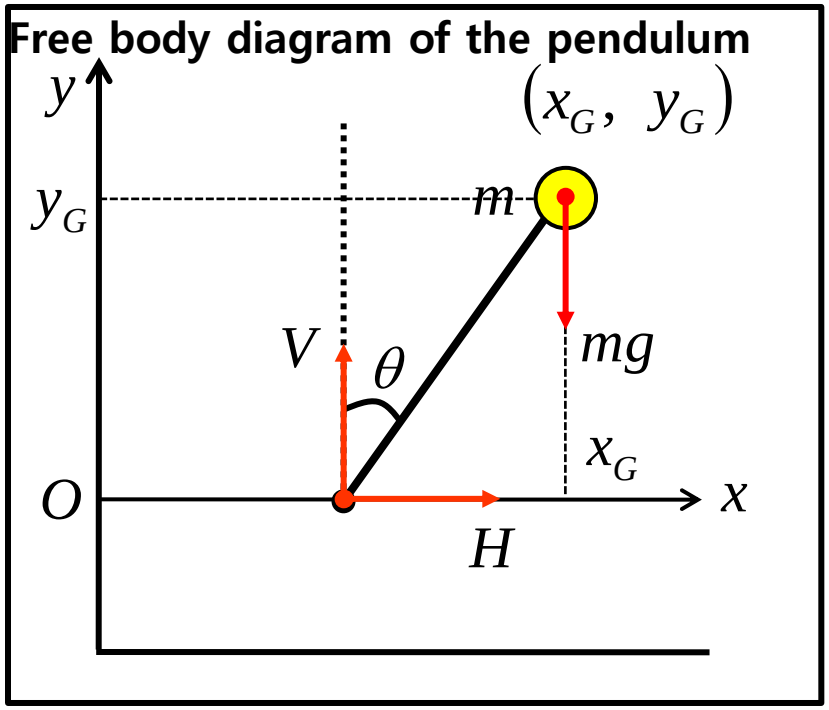
$$m\ddot{y}_C = 0$$

수레의 수직방향 운동은 없음

Inverted Pendulum

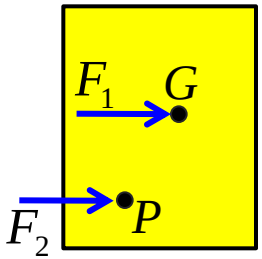
- 수학적 모델링

$$\sum F = m\frac{d^2x}{dt^2} \qquad \sum M = I\frac{d^2\theta}{dt^2}$$



(가정: 진자와 수레 사이의 막대의 무게는 무시)

c.f) Newton Equation



$$\sum \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = m\ddot{\mathbf{x}}_G$$

m: mass

G: center of mass

P: point on which the F_2 is exerted

3) 진자의 수평 운동

L.H.S : $\sum F = H$

R.H.S : $m\frac{d^2x_G}{dt^2}$

$$H = m\frac{d^2}{dt^2}x_G$$

4) 진자의 수직 운동

L.H.S : $\sum F = V - mg$

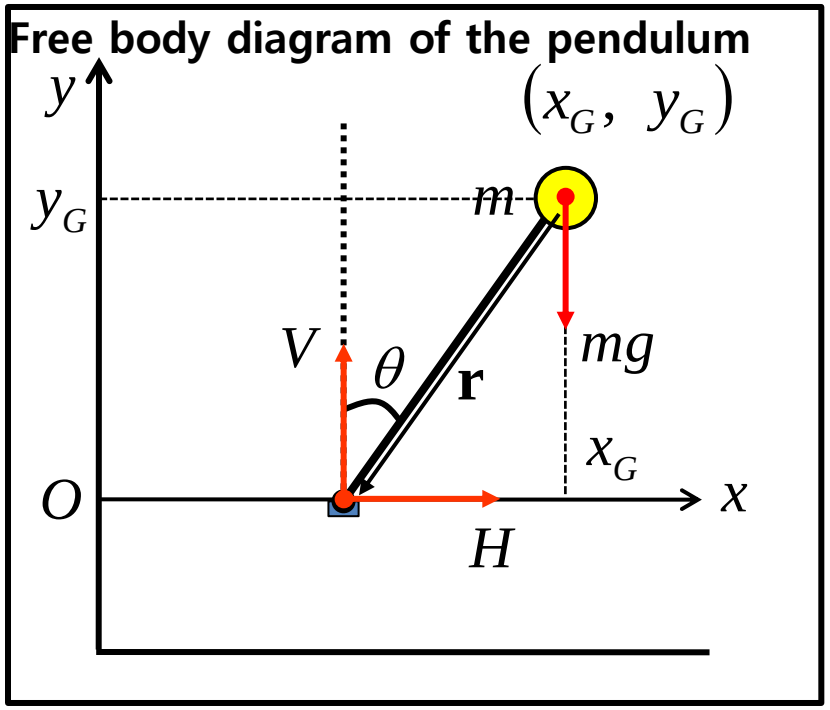
R.H.S : $m\frac{d^2y_G}{dt^2}$

$$V - mg = m\frac{d^2}{dt^2}y_G$$

Inverted Pendulum

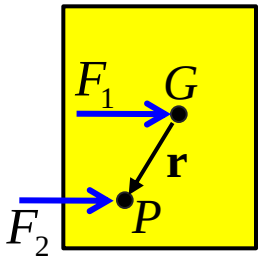
- 수학적 모델링

$$\sum F = m\frac{d^2x}{dt^2} \qquad \sum M = I\frac{d^2\theta}{dt^2}$$



(가정: 진자와 수레 사이의 막대의 무게는 무시)

c.f) Euler Equation about center of mass



$$\sum \mathbf{M} = \mathbf{r} \times \mathbf{F}_2 = I\ddot{\theta}$$

I : mass moment of inertia

G: center of mass

P: point on which the F_2 is exerted

5) 진자의 회전 운동 (반시계방향 +)

진자의 중심(x_G, y_G)을 회전 중심으로 설정

L.H.S : $\sum \mathbf{M} = \mathbf{r} \times \mathbf{H} + \mathbf{r} \times \mathbf{V}$

$$\mathbf{r} = (-l\sin\theta, -l\cos\theta, 0), \mathbf{H} = (H,0,0), \mathbf{V} = (0,V,0)$$

$$\sum \mathbf{M} = (0,0,Hl\cos\theta) + (0,0,-Vl\sin\theta)$$
$$= (0,0,Hl\cos\theta - Vl\sin\theta)$$

R.H.S : $\mathbf{I}\ddot{\theta}$

Euler Equation about center of mass

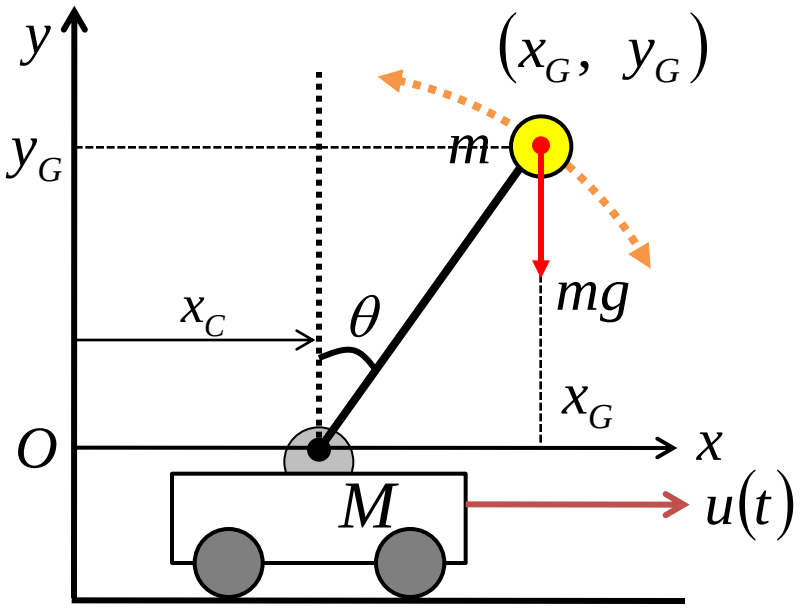
$$(0,0,Hl\cos\theta - Vl\sin\theta) = \mathbf{I}\ddot{\theta}$$

⇩ 평면운동이므로 z축 회전만 고려하면

$$\boxed{Hl\cos\theta - Vl\sin\theta = I\ddot{\theta}}$$

Inverted Pendulum

- 수학적 모델링



구속조건을 고려해야 함!!!!!

$$x_G = x_C + l\sin\theta$$

$$y_G = l\cos\theta$$

본 구속조건을 만족해야
진자 운동(Pendulum)을 하게 됨

1) 수레의 수평 운동

$$u(t) - H = M\ddot{x}_C$$

3) 진자의 수평 운동

$$H = m\frac{d^2}{dt^2}x_G$$

$$\Rightarrow H = m\frac{d^2}{dt^2}(x_C + l\sin\theta)$$

4) 진자의 수직 운동

$$V - mg = m\frac{d^2}{dt^2}y_G$$

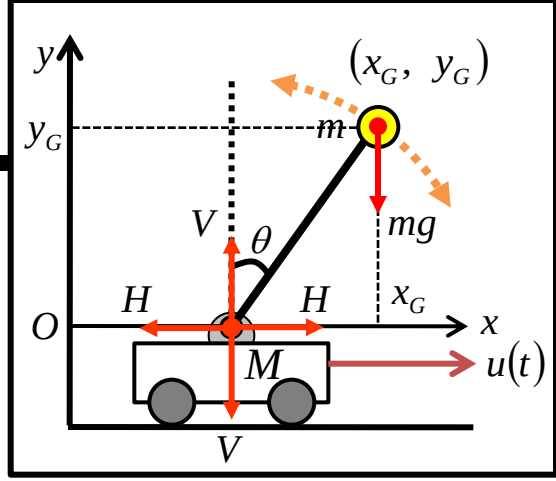
$$\Rightarrow V - mg = m\frac{d^2}{dt^2}(l\cos\theta)$$

5) 진자의 회전 운동 (반시계방향 +)

$$Hl\cos\theta - Vl\sin\theta = I\ddot{\theta}$$

Inverted Pendulum

- 수학적 모델링



$$u(t) - H = M\ddot{x}_C$$

$$H = m\frac{d^2}{dt^2}(x_C + l\sin\theta)$$

$$V - mg = m\frac{d^2}{dt^2}(l\cos\theta)$$

$$Hl\cos\theta - Vl\sin\theta = I\ddot{\theta}$$

Let $\theta \approx 0$ ➔ $\sin\theta \approx \theta,\ \cos\theta \approx 1$

$$u(t) - H = M\ddot{x}_C \longrightarrow u - H = M\ddot{x}_C\ ,\ u = u(t)$$

$$H = m\frac{d^2}{dt^2}(x_C + l\theta) \longrightarrow H = m\ddot{x}_C + ml\ddot{\theta}$$

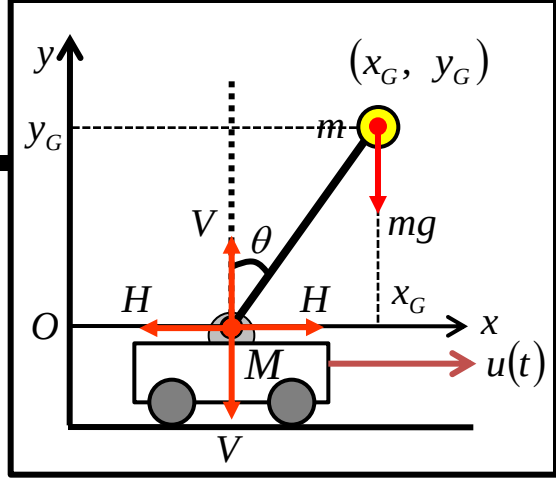
$$V - mg = m\frac{d^2}{dt^2}(l) \longrightarrow V - mg = 0$$

$$Hl - Vl\theta = I\ddot{\theta} \longrightarrow H - V\theta = 0$$

질량이 질량 중심에 모여 있으므로 I=0

Inverted Pendulum

- 상태 방정식으로의 변경



$$u - H = M\ddot{x}_C \quad \rightarrow (1)$$

$$H = m\ddot{x}_C + ml\ddot{\theta} \quad \rightarrow (2)$$

$$V - mg = 0 \quad \rightarrow (3)$$

$$H - V\theta = 0 \quad \rightarrow (4)$$

(3) 식을 (4) 식에 대입하면,

$$H - mg\theta = 0 \quad \rightarrow (4\text{-}1)$$

(4-1) 식을 (1) 식에 대입하면,

$$\boxed{u - mg\theta = M\ddot{x}_C} \quad \rightarrow (1\text{-}1)$$

(4-1) 식을 (2) 식에 대입하면,

$$mg\theta = m\ddot{x}_C + ml\ddot{\theta}$$

$$g\theta = \ddot{x}_C + l\ddot{\theta} \quad \rightarrow (2\text{-}1)$$

(1-1) 식을 (2-1) 식에 대입하면,

$$g\theta = \frac{u - mg\theta}{M} + l\ddot{\theta}$$

$$\boxed{(M+m)g\theta - u = Ml\ddot{\theta}} \rightarrow (2\text{-}2)$$

Inverted Pendulum

- 상태 방정식으로의 변경

$$u - mg\theta = M\ddot{x}_C$$

$$(M+m)g\theta - u = Ml\ddot{\theta}$$

$$\begin{cases} \ddot{x}_C = -\dfrac{mg}{M}\theta + \dfrac{1}{M}u \\ \ddot{\theta} = \dfrac{(M+m)g}{Ml}\theta - \dfrac{1}{Ml}u \end{cases}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{mg}{M}\theta + \frac{1}{M}u$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \frac{(M+m)g}{Ml}\theta - \frac{1}{Ml}u$$

상태 변수를 다음과 같이 설정한다.

$$x_1 = x_C \quad x_3 = \theta$$

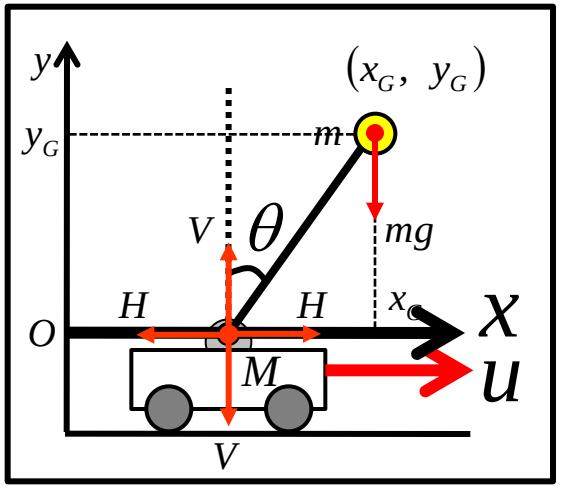
$$x_2 = \dot{x}_C \,, \quad x_4 = \dot{\theta}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{Ml} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{Ml} \end{bmatrix} u$$

Inverted Pendulum

- 상태 방정식으로의 변경

상태공간에 대한 내용은 Kreyszig책 4장 Systems of ODE를 참고한다. 여기서 중요한 것은 운동방정식이 아래와 같은 메트릭스 형태로 표현되었다는 것이다.



상태공간 (State Space)

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & -\frac{mg}{M} & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & \frac{(M+m)g}{Ml} & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{M}\\ 0\\ -\frac{1}{Ml}\end{bmatrix}u$$

u : x방향 힘

x_C : 수레의 x방향 변위

θ : Inverted Pendulum의 각도

$$x_1 = x_C,\ x_2 = \dot{x}_C,$$

$$x_3 = \theta,\ x_4 = \dot{\theta}$$

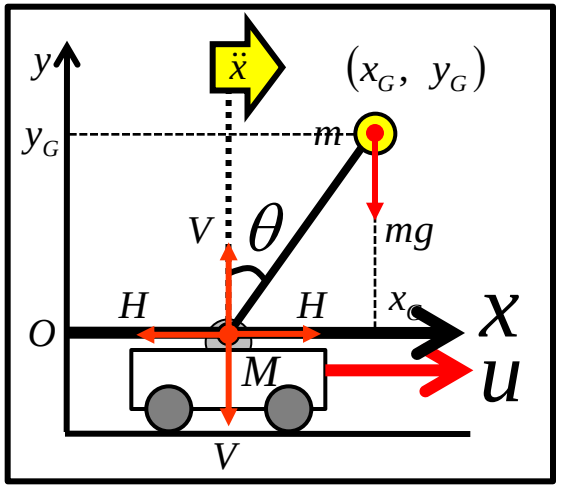
Let

$$M = 1,\ m = 0.6,\ g = 10,\ l = 1$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & -6 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 16 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ 1\\ 0\\ -1\end{bmatrix}u$$

Inverted Pendulum

- 상태 방정식으로의 변경



$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & -6 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 16 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+\begin{bmatrix}0\\ 1\\ 0\\ -1\end{bmatrix}u$$

$$\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}+\mathbf{B}u$$

u : x방향 힘

x_C: 수레의 x방향 변위

θ : Inverted Pendulum의 각도

$$x_1=x_C,\ x_2=\dot{x}_C,$$

$$x_3=\theta,\ x_4=\dot{\theta}$$

$$\dot{x}_1=x_2$$

$$\dot{x}_2=-6x_3+u$$

$$\dot{x}_3=x_4$$

$$\dot{x}_4=16x_3-u$$

x_1: 제어 가능한 x_2에 의해 제어 가능

x_2: 제어력 u에 의해 제어 가능

x_3: 제어 가능한 x_4에 의해 제어 가능

x_4: 제어력 u에 의해 제어 가능

메트릭스 A가 복잡해지면 판단하기가 쉽지 않음!!

메트릭스 A를 Jordan Form으로 변경하면 판단하기 쉬움!!

Inverted Pendulum

- Convert to Jordan Form (1/5)

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 16 & 0 \end{bmatrix}$$

$$\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda & -1 & 0 & 0 \\ 0 & \lambda & 6 & 0 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & -16 & \lambda \end{bmatrix}$$

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \lambda^2(\lambda^2 - 16) = 0$$

$$\therefore \lambda = 0,\ 4,\ -4$$

Inverted Pendulum

- Convert to Jordan Form (2/5)

$$\lambda\mathbf{I}-\mathbf{A}=\begin{bmatrix}\lambda & -1 & 0 & 0\\ 0 & \lambda & 6 & 0\\ 0 & 0 & \lambda & -1\\ 0 & 0 & -16 & \lambda\end{bmatrix}$$

1) $\lambda_1 = 0 \qquad (\lambda_1\mathbf{I}-\mathbf{A})\mathbf{x}_1 = 0$

$$\begin{bmatrix}0 & -1 & 0 & 0\\ 0 & 0 & 6 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & -16 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=0 \Rightarrow \begin{aligned} -x_2 &= 0\\ 6x_3 &= 0\\ -x_4 &= 0\\ -16x_3 &= 0\end{aligned} \qquad \therefore \mathbf{x}_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

2) $\lambda_1 = 0 \qquad (\lambda_1\mathbf{I}-\mathbf{A})\mathbf{x}_1 = 0 \qquad (\lambda_1\mathbf{I}-\mathbf{A})\mathbf{x}_2 = -\mathbf{x}_1 \left(\Leftrightarrow \mathbf{A}\mathbf{x}_2 = \mathbf{x}_1 + \lambda_1\mathbf{x}_2\right)$

$$(\lambda_1\mathbf{I}-\mathbf{A})^2\mathbf{x}_2 = 0$$

$$\begin{bmatrix}0 & 0 & -6 & 0\\ 0 & 0 & 0 & -6\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 16\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=0 \Rightarrow \begin{aligned} -6x_3 &= 0\\ -6x_4 &= 0\\ x_4 &= 0\\ 16x_4 &= 0\end{aligned} \qquad \therefore \mathbf{x}_2 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

Inverted Pendulum

- Convert to Jordan Form (3/5)

$$\lambda\mathbf{I}-\mathbf{A}=\begin{bmatrix}\lambda & -1 & 0 & 0\\ 0 & \lambda & 6 & 0\\ 0 & 0 & \lambda & -1\\ 0 & 0 & -16 & \lambda\end{bmatrix}$$

3) $\lambda_2 = 4 \qquad (\lambda_2\mathbf{I}-\mathbf{A})\mathbf{x}_3 = 0$

$$\begin{bmatrix}4 & -1 & 0 & 0\\ 0 & 4 & 6 & 0\\ 0 & 0 & 4 & -1\\ 0 & 0 & -16 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=0 \Rightarrow \begin{aligned}4x_1 - x_2 &= 0\\ 4x_2 + 6x_3 &= 0\\ 4x_3 - x_4 &= 0\\ -16x_3 + 4x_4 &= 0\end{aligned} \quad \therefore \mathbf{x}_3 = \begin{bmatrix}3\\ 12\\ -8\\ -32\end{bmatrix}$$

4) $\lambda_3 = -4 \qquad (\lambda_3\mathbf{I}-\mathbf{A})\mathbf{x}_4 = 0$

$$\begin{bmatrix}-4 & -1 & 0 & 0\\ 0 & -4 & 6 & 0\\ 0 & 0 & -4 & -1\\ 0 & 0 & -16 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=0 \Rightarrow \begin{aligned}-4x_1 - x_2 &= 0\\ -4x_2 + 6x_3 &= 0\\ -4x_3 - x_4 &= 0\\ -16x_3 - 4x_4 &= 0\end{aligned} \quad \therefore \mathbf{x}_4 = \begin{bmatrix}3\\ -12\\ -8\\ 32\end{bmatrix}$$

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Inverted Pendulum

- Convert to Jordan Form (4/5)

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 16 & 0 \end{bmatrix}$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 3 \\ 12 \\ -8 \\ -32 \end{bmatrix}, \mathbf{x}_4 = \begin{bmatrix} 3 \\ -12 \\ -8 \\ 32 \end{bmatrix} \Rightarrow \mathbf{T} = \begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 12 & -12 \\ 0 & 0 & -8 & -8 \\ 0 & 0 & -32 & 32 \end{bmatrix}$$

$$\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\,\mathbf{T} = \begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 12 & -12 \\ 0 & 0 & -8 & -8 \\ 0 & 0 & -32 & 32 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 16 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 12 & -12 \\ 0 & 0 & -8 & -8 \\ 0 & 0 & -32 & 32 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

Inverted Pendulum

- Convert to Jordan Form (5/5)

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & -6 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 16 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+\begin{bmatrix}0\\ 1\\ 0\\ -1\end{bmatrix}u$$

$$\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}+\mathbf{B}u$$

$$\times\mathbf{T}^{-1}$$

$$\mathbf{T}^{-1}\dot{\mathbf{x}}=\mathbf{T}^{-1}\mathbf{A}\mathbf{x}+\mathbf{T}^{-1}\mathbf{B}u$$

$$\mathbf{z}=\mathbf{T}^{-1}\mathbf{x}$$

$$\dot{\mathbf{z}}=\mathbf{T}^{-1}\dot{\mathbf{x}}$$

$$\dot{\mathbf{z}}=\mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{z}+\mathbf{T}^{-1}\mathbf{B}u$$

$$\dot{\mathbf{z}}=\mathbf{J}\mathbf{z}+\mathbf{T}^{-1}\mathbf{B}u$$

$$\mathbf{J}=\mathbf{T}^{-1}\mathbf{A}\mathbf{T}$$

$$\mathbf{S}=\begin{bmatrix}1 & 0 & 3 & 3\\ 0 & 1 & 12 & -12\\ 0 & 0 & -8 & -8\\ 0 & 0 & -32 & 32\end{bmatrix}$$

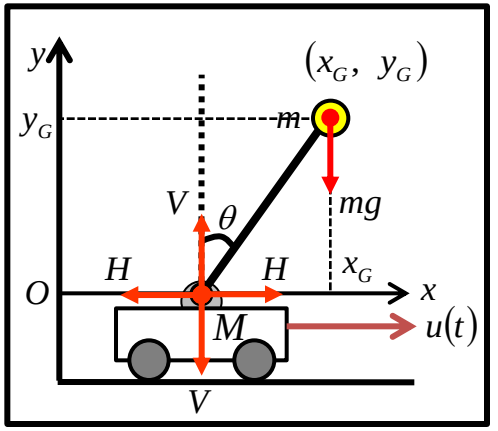
$$\mathbf{J}=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 4 & 0\\ 0 & 0 & 0 & -4\end{bmatrix}$$

Inverted Pendulum

- Controllability

$$\dot{\mathbf{z}} = \mathbf{J}\mathbf{z} + \mathbf{T}^{-1}\mathbf{B}u$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \frac{1}{40} \begin{bmatrix} 0 \\ 40 \\ 1 \\ -1 \end{bmatrix} u$$



Jordan Form으로 바뀌었으므로 Coupled된 변수들을 한눈에 알아볼 수 있음!!

본 예제에서는 z_1만 z_2의 영향을 받고 있고, 다른 변수들은 u에 의한 영향만 받고 있음!!

$\dot{z}_1 = z_2$ → z_1: 제어력의 영향은 받지 않으나 제어 가능한 z_2의 영향을 받음 ➔ 제어 가능

$\dot{z}_2 = u$ → z_2: 제어력 u에 의해 제어 가능

$\dot{z}_3 = 4z_3 + \frac{1}{40}u$ → z_3: 제어력 u에 의해 제어 가능

$\dot{z}_4 = -4z_4 - \frac{1}{40}u$ → z_4: 제어력 u에 의해 제어 가능

7-12. Singular Value Decomposition (SVD)

SINGULAR VALUE DECOMPOSITION (SVD)

*Anton, Busby, Contemporary Linear Algebra, Wiley, 2003, Ch.8.6, p.502

Problem Statement of Singular Value Decomposition (SVD)

Problem:

Given: $\mathbf{Ax} = \mathbf{b}$ $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 4 \end{bmatrix}$

Find: $\mathbf{x}$

The inverse matrix of $\mathbf{A}$ does not exist !!!

We try followings:

Instead of $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$, we solve $\mathbf{x}^{+} = \mathbf{A}^{+}\mathbf{b}$

, where $\mathbf{A}^{+}$ is pseudo inverse of $\mathbf{A}$

How to determine $\mathbf{A}^{+}$?

Similar Matrices &
Singular Value Decomposition (SVD)

SVD

Any m by n matrix A can be factored into

$$\underset{[m\times n]}{\mathbf{A}} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}$$

The columns of U(m by m) are eigenvectores of AA^T
The columns of V (n by n) are eigenvectores of A^TA.
The r singular values of the diagonal of Σ (m by m) are the square roots of the nonzero eigenvalues of both AA^T and A^TA.

Properties of A^TA

$\underset{[m\times n]}{\mathbf{A}}$ 1. Rank of A is r
Rank of A^TA is r $\Rightarrow$ A and A^TA have the same rank $_{[n\times n]}$

Howard Anton, Robert C. Busby, Contemporary Linear Algebra, Wiley, 2003, p.365

2.

$$\underset{[n\times n]}{\mathbf{A}^T\mathbf{A}} = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^T\mathbf{a}_1 & \mathbf{a}_1^T\mathbf{a}_2 & \cdots & \mathbf{a}_1^T\mathbf{a}_n \\ \mathbf{a}_2^T\mathbf{a}_1 & \mathbf{a}_2^T\mathbf{a}_2 & & \\ \vdots & & \ddots & \\ \mathbf{a}_n^T\mathbf{a}_1 & & & \mathbf{a}_1^T\mathbf{a}_1 \end{bmatrix}$$

$$\mathbf{a}_i^T\mathbf{a}_j = \mathbf{a}_i \cdot \mathbf{a}_j = \mathbf{a}_j \cdot \mathbf{a}_i = \mathbf{a}_j^T\mathbf{a}_i$$

$\mathbf{A}^T\mathbf{A}$ is a symmetric matrix.

Properties of A^TA

$\mathbf{A}_{[m\times n]}$

1. Rank of A is r
   Rank of A^TA is r $\Rightarrow$ A and A^TA have the same rank $_{[n\times n]}$ Howard Anton, Robert C. Busby, Contemporary Linear Algebra, Wiley, 2003, p.365

2. $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is a symmetric matrix.

3. A symmetric matrix is orthogonally diagonalizable. Howard Anton, Robert C. Busby, Contemporary Linear Algebra, Wiley, 2003, p.470

$$\mathbf{A}^{\mathrm{T}}\mathbf{A}=\mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{-1}$$,where $\boldsymbol{\Lambda}=\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}$, and λ_i is eigenvalues of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

4. Similar matrices have the same rank Howard Anton, Robert C. Busby, Contemporary Linear Algebra, Wiley, 2003, p.457

similar matrices A and B
$$\mathbf{A}=\mathbf{PBP}^{-1}$$

$\mathbf{A}^{\mathrm{T}}\mathbf{A}$ and $\boldsymbol{\Lambda}$ are similar matrices.

Therefore rank of $\boldsymbol{\Lambda}$ is r.

In other words, $\boldsymbol{\Lambda}=\begin{bmatrix}\lambda_1 & & & \\ & \ddots & & \\ & & \lambda_r & \\ & & & 0\end{bmatrix} \Leftrightarrow \lambda_i \neq 0, i=1,\ldots,r$ and $\lambda_i = 0, i=r+1,\ldots,n$

Properties of A^TA

$\mathbf{A}_{[m\times n]}$

1. Rank of A is r
   Rank of A^TA is r
   $\Leftrightarrow$ A and A^TA have the same rank $[n\times n]$
   Howard Anton, Robert C. Busby, Contemporary Linear Algebra, Wiley, 2003, p.365

2. $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is a symmetric matrix.

3. A symmetric matrix is orthogonally diagonalizable.
   Howard Anton, Robert C. Busby, Contemporary Linear Algebra, Wiley, 2003, p.470

4. Similar matrices have the same rank
   Howard Anton, Robert C. Busby, Contemporary Linear Algebra, Wiley, 2003, p.457

$\lambda_i \neq 0, i=1,\ldots,r$ and $\lambda_i = 0, i=r+1,\ldots,n$, where λ_i is eigenvalues of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

5. $\mathbf{Ax}\cdot\mathbf{Ax} = \|\mathbf{Ax}\|^2$

$$RHS : \|\mathbf{Ax}\|^2 \geq 0$$

$$LHS : \mathbf{Ax}\cdot\mathbf{Ax} = (\mathbf{Ax})^{\mathrm{T}}\mathbf{Ax} = \mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{Ax} = \mathbf{x}^{\mathrm{T}}\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)\mathbf{x}$$

$$\mathbf{x}^{\mathrm{T}}\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)\mathbf{x} \geq 0$$

$\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is semi-positive definite, whose eigenvalues are positive or zero.

$\lambda_i > 0, i=1,\ldots,r$ and $\lambda_i = 0, i=r+1,\ldots,n$, where λ_i is eigenvalues of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

Properties of A^TA

$\underset{[m\times n]}{\mathbf{A}}$

1. Rank of A is r
   Rank of A^TA is r
   ⇔ A and A^TA have the same rank $[n\times n]$
   Howard Anton, Robert C. Busby, Contemporary Linear Algebra, Wiley, 2003, p.365

2. $\mathbf{A}^T\mathbf{A}$ is a symmetric matrix.

3. A symmetric matrix is orthogonally diagonalizable.
   Howard Anton, Robert C. Busby, Contemporary Linear Algebra, Wiley, 2003, p.470

4. Similar matrices have the same rank
   Howard Anton, Robert C. Busby, Contemporary Linear Algebra, Wiley, 2003, p.457

5. $\mathbf{A}^T\mathbf{A}$ is semi-positive definite, whose eigenvalues are positive or zero.
   $\lambda_i \geq 0, i=1,\ldots,r$ and $\lambda_i = 0, i=r+1,\ldots,n$, where λ_i is eigenvalues of $\mathbf{A}^T\mathbf{A}$

6. A symmetric matrix has an orthonormal basis of eigenvectors
   Reference) 07-9 Linear Algebra Orthogonal, Diagonalization, Similar, Quadratic p. 43

$\mathbf{A}^T\mathbf{A}$ has r unit orthogonal eigenvectors $\underset{[n\times 1]}{\mathbf{v}_i}$

$$\underset{[n\times n][n\times 1]}{\mathbf{A}^T\mathbf{A}\mathbf{v}_i} = \underset{[n\times 1]}{\lambda_i \mathbf{v}_i}, \quad i=1,\ldots,r$$

,where λ_i is nonzero eigenvalue for $i=1,\ldots,r$

Singular Value Decomposition (SVD)

$\underset{[m\times n]}{\mathbf{A}}$

$\mathbf{A}^{\mathrm{T}}\mathbf{A}$ has r eigenvalues λ_i and r unit orthogonal eigenvectors $\underset{[n\times 1]}{\mathbf{v}_i}$

$$\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i,\ i = 1,\ldots,r$$

Now we consider the set of image vectors $\{\mathbf{A}\mathbf{v}_1 \quad \cdots \quad \mathbf{A}\mathbf{v}_r\}$

1) This is an orthogonal set, since

$$\mathbf{A}\mathbf{v}_i \cdot \mathbf{A}\mathbf{v}_j = (\mathbf{A}\mathbf{v}_i)^{\mathrm{T}} \mathbf{A}\mathbf{v}_j = \mathbf{v}_i^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{v}_j = \mathbf{v}_i^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{A})\mathbf{v}_j = \mathbf{v}_i \cdot (\mathbf{A}^{\mathrm{T}}\mathbf{A})\mathbf{v}_j = \mathbf{v}_i \cdot \lambda_j \mathbf{v}_j = \lambda_j \mathbf{v}_i \cdot \mathbf{v}_j = 0$$

$(\mathbf{A}^{\mathrm{T}}\mathbf{A})\mathbf{v}_j = \lambda_j \mathbf{v}_j$ $\qquad$ $\mathbf{v}_i \cdot \mathbf{v}_j = 0$

$$\mathbf{A}\mathbf{v}_i \cdot \mathbf{A}\mathbf{v}_i = (\mathbf{A}\mathbf{v}_i)^{\mathrm{T}} \mathbf{A}\mathbf{v}_i = \mathbf{v}_i^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{v}_i = \mathbf{v}_i^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{A})\mathbf{v}_i = \mathbf{v}_i \cdot \mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{v}_i = \mathbf{v}_i \cdot \lambda_i \mathbf{v}_i = \lambda_i \mathbf{v}_i \cdot \mathbf{v}_i = \lambda_i$$

$(\mathbf{A}^{\mathrm{T}}\mathbf{A})\mathbf{v}_j = \lambda_j \mathbf{v}_j$ $\qquad$ $\mathbf{v}_i \cdot \mathbf{v}_i = 1$

2) $\|\mathbf{A}\mathbf{v}_i\|^2 = \mathbf{A}\mathbf{v}_i \cdot \mathbf{A}\mathbf{v}_i = \lambda_i \Rightarrow \|\mathbf{A}\mathbf{v}_j\| = \sqrt{\lambda_i}$

If we define $\underset{[m\times 1]}{\mathbf{u}_i} = \dfrac{\underset{[m\times n][n\times 1]}{\mathbf{A}\mathbf{v}_i}}{\|\mathbf{A}\mathbf{v}_i\|} = \dfrac{\mathbf{A}\mathbf{v}_i}{\sqrt{\lambda_i}} = \dfrac{\mathbf{A}\mathbf{v}_i}{\sigma_i}$,where $\sigma_i = \sqrt{\lambda_i}$, $i = 1,\ldots,r$

$\mathbf{u}_1, \cdots, \mathbf{u}_r$ are orthonormal basis

Singular Value Decomposition (SVD)

$\underset{[m\times n]}{\mathbf{A}}$ | $\underset{[m\times 1]}{\mathbf{u}_i} = \frac{\mathbf{A}\mathbf{v}_i}{\sigma_i}$,where $\underset{[n\times 1]}{\mathbf{v}_1,\cdots,\mathbf{v}_r}$ are unit eigenvectors of $\mathbf{A}^T\mathbf{A}$ and orthonormal

$\sigma_i = \sqrt{\lambda_i}$ and λ_i is eigenvalues of $\mathbf{A}^T\mathbf{A}$

$$\mathbf{u}_i = \frac{\mathbf{A}\mathbf{v}_i}{\sigma_i}$$

$$\underset{[m\times n][n\times 1]}{\mathbf{A}\mathbf{v}_i} = \underset{[m\times 1]}{\mathbf{u}_i}\sigma_i \quad ,i=1,..,r$$

⇩ **Matrix form**

$$\underset{[m\times n]}{\mathbf{A}}\underset{[n\times r]}{\begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_r\end{bmatrix}} = \underset{[m\times r]}{\begin{bmatrix}\mathbf{u}_1 & \cdots & \mathbf{u}_r\end{bmatrix}}\underset{[r\times r]}{\begin{bmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_r\end{bmatrix}}$$

Singular Value Decomposition (SVD)

$$\underset{[m\times n]}{\mathbf{A}}\underset{[n\times r]}{\mathbf{V}}\begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_r\end{bmatrix}=\underset{[m\times r]}{\mathbf{U}}\underbrace{\begin{bmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_r\end{bmatrix}}_{\boldsymbol{\Sigma}[r\times r]}\begin{bmatrix}\mathbf{u}_1 & \cdots & \mathbf{u}_r\end{bmatrix}$$

여기에서 SVD, 즉 $\mathbf{A}=\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}$ 의 형태로 만들 수 있는가?

지금은 V가 [n by r] 이기 때문에 가능하지 않다.

따라서 V^{-1}가 존재하도록 [n by n]으로 만들어야 하며,
$V^{-1}=V^{T}$가 성립하도록 해야 한다.

즉 $\{\mathbf{v}_1 \cdots \mathbf{v}_r\}$과 서로 orthonormal한 basis $\{\mathbf{v}_{r+1} \cdots \mathbf{v}_n\}$을 찾아서
$\{\mathbf{v}_1 \cdots \mathbf{v}_r \ \mathbf{v}_{r+1} \cdots \mathbf{v}_n\}$를 구성한다.

⇨ $\{\mathbf{v}_1 \cdots \mathbf{v}_r \ \mathbf{v}_{r+1} \cdots \mathbf{v}_n\}$ is an extension of $\{\mathbf{v}_1 \cdots \mathbf{v}_r\}$ to an orthonormal basis.

Singular Value Decomposition (SVD)

$$\underset{[m\times n]}{\mathbf{A}}\underset{[n\times r]}{\begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_r\end{bmatrix}}=\underset{[m\times r]}{\begin{bmatrix}\mathbf{u}_1 & \cdots & \mathbf{u}_r\end{bmatrix}}\underset{[r\times r]}{\begin{bmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_r\end{bmatrix}}$$

$\{\mathbf{v}_1 \cdots \mathbf{v}_r \; \mathbf{v}_{r+1} \cdots \mathbf{v}_n\}$ is an extension of $\{\mathbf{v}_1 \cdots \mathbf{v}_r\}$ to an orthonormal basis.

$\{\mathbf{u}_1 \cdots \mathbf{u}_r \; \mathbf{u}_{r+1} \cdots \mathbf{u}_m\}$ is an extension of $\{\mathbf{u}_1 \cdots \mathbf{u}_r\}$ to an orthonormal basis.

$$\underset{[m\times n]}{\mathbf{A}}\underset{[n\times n]}{\begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_n\end{bmatrix}}=\underset{[m\times m]}{\begin{bmatrix}\mathbf{u}_1 & \cdots & \mathbf{u}_m\end{bmatrix}}\underset{[m\times n]}{\left[\begin{array}{ccc|c}\sigma_1 & & & \\ & \ddots & & 0_{k\times(n-k)} \\ & & \sigma_r & \\ \hline & 0_{(m-k)\times k} & & 0_{(m-k)\times(n-k)}\end{array}\right]}$$

Singular Value Decomposition (SVD)

$$\underset{[m\times n]}{\mathbf{A}}\begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_n\end{bmatrix}=\begin{bmatrix}\mathbf{u}_1 & \cdots & \mathbf{u}_m\end{bmatrix}\underbrace{\left[\begin{array}{ccc|c}\sigma_1 & & & \\ & \ddots & & 0_{k\times(n-k)} \\ & & \sigma_r & \\ \hline & 0_{(m-k)\times k} & & 0_{(m-k)\times(n-k)}\end{array}\right]}_{\boldsymbol{\Sigma}\,[m\times n]}$$

$$\mathbf{V}[n\times n] \qquad \mathbf{U}[m\times m]$$

$$\Rightarrow \quad \mathbf{AV}=\mathbf{U\Sigma}$$

$$\mathbf{A}=\mathbf{U\Sigma V}^{-1}$$

$$\mathbf{A}=\mathbf{U\Sigma V}^{\mathbf{T}}$$

$\{\mathbf{v}_1 \quad \cdots \quad \mathbf{v}_r \quad \mathbf{v}_{r+1} \quad \cdots \quad \mathbf{v}_n\}$ **is an orthonormal basis.**

Singular Value Decomposition (SVD)

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}$$

$$\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}(\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}})^{\mathrm{T}}$$

$$= \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}\mathbf{V}\boldsymbol{\Sigma}^{\mathrm{T}}\mathbf{U}^{\mathrm{T}}$$

$$= \mathbf{U}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\mathrm{T}}\mathbf{U}^{\mathrm{T}}$$

similarly $\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{V}\boldsymbol{\Sigma}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}$

Therefore $\mathbf{U}$ must be the eigenvector matrix for $\mathbf{A}\mathbf{A}^{\mathrm{T}}$

And $\mathbf{V}$ must be the eigenvector matrix for $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

$\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{v}_j = \sigma_j^2\mathbf{v}_j$, since $\mathbf{v}_j$ is the eigenvector of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

Multiply by $\mathbf{A}$ ⇨ $\mathbf{A}\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{v}_j = \sigma_j^2\mathbf{A}\mathbf{v}_j$

Therefore $\mathbf{A}\mathbf{v}_j$ is eigenvector of $\mathbf{A}\mathbf{A}^{\mathrm{T}}$

Length of $\mathbf{A}\mathbf{v}_j$ is σ_j, because

$$\mathbf{v}_j{}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{v}_j = \sigma_j^2\mathbf{v}_j{}^{\mathrm{T}}\mathbf{v}_j$$

$$(\mathbf{A}\mathbf{v}_j)^{\mathrm{T}}\mathbf{A}\mathbf{v}_j = \sigma_j^2\mathbf{v}_j\cdot\mathbf{v}_j$$

$$\mathbf{A}\mathbf{v}_j\cdot\mathbf{A}\mathbf{v}_j = \sigma_j^2$$

$\mathbf{u}_j$ is unit eigenvector of $\mathbf{A}\mathbf{A}^{\mathrm{T}}$

Therefore $\mathbf{u}_j = \mathbf{A}\mathbf{v}_j / \sigma_j$

앞서 본 자료에서 유도한 사항들이 어떤 의미를 가지는지 다시 한번 확인함
(Gilbert strang, Linear algebra and its application. pp.331 ~ 332)

Reference) 07-9 Linear Algebra Orthogonal, Diagonalization, Similar, Quadratic p. 43

Theorem 8.4.2 Symmetric Matrices
A symmetric matrix has an orthonormal basis of eigenvectors for R^n.

From example 8.4.1

$$\mathbf{A} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

$$\lambda_1 = 8, \quad \mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

normalize $\rightarrow \mathbf{x}_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^{\mathrm{T}}$

$$\lambda_2 = 2, \quad \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$$

normalize $\rightarrow \mathbf{x}_2 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^{\mathrm{T}}$

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_1^{\mathrm{T}} \mathbf{x}_2$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$= 0$$

So $\mathbf{x}_1$, $\mathbf{x}_2$ is an orthonormal basis of eigenvectors.

Singular Value Decomposition (SVD)

$$\mathbf{A}=\mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$

Example) Find the singular value decomposition of

$$\mathbf{A}=\begin{bmatrix}2 & 2\\ -1 & 1\end{bmatrix}$$

$\mathbf{u}_1,\cdots,\mathbf{u}_m$ in $\mathbf{U}$ are orthonormal basis for col(A)$=\dfrac{\mathbf{A}\mathbf{v}_j}{\|\mathbf{A}\mathbf{v}_j\|}=\dfrac{\mathbf{A}\mathbf{v}_j}{\sqrt{\lambda_i}}$,

$\mathbf{v}_1,\cdots,\mathbf{v}_n$ in $\mathbf{V}$ are unit eigenvectors of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

$\sigma_1,\ldots,\sigma_r$: **Square root of Eigenvalues of A^TA**

Compute $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ and its eigenvectors and then make them unit vectors:

$$\mathbf{A}^{\mathrm{T}}\mathbf{A}=\begin{bmatrix}2 & -1\\ 2 & 1\end{bmatrix}\begin{bmatrix}2 & 2\\ -1 & 1\end{bmatrix}=\begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix}$$

$$\det(\mathbf{A}^{\mathrm{T}}\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}5-\lambda & 3\\ 3 & 5-\lambda\end{vmatrix}=\lambda^2-10\lambda+25-9=\lambda^2-10\lambda+16=(\lambda-8)(\lambda-2)=0$$

$$\therefore \lambda_1=8,\ \lambda_2=2$$

$$(\mathbf{A}^{\mathrm{T}}\mathbf{A}-\lambda_1)\mathbf{v}_1=\begin{bmatrix}-3 & 3\\ 3 & -3\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix},\ \mathbf{v}_1=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$\therefore \sigma_1=\sqrt{8}=2\sqrt{2},\ \sigma_2=\sqrt{2}$$

$$(\mathbf{A}^{\mathrm{T}}\mathbf{A}-\lambda_2)\mathbf{v}_2=\begin{bmatrix}3 & 3\\ 3 & 3\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix},\ \mathbf{v}_2=\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\ 1\end{bmatrix}$$

$\mathbf{A}^{\mathrm{T}}\mathbf{A}$ **has unit eigenvectors** $\mathbf{v}_1=\dfrac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}$ $\mathbf{v}_2=\dfrac{1}{\sqrt{2}}\begin{bmatrix}-1\\ 1\end{bmatrix}$

Singular Value Decomposition (SVD)

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$

Example) Find the singular value decomposition of

$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

$\mathbf{u}_1, \cdots, \mathbf{u}_m$ in $\mathbf{U}$ are orthonormal basis for col(A)$= \dfrac{\mathbf{A}\mathbf{v}_j}{\|\mathbf{A}\mathbf{v}_j\|} = \dfrac{\mathbf{A}\mathbf{v}_j}{\sqrt{\lambda_i}}$,

$\mathbf{v}_1, \cdots, \mathbf{v}_n$ in $\mathbf{V}$ are unit eigenvectors of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

$\sigma_1, \ldots, \sigma_r$: **Square root of Eigenvalues of A^TA**

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{8} = 2\sqrt{2} \longrightarrow \mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{2} \longrightarrow \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\mathbf{A}\mathbf{v}_1 = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix} \longrightarrow \mathbf{u}_1 = \frac{\mathbf{A}\mathbf{v}_1}{\sqrt{\lambda_1}} = \frac{1}{2\sqrt{2}}\begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{A}\mathbf{v}_2 = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix} \longrightarrow \mathbf{u}_2 = \frac{\mathbf{A}\mathbf{v}_2}{\sqrt{\lambda_2}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Singular Value Decomposition (SVD)

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}$$

Example) Find the singular value decomposition of

$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

$\mathbf{u}_1, \cdots, \mathbf{u}_m$ in $\mathbf{U}$ are orthonormal basis for col(A)= $\dfrac{\mathbf{A}\mathbf{v}_j}{\|\mathbf{A}\mathbf{v}_j\|} = \dfrac{\mathbf{A}\mathbf{v}_j}{\sqrt{\lambda_i}}$,

$\mathbf{v}_1, \cdots, \mathbf{v}_n$ in $\mathbf{V}$ are unit eigenvectors of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

$\sigma_1, \ldots, \sigma_r$: **Square root of Eigenvalues of A^TA**

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

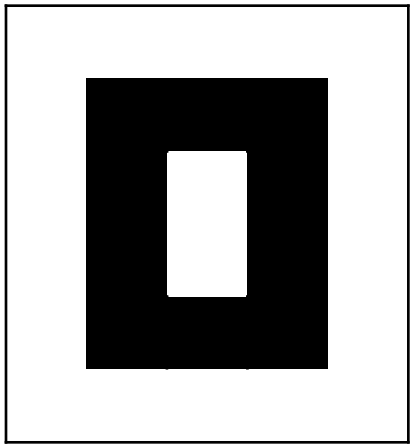
$$\sigma_1 = \sqrt{8} = 2\sqrt{2}, \ \sigma_2 = \sqrt{2}$$

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Application of SVD – 1) Image processing

Picture taken by a satellite



We want to send the picture to Earth.

The picture contains 6 by 5 pixels.

0	0	0	0	0
0	1	1	1	0
0	1	0	1	0
0	1	0	1	0
0	1	1	1	0
0	0	0	0	0

We need total 30 memories.

How to reduce the required number of memories?

Application of SVD – 4) Image processing

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

SVD

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{T}}$$

$$\mathbf{U} = \begin{bmatrix} 0 & 0 & 0 & 1.0000 & 0 & 0 \\ -0.5573 & 0.4352 & 0.7066 & 0 & -0.0268 & 0 \\ -0.4352 & -0.5573 & 0.0268 & 0 & 0.7066 & 0 \\ -0.4352 & -0.5573 & -0.0268 & 0 & -0.7066 & 0 \\ -0.5573 & 0.4352 & -0.7066 & 0 & 0.0268 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 \end{bmatrix}$$

$$\mathbf{\Sigma} = \begin{bmatrix} 3.0204 & 0 & 0 & 0 & 0 \\ 0 & 0.9364 & 0 & 0 & 0 \\ 0 & 0 & 0.0000 (= \varepsilon) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad 0 < \varepsilon \ll 1$$

$$\mathbf{V} = \begin{bmatrix} 0 & 0 & 0 & 1.0000 & 0 \\ -0.6572 & -0.2610 & 0.7071 & 0 & 0 \\ -0.3690 & 0.9294 & -0.0000 & 0 & 0 \\ -0.6572 & -0.2610 & -0.7071 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 \end{bmatrix}$$

Application of SVD – 4) Image processing

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$

$$\mathbf{U} = \begin{bmatrix} 0 & 0 & 0 & 1.0000 & 0 & 0 \\ -0.5573 & 0.4352 & 0.7066 & 0 & -0.0268 & 0 \\ -0.4352 & -0.5573 & 0.0268 & 0 & 0.7066 & 0 \\ -0.4352 & -0.5573 & -0.0268 & 0 & -0.7066 & 0 \\ -0.5573 & 0.4352 & -0.7066 & 0 & 0.0268 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 3.0204 & 0 & 0 & 0 & 0 \\ 0 & 0.9364 & 0 & 0 & 0 \\ 0 & 0 & 0.0000 (= \varepsilon) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad 0 < \varepsilon \ll 1$$

singular value 중에서 0이거나 아주 작은 값은 무시함

$$\mathbf{V} = \begin{bmatrix} 0 & 0 & 0 & 1.0000 & 0 \\ -0.6572 & -0.2610 & 0.7071 & 0 & 0 \\ -0.3690 & 0.9294 & -0.0000 & 0 & 0 \\ -0.6572 & -0.2610 & -0.7071 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 \end{bmatrix}$$

남겨진 singular value에 대응되는 eigenvector 만으로 matrix를 재구성

$$\mathbf{U}' = \begin{bmatrix} 0 & 0 \\ -0.5573 & 0.4352 \\ -0.4352 & -0.5573 \\ -0.4352 & -0.5573 \\ -0.5573 & 0.4352 \\ 0 & 0 \end{bmatrix}$$

$$\Sigma' = \begin{bmatrix} 3.0204 & 0 \\ 0 & 0.9364 \end{bmatrix}$$

$$\mathbf{V}' = \begin{bmatrix} 0 & 0 \\ -0.6572 & -0.2610 \\ -0.3690 & 0.9294 \\ -0.6572 & -0.2610 \\ 0 & 0 \end{bmatrix}$$

Application of SVD – 4) Image processing

$$\mathbf{U}' = \begin{bmatrix} 0 & 0 \\ -0.5573 & 0.4352 \\ -0.4352 & -0.5573 \\ -0.4352 & -0.5573 \\ -0.5573 & 0.4352 \\ 0 & 0 \end{bmatrix} \quad \mathbf{\Sigma}' = \begin{bmatrix} 3.0204 & 0 \\ 0 & 0.9364 \end{bmatrix} \quad \mathbf{V}' = \begin{bmatrix} 0 & 0 \\ -0.6572 & -0.2610 \\ -0.3690 & 0.9294 \\ -0.6572 & -0.2610 \\ 0 & 0 \end{bmatrix}$$

12 memories

4 memories

10 memories

$$\mathbf{U}'\mathbf{\Sigma}'\mathbf{V}'^{\mathbf{T}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1+\varepsilon & 1+\varepsilon & 1+\varepsilon & 0 \\ 0 & 1+\varepsilon & \varepsilon & 1+\varepsilon & 0 \\ 0 & 1+\varepsilon & \varepsilon & 1+\varepsilon & 0 \\ 0 & 1+\varepsilon & 1+\varepsilon & 1+\varepsilon & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \approx \mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

, where $\varepsilon = 10^{-15}$

Matrix A를 표현하기 위해서는 총 30개의 memory가 필요함

SVD후 작은 singular value들을 삭제하여
총 26개의 memory를 이용해 Matrix A와 거의 유사한 matrix를 복원할 수 있음

Matrix A의 크기가 클수록 더 큰 효과를 기대할 수 있음

Application of SVD – 2) Pseudo inverse

Problem:

Given: $\mathbf{Ax} = \mathbf{b}$ $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 4 \end{bmatrix}$

Find: $\mathbf{x}$

The inverse matrix of $\mathbf{A}$ does not exist.

Instead of $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$, we solve $\mathbf{x}^{+} = \mathbf{A}^{+}\mathbf{b}$

, where $\mathbf{A}^{+}$ is pseudo inverse of $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$

We define $\mathbf{A}^{+}$ as follows

$$\mathbf{A}^{+} = \mathbf{V}\Sigma^{+}\mathbf{U}^{\mathrm{T}} \text{ , where } \Sigma^{+} = \begin{bmatrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_n \end{bmatrix}$$

Application of SVD – 2) Pseudo inverse

$\mathbf{A}^+$ is pseudo inverse of $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}$

We define $\mathbf{A}^+$ as follows

$$\mathbf{A}^+ = \mathbf{V}\boldsymbol{\Sigma}^+\mathbf{U}^{\mathrm{T}} \text{ , where } \Sigma^+ = \begin{bmatrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_n \\ & & \end{bmatrix}$$

The matrix $\boldsymbol{\Sigma}$ is m by n, with r nonzero entries σ_i. Its pseudo inverse $\boldsymbol{\Sigma}^+$ is n by m, with r nonzero entries $1/\sigma_i$. All the blank spaces are zeros. Notice that $(\boldsymbol{\Sigma}^+)^+$ is $\boldsymbol{\Sigma}$ again.
That is like $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.
(Gilbert strang, Linear algebra and its application. pp.335)

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ & & \sigma_r \\ 0 & & 0 \end{bmatrix}, \boldsymbol{\Sigma}^+ = \begin{bmatrix} 1/\sigma_1 & & 0 \\ & \ddots & \\ & & 1/\sigma_r \\ 0 & & 0 \end{bmatrix}$$

Application of SVD – 2) Pseudo inverse

SVD: $\underset{[m\times n]}{\mathbf{A}} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}} = \underset{[m\times m]}{(\text{orthogonal})}\underset{[m\times n]}{(\text{diagonal})}\underset{[n\times n]}{(\text{orthogonal})}$

Pseudoinverse : $\underset{[n\times m]}{\mathbf{A}^{+}} = \mathbf{V}\boldsymbol{\Sigma}^{+}\mathbf{U}^{\mathrm{T}} = \underset{[n\times n]}{(\text{orthogonal})}\underset{[n\times m]}{(\text{diagonal})}\underset{[m\times m]}{(\text{orthogonal})}$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -0.1035 & 0.8302 & 0.0236 & 0.5472 \\ -0.3243 & 0.4414 & -0.4393 & -0.7120 \\ -0.5452 & 0.0525 & 0.8079 & -0.2176 \\ -0.7661 & -0.3364 & -0.3921 & 0.3824 \end{bmatrix} \begin{bmatrix} 4.1 & 0 \\ 0 & 1.0908 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -0.4242 & -0.9056 \\ 0.9056 & -0.4242 \end{bmatrix}$$

$$\mathbf{U} \qquad\qquad \boldsymbol{\Sigma} \qquad\qquad \mathbf{V}^{\mathrm{T}}$$

$$\mathbf{A}^{+} = \begin{bmatrix} -0.4242 & -0.4242 \\ 0.9056 & -0.9056 \end{bmatrix} \begin{bmatrix} 0.2439 & 0 & 0 & 0 \\ 0 & 0.9168 & 0 & 0 \end{bmatrix} \begin{bmatrix} -0.1035 & -0.3243 & -0.5452 & -0.7661 \\ 0.8302 & 0.4414 & 0.0525 & -0.3364 \\ 0.0236 & -0.4393 & 0.8079 & -0.3921 \\ 0.5472 & -0.7120 & -0.2176 & 0.3824 \end{bmatrix}$$

$$\mathbf{V} \qquad\qquad \boldsymbol{\Sigma}^{+} \qquad\qquad \mathbf{U}^{\mathrm{T}}$$

$$\mathbf{A}^{+} = \begin{bmatrix} 0.7 & 0.4 & 0.1 & -0.2 \\ -0.3 & -0.1 & 0.1 & 0.3 \end{bmatrix}$$

$$\mathbf{x}^{+} = \mathbf{A}^{+}\mathbf{b} = \begin{bmatrix} 0.7 & 0.4 & 0.1 & -0.2 \\ -0.3 & -0.1 & 0.1 & 0.3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$$

Application of SVD – 3) Least squares

$$\underset{[m\times n]}{\mathbf{A}}\ \underset{[n\times 1]}{\mathbf{x}}\ =\ \underset{[m\times 1]}{\mathbf{b}}$$

1) Ax = b의 유일한 해가 존재하지 않는 경우 (A^{-1}이 존재하지 않음)

① 해가 없는 경우 (불능, $p > n$) → 최적의 근사해 찾기

예제

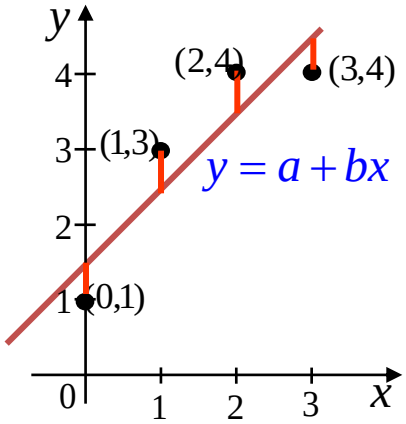
네 점을 지나는 직선 구하기

직선의 절편 : x
직선의 기울기 : y

$$a+0b=1,\ a+1b=3$$

$$a+2b=4,\ a+3b=4$$

$$\mathbf{Ax}=\begin{bmatrix}1&0\\1&1\\1&2\\1&3\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}1\\3\\4\\4\end{bmatrix}=\mathbf{b}$$



네 점을 지나는 직선은 없으므로, 오차의 제곱의 합이 가장 작은 직선을 구함. x = (1.5, 1)

Pseudo inverse A^+

①과 같이 exact solution x = A^{-1} b를 구할 수 없을 경우, Pseudo inverse를 사용하여, 각각의 목적에 맞는 해(이 경우 오차의 제곱이 가장 작게 하는 해)를 구할 수 있다.

$$\mathbf{x}^+=\mathbf{A}^+\mathbf{b}=\begin{bmatrix}0.7&0.4&0.1&-0.2\\-0.3&-0.1&0.1&0.3\end{bmatrix}\begin{bmatrix}1\\3\\4\\4\end{bmatrix}=\begin{bmatrix}1.5\\1\end{bmatrix}$$

$$\mathbf{A}^+=\mathbf{V}\boldsymbol{\Sigma}^+\mathbf{U}^{\mathrm{T}}=(\text{orthogonal})(\text{diagonal})(\text{orthogonal})$$

SVD

$$\underset{[m\times n]}{\mathbf{A}}=\underset{[m\times m]}{\mathbf{U}}\underset{[m\times n]}{\boldsymbol{\Sigma}}\underset{[n\times n]}{\mathbf{V}^{\mathrm{T}}}=(\text{orthogonal})(\text{diagonal})(\text{orthogonal})$$

$\mathbf{u}_1,\cdots,\mathbf{u}_m$ in $\mathbf{U}$ are eigenvectors of $\mathbf{AA}^{\mathrm{T}}$,

$\mathbf{v}_1,\cdots,\mathbf{v}_n$ in $\mathbf{V}$ are eigenvectors of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

$$\boldsymbol{\Sigma}=\begin{bmatrix}\sigma_1&&0\\&\ddots&\\0&&\sigma_r\end{bmatrix},\ \boldsymbol{\Sigma}^+=\begin{bmatrix}1/\sigma_1&&0\\&\ddots&\\0&&1/\sigma_r\end{bmatrix}$$

Application of SVD – 4) Numerical error

2) Ax = b의 유일 해 x = $A^{-1}b$ 를 구할 때, 오류가 생기는 경우

행렬 A의 역행렬 A^{-1}가 존재하는 경우,

즉 A는 $[n \times n]$ 행렬이고, rank가 n인 경우,

"A^{-1} = Pseudo inverse A^{+}"이다.

Proof)

$$\underset{[n\times n][n\times n]}{\mathbf{A}\mathbf{A}^{+}} = \underset{[n\times n][n\times n][n\times n]}{\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}} \underset{[n\times n][n\times n]}{\mathbf{V}\boldsymbol{\Sigma}^{+}} \underset{[n\times n]}{\mathbf{U}^{\mathrm{T}}} = \mathbf{U}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{+}\mathbf{U}^{\mathrm{T}} = \mathbf{U}\mathbf{U}^{\mathrm{T}} = \mathbf{I}$$

정의에 의하여 $\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ & & \sigma_r \\ 0 & & 0 \end{bmatrix}, \boldsymbol{\Sigma}^{+} = \begin{bmatrix} 1/\sigma_1 & & 0 \\ & \ddots & \\ & & 1/\sigma_r \\ 0 & & 0 \end{bmatrix}$

단, 이 경우에는 rank가 n이므로 $\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ & & \sigma_n \\ 0 & & \end{bmatrix}, \boldsymbol{\Sigma}^{+} = \begin{bmatrix} 1/\sigma_1 & & 0 \\ & \ddots & \\ & & 1/\sigma_n \\ 0 & & \end{bmatrix} \Rightarrow \boldsymbol{\Sigma}\boldsymbol{\Sigma}^{+} = \mathbf{I}$

Application of SVD – 4) Numerical error

2) Ax = b의 유일 해 x = $A^{-1}b$ 를 구할 때, 오류가 생기는 경우

행렬 A의 역행렬 A^{-1}가 존재하는 경우, "A^{-1} = Pseudo inverse A^{+}"이다.

$$\mathbf{Ax} = \mathbf{b}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1.0000000000000001 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 2.0000000000000001 \end{bmatrix}$$

$\mathbf{A}^{-1}$ 를 계산하기 위해 먼저 $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}$ 를 계산하면 아래와 같다.

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}} = \begin{bmatrix} -0.7071 & -0.7071 \\ -0.7071 & 0.7071 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & \varepsilon \end{bmatrix} \begin{bmatrix} -0.7071 & -0.7071 \\ -0.7071 & 0.7071 \end{bmatrix} \quad (0 < \varepsilon << 1)$$

$\mathbf{A}^{-1} = \mathbf{A}^{+} = \mathbf{V}\boldsymbol{\Sigma}^{+}\mathbf{U}^{\mathrm{T}}$ 이므로

$$\mathbf{A}^{+} = \mathbf{V}\boldsymbol{\Sigma}^{+}\mathbf{U}^{\mathrm{T}} = \begin{bmatrix} -0.7071 & -0.7071 \\ -0.7071 & 0.7071 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/\varepsilon \end{bmatrix} \begin{bmatrix} -0.7071 & -0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}$$

Application of SVD – 4) Numerical error

2) Ax = b의 유일 해 x = $A^{-1}b$ 를 구할 때, 오류가 생기는 경우

$$\mathbf{Ax}=\mathbf{b},\quad \begin{bmatrix}1 & 1\\ 1 & 1.0000000000000001\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}2\\ 2.0000000000000001\end{bmatrix}$$

$$\mathbf{A}^{+}=\mathbf{V}\boldsymbol{\Sigma}^{+}\mathbf{U}^{\mathrm{T}}=\begin{bmatrix}-0.7071 & -0.7071\\ -0.7071 & 0.7071\end{bmatrix}\begin{bmatrix}1/2 & 0\\ 0 & 1/\varepsilon\end{bmatrix}\begin{bmatrix}-0.7071 & -0.7071\\ -0.7071 & 0.7071\end{bmatrix}$$

그런데 $0<\varepsilon<<1$ 이므로 $1/\varepsilon$ 값이 무한대가 되어 수치적 에러가 발생한다.

따라서 아래와 같이 SVD에서 값이 작은 singular value를 0으로 만든 후 A^{+}를 계산한다.

$$\mathbf{A}=\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}=\begin{bmatrix}-0.7071 & -0.7071\\ -0.7071 & 0.7071\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 0\varepsilon\end{bmatrix}\begin{bmatrix}-0.7071 & -0.7071\\ -0.7071 & 0.7071\end{bmatrix}$$

$$\mathbf{A}^{+}=\mathbf{V}\boldsymbol{\Sigma}^{+}\mathbf{U}^{\mathrm{T}}=\begin{bmatrix}-0.7071 & -0.7071\\ -0.7071 & 0.7071\end{bmatrix}\begin{bmatrix}1/2 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}-0.7071 & -0.7071\\ -0.7071 & 0.7071\end{bmatrix}=\begin{bmatrix}0.25 & 0.25\\ 0.25 & 0.25\end{bmatrix}$$

$$\mathbf{x}^{+}=\mathbf{A}^{+}\mathbf{b}=\begin{bmatrix}0.25 & 0.25\\ 0.25 & 0.25\end{bmatrix}\begin{bmatrix}2\\ 2.0000000000000001\end{bmatrix}\approx\begin{bmatrix}1\\ 1\end{bmatrix}$$

참고 슬라이드

Singular Value Decomposition (SVD)

$$\underset{[m\times n]}{\mathbf{A}} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}$$

We define Σ as follows:

Rank of A is r

Rank of A^TA is r ⟺ A and A^TA have the same rank (p. 365)

$\mathbf{A}^{\mathrm{T}}\mathbf{A}$ has r eigenvalues λ_i

$$\underset{[m\times n]}{\boldsymbol{\Sigma}} = \begin{bmatrix} \sigma_1 & & & 0_{k\times(n-k)} \\ & \ddots & & \\ & & \sigma_r & \\ \hline 0_{(m-k)\times k} & & & 0_{(m-k)\times(n-k)} \end{bmatrix}$$

,where

$\sigma_i = \sqrt{\lambda_i}$

λ_i : Eigenvalues of A^TA

Matrix size에 대해서는 추후 설명

Singular Value Decomposition (SVD)

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1 & & & 0_{k\times(n-k)} \\ & \ddots & & \\ & & \sigma_r & \\ \hline 0_{(m-k)\times k} & & & 0_{(m-k)\times(n-k)} \end{bmatrix}$$,where $\sigma_i = \sqrt{\lambda_i}$ and λ_i is eigenvalues of A^TA

$\boldsymbol{\Sigma}$ = $[m\times n]$

We define V as follows:

$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}$, where $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are unit eigenvectors of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

1.

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} \mathbf{a}_1^{\mathrm{T}} \\ \mathbf{a}_2^{\mathrm{T}} \\ \vdots \\ \mathbf{a}_n^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^{\mathrm{T}}\mathbf{a}_1 & \mathbf{a}_1^{\mathrm{T}}\mathbf{a}_2 & \cdots & \mathbf{a}_1^{\mathrm{T}}\mathbf{a}_n \\ \mathbf{a}_2^{\mathrm{T}}\mathbf{a}_1 & \mathbf{a}_2^{\mathrm{T}}\mathbf{a}_2 & & \\ \vdots & & \ddots & \\ \mathbf{a}_n^{\mathrm{T}}\mathbf{a}_1 & & & \mathbf{a}_1^{\mathrm{T}}\mathbf{a}_1 \end{bmatrix}$$

$$\mathbf{a}_i^{\mathrm{T}}\mathbf{a}_j = \mathbf{a}_i \cdot \mathbf{a}_j = \mathbf{a}_j \cdot \mathbf{a}_i = \mathbf{a}_j^{\mathrm{T}}\mathbf{a}_i$$

$\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is symmetric matrix

2. $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are unit eigenvectors of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

Reference) 07-9 Linear Algebra Orthogonal, Diagonalization, Similar, Quadratic p. 43

A symmetric matrix has an orthonormal basis of eigenvectors .

Therefore, $\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_r & \mathbf{v}_{r+1} & \cdots & \mathbf{v}_n \end{bmatrix}$:orthonormal

Singular Value Decomposition (SVD)

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$

$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}$, where $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are unit eigenvectors of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ and orthonormal

$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}$,where $\sigma_i = \sqrt{\lambda_i}$ and λ_i is eigenvalues of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

We define $\mathbf{U}$ as follows:

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_m \end{bmatrix} \text{ , where } \mathbf{u}_j = \frac{\mathbf{A}\mathbf{v}_j}{\sigma_j}$$

and $\mathbf{u}_1, \cdots, \mathbf{u}_m$ are orthonormal basis.

Why are $\mathbf{u}_1, \cdots, \mathbf{u}_m$ orthonormal basis?

Singular Value Decomposition (SVD)

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}$$

We define Σ as follows:

Rank of A is r

Rank of A^TA is r ⟩ A and A^TA have the same rank p. 365

Rank of D is r ⟩ Similar matrices have the same rank p. 457

$\mathbf{A}^{\mathrm{T}}\mathbf{A}$ has r eigenvalues λ_i and r orthogonal eigenvectors $\mathbf{u}_i$

$$\mathbf{A}^{\mathbf{T}}\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i,\ i = 1,\ldots,r$$

$$\sigma_i = \sqrt{\lambda_i}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}$$

Singular Value Decomposition (SVD)

Rank of A is r
Rank of A^TA is r
Rank of D is r

A and A^TA have the same rank p. 365

Similar matrices have the same rank p. 457

$\mathbf{A}^T\mathbf{A}$ has r eigenvalues λ_i and r orthogonal eigenvectors $\mathbf{u}_i$

$$\mathbf{A}^T\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i, \; i = 1, \ldots, r$$

$$\sigma_i = \sqrt{\lambda_i}$$

Howard Anton, Robert C. Busby, Contemporary Linear Algebra, Wiley, 2003

$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{\mathrm{T}}$ $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is symmetric matrix

$[n\times m][m\times n]$

$[n\times n]$

Matrix is orthogonally diagonalizable if and only if it is symmetric.

p. 470

Rank of A is r

Rank of A^TA is r

Rank of D is r

A and A^TA have the same rank p. 365

Similar matrices have the same rank p. 457

$\mathbf{A}^{\mathrm{T}}\mathbf{A}$ has r eigenvalues λ_i and r orthogonal eigenvectors $\mathbf{u}_i$

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Singular Value Decomposition (SVD)

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}$$

We define Σ as follows:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}$$

,where

$$\sigma_i = \sqrt{\lambda_i}$$

λ_i **: Eigenvalues of A^TA**

Singular Value Decomposition (SVD)

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}$$,where $\sigma_i = \sqrt{\lambda_i}$ and λ_i is eigenvalues of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

We define V as follows:

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}$$, where $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are unit eigenvectors of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

1.

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} \mathbf{a}_1^{\mathrm{T}} \\ \mathbf{a}_2^{\mathrm{T}} \\ \vdots \\ \mathbf{a}_n^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^{\mathrm{T}}\mathbf{a}_1 & \mathbf{a}_1^{\mathrm{T}}\mathbf{a}_2 & \cdots & \mathbf{a}_1^{\mathrm{T}}\mathbf{a}_n \\ \mathbf{a}_2^{\mathrm{T}}\mathbf{a}_1 & \mathbf{a}_2^{\mathrm{T}}\mathbf{a}_2 & & \\ \vdots & & \ddots & \\ \mathbf{a}_n^{\mathrm{T}}\mathbf{a}_1 & & & \mathbf{a}_1^{\mathrm{T}}\mathbf{a}_1 \end{bmatrix}$$

$$\mathbf{a}_i^{\mathrm{T}}\mathbf{a}_j = \mathbf{a}_i \cdot \mathbf{a}_j = \mathbf{a}_j \cdot \mathbf{a}_i = \mathbf{a}_j^{\mathrm{T}}\mathbf{a}_i$$

$\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is symmetric matrix

2. $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are unit eigenvectors of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

Reference) 07-9 Linear Algebra Orthogonal, Diagonalization, Similar, Quadratic p. 43

A symmetric matrix has an orthonormal basis of eigenvectors .

Therefore, $\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}$:orthonormal

Singular Value Decomposition (SVD)

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$

$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}$, where $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are unit eigenvectors of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ and orthonormal

$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}$,where $\sigma_i = \sqrt{\lambda_i}$ and λ_i is eigenvalues of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

We define $\mathbf{U}$ as follows:

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_m \end{bmatrix} \text{ , where } \mathbf{u}_j = \frac{\mathbf{A}\mathbf{v}_j}{\sigma_j}$$

and $\mathbf{u}_1, \cdots, \mathbf{u}_m$ are orthonormal basis.

Why are $\mathbf{u}_1, \cdots, \mathbf{u}_m$ orthonormal basis?

Singular Value Decomposition (SVD)

Why are $\mathbf{u}_1, \cdots, \mathbf{u}_m$ orthonormal basis?

, where $\mathbf{u}_j = \dfrac{\mathbf{A}\mathbf{v}_j}{\sigma_j}$

$\mathbf{v}_1, \cdots, \mathbf{v}_n$ are unit eigenvectors of $\mathbf{A}^T\mathbf{A}$ and orthonormal

$\sigma_i = \sqrt{\lambda_i}$ **and** λ_i **is eigenvalues of** $\mathbf{A}^T\mathbf{A}$

Now we consider the set of image vectors

$$\{\mathbf{A}\mathbf{v}_1 \quad \cdots \quad \mathbf{A}\mathbf{v}_n\}$$

1) This is an orthogonal set, since

$$\mathbf{A}\mathbf{v}_i \cdot \mathbf{A}\mathbf{v}_j = (\mathbf{A}\mathbf{v}_i)^T \mathbf{A}\mathbf{v}_j = \mathbf{v}_i{}^T \mathbf{A}^T\mathbf{A}\mathbf{v}_j = \mathbf{v}_i{}^T (\mathbf{A}^T\mathbf{A})\mathbf{v}_j = \mathbf{v}_i \cdot (\mathbf{A}^T\mathbf{A})\mathbf{v}_j = \mathbf{v}_i \cdot \lambda_j \mathbf{v}_j = \lambda_j \mathbf{v}_i \cdot \mathbf{v}_j = 0$$

$(\mathbf{A}^T\mathbf{A})\mathbf{v}_j = \lambda_j \mathbf{v}_j$ $\qquad$ $\mathbf{v}_i \cdot \mathbf{v}_j = 0$

$$\mathbf{A}\mathbf{v}_i \cdot \mathbf{A}\mathbf{v}_i = (\mathbf{A}\mathbf{v}_i)^T \mathbf{A}\mathbf{v}_i = \mathbf{v}_i{}^T \mathbf{A}^T\mathbf{A}\mathbf{v}_i = \mathbf{v}_i{}^T (\mathbf{A}^T\mathbf{A})\mathbf{v}_i = \mathbf{v}_i \cdot \mathbf{A}^T\mathbf{A}\mathbf{v}_i = \mathbf{v}_i \cdot \lambda_i \mathbf{v}_i = \lambda_i \mathbf{v}_i \cdot \mathbf{v}_i = \lambda_i$$

$(\mathbf{A}^T\mathbf{A})\mathbf{v}_j = \lambda_j \mathbf{v}_j$ $\qquad$ $\mathbf{v}_i \cdot \mathbf{v}_i = 1$

2) $\|\mathbf{A}\mathbf{v}_i\|^2 = \mathbf{A}\mathbf{v}_i \cdot \mathbf{A}\mathbf{v}_i = \lambda_i \Rightarrow \|\mathbf{A}\mathbf{v}_j\| = \sqrt{\lambda_i} = \sigma_i$

Therefore $\mathbf{u}_1, \cdots, \mathbf{u}_m$ **are orthonormal basis**

, where $\{\mathbf{u}_1 \quad \cdots \quad \mathbf{u}_n\} = \left\{\dfrac{\mathbf{A}\mathbf{v}_1}{\|\mathbf{A}\mathbf{v}_1\|} \quad \cdots \quad \dfrac{\mathbf{A}\mathbf{v}_n}{\|\mathbf{A}\mathbf{v}_n\|}\right\} = \left\{\dfrac{\mathbf{A}\mathbf{v}_1}{\sqrt{\lambda_1}} \quad \cdots \quad \dfrac{\mathbf{A}\mathbf{v}_n}{\sqrt{\lambda_n}}\right\} = \left\{\dfrac{\mathbf{A}\mathbf{v}_1}{\sigma_1} \quad \cdots \quad \dfrac{\mathbf{A}\mathbf{v}_n}{\sigma_n}\right\}$

Similar Matrices & Singular Value Decomposition (SVD)

SVD

$$\underset{[m\times n]}{\mathbf{A}} = \underset{[m\times m]}{(\text{orthogonal})}\,\underset{[m\times n]}{(\text{diagonal})}\,\underset{[n\times n]}{(\text{orthogonal})}$$

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}, \text{ where } \mathbf{v}_1, \cdots, \mathbf{v}_n \text{ are unit eigenvectors of } \mathbf{A}^{\mathrm{T}}\mathbf{A} \text{ and orthonormal}$$

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}, \text{where } \sigma_i = \sqrt{\lambda_i} \text{ and } \lambda_i \text{ is eigenvalues of } \mathbf{A}^{\mathrm{T}}\mathbf{A}$$

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_m \end{bmatrix}, \text{ where } \mathbf{u}_j = \frac{\mathbf{A}\mathbf{v}_j}{\sigma_j}$$

and $\mathbf{u}_1, \cdots, \mathbf{u}_m$ are orthonormal basis.

$\text{rank } \mathbf{A} = r \le \min(n, m)$ $\mathbf{u}_1, \cdots, \mathbf{u}_m$ in $\mathbf{U}$ are eigenvectors of $\mathbf{A}\mathbf{A}^{\mathrm{T}}$,

$\mathbf{v}_1, \cdots, \mathbf{v}_n$ in $\mathbf{V}$ are eigenvectors of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$

Similar Matrices & Singular Value Decomposition (SVD)

SVD

$$\underset{[m\times n]}{\mathbf{A}} = \underset{[m\times m]}{(\text{orthogonal})}\underset{[m\times n]}{(\text{diagonal})}\underset{[n\times n]}{(\text{orthogonal})}$$

$\mathbf{V} = [\mathbf{v}_1 \quad \cdots \quad \mathbf{v}_n]$, **where** $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are unit eigenvectors of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ and orthonormal

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}$$,where $\sigma_i = \sqrt{\lambda_i}$ and λ_i is eigenvalues of A^TA

$\mathbf{U} = [\mathbf{u}_1 \quad \cdots \quad \mathbf{u}_m]$, where $\mathbf{u}_j = \dfrac{\mathbf{A}\mathbf{v}_j}{\sigma_j}$ and $\mathbf{u}_1, \cdots, \mathbf{u}_m$ are orthonormal basis.

$$\mathbf{A}\mathbf{v}_i = \sigma_i \mathbf{u}_i \Rightarrow \mathbf{A}\mathbf{V} = \mathbf{U}\mathbf{\Sigma} \Rightarrow \begin{aligned} \mathbf{A} &= \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{-1} \\ &= \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{T}} \end{aligned}$$ Since V is orthonormal.

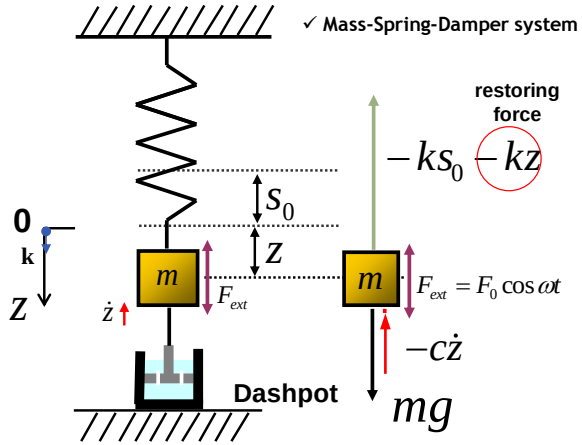
Chapter 8. PDEs(Partial Differential Equations)

1. ODEs vs PDEs
2. Classification of PDEs
3. Wave Equation
4. Heat Equation
5. Laplace's Equation
6. 2-D Heat Equation : Fourier Series in Two Variables

9.1 Linear ODE vs. Linear PDE (Classify !)

Linear ODE

Ex.)Forced Mass-Spring-Damping system

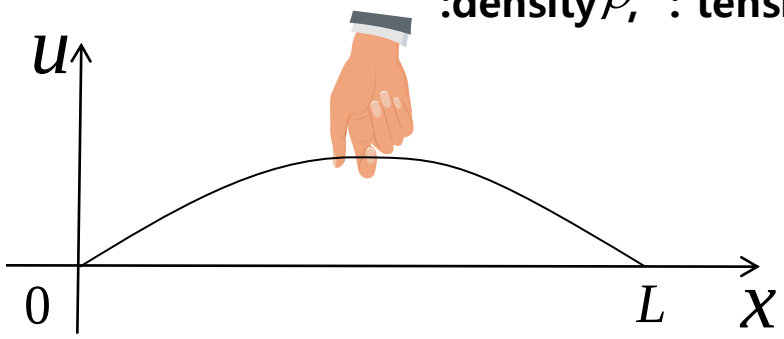


$$m\ddot{z}(t) + c\dot{z}(t) + kz(t) = F_0 \cos \omega t$$

Linear PDE

Ex.)1-D Wave Equation

:density ρ, : tension T



$$u_{xx}(x,t) = \frac{\rho}{T} u_{tt}(x,t)$$

time t — **independent variable** — x, t space & time

displacement of the mass at a time $z = z(t)$ — **dependent variable** — $u = u(x,t)$ Displacement of the string on a position(x) at a time(t)

velocity $\dot{z}(t) = \frac{dz(t)}{dt}$

acceleration $\ddot{z}(t) = \frac{d^2 z(t)}{dt^2}$

$u_{xx} = \frac{\partial^2 u(x,t)}{\partial x^2}$ Second derivative with respect to space

$u_{tt} = \frac{\partial^2 u(x,t)}{\partial t^2}$ acceleration

displacement of the mass at a time

$z = z(t)$ dependent variable

$u = u(x,t)$

Displacement of the string on a position(x) at a time(t)

conditions

initial condition

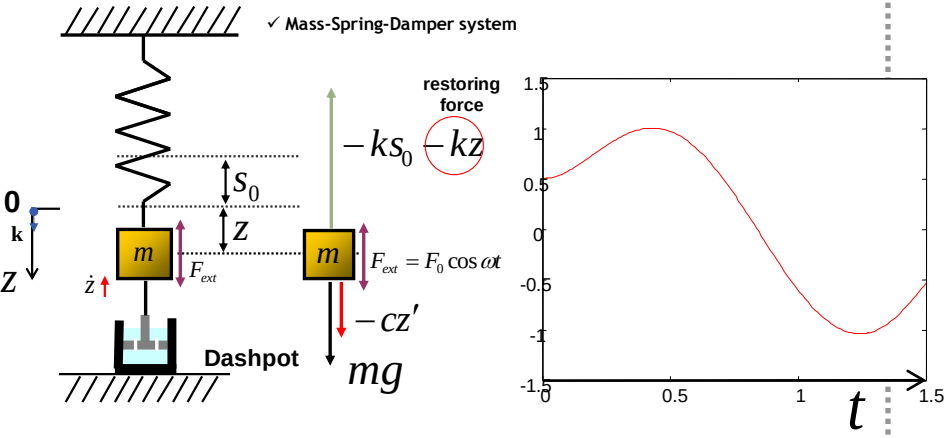
$z(0) = a, \dot{z}(0) = b$

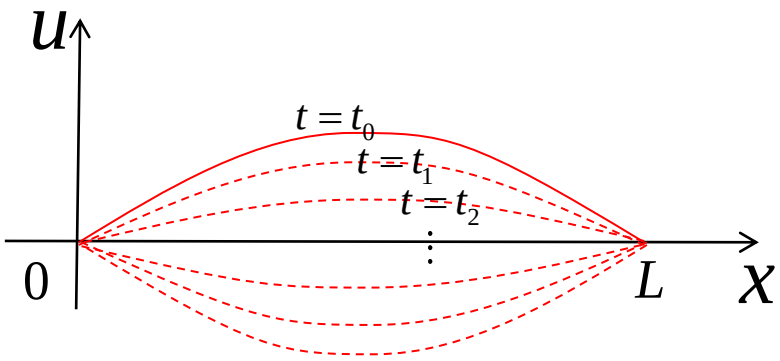
$u(x,0) = f(x), \frac{\partial u}{\partial t}(x,0) = g(x)$

initial condition

$u(0,t) = 0, u(L,t) = 0$

boundary condition





$m\ddot{z}(t) + c\dot{z}(t) + kz(t) = F_0 \cos \omega t$

$u_{xx}(x,t) = \frac{\rho}{T} u_{tt}(x,t)$

Partial Differential Equation(PDE)

EX) THE DEPENDENT VARIABLE u IS A FUNCTION OF MORE THAN TWO INDEPENDENT VARIABLES x, y, z, t

$$u = u(x, y, z, t)$$

$$u_{tt} = \alpha^2 (u_{xx} + u_{yy} + u_{zz})$$

Linear ODE vs Linear PDE (Classify !)

• **Linear ODE (Review)**

The dependent variable y and all its derivatives y′, y″,…,$y^{(n)}$ appear only to the first power, that is the power of each term involving y is 1.
The coefficients a_0 ,…,a_n of y′, y″,…,$y^{(n)}$ depend at most on the independent variable x.

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

ex) $my'' + cy' + ky = f(x)$ where, $y = y(x)$, $m, c, k = \text{constant}$

$y'' + \alpha y = 0$

$x^2y'' + xy' - \alpha y = 0$

$xy'' + y' + \alpha^2 y = 0$

$(1-x^2)y'' - 2xy + n(n+1)y = 0$ $n = 0,1,2\ldots$

$(1-y)y' + 2y = e^x$

$y'' + \sin y = e^x$

$y^{(4)} + y^2 = e^x$

Nonlinear ODE

• **Linear PDE**

The dependent variable and its partial derivatives appear only to the first power.

General form of a linear second-order partial differential equation :

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

8.2 CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS

General form of a linear second-order partial differential equation

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

$G(x,y)=0$ **Equation is said to be 'Homogeneous'**

Otherwise, nonhomogeneous

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Homogeneous

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial y}=xy$$

Nonhomogeneous

Classification of PDEs (Classify !)

☑ Classification of Equations

Definition 13.1 Classification of Equations

The linear second-order partial differential equation

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=0$$

Where A,B,C,D,E and F are real constants, is said to be

Hyperbolic if $B^2-4AC>0$

Parabolic if $B^2-4AC=0$

Elliptic if $B^2-4AC<0$

Example)

$$3\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial y}$$

$A=3, B=0, C=0$

$B^2-4AC=0$

Parabolic

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}$$

$A=1, B=0, C=-1$

$B^2-4AC=4>0$

Hyperbolic

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

$A=1, B=0, C=1$

$B^2-4AC=-4<0$

Elliptic

☑ Solution of Linear PDE

- Solution of linear PDE is function $u(x, y) = 0$ of two independent variables

- Our focus throughout will be on finding *particular solutions* of some of the important linear PDEs(Heat eq, Wave eq, Laplace's Eq, etc.)

- Notation

$$\frac{\partial u}{\partial x} = u_x, \quad \frac{\partial u}{\partial y} = u_y, \quad \frac{\partial^2 u}{\partial x^2} = u_{xx}, \quad \frac{\partial^2 u}{\partial y \partial x} = u_{xy}$$

Conditions: Initial Conditions

$$(1)\quad a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$$

$$(2)\quad k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t},\ k>0$$

$$(3)\quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

☑Initial Conditions

- Related to time(t)
- Since solution of equation (1) and (2) *depend on time t,* we can prescribe what happens at t=0, that is, we can give initial conditions

$$u(x,0)=f(x),\quad \left.\frac{\partial u}{\partial t}\right|_{t=0}=g(x)$$

Conditions : Boundary Conditions

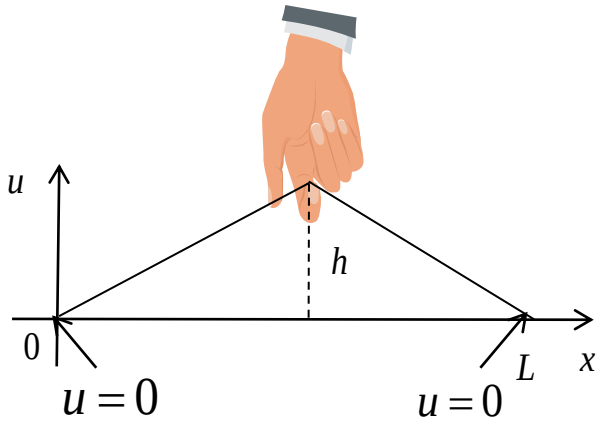
☑ Boundary Conditions

■ Related to position (*x*)

(1) $a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$

(2) $k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, k > 0$

(3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$



$$u(0,t) = 0, \quad u(L,t) = 0, \; t > 0$$

Three Types of Boundary Conditions

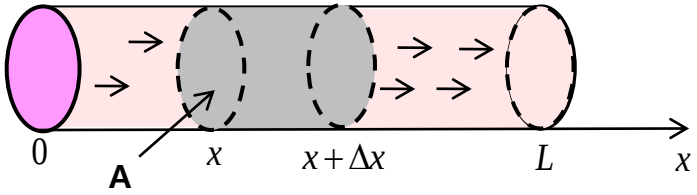
- **There are three types of boundary conditions associated with equation (1),(2), and (3).**

$$(i)\ u \qquad (ii)\ \frac{\partial u}{\partial n} \qquad (iii)\ \frac{\partial u}{\partial n} + hu$$

h : a constant

(Dirichlet condition) *(Neuman condition)* *(Robin condition)*

Example)



heat transfer from the lateral surface of a rod into a surrounding medium that is held at a constant temperature u_m

For a rod in figure,

$$(i)\ u(L,t) = u_0 \qquad (ii)\ \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0 \qquad (iii)\ \left.\frac{\partial u}{\partial x}\right|_{x=L} = -h(u(L,t) - u_m)$$

Boundary-Value Problems

Example 1: $a^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, \quad 0 < x < L, \ t > 0$

(1) $a^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$

(2) $k \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, k > 0$

(3) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

Subject to: $(BC) \ \ u(0,t) = 0, \ u(L,t) = 0, \ t > 0$

$$(IC) \ \ u(x,0) = f(x), \ \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \quad 0 < x < L$$

Example 2: $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \ 0 < y < b$

Subject to:
$$(BC)\begin{cases} \left.\dfrac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\dfrac{\partial u}{\partial x}\right|_{x=a} = 0, & 0 < y < b \\ u(x,0) = 0, \quad u(x,b) = f(x), & 0 < x < a \end{cases}$$

External Influences

(1) $a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$

(2) $k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, k > 0$

(3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

☑Variations

- **The PDE (1),(2),and (3) must be modified to take into consideration for internal or external influences acting on the physical system.**

heat transfer from the lateral surface of a rod into a surrounding medium that is held at a constant temperature u_m

external influences

$$k^2 \frac{\partial^2 u}{\partial x^2} + F(x,t,u,u_x) = \frac{\partial u}{\partial t} \quad \xrightarrow{\text{ex}} \quad k^2 \frac{\partial^2 u}{\partial x^2} - h(u - u_m) = \frac{\partial u}{\partial t}$$

Heat equation

external force | damping | restoring force

external influences

$$a^2 \frac{\partial^2 u}{\partial x^2} + F(x,t,u,u_x) = \frac{\partial^2 u}{\partial t^2} \quad \xrightarrow{\text{ex}} \quad a^2 \frac{\partial^2 u}{\partial x^2} + \underbrace{f(x,t) - c\frac{\partial u}{\partial t} - ku}_{F(x,t,u,u_t)} = \frac{\partial^2 u}{\partial t^2}$$

Wave equation

Superposition Principle of Linear PDEs

Theorem 13.1 **Superposition Principle**

If $u_1, u_2, \ldots, u_k$ **are solutions of a homogeneous linear partial differential equation, then the linear combination**

$$u = c_1 u_1 + c_2 u_2 + \cdots c_k u_k ,$$

where, the c_i, $i = 1,2,\ldots,k$ **are constants, is also a solution.**

Three Types of Basic PDEs

1-Dimension $\nabla^2 u = u_{xx}$
2-Dimension $\nabla^2 u = u_{xx} + u_{yy}$
3-Dimension $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$

Equations

Nonhomogeneous $F \neq 0$

The Heat Equation $u_t = \alpha^2 \nabla^2 u$, $u_t - \alpha^2 \nabla^2 u = F$

The Wave Equation $u_{tt} = c^2 \nabla^2 u$, $u_{tt} - c^2 \nabla^2 u = F$

Laplace's Equation $\nabla^2 u = 0$, $\nabla^2 u = F$ ←**Poisson's Equation**

Boundary Conditions

Linear Homogeneous BCs*

$$\alpha u_x(0,t) + \beta u(0,t) = 0$$
$$\gamma u_x(L,t) + \delta u(L,t) = 0$$
$$\alpha, \beta, \gamma, \delta : \text{constant}$$

(1-Dimension)
The Heat Equation
Initial Boundary Value Problem

$$u_t = \alpha^2 u_{xx} \quad , 0 < x < L,\ 0 < t < \infty$$
BCs: $u(0,t) = 0 \ , u(L,t) = 0 \quad , 0 < t < \infty$
IC: $u(x,0) = \phi(x) \quad , 0 \le x \le L$

(1-Dimension)
The Wave Equation
Initial Boundary Value Problem

$$u_{tt} = \alpha^2 u_{xx} \quad , 0 < x < L,\ 0 < t < \infty$$
BCs: $u(0,t) = 0 \ , u(L,t) = 0 \quad , 0 < t < \infty$
ICs: $u(x,0) = f(x) \ , u_t(x,0) = g(x) \quad , 0 \le x \le L$

(2-Dimension)
Laplace's Equation
Boundary Value Problem

$$u_{xx} + u_{yy} = 0 \quad , 0 < x < a,\ 0 < y < b$$
BCs: $u(0,y) = 0 \ , u(a,y) = 0 \quad , 0 < y < b$
$$u(x,0) = 0 \ , u(x,b) = f(x) \quad , 0 < x < a$$

*Farlow S.J., Partial Differential Equations for Scientists and Engineers, Dover, 1982, p33

Three Types of Basic PDEs

☑ Introduction

(1) $a^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$ **1-D Wave Equation (Hyperbolic Type)**

(2) $k \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, k > 0$ **1-D Heat Equation (Parabolic Type)**

(3) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ **2-D Laplace's Equation (Elliptic Type)**

cf*)

$\dfrac{\partial^2 u}{\partial t^2} - a^2 \dfrac{\partial^2 u}{\partial x^2} = 0 \Leftrightarrow \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ (Hyperbolic Curve)

$\dfrac{\partial u}{\partial t} = k \dfrac{\partial^2 u}{\partial x^2} \Leftrightarrow y = ax^2$ (Parabolic Curve)

$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \Leftrightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (Elliptic Curve)

*김찬중, 길잡이 공업수학, 제4판, 문운당, p451

Various Examples of PDEs[1)] (Classify !)

1-Dimension $\nabla^2 u = u_{xx}$
2-Dimension $\nabla^2 u = u_{xx} + u_{yy}$
3-Dimension $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$

Equations

The Heat Equation $u_t = \alpha^2 \nabla^2 u$, $u_t - \alpha^2 \nabla^2 u = F$

Nonhomogeneous $F \neq 0$

$u_t - \alpha^2 \nabla^2 u = u^2$ **nonhomogeneous?**

The Wave Equation $u_{tt} = c^2 \nabla^2 u$, $u_{tt} - c^2 \nabla^2 u = F$

Laplace's Equation $\nabla^2 u = 0$, $\nabla^2 u = F$ **←Poisson's Equation**

Navier-Stokes Equations

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \bullet \nabla)\mathbf{V}\right] = \rho\mathbf{F} - \nabla P + \mu\nabla^2\mathbf{V}$$

A nonlinear second order system of PDEs that models the motion of a viscous fluid

Linear Elastic Equations (Navier Equations)

$$\rho\frac{d\mathbf{V}}{dt} = \rho\mathbf{F} + (\lambda + G)\nabla(\nabla \bullet \mathbf{V}) + G\nabla^2\mathbf{V}$$

The equation models the propagation of vibrational waves in elastic solids

Maxwell's Equations

$$\nabla \bullet \mathbf{E} = 4\pi\rho$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{J}$$

$$\nabla \bullet \mathbf{B} = 0$$

A system of eight first-order linear PDEs (check !)

$Unknown: \mathbf{E} = (E_1, E_2, E_3), \mathbf{B} = (B_1, B_2, B_3)$

Electric Fields Magnetic Fields

*Betounes D.,Partial Differential Equations for Computational Science, Springer, 1998, p13

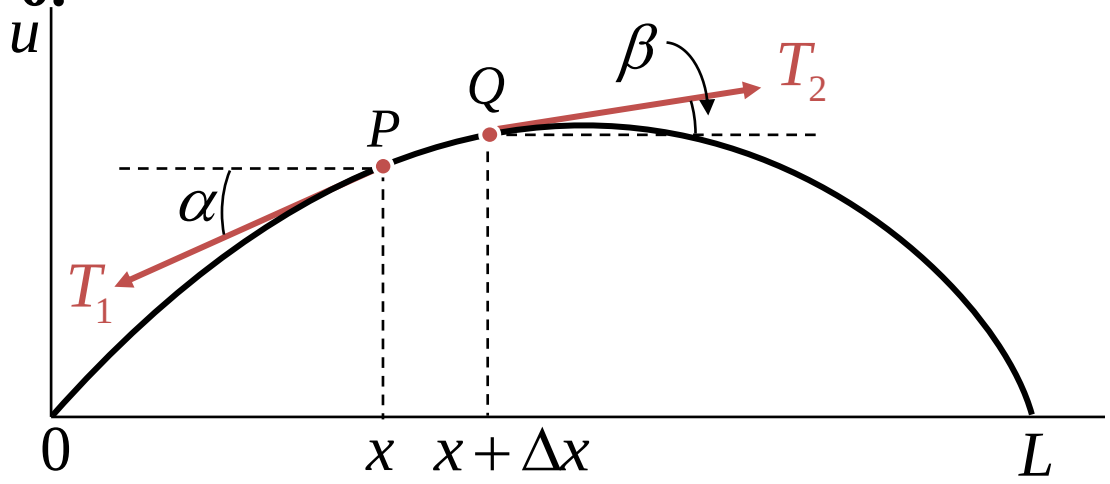
8.3 WAVE EQUATION

8.3 1-D Wave Equation ('Classify')

$$(1)\quad a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

We place the string along the x-axis, stretch it to length L, and fasten it at the ends $x = 0$ and $x = L$.

The problem is to determine the vibration of the string, that is to find its deflection $u(x,t)$ at any point x and at any time $t > 0$.



Mathematical Modeling of 1-D Wave Equation ('Configure')

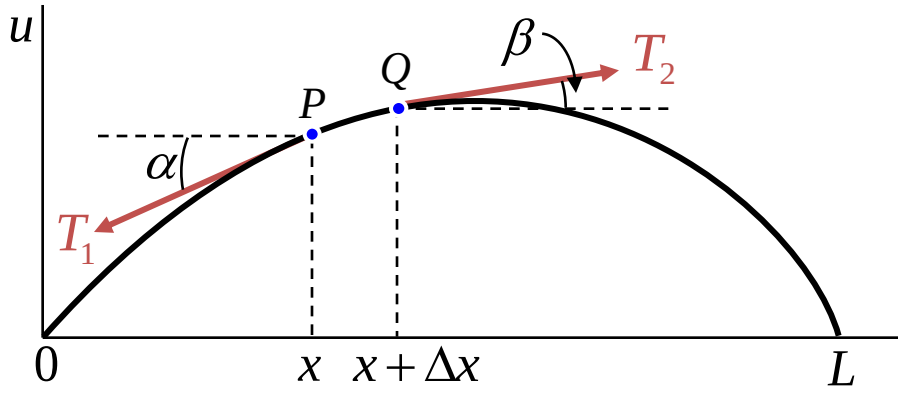
$$(1)\quad a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

Physical Assumptions

1. The mass of the string per unit length is constant ("homogeneous string). The string is perfectly elastic and does not offer any resistance to bending.

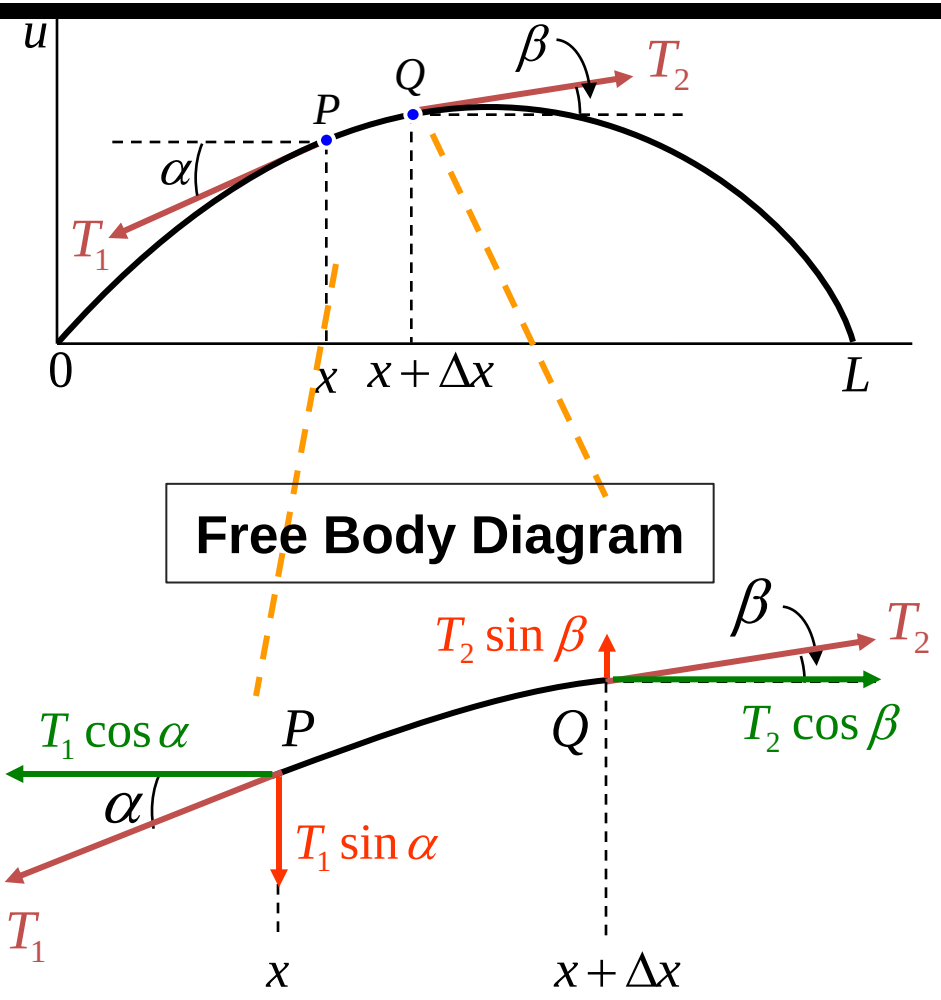
2. The tension caused by stretching the string before fastening it at the ends is so large that the action of the gravitational force on the string (trying to pull the string down a little) can be neglected.

3. The string performs small transverse motions in a vertical plane; that is, every particle of the string moves strictly vertically and so that the deflection and the slope at every point of the string always remain small in absolute value.



Mathematical Modeling of 1-D Wave Equation

$$(1)\quad a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$$



T_1, T_2 : tensions at the end point P, Q and they are directed along the tangents at the points.

Consider forces acting on a small portion of the string. That means construct a 'Free Body Diagram':

No motion in the horizontal direction.

$$\therefore T_2\cos\beta - T_1\cos\alpha = 0$$

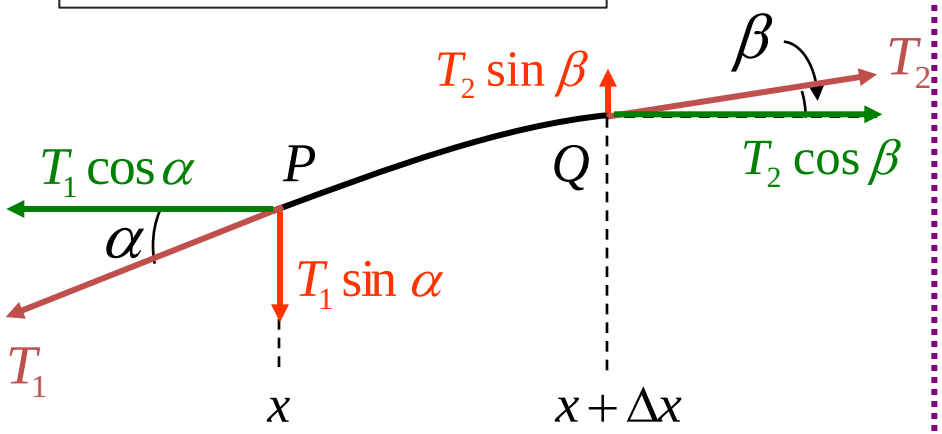
$$\therefore T_1\cos\alpha = T_2\cos\beta$$

$$= T = \text{constant}\cdots(1)$$

- Point of the string moves vertically.
- Net force acting on a small portion of the string:

$$\text{Net force} = T_2\sin\beta - T_1\sin\alpha$$

Free Body Diagram



- the mass of the small portion

$$\Delta m = \rho \Delta x$$

ρ : mass of the undeflected string per unit length

Δx :length of the portion of the undeflected string

- the acceleration

$$a = \frac{\partial^2 u}{\partial t^2}, \text{ where } u(x,t) \text{ is deflection}$$

- Apply Newton's 2nd law to the free body diagram,

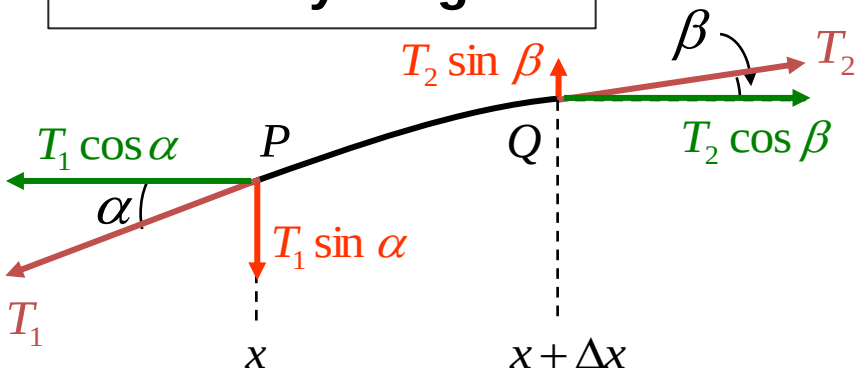
$$ma: \quad \Delta m a = \rho \Delta x \frac{\partial^2 u}{\partial t^2}$$

$$F: \quad \text{Net force} = T_2 \sin\beta - T_1 \sin\alpha$$

$$\therefore \rho \Delta x \frac{\partial^2 u}{\partial t^2} = T_2 \sin\beta - T_1 \sin\alpha$$

$$T_1 \cos\alpha = T_2 \cos\beta = T = \text{constant} \cdots (1)$$

Free Body Diagram



$$T_2 \sin\beta - T_1 \sin\alpha = \rho \Delta x \frac{\partial^2 u}{\partial t^2}$$

Dividing both sides by (1)

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\therefore \tan\beta - \tan\alpha = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\because \tan\alpha = \left(\frac{\partial u}{\partial x}\right)_x, \ \tan\beta = \left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}$$

$$\frac{1}{\Delta x}\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}$$

Let $\Delta x \to 0$, then we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}, \text{ where } c^2 = \frac{T}{\rho}$$

Method of separating variables('Divide')

(1) $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ **b/c** $u(0,t) = 0,\ u(L,t) = 0,\ \text{for all } t$

i/c $u(x,0) = f(x)$, $u_t(x,0) = g(x)$ $\quad (0 \le x \le L)$

Although there are several methods that can be tried to find particular solutions of a linear PDE, in the method of separation of variables we seek to find a particular solution of the form of a product of a function of *x* and a function of *t*,

$$u(x,t) = X(x)T(t) \ldots (1)$$

which is a product of two functions, each depending only on one of the variables *x* and *t*.

$$\therefore \frac{\partial^2 u}{\partial t^2} = X\ddot{T}, \quad \frac{\partial^2 u}{\partial x^2} = X''T \quad \text{, where } \ddot{T} = \frac{d^2 T}{dt^2}, \quad X'' = \frac{d^2 X}{dx^2}$$

$$\therefore \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \Rightarrow \quad X\ddot{T} = c^2 X''T$$

Method of separating variables('Divide')

(1) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

$$X\ddot{T} = c^2 X'' T$$

Dividing by $c^2 XT$,

$$\frac{X\ddot{T}}{c^2 XT} = \frac{c^2 X'' T}{c^2 XT}$$

$$\therefore \frac{\ddot{T}}{c^2 T} = \frac{X''}{X}$$

Left hand side is depending only on *t* and right hand side is depending only on *x*.

Let's consider, for example,

$$\frac{\ddot{T}}{c^2 T} = a_1 t + b_1, \quad \frac{X''}{X} = a_2 x + b_2$$

$$\therefore a_1 t + b_1 = a_2 x + b_2$$

The variables *t* and *x* are independent. Thus a_1 and a_2 must be zero to satisfy the equation with respect to all *t* and *x*.

$$\therefore b_1 = b_2$$

That means both sides must be constant

Solve ('Conquer')

$$\frac{\ddot{T}}{c^2 T} = \frac{X''}{X}$$

Both sides must be constant

$$\therefore \frac{\ddot{T}}{c^2 T} = \frac{X''}{X} = -\lambda$$

Multiplying by the denominators gives two ordinary differential equations

$$X'' + \lambda X = 0 \cdots (2)$$

$$\ddot{T} + c^2 \lambda T = 0 \cdots (3)$$

$\Rightarrow$ **Solve 2nd order ODEs**

b/c : $u(0,t)=0,\ u(L,t)=0,\ \text{for all } t$

$$u(x,t)=X(x)T(t)\ldots(1)$$

$$X''+\lambda X=0\cdots(2)$$

$$\ddot{T}+c^2\lambda T=0\cdots(3)$$

- **Boundary condition**

$$u(0,t)=X(0)T(t)=0,$$

$$u(L,t)=X(L)T(t)=0,$$

$$T(t)\neq 0$$

otherwise, must $u(x,t)=0$

That is trivial solution→ no interest

$$\therefore X(0)=X(L)=0$$

Consider three cases according to the sign of *λ*:

Case 1: *λ* = 0

$$(2): X''+\lambda X=0 -> X''=0$$

$$\therefore X(x)=Ax+B$$

$$\left.\begin{array}{l}\text{Boundary condition}\\ X(0)=A\cdot 0+B=0\\ X(L)=A\cdot L+B=0\end{array}\right\} A=B=0$$

$$\therefore X(x)=0$$

trivial solution -> no interest

$$X'' + \lambda X = 0$$

Boundary condition

$$X(0) = X(L) = 0$$

Case 2: $\lambda = -\alpha^2 < 0$

$$X'' - \alpha^2 X = 0$$

$$X(x) = Ae^{\alpha x} + Be^{-\alpha x}$$

Boundary condition

$$X(0) = A + B = 0, \quad B = -A$$

$$\therefore X(x) = A(e^{\alpha x} - e^{-\alpha x})$$

$$X(L) = A(e^{\alpha L} - e^{-\alpha L}) = 0$$

if $A = 0 \rightarrow B = 0$ ⇨ $X(x) = 0$

trivial solution → no interest

if $(e^{\alpha L} - e^{-\alpha L}) = 0$

$$e^{\alpha L} - \frac{1}{e^{\alpha L}} = \frac{e^{\alpha L} \cdot e^{\alpha L} - 1}{e^{\alpha L}} = \frac{e^{2\alpha L} - 1}{e^{\alpha L}} = 0$$

$$\rightarrow e^{2\alpha L} = 1,$$

$$\rightarrow \alpha = 0 \ (\because L \neq 0) \Rightarrow X(x) = 0$$

trivial solution → no interest

Case 3: $\lambda = \alpha^2 > 0$

$$X'' + \alpha^2 X = 0$$

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$$

Boundary condition

$$X(0) = c_1 = 0$$

$$\therefore X(x) = c_2 \sin \alpha x$$

$$X(L) = c_2 \sin \alpha L = 0$$

$c_2 \neq 0$ **(otherwise** $X(x) = 0$ **:** trivial solution → no interest**).**

Thus, $\sin \alpha L = 0$

$$\therefore \alpha L = n\pi \ , (n = 1, 2, \cdots)$$

Eigenvalues and Eigenfunctions

$$X'' + \lambda X = 0$$

Thus, $\sin \alpha L = 0$

$$\therefore \alpha L = n\pi \ , (n = 1,2,\cdots)$$

Eigenvalues of problem: $\lambda_n = \alpha_n{}^2 = \dfrac{n^2\pi^2}{L^2}$

Eigenfunctions : $\sin \alpha_n x, \quad n = 1,2,3 \ldots -> X(x) = c_2 \sin \alpha_n x$

$$X'' + \lambda X = 0$$

Boundary condition

$$X(0) = X(L) = 0$$

$$X(x) = c_2 \sin \alpha x$$

$$\alpha = \frac{n\pi}{L}, \ (n = 1,2,\cdots)$$

We obtain many solutions $X(x) = X_n(x)$, where

$$X_n(x) = c_2 \sin \frac{n\pi}{L} x, \ \ (n = 1,2,\cdots)$$

For negative integer *n* , we obtain essentially the same solutions, except for a minus sign constant , because of sin(-α) = -sin(α).
-> Explain in detail !! And if *n* is zero, X_n becomes zero, so it cannot be a basis.

From 'Divide' to 'Synthesize'

$$X_n(x) = A_n \sin\frac{n\pi}{L}x$$

$\lambda = \alpha^2$

$$\alpha = \frac{n\pi}{L}\ ,\ (n = 1,2,\cdots)$$

$$\lambda_n = \alpha_n{}^2 = \frac{n^2\pi^2}{L^2}$$

$$\ddot{T} + (\alpha c)^2 T = 0$$

$$u(x,t) = X(x)T(t)\ldots(1)$$

$$X'' + \lambda X = 0\cdots(2)$$

$$\ddot{T} + c^2\lambda T = 0\cdots(3)$$

$$u(0,t) = 0,\ u(L,t) = 0, \text{for all } t$$

$$T_n(t) = c_3 \cos\frac{n\pi}{L}ct + c_4 \sin\frac{n\pi}{L}ct$$

Hence solutions of (1) satisfying (2) are $u_n(x,t) = T_n(t)X_n(x)$, written out

$$u_n = (A_n \cos\frac{n\pi c}{L}t + B_n \sin\frac{n\pi c}{L}t)\cdot\sin\frac{n\pi}{L}x\ ,\ basis\ \ solution$$

By linear combination of the basis solutions, we have entire solution:

$$u\ (x,t) = \sum_{n=1}^{\infty}(A_n \cos\frac{n\pi c}{L}t + B_n \sin\frac{n\pi c}{L}t)\cdot\sin\frac{n\pi}{L}x$$

Fourier Series

One-dimensional wave equation : $u_{tt} = c^2 u_{xx}$

General Solution : $u(x,t) = \sum_{n=1}^{\infty} (A_n \cos\frac{n\pi c}{L} t + B_n \sin\frac{n\pi c}{L} t) \sin\frac{n\pi}{L} x$

To determine the arbitrary constants An and Bn, we apply the initial conditions to the general solution: $u(x,0) = f(x), \quad u_t(x,0) = g(x)$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L} x = f(x)$$

Fourier sine series of *f* (*x*)

$$\therefore A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L} dx,$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \left[\sum_{n=1}^{\infty} (-A_n \frac{n\pi c}{L}\sin\frac{n\pi c}{L} t + B_n \frac{n\pi c}{L}\cos\frac{n\pi c}{L} t)\sin\frac{n\pi}{L} x\right]_{t=0}$$

$$= \sum_{n=1}^{\infty} (B_n \frac{n\pi c}{L})\sin\frac{n\pi}{L} x = g(x)$$

Fourier sine series of *g*(*x*)

$$\therefore B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L} dx,$$

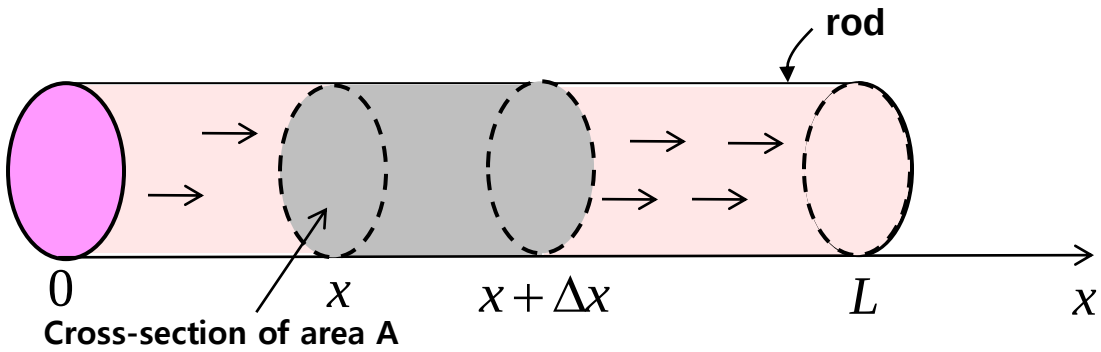
8.4 Heat Equation (Classify !)

$$(2)\quad k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t},k>0$$

8.4 Derivation of 1-D Heat Equation

Assumptions

- The flow of heat within the rod takes place only in the x-direction
- The lateral, or curved, surface of the rod is insulated; that is , no heat escapes from this surface
- No heat is being generated within rod
- The rod is homogeneous; that is, its mass per unit volume ρ is constant
- The specific heat(비열)* Cp and thermal conductivity(열전도도)** K of material of the rod are constants



One dimension flow of heat

* Oxtoby, Principles of Modern Chemistry, 6th Ed.,(year?) Thomson, Index 1.25, "Specific heat capacity : The amount of heat required to raise the temperature of one gram of a substance by one kelvin at constant pressure"(Check !!!)

** 여상도, 열역학 개념의 해설, 청문각, 2006, p18 "온도가 동일한 두 물체와 우리의 손이 닿았을 때 그 차갑고 뜨거운 정도가 다른 이유는, 두 물체의 온도가 다르기 때문이 아니라 우리 손에서 물체로 이동하는 열의 전달 속도가 다르기 때문이다. 열전도도가 큰 철판이 열전도도가 작은 나무판에 비해 훨씬 빨리 손으로부터 열을 빼앗아 간다."

Mathematical Modeling (Configure !)

$$(2)\quad k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}, k>0$$

■ Empirical laws of heat conduction

(i) The quantity of heat(열량) Q in an element of mass(질량소) m is

$$Q = C_p\, m\, u$$

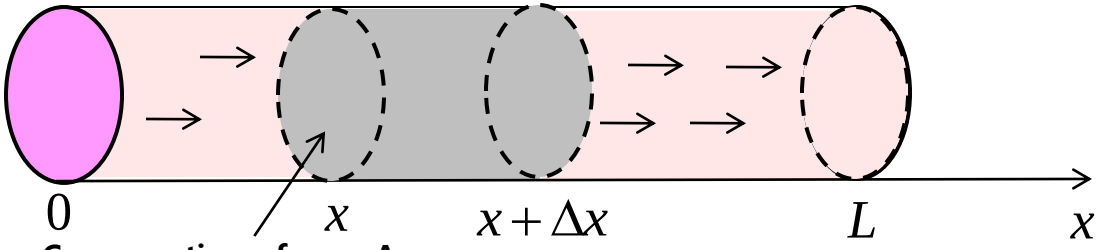
specific heat(비열) : *Cp* .
temperature of the element : *u*

(ii) The rate of heat flow(열흐름율) Q_t through the cross-section indicated in Figure is proportional to the area A of the cross-section and the partial derivative of the temperature with respect to x

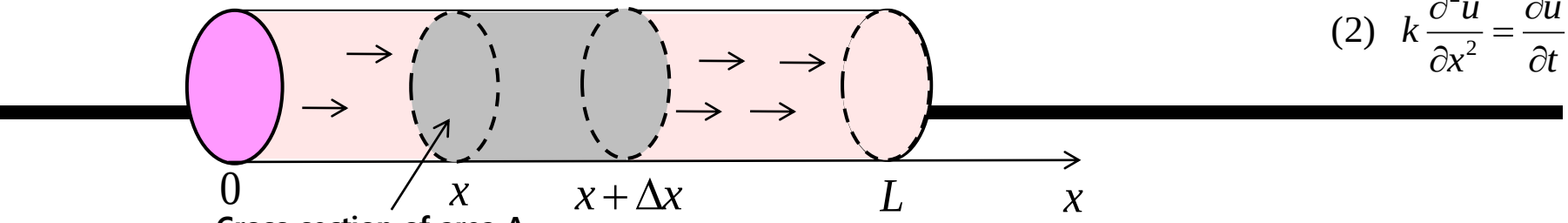
$$Q_t = -KAu_x$$

thermal conductivity(열전도도) : *K*

Heat flows in the direction of decreasing temperature



One dimension flow of heat



One dimension flow of heat

$$(2)\quad k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}, k>0$$

$$(i)\ Q = C_P\, m\, u$$

↓ Substitute $m = \rho A \Delta x$

$$Q = C_P\, \rho\, A \Delta x\, u$$

↓ Differentiate with respect to time

$$Q_t = C_P\, \rho\, A \Delta x\, u_t \cdots (1)$$

$$(ii)\ Q_t = -KAu_x$$

Net flow rate

$$-KAu_x(x,t) - [-KAu_x(x+\Delta x,t)]$$

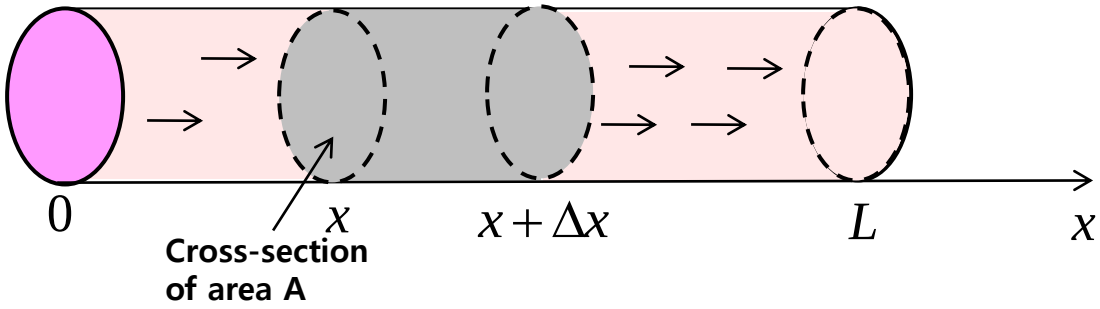
$$= KA[u_x(x+\Delta x,t) - u_x(x,t)] \cdots (2)$$

From (1) and (2) : $KA[u_x(x+\Delta x,t) - u_x(x,t)] = C_P\, \rho\, A \Delta x\, u_t$

$$\frac{K}{C_P\,\rho}\,\frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x} = u_t$$

Mathematical Model

(2) $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, k > 0$



$$\frac{K}{C_P \rho}\frac{u_x(x+\Delta x,t)-u_x(x,t)}{\Delta x} = u_t$$

As $\Delta x \to 0$, $\quad \dfrac{K}{C_P \rho} u_{xx} = u_t$

Let $\dfrac{K}{C_P \rho} = k$, $\quad \boxed{k u_{xx} = u_t}$

(***k*** : thermal diffusivity) 열확산율

Solution of 1-D Heat Equation: Divide !

(2) $k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}, k>0$

$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}\cdots(1)$$

Boundary conditions

$$u(0,t)=0,\quad u(L,t)=0$$

$$\text{for all } t$$

initial conditions

$$u(x,0)=f(x)$$

$$where,\ f(x)\ \text{is given}$$

Assume the solution as follows:

$$u(x,t)=X(x)T(t)$$

$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}$$

$$k\frac{\partial^2 X(x)}{\partial x^2}T(t)=X(x)\frac{\partial T(t)}{\partial t}$$

$$kX''(x)T(t)=X(x)\dot{T}(t)$$

Dividing both sides by $k\,X(x)\,T(t)$

$$\frac{X''(x)}{X(x)}=\frac{\dot{T}(t)}{kT(t)}=-\lambda$$

If $-\lambda\geq 0$, $X(x)$ **becomes trivial solution.**

Thus, $-\lambda$ **must be negative, i.e.,**
λ **must be positive.** **(ref. 1-D Wave Equation)**

Conquer !

$$(2)\quad k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t},\ k>0$$

Boundary conditions

$$u(0,t)=0,\quad u(L,t)=0$$

initial condition

$$u(x,0)=f(x)$$

$$\frac{X''(x)}{X(x)}=\frac{\dot{T}(t)}{kT(t)}=-\lambda$$

Step 1. Two ODEs

$$X''(x)+\lambda X(x)=0\cdots(2)$$

$$\dot{T}(t)+k\lambda T(t)=0\cdots(3)$$

Step 2. Satisfying the boundary conditions

- **general solution of (2)**

$$X(x)=c_1\cos\alpha x+c_2\sin\alpha x,\ \lambda=\alpha^2$$

- **boundary conditions**

$$u(x,t)=X(x)T(t)$$

$$u(0,t)=X(0)\cdot T(t)=0$$
$$u(L,t)=X(L)\cdot T(t)=0$$

if $T(t)\equiv 0$, u is a trivial solution.

$$\therefore X(0)=c_1=0$$

$$X(x)=c_2\sin\alpha x$$

$$X(L)=c_2\sin\alpha L=0$$

$c_2\neq 0$ (**otherwise** $X(x)=0$: **trivial solution → no interest).**

Thus, $\sin\alpha L=0$

$$\therefore \alpha L=n\pi\ ,(n=1,2,\cdots)$$

$$\alpha=\frac{n\pi}{L}\ ,\ (n=1,2,\cdots)$$

$$(2)\quad k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}, k>0$$

$$X(x)=c_2\sin\alpha x$$

$$\therefore \alpha=\frac{n\pi}{L}\ ,\ (n=1,2,\cdots)$$

Setting $c_2 = 1$, we obtain many solutions $X(x) = X_n(x)$, where

$$X_n(x)=\sin\frac{n\pi}{L}x,\ \ (n=1,2,\cdots)$$

$$\dot{T}(t)+k\lambda T(t)=0\cdots(3)$$

$$\lambda_n=\alpha_n{}^2$$

- **general solution of (3) :**

$$T(t)=T_n(t)=c_3e^{-k\frac{n^2\pi^2}{L^2}t},(n=1,2,\cdots)$$

$$\therefore u_n(x,t)=X_n(x)T_n(t)$$

$$=A_n\sin\frac{n\pi x}{L}e^{-k\frac{n^2\pi^2}{L^2}t}$$

Synthesize !

initial condition $u(x,0) = f(x)$

(2) $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, k > 0$

$$u_n(x,t) = A_n \sin\frac{n\pi x}{L} e^{-k\frac{n^2\pi^2}{L^2}t}$$

$$(n = 1,2,\cdots),$$

$$\therefore \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f(x)$$

Fourier sine series of $f(x)$

Step 3. Solution of the entire problem

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} e^{-k\frac{n^2\pi^2}{L^2}t}$$

$$\therefore A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

- **initial condition**

$$u(x,0) = f(x)$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} e^{-k\frac{n^2\pi^2}{L^2}t}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

8.5 LAPLACE'S EQUATION

(3) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

- Laplace's equation occurs in time-independent problems (steady-state) involving *potentials* such as electrostatic, gravitational, and *velocity in fluid mechanics*.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

$$\nabla^2 u=0$$

Examples of Laplace's Equation

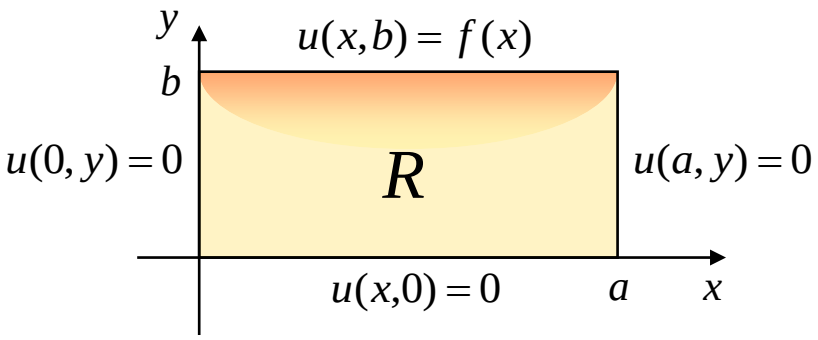
$$(3)\quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Eng. Math., Chapter 12.6[1)]
(2-D Heat equation of Steady state)

① Governing Equation :

$$\nabla^2 u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

② Boundary condition : (Dirichlet B.C.[1)])



Laplace's Eq. with Different BC Conditions

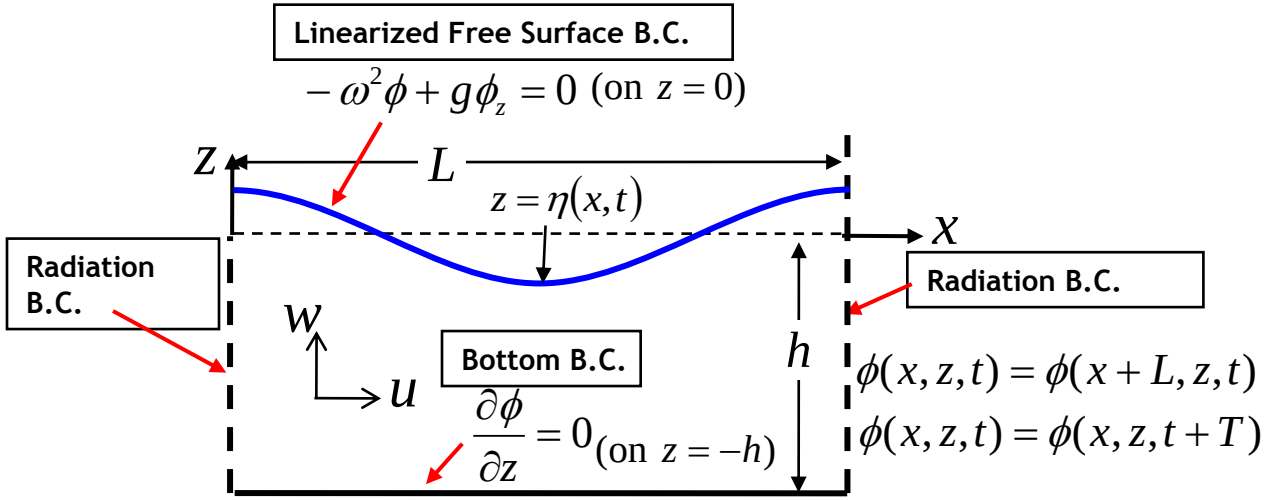
Ocean Wave Equation

① Governing Equation :

$$\nabla^2\phi=0$$

$$\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}=0\right)$$

② Boundary condition(B.C.) : (Robin B.C.[1)])



1) Erwin Kreyszig, Advanced Engineering Mathematics 10th ,Wiley,2011,Ch12.6(p 564)

Laplace's Equation:
Steady Two-Dimensional Heat Problem

Divide !

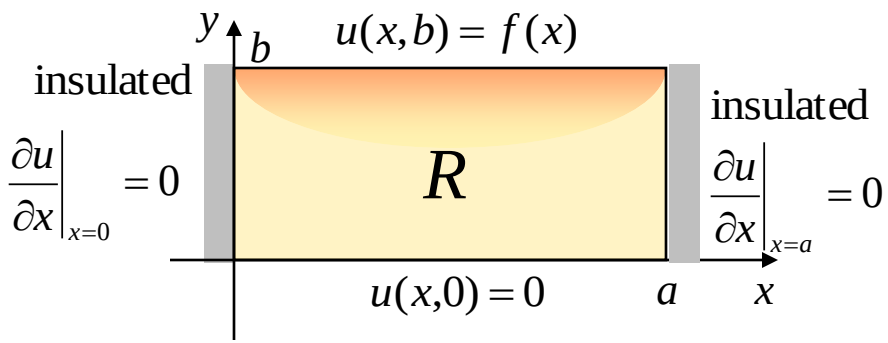
$$(3)\quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Two-Dimensional Heat Problem

$$\frac{\partial u}{\partial t}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

Steady $\Rightarrow \quad \frac{\partial u}{\partial t}=0$

$$\nabla^2 u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$



Assume the solution as follows:

$$u(x,y)=X(x)Y(y)$$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 X}{\partial x^2}Y(y)+X(x)\frac{\partial^2 Y}{\partial y^2}=0$$

separating variables,

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=-\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=-\lambda$$

Two ODEs

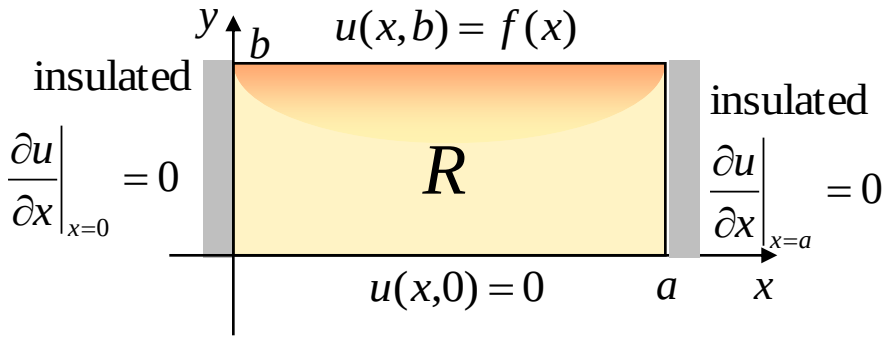
$$X''+\lambda X=0$$

$$Y''-\lambda Y=0$$

Conquer !

$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0$$

$$u(x,y) = X(x)Y(y)$$



$$X'' + \lambda X = 0$$

1) $\lambda = 0$

$$X'' = 0 \Rightarrow X(x) = c_1 x + c_2$$

- boundary condition: 1,2

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = X'(0)Y(y) = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=a} = X'(a)Y(y) = 0$$

$$\therefore X'(0) = X'(a) = 0$$

$$X'(0) = c_1 = 0$$

For any c_2 ,the second b.c. $X'(a) = 0$ is satisfied

Thus, $c_2 \neq 0$, $X(x) = c_2$: nontrivial solution!

2) $\lambda = -\alpha^2 < 0$

$$X'' - \alpha^2 X = 0$$

$$X(x) = c_3 e^{\alpha x} + c_4 e^{-\alpha x}$$

- boundary condition

$$\therefore X'(0) = X'(a) = 0$$

$$X'(0) = (c_3 - c_4)\alpha = 0 \ \therefore c_3 = c_4$$

$$X'(a) = c_3\alpha(e^{\alpha a} - e^{-\alpha a}) = 0$$

if $c_3 = 0 \rightarrow c_4 = 0$ ⇨ $X(x) = 0$

trivial solution → no interest

if $(e^{\alpha a} - e^{-\alpha a}) = 0$

$$e^{\alpha a} - \frac{1}{e^{\alpha a}} = \frac{e^{\alpha a}\cdot e^{\alpha a} - 1}{e^{\alpha a}} = \frac{e^{2\alpha a} - 1}{e^{\alpha a}} = 0$$

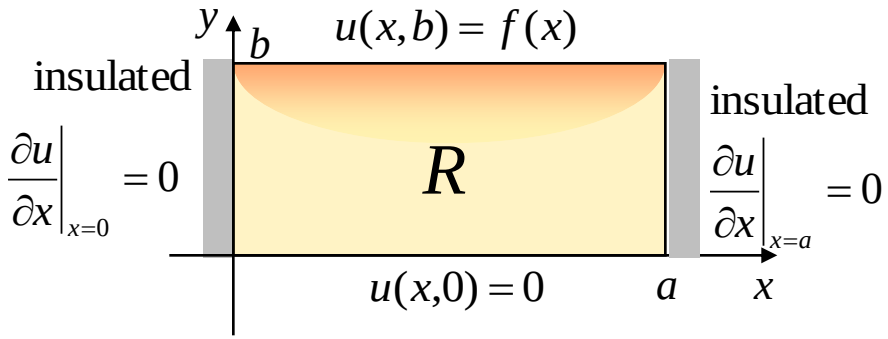
$$\rightarrow e^{2\alpha a} = 1,$$

$$\rightarrow \alpha = 0 \ (\because a \neq 0) \Rightarrow X(x) = 0$$

trivial solution → no interest

$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0$$

When $\lambda = 0$, $X(x) = c_2$: nontrivial solution!



$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \qquad \left.\frac{\partial u}{\partial x}\right|_{x=a} = 0$$

$$X'' + \lambda X = 0$$

3) $\lambda = \alpha^2 > 0$

$$X'' + \alpha^2 X = 0$$

$$X(x) = c_5 \cos \alpha x + c_6 \sin \alpha x$$

- **general solution**

$$X(x) = c_5 \cos \alpha x + c_6 \sin \alpha x$$

$$X'(x) = -c_5 \alpha \sin \alpha x + c_6 \alpha \cos \alpha x$$

- **boundary condition: 1,2**

$$\therefore X'(0) = X'(a) = 0$$

$$X'(0) = c_6 \alpha = 0 \ \therefore \ c_6 = 0$$

$$\therefore X(x) = c_5 \cos \alpha x$$

$$X'(x) = -c_5 \alpha \sin \alpha x$$

$$X'(a) = -c_5 \alpha \sin \alpha a = 0 \qquad \therefore \sin \alpha a = 0$$

$$a\alpha = n\pi, \quad \alpha = \frac{n\pi}{a}, \ (n = 1, 2, \cdots)$$

$$\therefore \lambda_n = \left(\frac{n\pi}{a}\right)^2, \ (n = 1, 2, \cdots)$$

$$\therefore X(x) = X_n(x) = c_5 \cos \frac{n\pi}{a} x, \quad (n = 1, 2, \cdots)$$

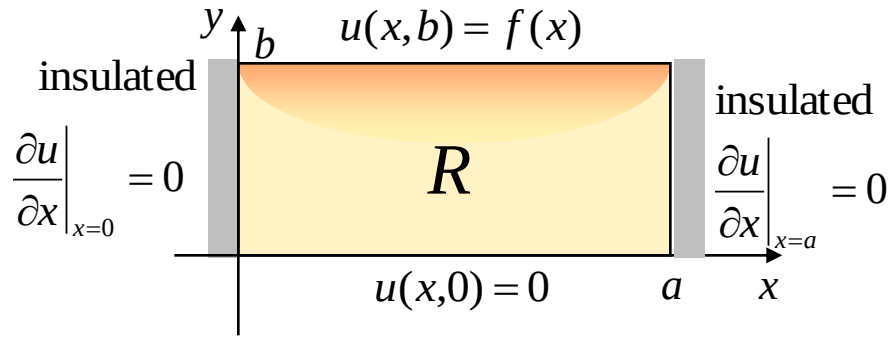
by corresponding $\lambda_0 = 0$ with $n = 0$

$$\begin{cases} X(x) = c_2, \ n = 0 \\ X_n(x) = c_5 \cos \dfrac{n\pi}{a} x, \quad (n = 1, 2, \cdots) \end{cases}$$

$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0$$

$$\lambda = \left(\frac{n\pi}{a}\right)^2, \ (n = 1, 2, \cdots)$$

$$\begin{cases} X(x) = c_2, \ n = 0 \\ X_n(x) = \cos\dfrac{n\pi}{a}x, \quad (n = 1, 2, \cdots) \end{cases}$$



$$Y'' - \lambda Y = 0$$

First, for $\lambda_0 = 0 \ (n = 0)$

$$Y'' = 0 \Rightarrow Y(y) = c_7 y + c_8$$

- **Apply boundary condition 3**

$$u(x,0) = X(x)Y(0) = 0$$

$$\therefore Y(0) = 0 \quad \Rightarrow Y(0) = c_8 = 0$$

$$\therefore Y(y) = c_7 y \ : \text{nontrivial solution!}$$

Second, for $\lambda_n = \left(\dfrac{n\pi}{a}\right)^2, \ (n = 1, 2, \cdots)$

$$Y'' - \frac{n^2\pi^2}{a^2}Y = 0$$

- **general solution**

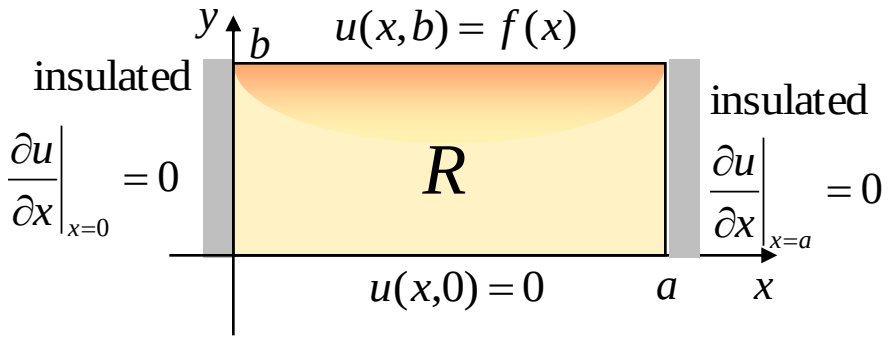
$$Y(y) = Y_n(y) = c_9 e^{n\pi y/a} + c_{10} e^{-n\pi y/a}$$

$$Y(0) = c_9 + c_{10} = 0 \qquad \therefore c_{10} = -c_9$$

$$\therefore Y_n(y) = c_9(e^{n\pi y/a} - e^{-n\pi y/a}) = 2c_9 \sinh\frac{n\pi y}{a}$$

$$\begin{cases} Y(y) = c_7 y, \quad n = 0 \\ Y_n(y) = c_9^* \sinh\dfrac{n\pi y}{a}, \quad (n = 1, 2, \cdots), c_9^* = 2c_9 \end{cases}$$

Synthesize !



$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0$$

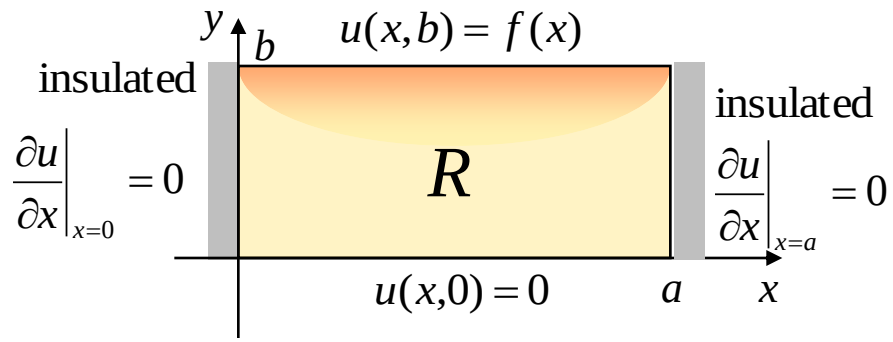
$$\begin{cases} X(x) = c_2, & n = 0 \\ X_n(x) = c_5 \cos\dfrac{n\pi}{a}x, & (n = 1, 2, \cdots) \end{cases} \qquad \begin{cases} Y(y) = c_7 y, & n = 0 \\ Y_n(y) = c_9^* \sinh\dfrac{n\pi y}{a}, & (n = 1, 2, \cdots) \end{cases}$$

$$\therefore u_n(x,y) = X_n(x)Y_n(y) = \begin{cases} A_0^* y, & (n = 0) \\ A_n^* \sinh\dfrac{n\pi y}{a}\cos\dfrac{n\pi x}{a}, & (n = 1, 2, \ldots) \end{cases}$$

$$where, \quad A_0^* = c_2 c_7, \quad A_n^* = c_5 c_9^*$$

Fourier cosine series where, $(-p, p)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x \qquad a_0 = \frac{2}{p}\int_0^p f(x)dx, \quad a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}xdx$$



$$u_n(x,y) = X_n(x)Y_n(y)$$

$$= \begin{cases} A_0^* y, & (n=0) \\ A_n^* \sinh\frac{n\pi y}{a}\cos\frac{n\pi x}{a}, & (n=1,2,\ldots) \end{cases}$$

By Superposition

$$u(x,t) = A_0^* y + \sum_{n=1}^{\infty} A_n^* \sinh\frac{n\pi y}{a}\cos\frac{n\pi x}{a}$$

- **Apply boundary condition 4**

$$u(x,b) = X(x)Y(b) = f(x)$$

$$u(x,b) = f(x)$$

$$= A_0^* b + \sum_{n=1}^{\infty} A_n^* \sinh\frac{n\pi b}{a}\cos\frac{n\pi x}{a}$$

$$= A_0^* b + \sum_{n=1}^{\infty}\left(A_n^* \sinh\frac{n\pi b}{a}\right)\cos\frac{n\pi x}{a}$$

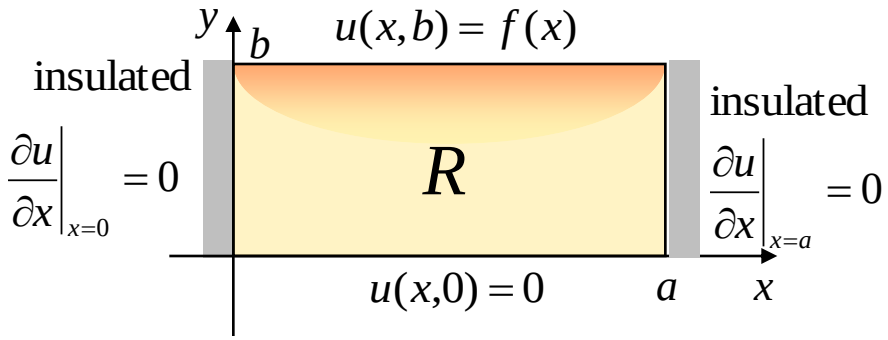
Fourier cosine series of $f(x)$

$$A_0^* b = \frac{1}{2}a_0 \quad and \quad a_n = A_n^* \sinh\frac{n\pi b}{a}$$

$$A_0^* b = \frac{1}{a}\int_0^a f(x)dx \Rightarrow A_0^* = \frac{1}{ab}\int_0^a f(x)dx$$

$$A_n^* \sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}dx$$

$$\therefore A_n^* = \frac{2}{a\sinh(n\pi b/a)}\int_0^a f(x)\cos\frac{n\pi x}{a}dx$$

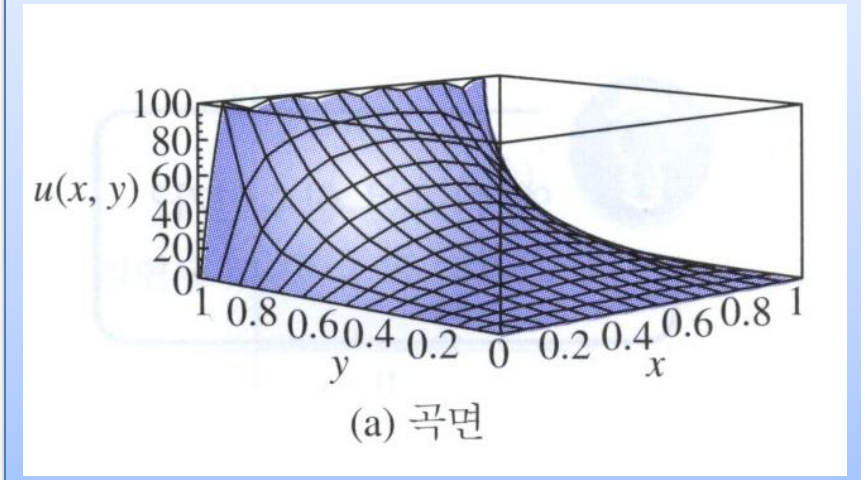


$$u(x,t) = A_0^* y + \sum_{n=1}^{\infty} A_n^* \sinh\frac{n\pi y}{a}\cos\frac{n\pi x}{a}$$

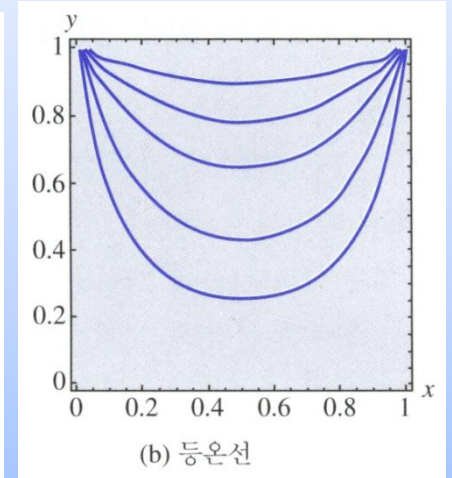
$$A_0^* = \frac{1}{ab}\int_0^a f(x)dx$$

$$A_n^* = \frac{2}{a\sinh(n\pi b/a)}\int_0^a f(x)\cos\frac{n\pi x}{a}dx$$

Ex) f(x)=100, a=1, b=1



(a) 곡면



(b) 등온선

Laplace's Equation: Dirichlet Problem in a Rectangle R

- **Laplace's equation**

(Steady Two-Dimensional Heat Problem)

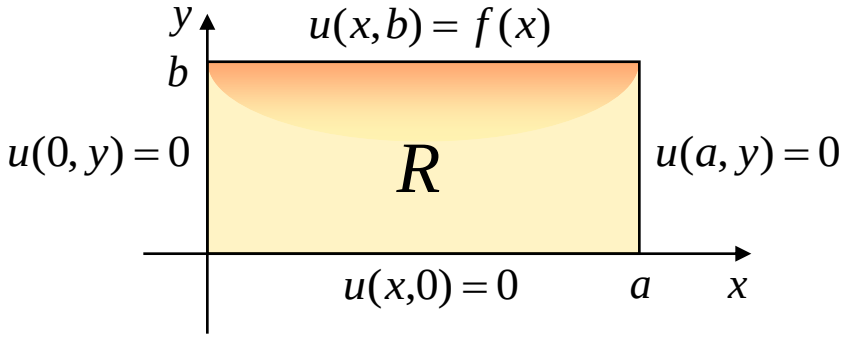
Two-Dimensional Heat Problem

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Steady $\Rightarrow \quad \frac{\partial u}{\partial t} = 0$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Dirichlet Problem in a Rectangle R



$$u(x,y) = F(x)G(y)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 F}{\partial x^2}G(y) + F(x)\frac{\partial^2 G}{\partial y^2} = 0$$

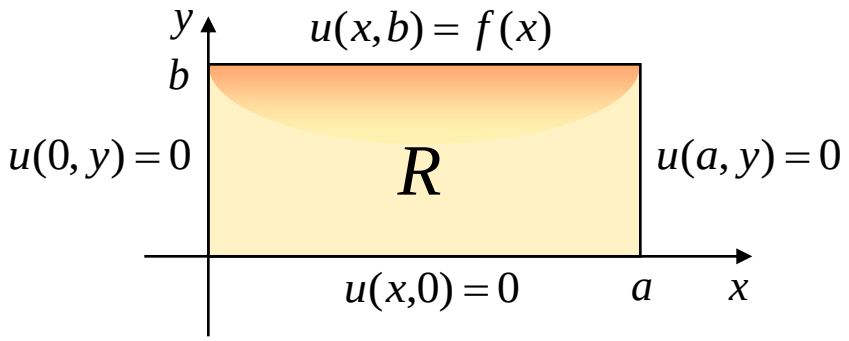
separating variables,

$$\frac{1}{F}\frac{\partial^2 F}{\partial x^2} = -\frac{1}{G}\frac{\partial^2 G}{\partial y^2} = -\lambda \ < 0$$

Two ODEs

$$F'' + \lambda F = 0$$

$$G'' - \lambda G = 0$$



$$F'' + \lambda F = 0$$

- **general solution**

$$F(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

- Apply boundary condition 1,2

$$u(0, y) = F(0)G(y) = 0$$

$$u(a, y) = F(a)G(y) = 0$$

$$\therefore F(0) = F(a) = 0$$

$$F(0) = A = 0$$

$$\therefore F(x) = B\sin\sqrt{\lambda}x$$

$$F(a) = B\sin\sqrt{\lambda}a = 0$$

$$\therefore \sin\sqrt{\lambda}a = 0$$

$$a\sqrt{\lambda} = n\pi, \quad \sqrt{\lambda} = \frac{n\pi}{a}, \ (n = 1, 2, \cdots)$$

$$\therefore \lambda = \left(\frac{n\pi}{a}\right)^2, \ (n = 1, 2, \cdots)$$

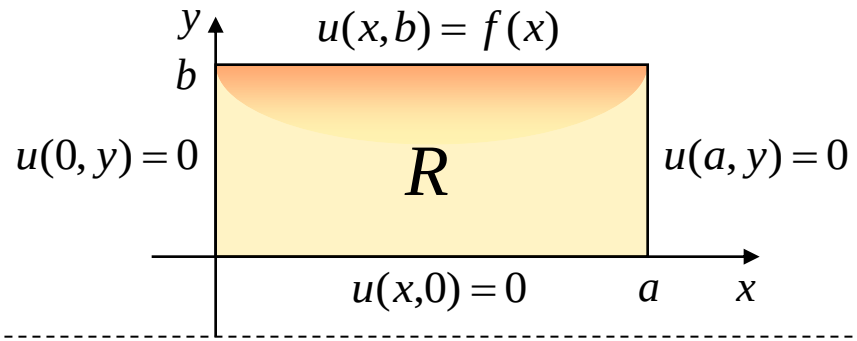
Setting $B = 1$,

$$\therefore F(x) = F_n(x)$$

$$= \sin\frac{n\pi}{a}x, \quad (n = 1, 2, \cdots)$$

$F'' + \lambda F = 0, \quad G'' - \lambda G = 0$

$\lambda = \left(\frac{n\pi}{a}\right)^2, \ (n = 1, 2, \cdots)$ $F_n(x) = \sin\frac{n\pi}{a}x$



$$G'' - \lambda G = 0, \qquad \lambda = \left(\frac{n\pi}{a}\right)^2, \ (n = 1,2,\cdots)$$

$$\therefore G'' - \frac{n^2\pi^2}{a^2}G = 0$$

- **general solution**

$$G(y) = G_n(y) = A_n e^{n\pi y/a} + B_n e^{-n\pi y/a}$$

- **Apply boundary condition 3**

$$u(x,0) = F(x)G(0) = 0$$

$$\therefore G(0) = A_n + B_n = 0$$

$$\therefore B_n = -A_n$$

$$\therefore G_n(y) = A_n(e^{n\pi y/a} - e^{-n\pi y/a})$$

$$= 2A_n \sinh\frac{n\pi y}{a}$$

$$\therefore G_n(y) = A_n^* \sinh\frac{n\pi y}{a}$$

where $A_n^* = 2A_n$

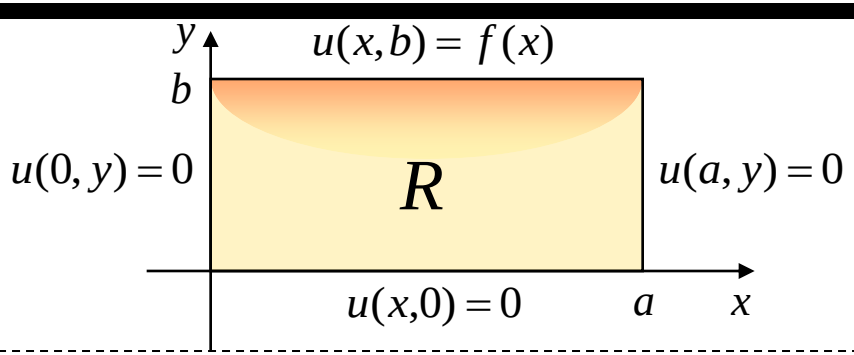
$$\therefore u_n(x,y) = F_n(x)G_n(y)$$

$$= A_n^* \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

Laplace's Equation

Fourier sine series where, $(-p, p)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x \qquad b_n = \frac{2}{p}\int_0^p f(x) \sin \frac{n\pi}{p} x dx$$



$$u_n(x,y) = A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}$$

By Superposition

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,y)$$

$$= \sum_{n=1}^{\infty} A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}$$

- **Apply boundary condition 4**

$$u(x,b) = F(x)G(b) = f(x)$$

$$u(x,b) = f(x)$$

$$= \sum_{n=1}^{\infty} A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi b}{a}$$

$$= \sum_{n=1}^{\infty} \left(A_n^* \sinh \frac{n\pi b}{a} \right) \sin \frac{n\pi x}{a}$$

Fourier sine series of $f(x)$

$$\therefore A_n^* \sinh \frac{n\pi b}{a}$$

$$= \frac{2}{a}\int_0^a f(x) \sin \frac{n\pi x}{a} dx$$

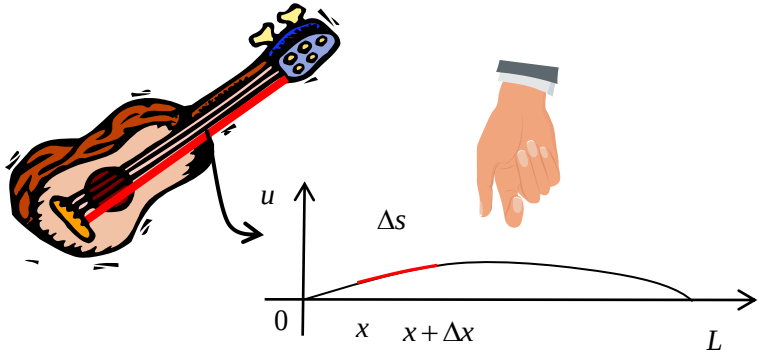
$$\therefore A_n^* = \frac{2}{a \sinh(n\pi b/a)} \int_0^a f(x) \sin \frac{n\pi x}{a} dx$$

8.6 2-D HEAT PDE :

FOURIER SERIES IN TWO VARIABLES

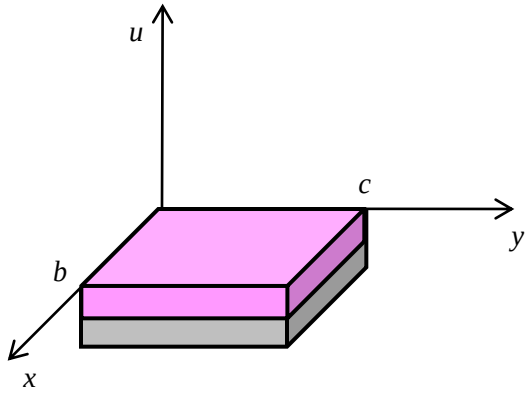
Wave equation in Two dimensions

<1-D wave equation>



$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

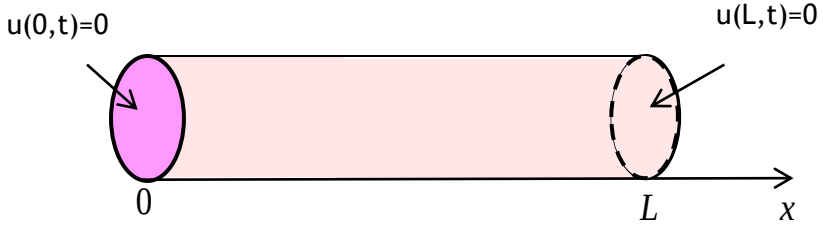
<2-D wave equation>



$$a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial^2 u}{\partial t^2}$$

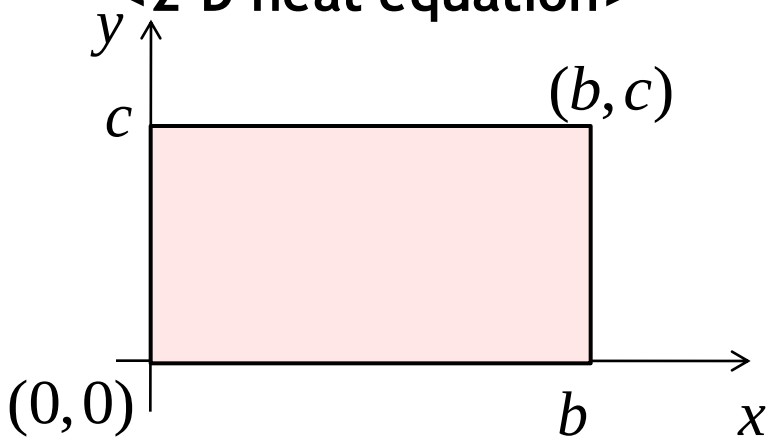
8.6 Heat equation in Two dimensions

<1-D heat equation>



$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},\ 0 < x < L,\ t > 0$$

<2-D heat equation>

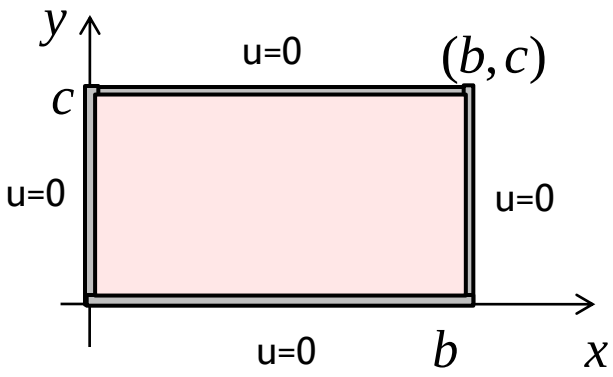


$$k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial u}{\partial t}$$

Divide !

☑ Example 1: Temperatures in a Plate

$$k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial u}{\partial t},\ 0<x<b,\ 0<y<c, t>0$$



subject to

$$u(0,y,t)=0,\ u(b,y,t)=0,\ 0<y<c,\ t>0$$
$$u(x,0,t)=0,\ u(x,c,t)=0,\ 0<x<b,\ t>0$$
$$u(x,y,0)=f(x,y),\ 0<x<b,\ 0<y<c$$

Separating variables

$$u(x,y,t)=X(x)Y(y)T(t) \longrightarrow k\left(u_{xx}+u_{yy}\right)=u_t$$

$$k(X''YT+XY''T)=XYT'$$

dividing by $kXYT$

$$\frac{X''}{X}=-\frac{Y''}{Y}+\frac{T'}{kT}=-\lambda \quad \Rightarrow \quad X''+\lambda X=0$$

$$\frac{Y''}{Y} = \frac{T'}{kT} + \lambda \qquad X'' + \lambda X = 0$$

$$u(0, y, t) = 0,\ u(b, y, t) = 0,\ 0 < y < c,\ t > 0$$
$$u(x, 0, t) = 0,\ u(x, c, t) = 0,\ 0 < x < b,\ t > 0$$
$$u(x, y, 0) = f(x, y),\ 0 < x < b,\ 0 < y < c$$

Introduce another separation constant,

$$\frac{Y''}{Y} = -\mu, \text{ then } \frac{T'}{kT} + \lambda = -\mu \Rightarrow Y'' + \mu Y = 0,\ T' + k(\lambda + \mu)T = 0$$

$$\begin{cases} X'' + \lambda X = 0 \\ Y'' + \mu Y = 0 \\ T' + k(\lambda + \mu)T = 0 \end{cases}$$

Boundary conditions

$$u(0, y, t) = X(0)Y(y)T(t) = 0$$
$$u(b, y, t) = X(b)Y(y)T(t) = 0$$

imply $X(0) = 0,\ X(b) = 0$

Otherwise $(Y(y) = 0 \text{ or } T(t) = 0)$, u(x,y,t) : trivial solution

$$X'' + \lambda X = 0,\ X(0) = 0,\ X(b) = 0$$

$$Y'' + \mu Y = 0,\ Y(0) = 0,\ Y(c) = 0$$

Two Sturm-Liouville Boundary Value problems

$$u(x,y,t) = X(x)Y(y)T(t)$$

$$\begin{cases} X'' + \lambda X = 0 \\ Y'' + \mu Y = 0 \\ T' + k(\lambda + \mu)T = 0 \end{cases}$$

$$X(0) = 0,\ X(b) = 0$$
$$Y(0) = 0,\ Y(c) = 0$$

$$X'' + \lambda X = 0,\ X(0) = 0,\ X(b) = 0$$

$$\lambda_m = \frac{m^2\pi^2}{b^2} \quad \therefore X(x) = c_2 \sin\frac{m\pi}{b}x,\ m = 1,2,3,\ldots$$

$$Y'' + \mu Y = 0,\ Y(0) = 0,\ Y(c) = 0$$

$$\mu_n = \frac{n^2\pi^2}{c^2} \quad \therefore Y(y) = c_4 \sin\frac{n\pi}{c}y,\ n = 1,2,3,\ldots$$

Solving

$$T' + k(\lambda + \mu)T = 0 \Rightarrow T(t) = c_5 e^{-k[\lambda+\mu]t} = c_5 e^{-k[(m\pi/b)^2 + (n\pi/c)^2]t}$$

$$u_{mn}(x,y,t) = X(x)Y(y)T(t)$$

$$= A_{mn}\, e^{-k[(m\pi/b)^2 + (n\pi/c)^2]t} \sin\frac{m\pi}{b}x \sin\frac{n\pi}{c}y$$

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\, e^{-k[(m\pi/b)^2 + (n\pi/c)^2]t} \sin\frac{m\pi}{b}x \sin\frac{n\pi}{c}y$$

$$u(0,y,t)=0,\ u(b,y,t)=0,\ 0<y<c,\ t>0$$
$$u(x,0,t)=0,\ u(x,c,t)=0,\ 0<x<b,\ t>0$$
$$u(x,y,0)=f(x,y),\ 0<x<b,\ 0<y<c$$

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}e^{-k[(m\pi/b)^2+(n\pi/c)^2]t}\sin\frac{m\pi}{b}x\sin\frac{n\pi}{c}y$$

Initial condition:

$$u(x,y,0)=f(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{m\pi x}{b}\sin\frac{n\pi y}{c}$$

"Double Sine Series(Sine Series in Two Variables)"

$$u(x,y,0)=f(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{m\pi x}{b}\sin\frac{n\pi y}{c}$$

setting $K_m(y)=\sum_{n=1}^{\infty}A_{mn}\sin\frac{n\pi y}{c}$,

then, $f(x,y)=\sum_{m=1}^{\infty}K_m(y)\sin\frac{m\pi x}{b}$ $\Rightarrow$ $\therefore K_m(y)=\frac{2}{b}\int_0^b f(x,y)\sin\frac{m\pi x}{b}dx$

Fourier sine series

Fourier sine series Coefficients

$$u(x,y,0) = f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{b} \sin\frac{n\pi y}{c}$$

"Double Sine Series(Sine Series in Two Variables)"

setting $K_m(y) = \sum_{n=1}^{\infty} A_{mn} \sin\frac{n\pi y}{c}$,

Fourier sine series

$$\therefore A_{mn} = \frac{2}{c}\int_0^c K_m(y) \sin\frac{n\pi y}{c} dy$$

Fourier sine series Coefficients

then, $f(x,y) = \sum_{m=1}^{\infty} K_m(y) \sin\frac{m\pi x}{b}$ ⇒ $\therefore K_m(y) = \frac{2}{b}\int_0^b f(x,y) \sin\frac{m\pi x}{b} dx$

$$\therefore u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{b} \sin\frac{n\pi y}{c}$$

$$\text{where, } A_{mn} = \frac{4}{bc}\int_0^c\int_0^b f(x,y) \sin\frac{m\pi x}{b} dx \sin\frac{n\pi y}{c} dy$$

8-7. Laplacian in Polar Coordinates

Problems in Polar Coordinates

☑ Laplacian in Polar Coordinates

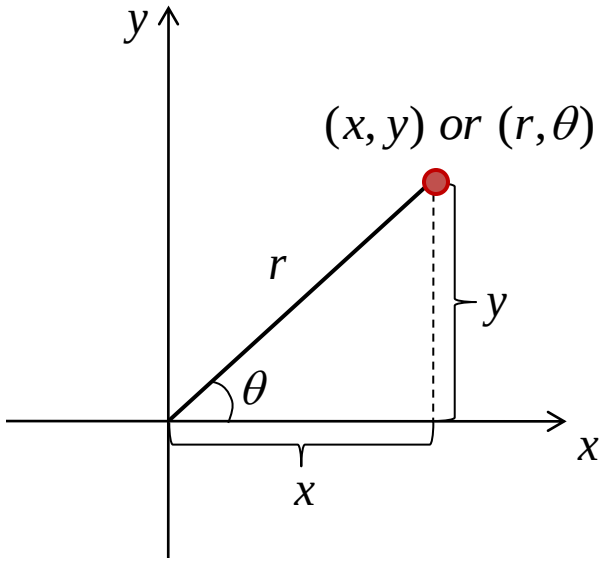
■ Relation between polar coordinates and rectangular coordinates are given by

polar coordinate(r,θ) → rectangular coordinate (x,y)

$$x = r\cos\theta, \quad y = r\sin\theta$$

rectangular coordinate (x,y) → polar coordinate(r,θ)

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$$



Problems in Polar Coordinates

☑ Laplacian in Polar Coordinates

- It is possible to convert the 2-D Laplacian of the function u, $\nabla^2 u = u_{xx} + u_{yy}$ into polar coordinate.

Theorem 9.5 **Chain Rule**

If $z = f(u,v)$ is differentiable and $u = g(x,y)$ and $v = h(x,y)$ have continuous first partial derivatives, then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$z = f(u,v) = f(g(x,y), h(x,y))$$

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = u_r r_x + u_\theta \theta_x$$

$$u_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right)\frac{\partial r}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial}{\partial x}\left(\frac{\partial \theta}{\partial x}\right)$$

$$= \left(\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\right)\frac{\partial \theta}{\partial x}\right)\frac{\partial r}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2} + \left(\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial \theta}{\partial x}\right)\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2}$$

$$= \left(\frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial \theta \partial r}\frac{\partial \theta}{\partial x}\right)\frac{\partial r}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2} + \left(\frac{\partial^2 u}{\partial r \partial \theta}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial \theta^2}\frac{\partial \theta}{\partial x}\right)\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2}$$

$$= (u_{rr} r_x + u_{r\theta}\theta_x) r_x + u_r r_{xx} + (u_{\theta r} r_x + u_{\theta\theta}\theta_x)\theta_x + u_\theta \theta_{xx}$$

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$$

☑ Laplacian in Polar Coordinates

$$\nabla^2 u = u_{xx} + u_{yy}$$

$$u_{xx} = (u_{rr} r_x + u_{r\theta}\theta_x) r_x + u_r r_{xx} + (u_{\theta r} r_x + u_{\theta\theta}\theta_x)\theta_x + u_\theta \theta_{xx}$$

$$r_x = \frac{\partial}{\partial x}(\sqrt{x^2+y^2}) = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}} = \frac{x}{r}$$

$$u_x = u_r r_x + u_\theta \theta_x$$

$$\theta_x = \frac{\partial}{\partial x}\left(\arctan\frac{y}{x}\right) = \frac{1}{1+(\frac{y}{x})^2}(-\frac{y}{x^2}) = -\frac{y}{r^2}$$

$$r_{xx} = \frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \frac{\partial}{\partial x}\left(x\frac{1}{\sqrt{x^2+y^2}}\right) = \frac{1}{\sqrt{x^2+y^2}} + x\left(-\frac{1}{2}\frac{2x}{\left(\sqrt{x^2+y^2}\right)^3}\right) = \frac{x^2+y^2-x^2}{\left(\sqrt{x^2+y^2}\right)^3} = \frac{y^2}{r^3}$$

$$\theta_{xx} = \frac{\partial}{\partial x}\left(-\frac{y}{r^2}\right) = \frac{\partial}{\partial x}\left(-\frac{y}{x^2+y^2}\right) = -(-1)\frac{2xy}{\left(x^2+y^2\right)^2} = \frac{2xy}{r^4}$$

$$u_{xx} = (u_{rr} r_x + u_{r\theta}\theta_x) r_x + u_r r_{xx} + (u_{\theta r} r_x + u_{\theta\theta}\theta_x)\theta_x + u_\theta \theta_{xx}$$

$$\therefore u_{xx} = (u_{rr}\frac{x}{r} + u_{r\theta}(-\frac{y}{r^2}))\frac{x}{r} + u_r \frac{y^2}{r^3}$$
$$+ (u_{\theta r}\frac{x}{r} + u_{\theta\theta}(-\frac{y}{r^2}))(-\frac{y}{r^2}) + u_\theta \frac{2xy}{r^4}$$

$$= u_{rr}\frac{x^2}{r^2} - u_{r\theta}\frac{xy}{r^3} + u_r \frac{y^2}{r^3} - u_{\theta r}\frac{xy}{r^3} + u_{\theta\theta}\frac{y^2}{r^4} + u_\theta \frac{2xy}{r^4}$$

$$= \frac{x^2}{r^2}u_{rr} - 2\frac{xy}{r^3}u_{r\theta} + \frac{y^2}{r^4}u_{\theta\theta} + \frac{y^2}{r^3}u_r + 2\frac{xy}{r^4}u_\theta$$

$$r_x = \frac{x}{r} \qquad \theta_x = -\frac{y}{r^2}$$

$$r_{xx} = \frac{y^2}{r^3} \qquad \theta_{xx} = \frac{2xy}{r^4}$$

In the similar way for y

$$u_{yy} = \frac{y^2}{r^2}u_{rr} + 2\frac{xy}{r^3}u_{r\theta} + \frac{x^2}{r^4}u_{\theta\theta} + \frac{x^2}{r^3}u_r - 2\frac{xy}{r^4}u_\theta$$

$$r_y = \frac{y}{r} \qquad \theta_y = \frac{x}{r^2}$$

$$r_{yy} = \frac{x^2}{r^3} \qquad \theta_{yy} = -\frac{2xy}{r^4}$$

$$\therefore \nabla^2 u = u_{xx} + u_{yy}$$

$$= \frac{x^2+y^2}{r^2} u_{rr} + \frac{x^2+y^2}{r^4} u_{\theta\theta} + \frac{x^2+y^2}{r^3} u_r = \frac{r^2}{r^2} u_{rr} + \frac{r^2}{r^4} u_{\theta\theta} + \frac{r^2}{r^3} u_r$$

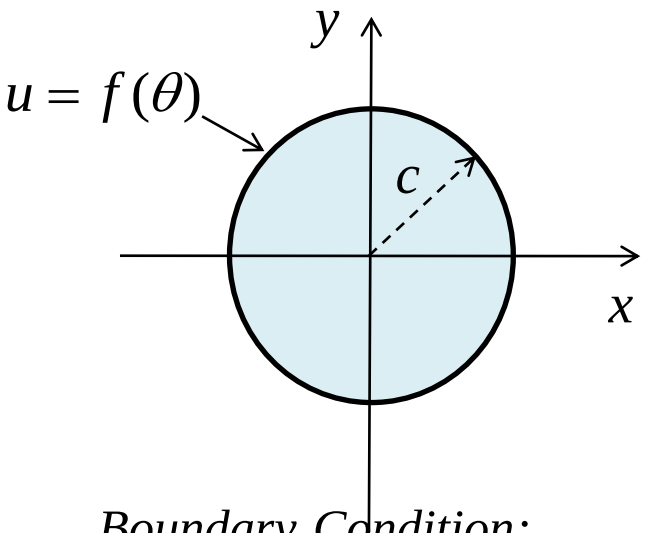
$$= u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$$

Laplace equation in Polar Coordinates

$$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

☑ Example

Solve Laplace's equation for the steady-state temperature u



$$Boundary\ Condition:$$

$$u(c,\theta) = f(\theta),\ 0 < \theta < 2\pi$$

Assume that two faces of the plate are insulated

- **Temperature should be continuous and therefore bounded inside the circle r=c**
- $u(r,\theta) = u(r,\theta + 2\pi)$

Separation Variables

➔ **If we assume** u **as** $u = R(r)\Theta(\theta)$

then $\Theta(\theta)$ **needs to be 2π-periodic**

$$\nabla^2 u = 0$$

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

Steady Temperatures in a Circular Plate

$$u = R(r)\Theta(\theta)$$

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

Divide by Θ and rewrite

$$R'' + \frac{1}{r}R' = -\frac{1}{r^2}R\frac{\Theta''}{\Theta}$$

Divide by $\frac{R}{r^2}$

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta}$$

Separating constant

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

$$r^2R'' + rR' - \lambda R = 0 \qquad \Theta'' + \lambda\Theta = 0 \ , \Theta(\theta + 2\pi) = \Theta(\theta)$$

Two ODEs

$$u = R(r)\Theta(\theta)$$

$$r^2R'' + rR' - \lambda R = 0$$

$$\Theta'' + \lambda\Theta = 0$$

Three possible solution of equation

$\Theta'' + \lambda\Theta = 0$ are

$\lambda = 0 \Rightarrow \Theta(\theta) = c_1 + c_2\theta$

$\lambda = -\alpha^2 < 0 \Rightarrow \Theta(\theta) = c_3\cosh\alpha\theta + c_4\sinh\alpha\theta$

$\lambda = \alpha^2 > 0 \Rightarrow \Theta(\theta) = c_5\cos\alpha\theta + c_6\sin\alpha\theta$

$1)\lambda = 0$

$\Theta(\theta)$ needs to be 2π-periodic

$$\Theta(\theta) = c_1 + c_2\theta$$

c_2 Should be zero to be periodic

$$\therefore \Theta(\theta) = c_1 \quad , c_1 \neq 0$$

can be assigned any period

$\lambda = 0$: eigenvalue

$2)\lambda = -\alpha^2 < 0$

$$\Theta(\theta) = c_3\cosh\alpha\theta + c_4\sinh\alpha\theta$$

$c_3 = c_4 = 0$:Trivial Solution

: Θ: not periodi

$unless \quad c_3 = c_4 = 0$

We dismiss the case : $\lambda = -\alpha^2 < 0$

$3)\lambda = \alpha^2 > 0$

$$\Theta(\theta) = c_5\cos\alpha\theta + c_6\sin\alpha\theta$$

When $\alpha = n$, Θ : 2π-periodic

$$\therefore \Theta(\theta) = c_5\cos n\theta + c_6\sin n\theta$$

$$, \lambda_n = \alpha_n^2 = n^2, n = 1, 2, 3\ldots$$: eigenvalue

$u = R(r)\Theta(\theta)$ $\quad r^2R'' + rR' - \lambda R = 0$, $\Theta'' + \lambda\Theta = 0$

☑

$$\Theta'' + \lambda\Theta = 0$$

$$\begin{cases} \Theta(\theta) = c_1 & , \lambda = 0 \\ \Theta(\theta) = c_5\cos n\theta + c_6 \sin n\theta & , \lambda_n = n^2, n = 1,2,3\ldots \end{cases}$$

when $\lambda_n = n^2, n = 0,1,2,3\ldots$

$$r^2R'' + rR' - \lambda R = 0$$

$$r^2R'' + rR' - n^2R = 0$$:**Cauchy –Euler Equation**

when $n = 0$ **solution of the equation is** $R(r) = c_7 + c_8 \ln r$

when $n = 1,2,3..$ **solution of the equation is** $R(r) = c_9 r^n + c_{10} r^{-n}$

As temperature should be bounded Inside the circle r=c

For r=0, $c_8 = 0, c_{10} = 0$

$$\therefore R(r) = c_7 \quad , \lambda = 0$$

$$R(r) = c_9 r^n \ , \lambda_n = n^2, n = 1,2,3\ldots$$

$$\therefore u = R(r)\Theta(\theta)$$

$$for\ n = 0 : u_o = A_0,\ (A_0 = c_1c_7)$$

$$for\ \ n = 1,2,\ldots$$

$$u_n = r^n(A_n \cos n\theta + B_n \sin n\theta),$$
$$, A_n = c_5c_9,\ B_n = c_6c_9$$

$$\therefore u(r,\theta) = R(r)\Theta(\theta) = A_0 + \sum_{n=1}^{\infty} r^n(A_n \cos n\theta + B_n \sin n\theta)$$

☑

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

$$\textit{Boundary Condition}: \; u(c,\theta) = f(\theta),\; 0 < \theta < 2\pi$$

$$f(\theta) = A_0 + \sum_{n=1}^{\infty} c^n (A_n \cos n\theta + B_n \sin n\theta)$$

Fourier Series , $p=\pi$, $0<\theta<2\pi$

$$\therefore A_0 = \frac{a_0}{2} = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta$$

$$c_n A_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta \quad \Rightarrow \therefore A_n = \frac{1}{c^n \pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta$$

$$c^n B_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta \quad \Rightarrow \therefore B_n = \frac{1}{c^n \pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta$$

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right), (-p, p)$$

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}dx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}dx$$

8-8. Bessel's Equation

Radial vibrations of Circular Membrane (Polar Coordinates)

Steady Temperatures in a Circular Cylinder (Cylindrical coordinates)

Problems in Polar Coordinates and Cylindrical Coordinates : Radial Symmetry

☑

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

	In rectangular coordinate	In polar coordinate
2-D heat equation	$k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial u}{\partial t}$	$k\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)=\frac{\partial u}{\partial t}$
2-D wave equation	$a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial^2 u}{\partial t^2}$	$a^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)=\frac{\partial^2 u}{\partial t^2}$

To solve equations by separation of variables we must define

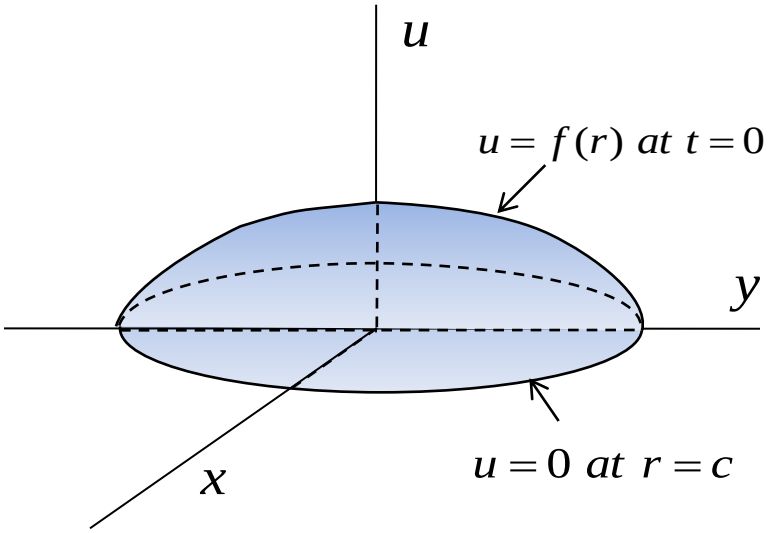
$$u = R(r)\Theta(\theta)T(t)$$

When u is independent of the angular coordinate θ (radial symmetry), $\frac{\partial u}{\partial \theta}=0$

radial vibration

$$k\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)=\frac{\partial u}{\partial t} \qquad a^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)=\frac{\partial^2 u}{\partial t^2}$$

☑ Radial vibrations of Circular Membrane



Free undamped radial vibrations of a thin circular membrane (vibrating drumhead)

Assume that

-the displacements are small
-the motion is in a direction perpendicular to xy-plane (transverse vibrations)

$$a^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)=\frac{\partial^2 u}{\partial t^2},\ 0<r<c,\ t>0$$

$$BC: u(c,t)=0,\quad t>0$$

$$IC: u(r,0)=f(r),\ \left.\frac{\partial u}{\partial t}\right|_{t=0}=g(r),\ 0<r<c$$

$$a^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)=\frac{\partial^2 u}{\partial t^2}$$

Substituting

$$u(r,t)=R(r)T(t)$$

Separation of variables ⇨

$$\frac{R''+\frac{1}{r}R}{R}=\frac{T''}{a^2T}=-\lambda$$

$$rR''+R'+\lambda rR=0 \quad \cdots(1)$$

$$T''+a^2\lambda T=0 \quad \cdots\cdots(2)$$

Equation (1) is the parametric Bessel differential equation of order $\nu=0$

$$rR''+R'+\alpha^2 rR=0 \quad (\lambda=\alpha^2,\alpha>0)$$

General solution of equation (1) is

$$R(r)=c_1J_0(\alpha r)+c_2Y_0(\alpha r)$$

$$T''+(a\alpha)^2T=0 \quad \cdots\cdots(2)$$

General solution of equation (2) is

$$T(t)=c_3\cos a\alpha t+c_4\sin a\alpha t$$

☑ **DEs solvable in terms of Bessel Functions**

$$x^2y''+xy'+(\alpha^2x^2-\nu^2)y=0$$

To transform into the Bessel equation, let $t=\alpha x$

$$t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}+(t^2-\nu^2)y=0$$

"Parametric Bessel equation of order ν**"**

$$x^2y''+xy'+(\alpha^2x^2-\nu^2)y=0$$

The general solution : $y=c_1J_\nu(\alpha x)+c_2Y_\nu(\alpha x)(\alpha>0)$

☑

$$u(r,t) = R(r)T(t)$$

$$R(r) = c_1 J_0(\alpha r) + c_2 Y_0(\alpha r)$$

$$T(t) = c_3 \cos a\alpha t + c_4 \sin a\alpha t$$

$u(r,t)$ **Should be bounded at r=0,**

And $Y_0(\alpha r) \to -\infty$ as $r \to 0^+$

so c_2 **must be zero** $R(r) = c_1 J_0(\alpha r)$

From boundary condition, $u(c,t) = R(c)T(t) = 0$

$T(t) \neq 0$ **otherwise u(r,t) : trivial solution, So**

$$R(c) = c_1 J_0(\alpha c) = 0$$

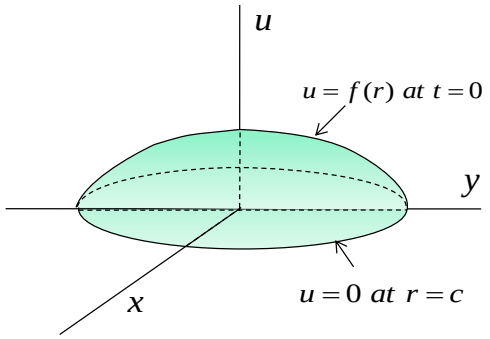
$c_1 \neq 0$ **otherwise R(r) : trivial solution, So**

$$J_0(\alpha c) = 0$$

If $x_n = \alpha_n c$ **are positive roots** $\alpha_n = \dfrac{x_n}{c}$

$$\lambda_n{}^2 = \alpha_n{}^2 = \frac{x_n{}^2}{c^2}$$

and eigenfunctions are $c_1 J_0(\alpha_n r),\ n = 1, 2, 3\ldots$



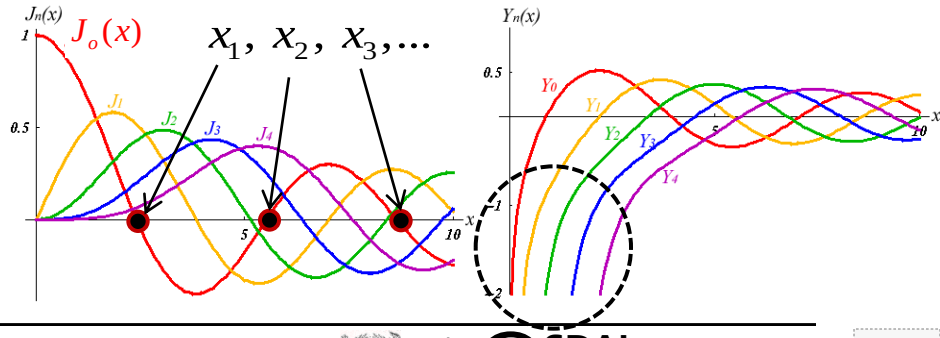
$$a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = \frac{\partial^2 u}{\partial t^2},\ 0 < r < c,\ t > 0$$

$$u(c,t) = 0,\quad t > 0$$

$$u(r,0) = f(r),\ \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(r),\ 0 < r < c$$

Bessel functions

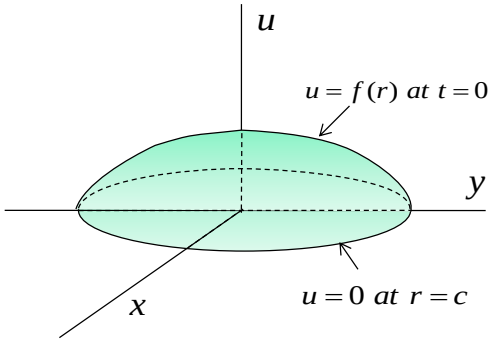
$J_n(x)$ $Y_n(x)$



$$u(r,t) = R(r)T(t)$$
$$R(r) = c_1 J_0(\alpha r)$$
$$T(t) = c_3 \cos a\alpha t + c_4 \sin a\alpha t$$

$$\begin{aligned} u_n(r,t) &= R(r)T(t) \\ &= c_1 J_0(\alpha_n r)\left(c_3 \cos a\alpha_n t + c_4 \sin a\alpha_n t\right) \\ &= \left(A_n \cos a\alpha_n t + B_n \sin a\alpha_n t\right) J_0(\alpha_n r) \end{aligned}$$

$$\therefore u(r,t) = \sum_{n=1}^{\infty}\left(A_n \cos a\alpha_n t + B_n \sin a\alpha_n t\right) J_0(\alpha_n r)$$



$$a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = \frac{\partial^2 u}{\partial t^2},\ 0 < r < c,\ t > 0$$

$$u(c,t) = 0,\quad t > 0$$

$$u(r,0) = f(r),\ \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(r),\ 0 < r < c$$

$$u(r,t) = R(r)T(t)$$

$$u(r,t) = \sum_{n=1}^{\infty} \left(A_n \cos a\alpha_n t + B_n \sin a\alpha_n t \right) J_0(\alpha_n r)$$

Given initial condition determine the A_n, B_n

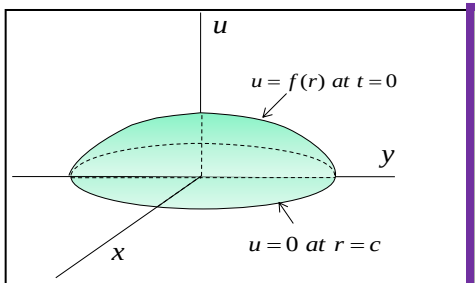
at $t = 0$, $\quad u(r,0) = f(r)$ **gives**

$$f(r) = \sum_{n=1}^{\infty} A_n J_0(\alpha_n r)$$

Fourier-Bessel Series

This is the case that α_n are defined by $J_0(\alpha_n c) = 0$

$$\therefore A_n = \frac{2}{c^2 J_1^2(\alpha_n c)} \int_0^c r J_0(\alpha_n r) f(r) dr$$



$$a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = \frac{\partial^2 u}{\partial t^2},\ 0 < r < c,\ t > 0$$

$$u(c,t) = 0,\quad t > 0$$

$$u(r,0) = f(r),\ \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(r),\ 0 < r < c$$

$R(r) = c_1 J_0(\alpha r)$

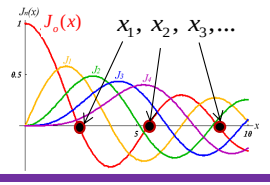
From boundary condition, $c_1 J_0(\alpha c) = 0$

$c_1 \neq 0$ otherwise R(r) : trivial solution, So $J_0(\alpha c) = 0$

If $x_n = \alpha_n c$ are positive roots $\alpha_n = \frac{x_n}{c}$

$$\lambda_n^2 = \alpha_n^2 = \frac{x_n^2}{c^2}$$

and eigenfunctions are $c_1 J_0(\alpha_n r)$



Fourier-Bessel Series

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x)$$

Definition 12.8

Fourier-Bessel Series

The Fourier-Bessel series of a function f defined on the interval $(0, b)$ is given by

* 1) α_i are defined by $J_n(\alpha_i b) = 0$

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \qquad , c_i = \frac{2}{b^2 J_{n+1}^2(\alpha_i b)} \int_0^b x J_n(\alpha_i x) f(x) dx$$

2) α_i are defined by $hJ_n(\alpha b) + \alpha b J_n'(\alpha b) = 0$

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \quad , c_i = \frac{2\alpha_i^2}{(\alpha_i^2 b^2 - n^2 + h^2) J_n^2(\alpha_i b)} \int_0^b x J_n(\alpha_i x) f(x) dx$$

3) α_i are defined by $J_n'(\alpha b) = 0$

$$f(x) = c_1 + \sum_{i=2}^{\infty} c_i J_n(\alpha_i x) \qquad , c_1 = \frac{2}{b^2} \int_0^b x f(x) dx$$

$$, c_i = \frac{2}{b^2 J_0^2(\alpha_i b)} \int_0^b x J_n(\alpha_i x) f(x) dx$$

$$u(r,t) = R(r)T(t)$$

$$u(r,t) = \sum_{n=1}^{\infty} \left(A_n \cos a\alpha_n t + B_n \sin a\alpha_n t \right) J_0(\alpha_n r)$$

Given initial condition determine the A_n, B_n

at $t = 0$, $\quad u_t(r,0) = g(x)$ **gives**

$$u_t(r,t) = \left[\sum_{n=1}^{\infty} a\alpha_n \left(-A_n \sin a\alpha_n t + B_n \cos a\alpha_n t \right) J_0(\alpha r) \right]_{t=0} = g(r)$$

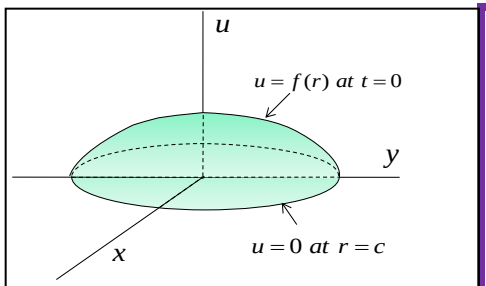
$$g(r) = \sum_{n=1}^{\infty} a\alpha_n B_n J_0(\alpha_n r)$$

Fourier-Bessel Series

This is the case that α_n are defined by $J_0(\alpha_n c) = 0$

$$a\alpha_n B_n = \frac{2}{c^2 J_1^2(\alpha_n c)} \int_0^c r J_0(\alpha_n r) g(r) dr$$

$$\therefore B_n = \frac{2}{a\alpha_n c^2 J_1^2(\alpha_n c)} \int_0^c r J_0(\alpha_n r) g(r) dr$$



$$a^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right) = \frac{\partial^2 u}{\partial t^2}, \quad 0 < r < c, \quad t > 0$$

$$u(c,t) = 0, \quad t > 0$$

$$u(r,0) = f(r), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(r), \quad 0 < r < c$$

$$R(r) = c_1 J_0(\alpha r)$$

From boundary condition, $c_1 J_0(\alpha c) = 0$

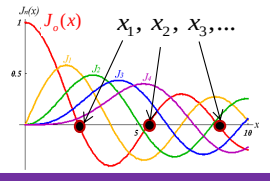
$c_1 \neq 0$ otherwise R(r) : trivial solution, So

$$J_0(\alpha c) = 0$$

If $x_n = \alpha_n c$ are positive roots $\alpha_n = \frac{x_n}{c}$

$$\lambda_n^2 = \alpha_n^2 = \frac{x_n^2}{c^2}$$

and eigenfunctions are $c_1 J_0(\alpha_n r)$



Fourier-Bessel Series

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x)$$

Definition 12.8

Fourier-Bessel Series

The Fourier-Bessel series of a function f defined on the interval $(0, b)$ is given by

1) α_i are defined by $J_n(\alpha_i b) = 0$

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \qquad , c_i = \frac{2}{b^2 J_{n+1}^2(\alpha_i b)} \int_0^b x J_n(\alpha_i x) f(x) dx$$

2) α_i are defined by $hJ_n(\alpha b) + \alpha b J_n'(\alpha b) = 0$

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \quad , c_i = \frac{2\alpha_i^2}{(\alpha_i^2 b^2 - n^2 + h^2) J_n^2(\alpha_i b)} \int_0^b x J_n(\alpha_i x) f(x) dx$$

3) α_i are defined by $J_n'(\alpha b) = 0$

$$f(x) = c_1 + \sum_{i=2}^{\infty} c_i J_n(\alpha_i x) \qquad , c_1 = \frac{2}{b^2} \int_0^b x f(x) dx$$

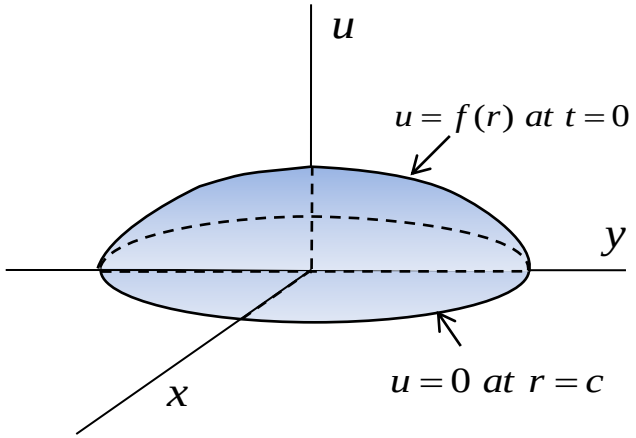
$$, c_i = \frac{2}{b^2 J_0^2(\alpha_i b)} \int_0^b x J_n(\alpha_i x) f(x) dx$$

Solution

$$u(r,t)=\sum_{n=1}^{\infty}\left(A_n\cos a\alpha_n t+B_n\sin a\alpha_n t\right)J_0(\alpha_n r)$$

$$,A_n=\frac{2}{c^2J_1{}^2(\alpha_n c)}\int_0^c rJ_0(\alpha_n r)\,f(r)dr$$

$$,B_n=\frac{2}{a\alpha_n c^2J_1{}^2(\alpha_n c)}\int_0^c rJ_0(\alpha_n r)g(r)dr$$



$$a^2\left(\frac{\partial^2u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)=\frac{\partial^2u}{\partial t^2},\ 0<r<c,\ t>0$$

$$u(c,t)=0,\quad t>0$$

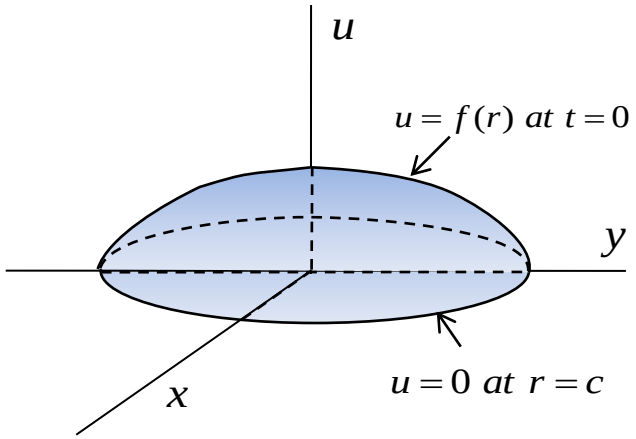
$$u(r,0)=f(r),\ \left.\frac{\partial u}{\partial t}\right|_{t=0}=g(r),\ 0<r<c$$

Solution

$$u(r,t)=\sum_{n=1}^{\infty}\left(A_n\cos a\alpha_n t+B_n\sin a\alpha_n t\right)J_0(\alpha_n r)$$

$$,A_n=\frac{2}{c^2J_1{}^2(\alpha_n c)}\int_0^c rJ_0(\alpha_n r)\,f(r)dr$$

$$,B_n=\frac{2}{a\alpha_n c^2J_1{}^2(\alpha_n c)}\int_0^c rJ_0(\alpha_n r)g(r)dr$$

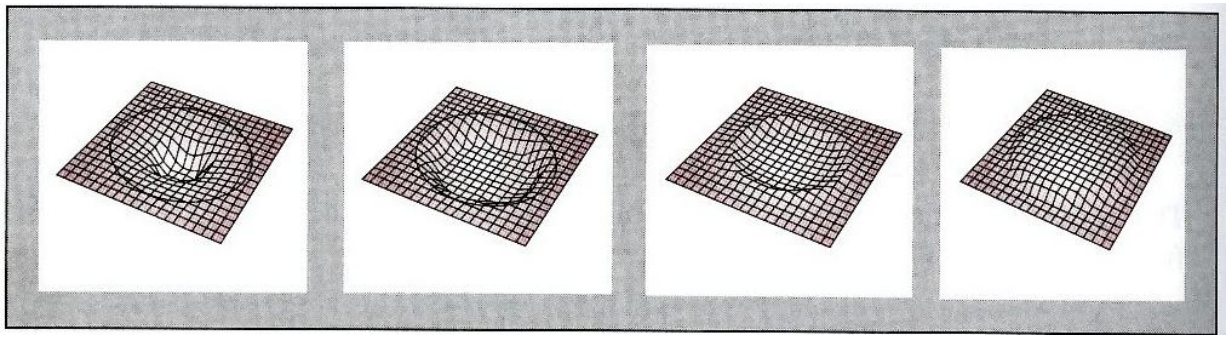


$$a^2\left(\frac{\partial^2u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)=\frac{\partial^2u}{\partial t^2},\ 0<r<c,\ t>0$$

$$u(c,t)=0,\quad t>0$$

$$u(r,0)=f(r),\ \left.\frac{\partial u}{\partial t}\right|_{t=0}=g(r),\ 0<r<c$$

Vibrating Drumhead images*



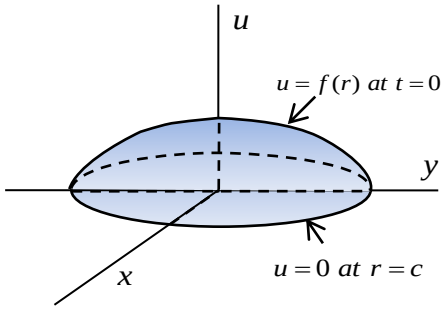
*Zill D.G, Gullen M.R., Advanced Engineering Mathematics, The Third Edition, Jones and Bartlett, 2006, p736 Figure 14.9

Solution

$$u(r,t)=\sum_{n=1}^{\infty}\left(A_n\cos a\alpha_n t+B_n\sin a\alpha_n t\right)J_0(\alpha_n r)$$

$$,A_n=\frac{2}{c^2 J_1{}^2(\alpha_n c)}\int_0^c rJ_0(\alpha_n r)\,f(r)dr$$

$$,B_n=\frac{2}{a\alpha_n c^2 J_1{}^2(\alpha_n c)}\int_0^c rJ_0(\alpha_n r)g(r)dr$$



$$a^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)=\frac{\partial^2 u}{\partial t^2},\ 0<r<c,\ t>0$$

$$u(c,t)=0,\quad t>0$$

$$u(r,0)=f(r),\ \left.\frac{\partial u}{\partial t}\right|_{t=0}=g(r),\ 0<r<c$$

Standing waves

$$u_n(r,t)=\left(A_n\cos a\alpha_n t+B_n\sin a\alpha_n t\right)J_0(\alpha_n r)$$

For n=1,2,3,…Standing waves are graph of $J_0(\alpha_n r)$ with

time varying amplitude $A_n\cos a\alpha_n t+B_n\sin a\alpha_n t$

Nodal Lines : set of points on a standing wave where there is no motion

Solution

$$u(r,t)=\sum_{n=1}^{\infty}\left(A_n\cos a\alpha_n t+B_n\sin a\alpha_n t\right)J_0(\alpha_n r)$$

$$,A_n=\frac{2}{c^2J_1^{\,2}(\alpha_n c)}\int_0^c rJ_0(\alpha_n r)f(r)dr$$

$$,B_n=\frac{2}{a\alpha_n c^2J_1^{\,2}(\alpha_n c)}\int_0^c rJ_0(\alpha_n r)g(r)dr$$

Standing waves

$$u_n(r,t)=\left(A_n\cos a\alpha_n t+B_n\sin a\alpha_n t\right)J_0(\alpha_n r)$$

For n=1,2,3,...Standing waves are graph of $J_0(\alpha_n r)$ with time varying amplitude $A_n\cos a\alpha_n t+B_n\sin a\alpha_n t$

Zeros of each standing wave are the roots of

$$J_0(\alpha_n r)=J_0\left(\frac{x_n}{c}r\right)=0$$

For n=1

$$J_0\left(\frac{x_1}{c}r\right)=0 \qquad \frac{2.4}{c}r=2.4\ \ (r=c)$$

$$\Rightarrow r=c$$

For n=2

$$J_0\left(\frac{x_2}{c}r\right)=0 \qquad \frac{5.5}{c}r=2.4,\ and\ \frac{5.5}{c}r=5.5$$

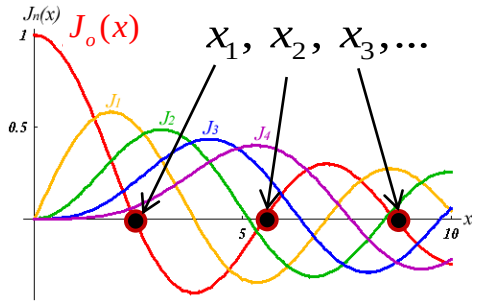
$$\Rightarrow r\doteq 0.44c, c$$

For n=3

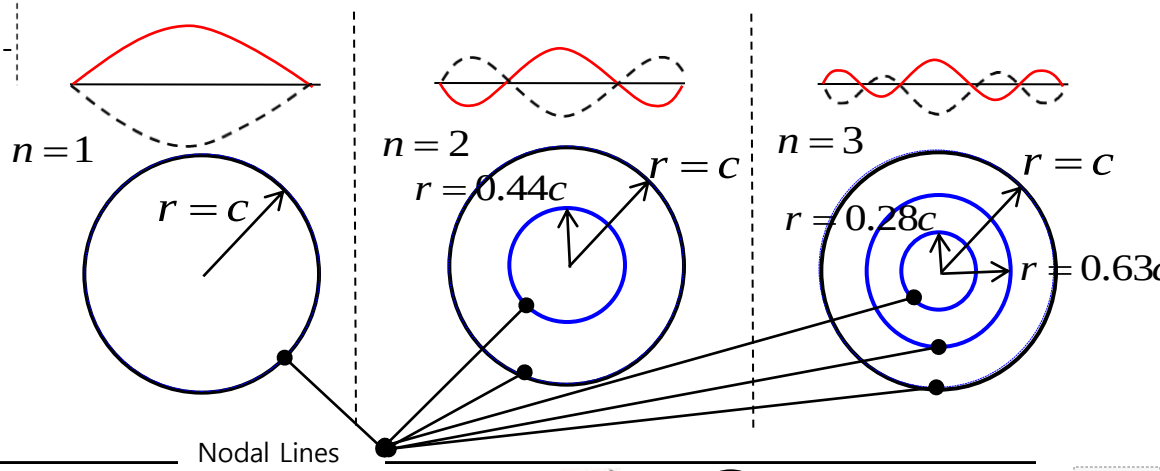
$$J_0\left(\frac{x_3}{c}r\right)=0 \qquad \frac{8.7}{c}r=2.4,\ \frac{8.7}{c}r=5.5\ and\ \frac{8.7}{c}r=8.7$$

$$\Rightarrow r\doteq 0.28c, 0.63c, c$$

Bessel functions $J_n(x)$



$$x_1\approx 2.4$$
$$x_2\approx 5.5$$
$$x_3\approx 8.7$$



Nodal Lines : set of points on a standing wave where there is no motion

Solution

$$u(r,t)=\sum_{n=1}^{\infty}\left(A_n\cos a\alpha_n t+B_n\sin a\alpha_n t\right)J_0(\alpha_n r)$$

$$,A_n=\frac{2}{c^2J_1{}^2(\alpha_n c)}\int_0^c rJ_0(\alpha_n r)f(r)dr$$

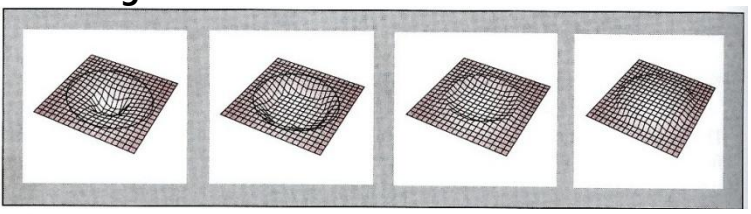
$$,B_n=\frac{2}{a\alpha_n c^2J_1{}^2(\alpha_n c)}\int_0^c rJ_0(\alpha_n r)g(r)dr$$

Standing waves

$$u_n(r,t)=\left(A_n\cos a\alpha_n t+B_n\sin a\alpha_n t\right)J_0(\alpha_n r)$$

For n=1,2,3,...Standing waves are graph of $J_0(\alpha_n r)$ **with time varying amplitude** $A_n\cos a\alpha_n t+B_n\sin a\alpha_n t$

Vibrating Drumhead



Superposition of standing waves for n=1,2,3,...

Zeros of each standing wave are the roots of

$$J_0(\alpha_n r)=J_0\left(\frac{x_n}{c}r\right)=0$$

For n=1

$$J_0\left(\frac{x_1}{c}r\right)=0 \qquad \frac{2.4}{c}r=2.4\ \ (r=c)$$

$$\Rightarrow r=c$$

For n=2

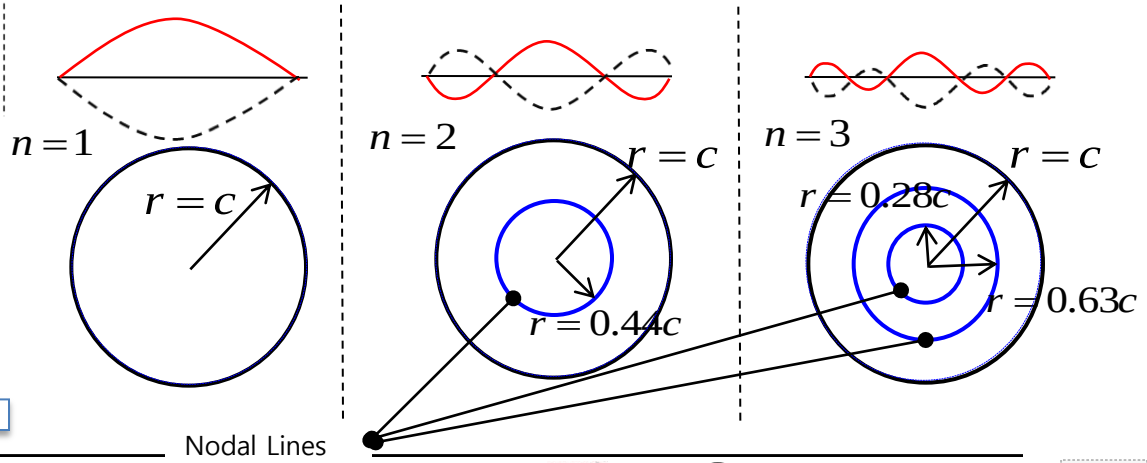
$$J_0\left(\frac{x_2}{c}r\right)=0 \qquad \frac{5.5}{c}r=2.4,\ and\ \frac{5.5}{c}r=5.5$$

$$\Rightarrow r\doteq 0.44c, c$$

For n=3

$$J_0\left(\frac{x_3}{c}r\right)=0 \qquad \frac{8.7}{c}r=2.4,\ \frac{8.7}{c}r=5.5\ and\ \frac{8.7}{c}r=8.7$$

$$\Rightarrow r\doteq 0.28c, 0.63c, c$$



Nodal Lines

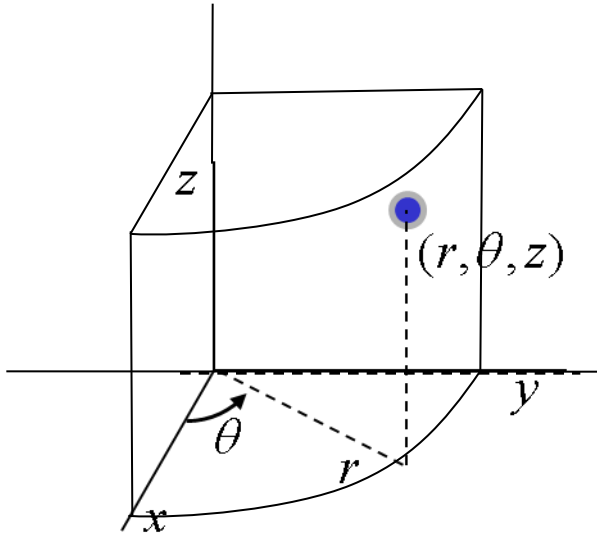
Laplacian in Cylindrical Coordinates: Bessel Equation

☑ Laplacian in Cylindrical Coordinates

$$\nabla^2 u(r,\theta) = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

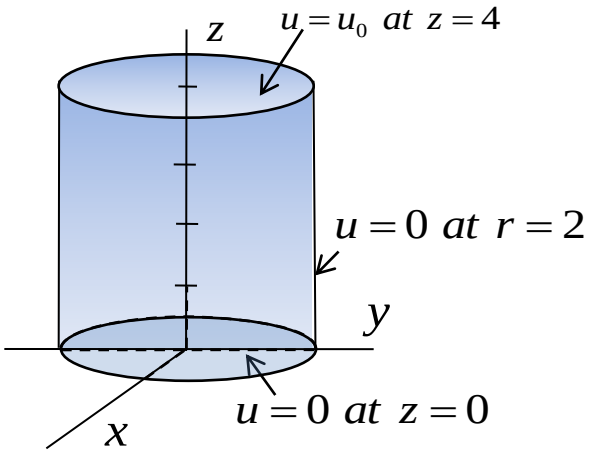


Laplacian in cylindrical coordinates is

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

Problems in Polar Coordinates and Cylindrical Coordinates : Bessel Equation

☑ Steady Temperatures in a Circular Cylinder



Boundary conditions suggest that the temperature u has radial symmetry

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

Accordingly, u(r,z) is determined from

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0,$$

$$u(2,z) = 0, \quad 0 < z < 4$$

$$u(r,0) = 0,\ u(r,4) = u_0, \quad 0 < r < 2$$

Using $u(r,z)=R(r)Z(z)$ **and product method gives**

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0 \Rightarrow R''Z+\frac{1}{r}R'Z+RZ''=0$$

⇩ **dividing by RZ**

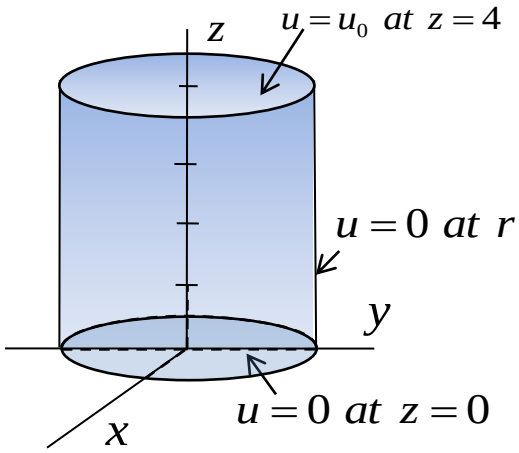
$$\frac{R''+\frac{1}{r}R'}{R}=-\frac{Z''}{Z} \Leftarrow R''Z+\frac{1}{r}R'Z=-RZ''$$

separating variables

$$\frac{R''+\frac{1}{r}R'}{R}=-\frac{Z''}{Z}=-\lambda$$

$$rR''+R'+\lambda rR=0 \quad \cdots(1)$$

$$Z''-\lambda Z=0 \quad \cdots(2)$$



$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0,$$

$$u(2,z)=0,\ 0<z<4$$

$$u(r,0)=0,\ u(r,4)=u_0,\ 0<r<2$$

$$rR'' + R' + \lambda rR = 0 \quad \cdots(1)$$

Equation (1) is the parametric Bessel differential equation of order 0

$$\therefore R(r) = c_1 J_0(\alpha r) + c_2 Y_0(\alpha r)$$

$$(\lambda = \alpha^2, \alpha > 0)$$

$$Z'' - \lambda Z = 0 \quad \cdots(2)$$

when $\lambda = -\alpha^2,\ \alpha > 0 \Rightarrow Z'' - \alpha^2 Z = 0$

Let: $Z = e^{mz} \longrightarrow (m^2 - \alpha^2)e^{mz} = 0$

$$\longrightarrow m_1 = \alpha,\ m_2 = -\alpha$$

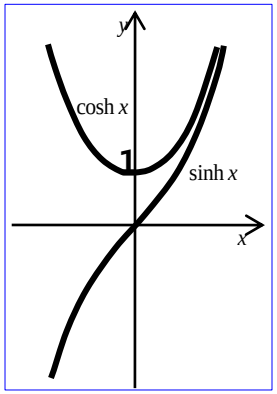
Linear combination of $e^{\alpha z}$ and $e^{-\alpha z}$

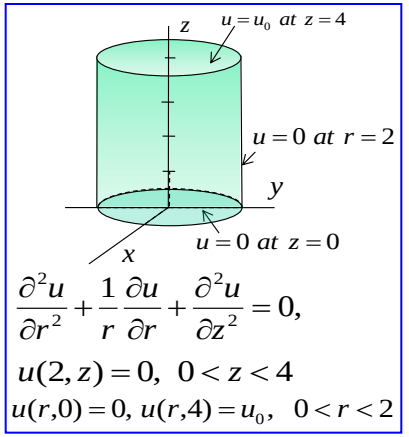
(basis transformation)

$$\frac{e^{\alpha z} + e^{-\alpha z}}{2} = \cosh \alpha z, \quad \frac{e^{\alpha z} - e^{-\alpha z}}{2} = \sinh \alpha z$$

General solution of equation (2) is

$$\therefore Z(z) = c_3 \cosh az + c_4 \sinh az$$





$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0,$$

$$u(2, z) = 0, \quad 0 < z < 4$$

$$u(r, 0) = 0,\ u(r, 4) = u_0, \quad 0 < r < 2$$

$$u(r, z) = R(r)Z(z)$$

$$rR'' + R' + \lambda rR = 0 \quad \cdots(1)$$

$$Z'' - \lambda Z = 0 \quad \cdots(2)$$

* ☑ **DEs solvable in terms of Bessel Functions**

$$x^2 y'' + xy' + (\alpha^2 x^2 - \nu^2)y = 0$$

To transform into the Bessel equation, let $t = \alpha x \quad (\alpha > 0)$

$$t^2 \frac{d^2 y}{dt^2} + t\frac{dy}{dt} + (t^2 - \nu^2)y = 0$$

"Parametric Bessel equation of order ν"

$$x^2 y'' + xy' + (\alpha^2 x^2 - \nu^2)y = 0$$

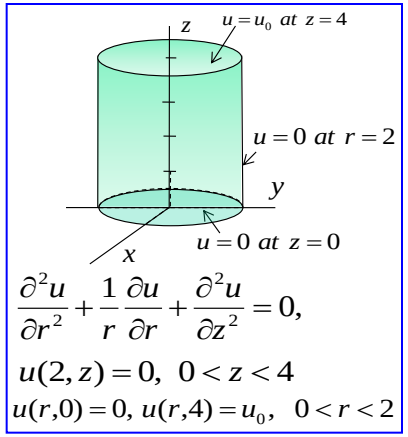
The general solution : $y = c_1 J_\nu(\alpha x) + c_2 Y_\nu(\alpha x)$

$$R(r) = c_1 J_0(\alpha r) + c_2 Y_0(\alpha r)$$
$$Z(z) = c_3 \cosh az + c_4 \sinh az \qquad (\lambda = \alpha^2, \alpha > 0)$$

$u(r,t)$ **Should be bounded at r=0,**

And $Y_0(\alpha r) \to -\infty$ as $r \to 0^+$ **so** c_2 **must be 0**

$$\therefore R(r) = c_1 J_0(\alpha r)$$

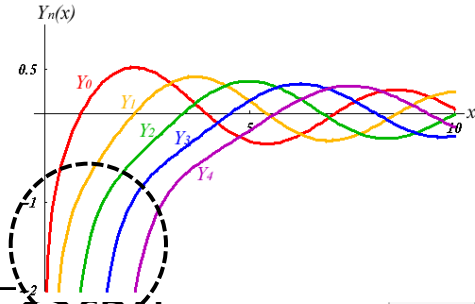


$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0,$$
$$u(2,z) = 0, \quad 0 < z < 4$$
$$u(r,0) = 0, \; u(r,4) = u_0, \quad 0 < r < 2$$

$$u(r,z) = R(r)Z(z)$$
$$rR'' + R' + \lambda rR = 0 \quad \cdots(1)$$
$$Z'' - \lambda Z = 0 \quad \cdots(2)$$

Bessel functions

$$Y_n(x)$$



$$R(r) = c_1 J_0(\alpha r) + c_2 Y_0(\alpha r)$$
$$Z(z) = c_3 \cosh az + c_4 \sinh az \qquad (\lambda = \alpha^2, \alpha > 0)$$

$$R(r) = c_1 J_0(\alpha r)$$

From **boundary condition**, $u(2,z) = R(2)Z(z) = 0$

$Z(z) \neq 0$ **otherwise u(r,z) : trivial solution, So**

$$R(2) = c_1 J_0(2\alpha) = 0$$

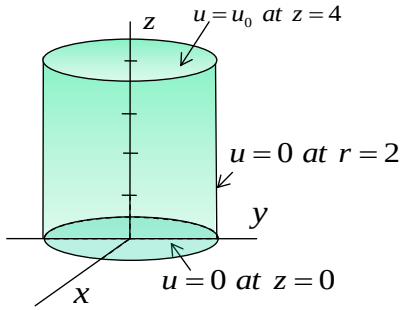
$c_1 \neq 0$ **otherwise R(r) : trivial solution, So**

$$J_0(2\alpha) = 0$$

If $x_n = 2\alpha_n$ **are positive roots** $\alpha_n = \dfrac{x_n}{2}$

$$\lambda_n{}^2 = \alpha_n{}^2 = \frac{x_n{}^2}{2^2}$$

and eigenfunctions are $c_1 J_0(\alpha_n r), \quad n = 1, 2, 3\ldots$



$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0,$$

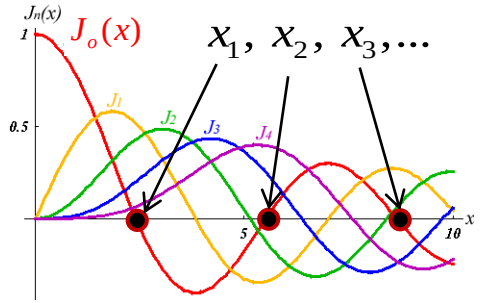
$u(2,z) = 0, \quad 0 < z < 4$

$u(r,0) = 0, \; u(r,4) = u_0, \quad 0 < r < 2$

$$u(r,z) = R(r)Z(z)$$
$$rR'' + R' + \lambda rR = 0 \quad \cdots(1)$$
$$Z'' - \lambda Z = 0 \quad \cdots(2)$$

Bessel functions

$J_n(x)$



$$R(r) = c_1 J_0(\alpha r) + c_2 Y_0(\alpha r)$$
$$Z(z) = c_3 \cosh az + c_4 \sinh az \qquad (\lambda = \alpha^2, \alpha > 0)$$

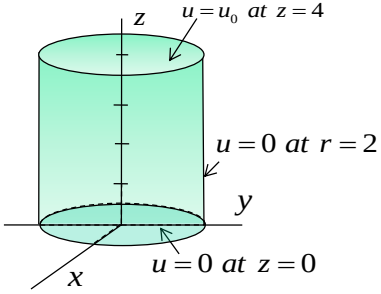
$$R(r) = c_1 J_0(\alpha r) \qquad \lambda_n{}^2 = \alpha_n{}^2 = \frac{x_n{}^2}{2^2}, n = 1, 2, 3, ..$$

From boundary condition, $u(r,0) = R(r)Z(0) = 0$

$R(r) \neq 0$ **otherwise u(r,z) : trivial solution, So**

$$Z(0) = c_3 = 0$$

$$\therefore Z(z) = c_4 \sinh az$$



$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0,$$

$$u(2,z) = 0, \quad 0 < z < 4$$

$$u(r,0) = 0, \; u(r,4) = u_0, \quad 0 < r < 2$$

$$u(r,z) = R(r)Z(z)$$

$$rR'' + R' + \lambda rR = 0 \quad \cdots(1)$$

$$Z'' - \lambda Z = 0 \quad \cdots(2)$$

$$u(r,z) = R(r)Z(z)$$

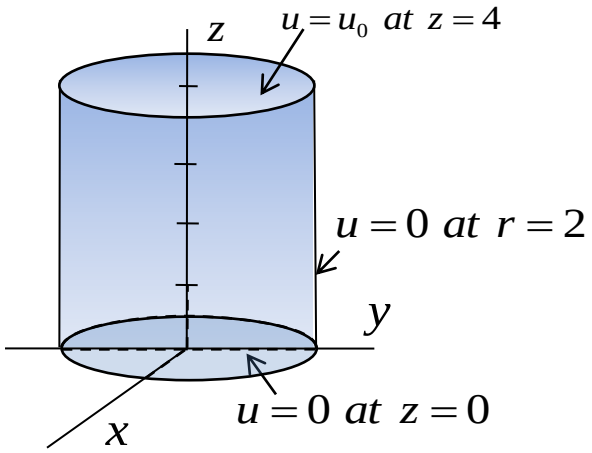
$$R(r) = c_1 J_0(\alpha r)$$

$$Z(z) = c_4 \sinh az$$

$$\therefore u_n = R(r)Z(z) = A_n \sinh \alpha_n z \, J_0(\alpha_n r)$$

$$\lambda_n{}^2 = \alpha_n{}^2 = \frac{x_n{}^2}{2^2}, n = 1, 2, 3, ..$$

$$u(r,z) = \sum_{n=1}^{\infty} A_n \sinh \alpha_n z \, J_0(\alpha_n r)$$



$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0,$$

$$u(2,z) = 0, \quad 0 < z < 4$$

$$u(r,0) = 0, \; u(r,4) = u_0, \quad 0 < r < 2$$

$$u(r,z)=\sum_{n=1}^{\infty}A_n \sinh \alpha_n z\, J_0(\alpha_n r)$$

$$\lambda_n{}^2=\alpha_n{}^2=\frac{x_n{}^2}{2^2},\ n=1,2,3,..$$

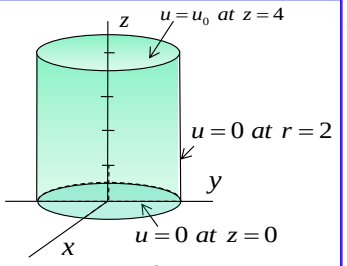
From **boundary condition** $u(r,4)=u_0,\ 0<r<2$

$$u_0=\sum_{n=1}^{\infty}A_n \sinh 4\alpha_n J_0(\alpha_n r)$$

Fourier-Bessel series

This is the case that α_n are defined by $J_0(\alpha_n c)=0$

$$\therefore A_n \sinh 4\alpha_n=\frac{2u_0}{2^2 J_1{}^2(2\alpha_n)}\int_0^2 rJ_0(\alpha_n r)dr$$



$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0,$$

$$u(2,z)=0,\ 0<z<4$$

$$u(r,0)=0,\ u(r,4)=u_0,\ 0<r<2$$

From boundary condition,

$u(2,t)=R(2)Z(z)=0$

$Z(z)\neq 0$ otherwise u(r,z) : trivial solution, So

$R(2)=c_1 J_0(2\alpha)=0$

$c_1\neq 0$ otherwise R(r) : trivial solution, So

$J_0(2\alpha)=0$

If $x_n=2\alpha_n$ are positive roots $\alpha_n=\frac{x_n}{2}$

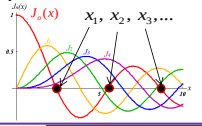
$$\lambda_n{}^2=\alpha_n{}^2=\frac{x_n{}^2}{2^2}$$

and eigenfunctions are

$c_1 J_0(\alpha_n r),\ n=1,2,3...$

Bessel functions

$J_n(x)$



Fourier-Bessel Series

$$f(x)=\sum_{i=1}^{\infty}c_i J_n(\alpha_i x)$$

Definition 12.8

Fourier-Bessel Series

The Fourier-Bessel series of a function f defined on the interval $(0,b)$ is given by

1) α_i are defined by $J_n(\alpha_i b)=0$

$$f(x)=\sum_{i=1}^{\infty}c_i J_n(\alpha_i x),\quad c_i=\frac{2}{b^2 J_{n+1}^2(\alpha_i b)}\int_0^b xJ_n(\alpha_i x)f(x)dx$$

2) α_i are defined by $hJ_n(\alpha b)+\alpha b J_n'(\alpha b)=0$

$$f(x)=\sum_{i=1}^{\infty}c_i J_n(\alpha_i x),\quad c_i=\frac{2\alpha_i^2}{(\alpha_i^2 b^2-n^2+h^2)J_n^2(\alpha_i b)}\int_0^b xJ_n(\alpha_i x)f(x)dx$$

3) α_i are defined by $J_n'(\alpha b)=0$

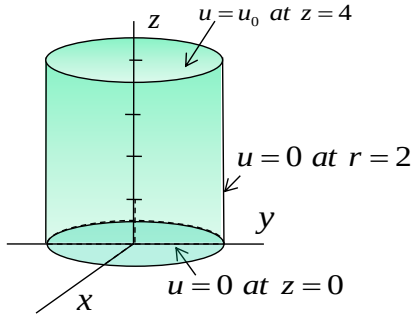
$$f(x)=c_1+\sum_{i=2}^{\infty}c_i J_n(\alpha_i x)$$

$$c_1=\frac{2}{b^2}\int_0^b xf(x)dx$$

$$c_i=\frac{2}{b^2 J_0^2(\alpha_i b)}\int_0^b xJ_n(\alpha_i x)f(x)dx$$

$$u(r,z)=\sum_{n=1}^{\infty}A_n \sinh \alpha_n z\, J_0(\alpha_n r)$$

$$\lambda_n{}^2=\alpha_n{}^2=\frac{x_n{}^2}{2^2},\ n=1,2,3,..$$



$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0,$$

$$u(2,z)=0,\quad 0<z<4$$

$$u(r,0)=0,\ u(r,4)=u_0,\quad 0<r<2$$

From boundary condition $u(r,4)=u_0,\ 0<r<2$

$$u_0=\sum_{n=1}^{\infty}A_n \sinh 4\alpha_n J_0(\alpha_n r)$$ **Fourier-Bessel series**

$$A_n \sinh 4\alpha_n=\frac{2u_0}{2^2 J_1{}^2(2\alpha_n)}\int_0^2 rJ_0(\alpha_n r)dr$$

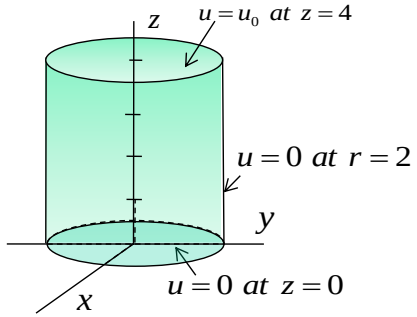
Substitute $t=\alpha_n r \qquad dr=dt/\alpha_n,\ r=t/\alpha_n,$

$$\frac{2u_0}{2^2 J_1{}^2(2\alpha_n)}\int_0^2 rJ_0(\alpha_n r)dr=\frac{2u_0}{2^2 J_1{}^2(2\alpha_n)}\int_0^{2\alpha_n}\frac{t}{\alpha_n}J_0(t)\frac{dt}{\alpha_n}$$

$$=\frac{2u_0}{2^2\alpha_n^2 J_1{}^2(2\alpha_n)}\int_0^{2\alpha_n}tJ_0(t)dt$$

$$u(r,z)=\sum_{n=1}^{\infty}A_n \sinh \alpha_n z\, J_0(\alpha_n r)$$

$$\lambda_n{}^2=\alpha_n{}^2=\frac{x_n{}^2}{2^2},\ n=1,2,3,..$$



$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0,$$

$$u(2,z)=0,\quad 0<z<4$$

$$u(r,0)=0,\ u(r,4)=u_0,\quad 0<r<2$$

From **boundary condition** $u(r,4)=u_0,\ 0<r<2$

$$u_0=\sum_{n=1}^{\infty}A_n \sinh 4\alpha_n J_0(\alpha_n r)$$

Fourier-Bessel series

$$A_n \sinh 4\alpha_n=\frac{2u_0}{2^2 J_1{}^2(2\alpha_n)}\int_0^2 rJ_0(\alpha_n r)dr$$

$$=\frac{2u_0}{2^2\alpha_n^2 J_1{}^2(2\alpha_n)}\int_0^{2\alpha_n} tJ_0(t)dt$$

$$=\frac{u_0}{2\alpha_n^2 J_1{}^2(2\alpha_n)}\int_0^{2\alpha_n}\frac{d}{dt}[tJ_1(t)]dt$$

$$=\frac{u_0}{2\alpha_n^2 J_1{}^2(2\alpha_n)}\left[tJ_1(t)\right]_0^{2\alpha_n}=\frac{u_0}{2\alpha_n^2 J_1{}^2(2\alpha_n)}2\alpha_n J_1(2\alpha_n)=\frac{u_0}{\alpha_n J_1(2\alpha_n)}$$

Substitute $t=\alpha_n r$

$$\frac{d}{dt}[tJ_1(t)]=tJ_0(t)$$

Recall

$$\frac{d}{dx}\left[x^n J_n(x)\right]=x^n J_{n-1}(x)$$

$$u(r,z)=\sum_{n=1}^{\infty}A_n \sinh \alpha_n z\, J_0(\alpha_n r)$$

$$\lambda_n{}^2=\alpha_n{}^2=\frac{x_n{}^2}{2^2},\ n=1,2,3,..$$

From **boundary condition** $u(r,4)=u_0,\ 0<r<2$

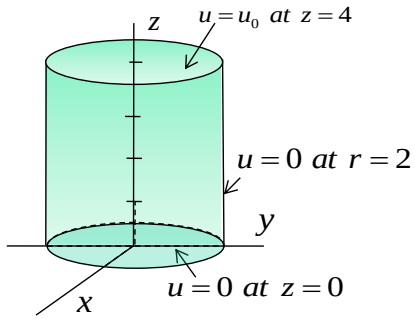
$$u_0=\sum_{n=1}^{\infty}A_n \sinh 4\alpha_n J_0(\alpha_n r)$$

Fourier-Bessel series

$$A_n \sinh 4\alpha_n=\frac{u_0}{\alpha_n J_1(2\alpha_n)}$$

$$\therefore A_n=\frac{u_0}{\alpha_n \sinh 4\alpha_n J_1(2\alpha_n)}$$

$$\therefore u(r,z)=u_0\sum_{n=1}^{\infty}\frac{\sinh \alpha_n z}{\alpha_n \sinh 4\alpha_n J_1(2\alpha_n)}J_0(\alpha_n r)$$



$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0,$$

$$u(2,z)=0,\ \ 0<z<4$$

$$u(r,0)=0,\ u(r,4)=u_0,\ \ 0<r<2$$

[Reference]

Bessel’ s Equation : example

Example

$$y'' + \frac{1-2a}{x} y' + (b^2 c^2 x^{2c-2} + \frac{a^2 - p^2 c^2}{x^2}) y = 0,\ p \geq 0,\ b > 0 \cdots (1)$$

$$z = bx^c,\ y(x) = \left(\frac{z}{b}\right)^{a/c} w(z) \cdots (2)$$

$$z^2 \frac{d^2 w(z)}{dz^2} + z \frac{dw(z)}{dz} + (z^2 - p^2) w = 0 \cdots (3)$$

$$y = x^a \left[c_1 J_p(bx^c) + c_2 J_{-p}(bx^c) \right] \cdots (4)$$

식 (1)의 꼴의 방정식의 경우 (2)의 관계식을 이용하여 (3)과 같은 Bessel Equation 형식으로 바꿀 수 있다. 식 (3)에서 구한 Bessel Function을 이용하여 원래 식(1)의 해를 구하면 식 (4)와 같다. 이 과정을 이용하여 $4x^2 y'' - 4xy' + (16x^2 + 3) y = 0$ 의 해를 찾으려고 한다.

Physical meaning?

Example

$$y'' + \frac{1-2a}{x} y' + (b^2 c^2 x^{2c-2} + \frac{a^2 - p^2 c^2}{x^2}) y = 0,\ p \geq 0,\ b > 0 \cdots (1)$$

$$z = bx^c,\ y(x) = \left(\frac{z}{b}\right)^{a/c} w(z) \cdots (2)$$

$$z^2 \frac{d^2 w(z)}{dz^2} + z \frac{dw(z)}{dz} + (z^2 - p^2) w = 0 \cdots (3)$$

$$y = x^a \left[c_1 J_p(bx^c) + c_2 J_{-p}(bx^c) \right] \cdots (4)$$

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식 (3)에서 구한 Bessel Function을 이용하여 원래 식(1)의 해를 구하면 식 (4)와 같다.

이 과정을 이용하여 $4x^2 y'' - 4xy' + (16x^2 + 3) y = 0$ 의 해를 찾으려고 한다.

문제 4) a,b,c,p에 해당하는 값을 각각 구하시오. (10점)

주어진 문제를 식 (1)과 같이 변형하고 비교한다.

$$y'' + \left(-\frac{1}{x}\right) y' + (4 + \frac{3/4}{x^2}) y = 0$$

$$1 - 2a = -1 \rightarrow a = 1$$

$$2c - 2 = 0 \rightarrow c = 1$$

$$b^2 c^2 = 4 \rightarrow b = 2$$

$$a^2 - p^2 c^2 = \frac{3}{4} \rightarrow p = \frac{1}{2}$$

Example

$$y'' + \frac{1-2a}{x} y' + (b^2 c^2 x^{2c-2} + \frac{a^2 - p^2 c^2}{x^2}) y = 0,\ p \geq 0,\ b > 0 \cdots (1)$$

$$z = bx^c,\ y(x) = \left(\frac{z}{b}\right)^{a/c} w(z) \cdots (2)$$

$$z^2 \frac{d^2 w(z)}{dz^2} + z \frac{dw(z)}{dz} + (z^2 - p^2) w = 0 \cdots (3)$$

$$y = x^a \left[c_1 J_p(bx^c) + c_2 J_{-p}(bx^c) \right] \cdots (4)$$

식 (1)의 꼴의 방정식의 경우 (2)의 관계식을 이용하여 (3)과 같은 Bessel Equation 형식으로 바꿀 수 있다.
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이 과정을 이용하여 $4x^2 y'' - 4xy' + (16x^2 + 3) y = 0$ 의 해를 찾으려고 한다.

문제 5) 해를 구하되 해가 sine, cosine, x의 거듭 제곱 등과 같은 elementary function의 형태로 구하시오. (20점)

$a = 1, c = 1, b = 2, p = \frac{1}{2}$ 이므로 식(4)의 해의 형태는 다음과 같다.

$$y = x[c_1 J_{1/2}(2x) + c_1 J_{-1/2}(2x)]$$

Example

$$y'' + \frac{1-2a}{x} y' + (b^2 c^2 x^{2c-2} + \frac{a^2 - p^2 c^2}{x^2}) y = 0,\ p \geq 0,\ b > 0 \cdots (1)$$

$$z = bx^c,\ y(x) = \left(\frac{z}{b}\right)^{a/c} w(z) \cdots (2)$$

$$z^2 \frac{d^2 w(z)}{dz^2} + z \frac{dw(z)}{dz} + (z^2 - p^2) w = 0 \cdots (3)$$

$$y = x^a \left[c_1 J_p(bx^c) + c_2 J_{-p}(bx^c) \right] \cdots (4)$$

식 (1)의 꼴의 방정식의 경우 (2)의 관계식을 이용하여 (3)과 같은 Bessel Equation 형식으로 바꿀 수 있다.
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$J_{1/2}$ 을 구하면

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

$$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\frac{1}{2}+n)} \left(\frac{x}{2}\right)^{2n+1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\frac{(2n+1)!}{2^{2n+1} n!}\sqrt{\pi}} \left(\frac{x}{2}\right)^{2n+1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!\sqrt{\pi}} \sqrt{\frac{2}{x}} \frac{x^{2n+1}}{2^{2n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\sqrt{\pi}} \sqrt{\frac{2}{x}} x^{2n+1} = \sqrt{\frac{2}{x\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sqrt{\frac{2}{x\pi}} \sin x$$

Example

$$y'' + \frac{1-2a}{x} y' + (b^2 c^2 x^{2c-2} + \frac{a^2 - p^2 c^2}{x^2}) y = 0,\ p \geq 0,\ b > 0 \cdots (1)$$

$$z = bx^c,\ y(x) = \left(\frac{z}{b}\right)^{a/c} w(z) \cdots (2)$$

$$z^2 \frac{d^2 w(z)}{dz^2} + z \frac{dw(z)}{dz} + (z^2 - p^2) w = 0 \cdots (3)$$

$$y = x^a \left[c_1 J_p(bx^c) + c_2 J_{-p}(bx^c) \right] \cdots (4)$$

식 (1)의 꼴의 방정식의 경우 (2)의 관계식을 이용하여 (3)과 같은 Bessel Equation 형식으로 바꿀 수 있다.
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문제 5) 해를 구하되 해가 sine, cosine, x의 거듭 제곱 등과 같은 elementary function의 형태로 구하시오. (20점)

$J_{1/2}$ 을 구하면

$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1-\nu+n)} \left(\frac{x}{2}\right)^{2n-\nu}$$

$$J_{-1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1-\frac{1}{2}+n)} \left(\frac{x}{2}\right)^{2n-1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \frac{(2n-1)!}{2^{2n-1}(n-1)!}\sqrt{\pi}} \left(\frac{x}{2}\right)^{2n-1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{2n(2n-1)!\sqrt{\pi}} \sqrt{\frac{2}{x}} \frac{x^{2n}}{2^{2n}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!\sqrt{\pi}} \sqrt{\frac{2}{x}} x^{2n} = \sqrt{\frac{2}{x\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \sqrt{\frac{2}{x\pi}} \cos x$$

Example

$$y'' + \frac{1-2a}{x} y' + (b^2 c^2 x^{2c-2} + \frac{a^2 - p^2 c^2}{x^2}) y = 0,\ p \geq 0,\ b > 0 \cdots (1)$$

$$z = bx^c,\ y(x) = \left(\frac{z}{b}\right)^{a/c} w(z) \cdots (2)$$

$$z^2 \frac{d^2 w(z)}{dz^2} + z \frac{dw(z)}{dz} + (z^2 - p^2) w = 0 \cdots (3)$$

$$y = x^a \left[c_1 J_p(bx^c) + c_2 J_{-p}(bx^c) \right] \cdots (4)$$

Physical meaning? -> damping (opposite sign)*

식 (1)의 꼴의 방정식의 경우 (2)의 관계식을 이용하여 (3)과 같은 Bessel Equation 형식으로 바꿀 수 있다.
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문제 5) 해를 구하되 해가 sine, cosine, x의 거듭 제곱 등과 같은 elementary function의 형태로 구하시오. (20점)

$$\therefore y = x[c_1 J_{1/2}(2x) + c_1 J_{-1/2}(2x)]$$

$$= x\left[c_1 \sqrt{\frac{2}{x\pi}} \sin(2x) + c_2 \sqrt{\frac{2}{x\pi}} \cos(2x) \right]$$

$$= \sqrt{x}\left[c_1 \sin(2x) + c_2 \cos(2x) \right]$$ (diverse)

$$J_{1/2}(x) = \sqrt{\frac{2}{x\pi}} \sin x,\ J_{-1/2}(x) = \sqrt{\frac{2}{x\pi}} \cos x$$

* See also Kreyszig E., Advanced Engineering Mathematics, 9th edition, Wiley, 2006, p157 example 5

8-9. Problems in Spherical Coordinates: Legendre's Equation

Problems in Spherical Coordinates : Legendre's Equation

$$\text{rectangular coord.}: \nabla^2 u = u_{xx} + u_{yy}$$

$$\text{polar coord.}: \nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$$

$$\text{cylindrical coord.}: \nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

☑ Laplacian in Spherical Coordinates

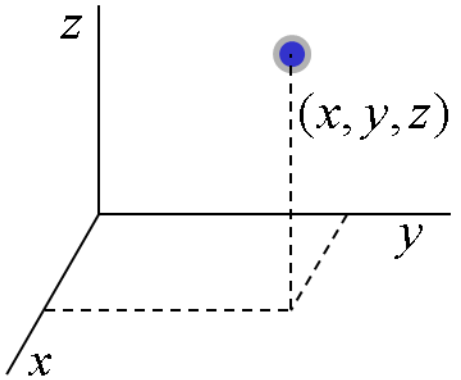
$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

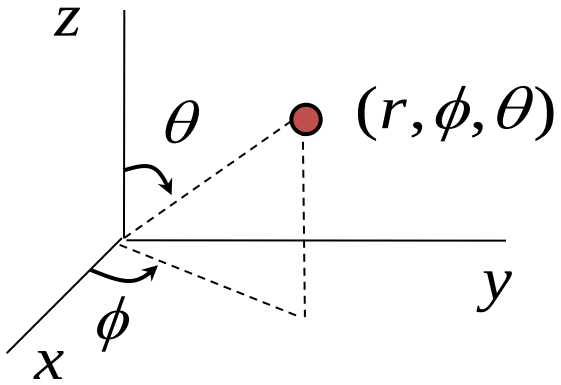
$$z = r\cos\theta$$

Laplacian in Spherical coordinates* is

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}$$



Cartesian coordinates



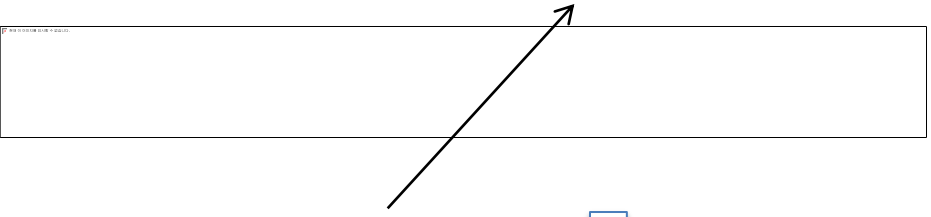
Spherical coordinates

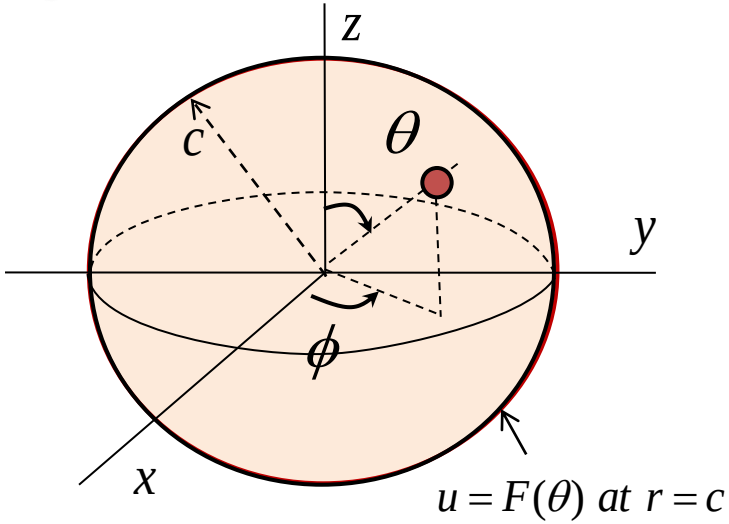
*http://planetmath.org/encyclopedia/%3chttp:/planetmath.org/?method=l2h&from=collab&id=76&op=getobj

Problems in Spherical Coordinates : Legendre's Equation

☑ Steady Temperatures in Sphere

Find the steady-state temperature u(r,θ) in the sphere
(we shall consider only a few of the simpler problems that are independent of the azimuthal angle ϕ



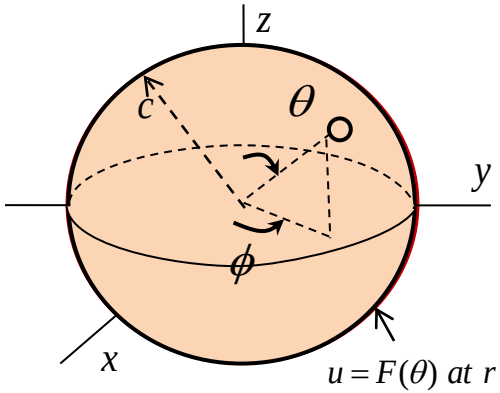


The temperature is determined from

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0,$$

$$0 < r < c, \quad 0 < \theta < \pi$$

$$u(c,\theta) = F(\theta), \quad 0 < \theta < \pi$$



$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}=0,$$

$$0<r<c,\ \ 0<\theta<\pi$$

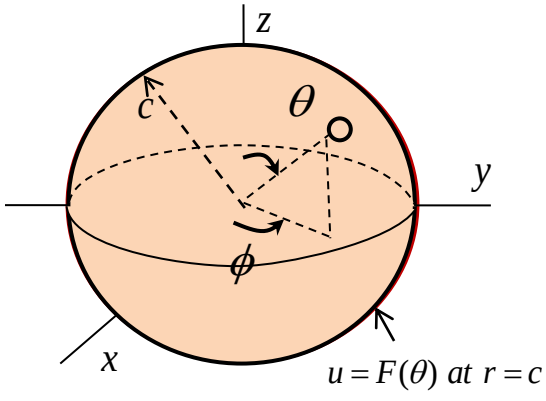
$$u(c,\theta)=F(\theta),\ \ 0<\theta<\pi$$

Product Method

$$u(r,\theta)=R(r)\Theta(\theta)$$

$$R''\Theta+\frac{2}{r}R'\Theta+\frac{1}{r^2}R\Theta''+\frac{\cot\theta}{r^2}R\Theta'=0$$ **Dividing by** $$\frac{R\Theta}{r^2}$$

$$\frac{r^2R''}{R}+2r\frac{R'}{R}+\frac{\Theta''}{\Theta}+\cot\theta\frac{\Theta'}{\Theta}=0 \quad \Rightarrow \quad \frac{r^2R''+2rR'}{R}=-\frac{\Theta''+\cot\theta\,\Theta'}{\Theta}=\lambda$$



$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}=0,$$

$$0<r<c,\ \ 0<\theta<\pi$$

$$u(c,\theta)=F(\theta),\ \ 0<\theta<\pi$$

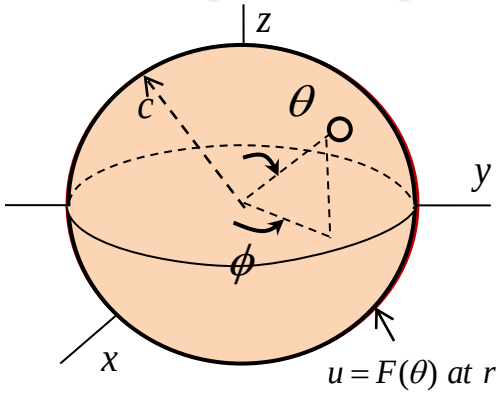
$$\frac{r^2R''+2rR'}{R}=-\frac{\Theta''+\cot\theta\,\Theta'}{\Theta}=\lambda$$

$$r^2R''+2rR'-\lambda R=0\cdots(1)$$

$$\sin\theta\,\Theta''+\cos\theta\,\Theta'+\lambda\sin\theta\,\Theta=0\cdots(2)$$

Problems in Spherical Coordinates : Legendre Polynomials

☑ Steady Temperatures in Sphere



$u = F(\theta)$ at $r = c$

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0,$$

$$0 < r < c, \quad 0 < \theta < \pi$$

$$u(c,\theta) = F(\theta), \quad 0 < \theta < \pi$$

Legendre's equation*

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

Legendre Polynomials

$$P_0(x) = 1 \qquad P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$\vdots$$

$$r^2 R'' + 2rR' - \lambda R = 0 \cdots (1)$$

$$\sin\theta\,\Theta'' + \cos\theta\,\Theta' + \lambda\sin\theta\,\Theta = 0 \cdots (2)$$

After substitute $x = \cos\theta, \ 0 \le \theta \le \pi$

$$(2): (1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \lambda\Theta = 0, -1 \le x \le 1$$ **(Legendre's equation)**

$$\lambda^2 = n(n+1), \ n = 0,1,2,\ldots \qquad \Theta = P_n(x) = P_n(\cos\theta)$$

$$x = \cos\theta$$

$$\Theta' = \frac{d\Theta}{d\theta} = \frac{d\Theta}{dx}\frac{dx}{d\theta}$$

$$\Theta'' = \frac{d}{d\theta}\frac{d\Theta}{d\theta} = \frac{d}{d\theta}\left(\frac{d\Theta}{dx}\cdot\frac{dx}{d\theta}\right)$$

$$= \left(\frac{d}{dx}\frac{d\Theta}{dx}\right)\frac{dx}{d\theta}\cdot\frac{dx}{d\theta} + \frac{d\Theta}{dx}\frac{d^2x}{d\theta^2}$$

$$= \frac{d^2\Theta}{dx^2}\cdot\left(\frac{dx}{d\theta}\right)^2 + \frac{d\Theta}{dx}\frac{d^2x}{d\theta^2}$$

Definition 12.9

Fourier-Legendre Series

The Fourier-Legendre series of a function f defined on the interval $(-1,1)$ is given by

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x)\ , c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx$$

$$x = \cos\theta$$

$$F(\theta) = \sum_{n=0}^{\infty} c_n P_n(\cos\theta), c_n = \frac{2n+1}{2}\int_{0}^{\pi} F(\theta)P_n(\cos\theta)\sin\theta d\theta$$

$$F(\theta) = f(\cos\theta) = f(x)$$

$$\sin\theta\,\Theta'' + \cos\theta\,\Theta' + \lambda\sin\theta\,\Theta = 0 \cdots (2)$$

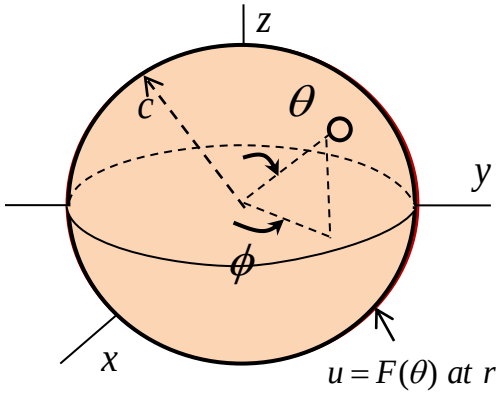
$$\sin\theta\left(\frac{d^2\Theta}{dx^2}\sin^2\theta + \frac{d\Theta}{dx}(-\cos\theta)\right) + \cos\theta\frac{d\Theta}{dx}(-\sin\theta) + \lambda\sin\theta\,\Theta = 0$$

$$\sin\theta\left(\frac{d^2\Theta}{dx^2}\sin^2\theta - 2\cos\theta\frac{d\Theta}{dx} + \lambda\Theta\right) = 0$$

$$\frac{d^2\Theta}{dx^2}\sin^2\theta + -2\cos\theta\frac{d\Theta}{dx} + \lambda\Theta = 0$$

$$(1-\cos^2\theta)\frac{d^2\Theta}{dx^2} + -2\cos\theta\frac{d\Theta}{dx} + \lambda\Theta = 0$$

$$(1-x^2)\frac{d^2\Theta}{dx^2} + -2x\frac{d\Theta}{dx} + \lambda\Theta = 0$$



$$u = F(\theta) \text{ at } r = c$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0,$$

$$0 < r < c, \quad 0 < \theta < \pi$$

$$u(c,\theta) = F(\theta), \quad 0 < \theta < \pi$$

$$r^2 R'' + 2rR' - \lambda R = 0 \cdots (1)$$

$$\sin\theta\,\Theta'' + \cos\theta\,\Theta' + \lambda \sin\theta\,\Theta = 0 \cdots (2)$$

$$(2): (1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \lambda\Theta = 0, \quad -1 \le x \le 1$$

$$\lambda^2 = n(n+1), \; n = 0,1,2,\ldots$$

$$\Theta = P_n(x) = P_n(\cos\theta)$$

Definition 12.9 — **Fourier-Legendre Series**

The Fourier-Legendre series of a function f defined on the interval $(-1,1)$ is given by

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x), \quad c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx$$

•In an alternative form, let $x = \cos\theta$

$$F(\theta) = \sum_{n=0}^{\infty} c_n P_n(\cos\theta), \; c_n = \frac{2n+1}{2}\int_0^{\pi} F(\theta)P_n(\cos\theta)\sin\theta\,d\theta$$

$$F(\theta) = f(\cos\theta) = f(x)$$

Legendre's equation**$(1-x^2)y'' - 2xy' + n(n+1)y = 0$

Legendre Polynomials

$$P_0(x) = 1 \qquad P_1(x) = x$$

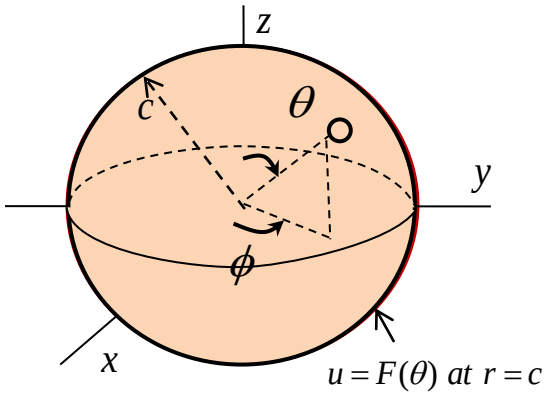
$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$\vdots$$

$$\Theta = P_n(x) = P_n(\cos\theta)$$

$$u(r,\theta) = R(r)\Theta(\theta)$$



$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0,$$

$$0 < r < c, \quad 0 < \theta < \pi$$

$$u(c,\theta) = F(\theta), \quad 0 < \theta < \pi$$

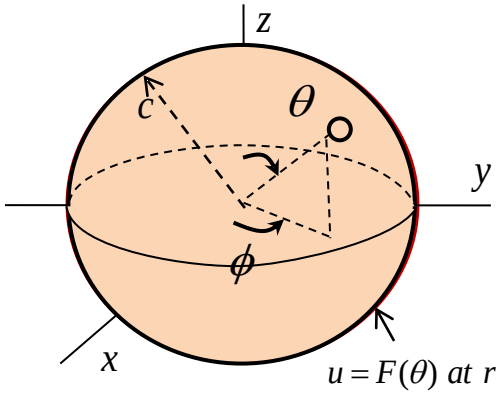
$$r^2 R'' + 2rR' - \lambda R = 0 \cdots (1)$$

Equation (1) is Cauchy-Euler equation,

$$R(r) = c_1 r^n + c_2 r^{-(n+1)}$$

$$\therefore u_n(r,\theta) = A_n r^n P_n(\cos\theta)$$

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$$



$u = F(\theta)$ at $r = c$

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0,$$

$$0 < r < c, \quad 0 < \theta < \pi$$

$$u(c,\theta) = F(\theta), \quad 0 < \theta < \pi$$

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$$

Boundary Condition: At $r = c$,

$$F(\theta) = \sum_{n=0}^{\infty} A_n c^n P_n(\cos\theta)$$ **(Fourier-Legendre series)**

Definition 12.9 Fourier-Legendre Series

The Fourier-Legendre series of a function f defined on the interval $(-1,1)$ is given by

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x), \quad c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx$$

- In an alternative form, let $x = \cos\theta$

$$F(\theta) = \sum_{n=0}^{\infty} c_n P_n(\cos\theta), \quad c_n = \frac{2n+1}{2}\int_0^{\pi} F(\theta)P_n(\cos\theta)\sin\theta d\theta$$

$$F(\theta) = f(\cos\theta) = f(x)$$

Fourier-Legendre series coefficients :

$$A_n = \frac{2n+1}{2c^n}\int_0^{\pi} F(\theta)P_n(\cos\theta)\sin\theta\, d\theta$$

$$\therefore u(r,\theta) = \sum_{n=0}^{\infty}\left(\frac{2n+1}{2}\int_0^{\pi} F(\theta)P_n(\cos\theta)\sin\theta\, d\theta\right)\left(\frac{r}{c}\right)^n P_n(\cos\theta)$$

CHAPTER 9. VECTOR CALCULUS

DIFFERENTIAL GEOMETRY & CURVILINEAR MOTION

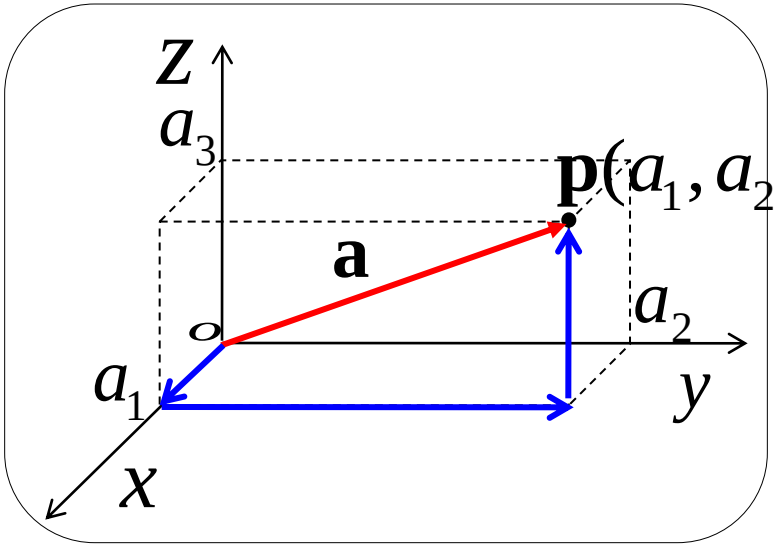
VECTOR FUNCTIONS, VECTOR FIELDS

LINE, DOUBLE AND TRIPLE INTEGRALS

GREEN'S, STOKES' AND DIVERGENCE THEOREM

Vectors in 3D Space

Definition



•Vector

$$\mathbf{a} = (a_1, a_2, a_3)$$
$$= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

•Magnitude : 'norm'

$$\|\mathbf{a}\| = \sqrt{a_1^{\,2} + a_2^{\,2} + a_3^{\,2}}$$

$\mathbf{i} = (1,0,0)$ Unit movement in the direction of x

$\mathbf{j} = (0,1,0)$ Unit movement in the direction of y

$\mathbf{k} = (0,0,1)$ Unit movement in the direction of z

•Unit Vector of $\mathbf{a}$

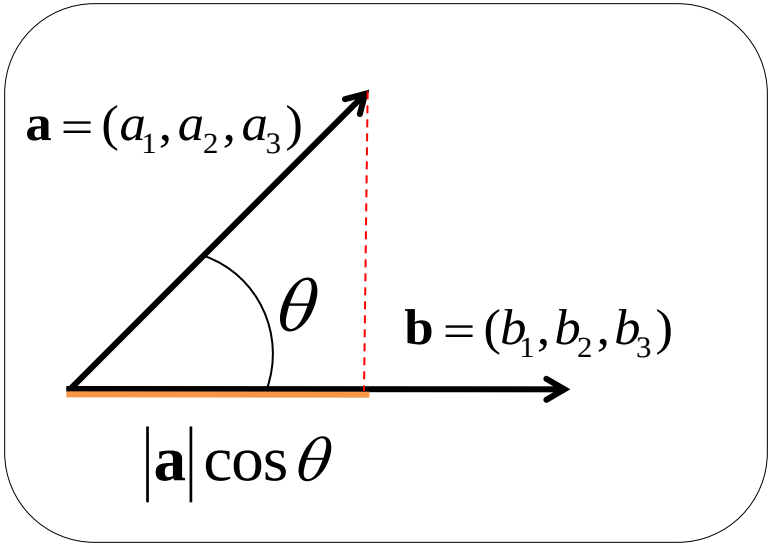
$$\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$$

Direction : $\mathbf{a}$

Magnitude : 1

Vectors in 3D Space

Scalar (Dot) Product



•Scalar (Dot) Product

$$\mathbf{a}\bullet\mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\bullet(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$

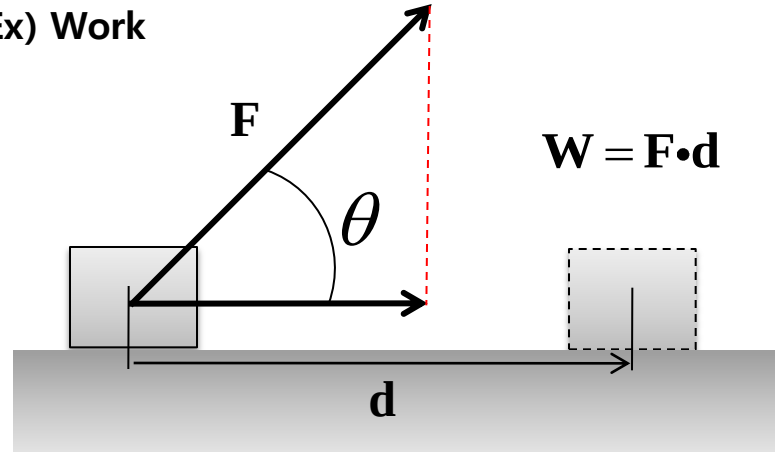
in geometry

$$\mathbf{a}\bullet\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$$

$$ex)\ \mathbf{i}\bullet\mathbf{i} = \|\mathbf{i}\|\|\mathbf{i}\|\cos 0 = 1$$

$$\mathbf{i}\bullet\mathbf{j} = \|\mathbf{i}\|\|\mathbf{j}\|\cos\frac{\pi}{2} = 0$$

Ex) Work



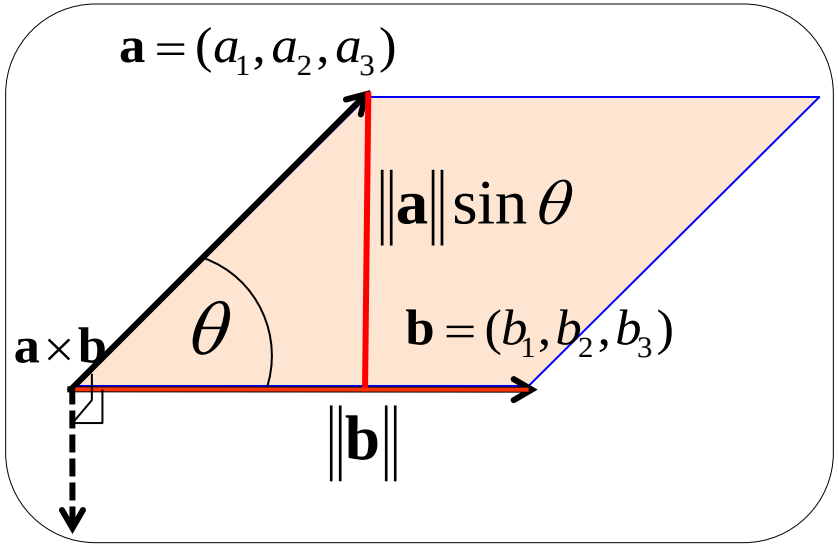
$$\mathbf{W} = \mathbf{F}\bullet\mathbf{d}$$

in component form

$$\mathbf{a}\bullet\mathbf{b} = a_1b_1 + a_2b_2 + a_2b_2$$

Vectors in 3D Space

Vector (Cross) Product



•Vector (Cross) Product

$$\mathbf{a}\times\mathbf{b}=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\times(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})$$

in geometry

$$\|\mathbf{a}\times\mathbf{b}\|=\|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$$

$$ex)\ \mathbf{i}\times\mathbf{i}=\|\mathbf{i}\|\|\mathbf{i}\|\sin 0=0$$

$$\mathbf{i}\times\mathbf{j}=\mathbf{k},\ \ \mathbf{j}\times\mathbf{k}=\mathbf{i},\ \mathbf{k}\times\mathbf{i}=\mathbf{j}$$

$$\mathbf{j}\times\mathbf{i}=-\mathbf{k},\ \ \mathbf{k}\times\mathbf{j}=-\mathbf{i},\ \mathbf{i}\times\mathbf{k}=-\mathbf{j}$$

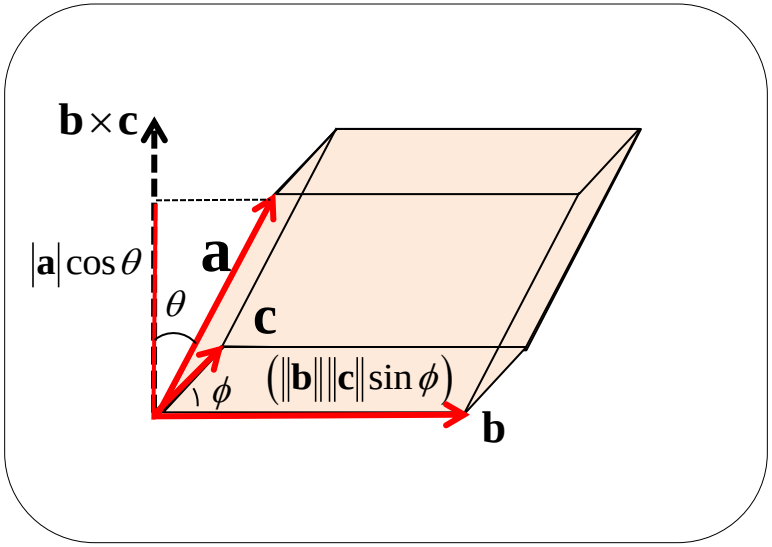
in component form

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}$$

$$=(a_2b_3-a_3b_2)\mathbf{i}+(a_3b_1-a_1b_3)\mathbf{j}+(a_1b_2-a_2b_1)\mathbf{k}$$

Vectors in 3D Space

Triple Scalar Product



•Triple Scalar Product

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
$$= a_1(b_2c_3 - b_3c_2)$$
$$+ a_2(b_3c_1 - b_1c_3)$$
$$+ a_3(b_1c_2 - b_2c_1)$$

in geometry

$$\|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\| = \|\mathbf{a}\|\left(\|\mathbf{b}\|\|\mathbf{c}\|\sin\phi\right)\cos\theta$$

Vectors in 3D Space

Properties of Scalar (Dot) Product

$$\mathbf{a}\bullet\mathbf{b}=0 \text{ if } \mathbf{a}=0 \text{ or } \mathbf{b}=0$$

$$\mathbf{a}\bullet\mathbf{b}=\mathbf{b}\bullet\mathbf{a}$$

$$\mathbf{a}\bullet(\mathbf{b}+\mathbf{c})=\mathbf{a}\bullet\mathbf{b}+\mathbf{a}\bullet\mathbf{c}$$

$$\mathbf{a}\bullet(k\mathbf{b})=(k\mathbf{a})\bullet\mathbf{b}=k(\mathbf{a}\bullet\mathbf{b}) \quad k:\textit{scalar}$$

$$\mathbf{a}\bullet\mathbf{a}=|\mathbf{a}|^2\geq 0$$

Properties of Vector (Cross) Product

$$\mathbf{a}\times\mathbf{b}=0 \text{ if } \mathbf{a}=0 \text{ or } \mathbf{b}=0$$

$$\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$$

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$$

$$(\mathbf{a}+\mathbf{b})\times\mathbf{c}=\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$$

$$\mathbf{a}\times(k\mathbf{b})=(k\mathbf{a})\times\mathbf{b}=k(\mathbf{a}\times\mathbf{b}) \quad k:\textit{scalar}$$

$$\mathbf{a}\times\mathbf{a}=0$$

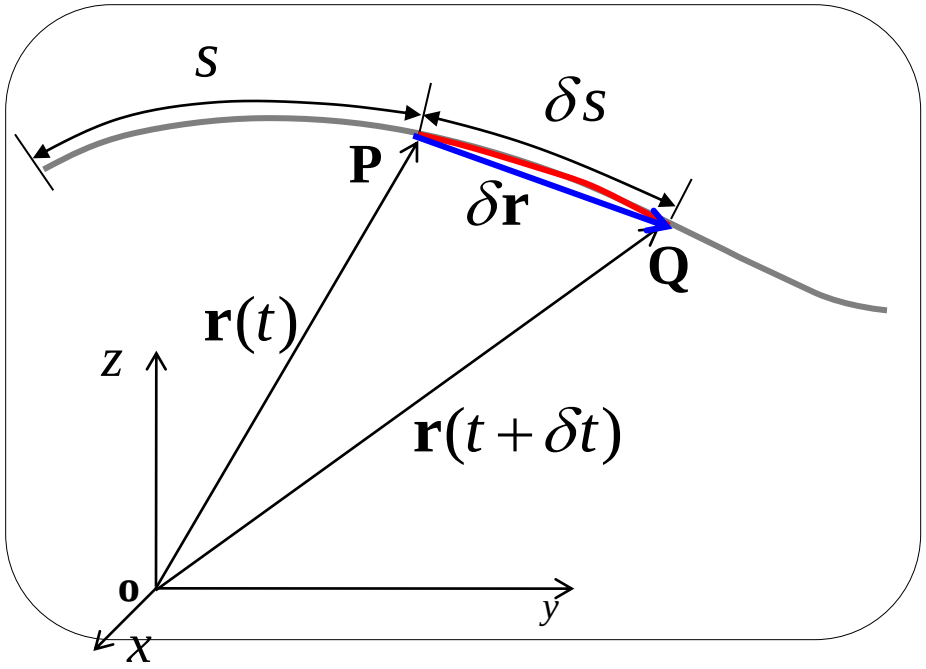
$$\mathbf{a}\bullet(\mathbf{a}\times\mathbf{b})=0$$

$$\mathbf{b}\bullet(\mathbf{a}\times\mathbf{b})=0$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\bullet\mathbf{c})\mathbf{b}-(\mathbf{a}\bullet\mathbf{b})\mathbf{c}$$

Properties of a Curve

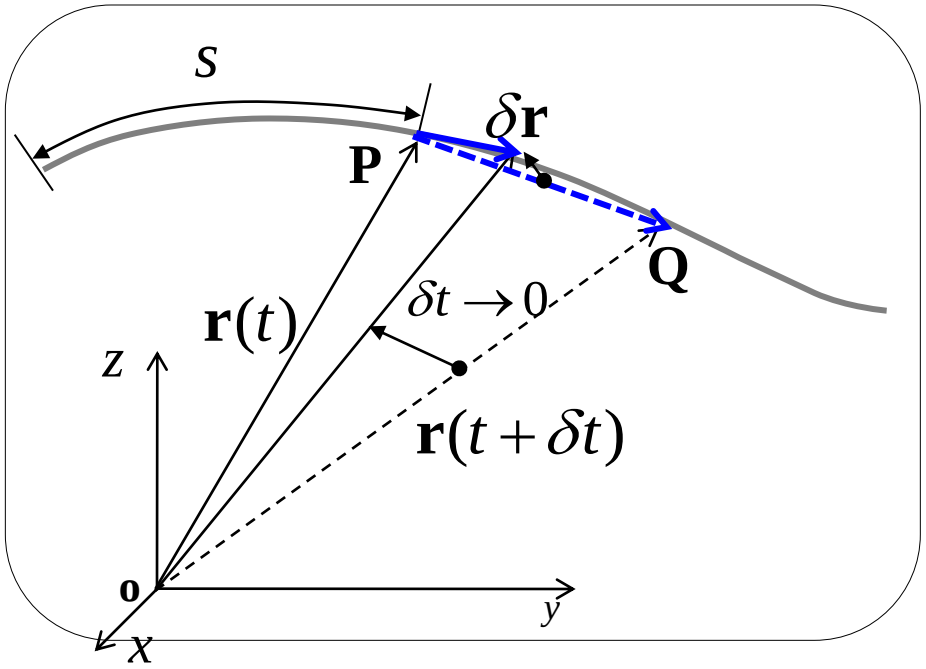
Tangent vector



The chord PQ

$$\delta\mathbf{r} = \mathbf{r}(t+\delta t) - \mathbf{r}(t)$$

The arc length s



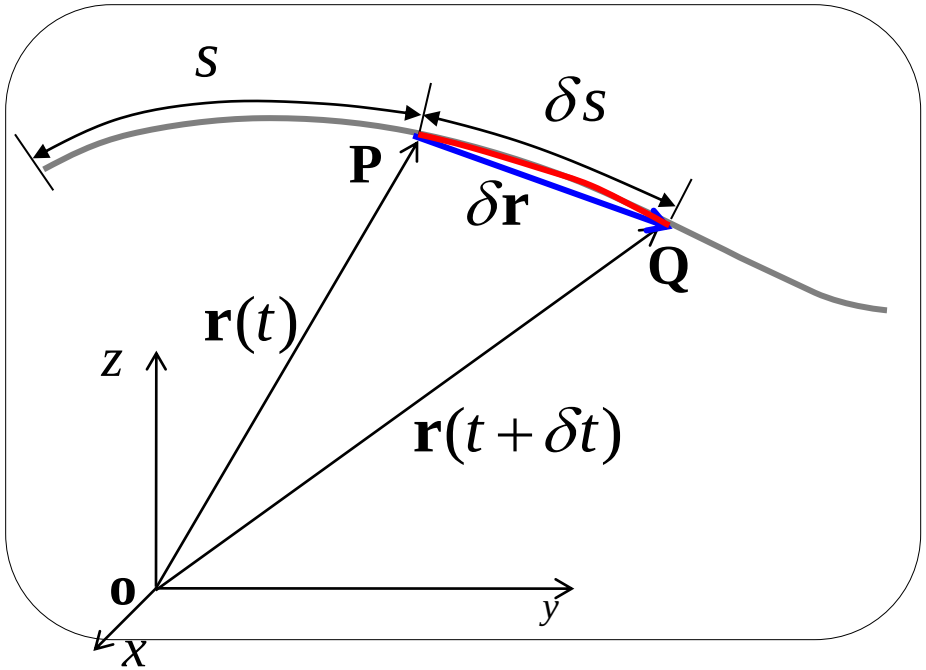
The tangent vector at P

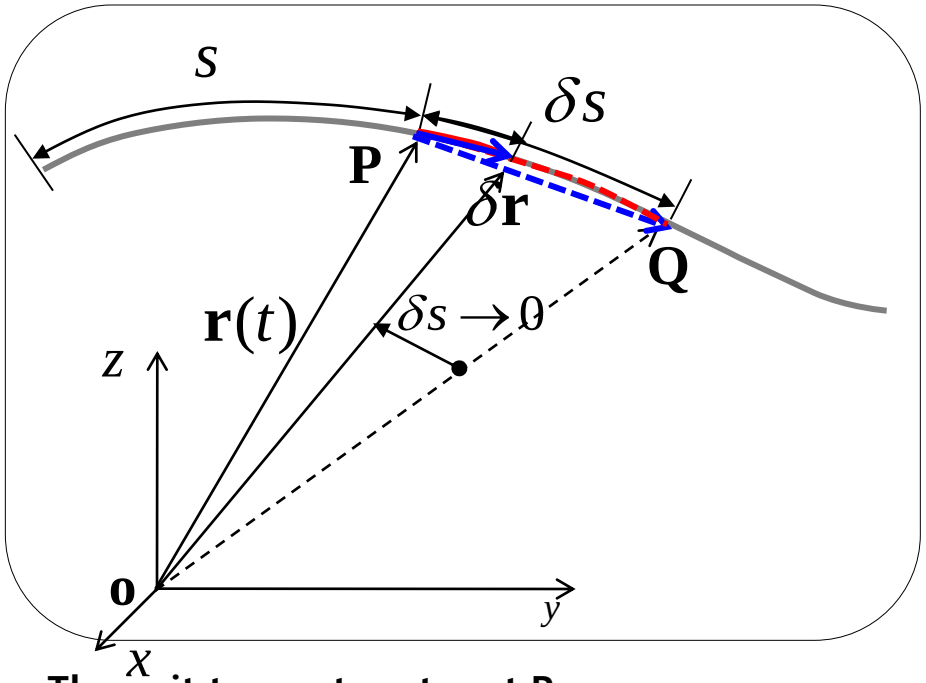
$$\lim_{\delta t \to 0} \frac{\delta\mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}$$

Properties of a Curve

Unit Tangent Vector

If choose s as parameter, then
chord length and arc length become equal in the limit





The chord PQ

$$\delta\mathbf{r} = \mathbf{r}(t+\delta t) - \mathbf{r}(t)$$

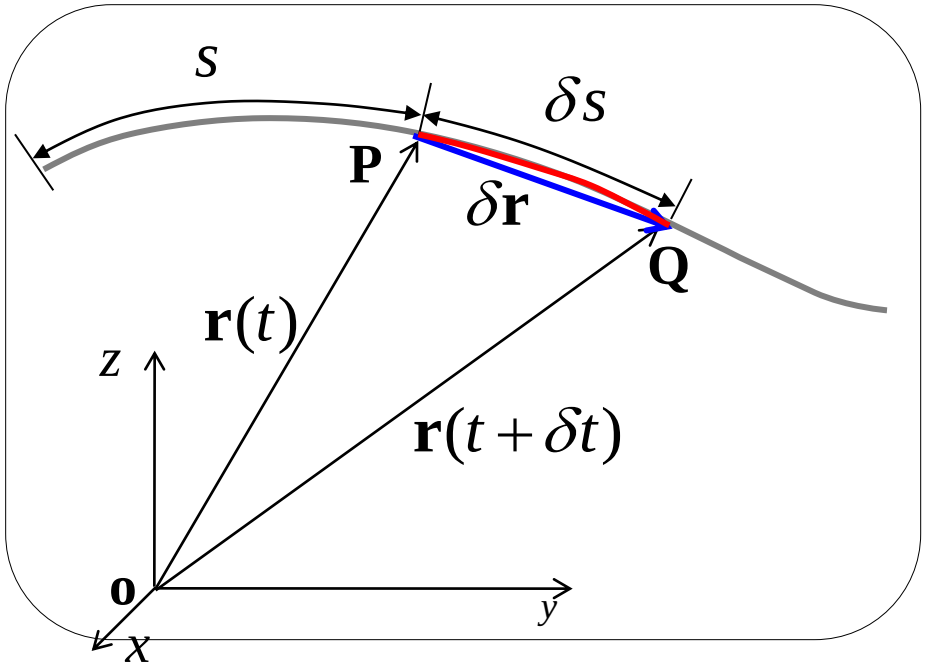
The arc length s

The unit tangent vector at P

$$\lim_{\delta s \to 0} \frac{\delta \mathbf{r}}{\delta s} = \frac{d\mathbf{r}}{ds} = \mathbf{T}$$

Properties of a Curve

Tangent Vector and Unit Tangent Vector



tangent vector at P

$$\lim_{\delta t \to 0} \frac{\delta \mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}$$

unit tangent vector at P

$$\lim_{\delta s \to 0} \frac{\delta \mathbf{r}}{\delta s} = \frac{d\mathbf{r}}{ds} = \mathbf{T}$$

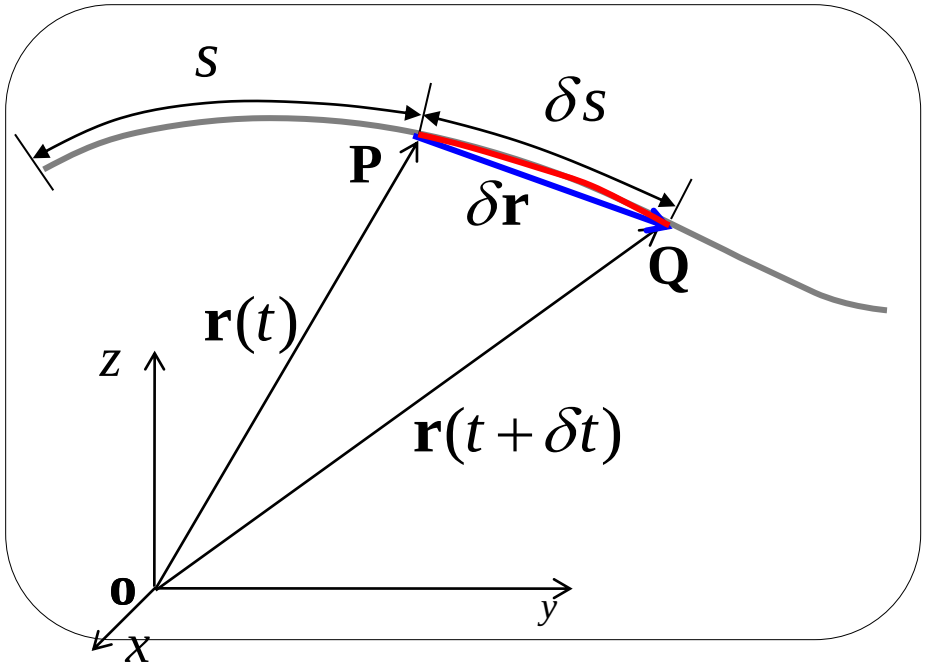
The chord PQ

$$\delta \mathbf{r} = \mathbf{r}(t+\delta t) - \mathbf{r}(t)$$

The arc length s

Properties of a Curve

Tangent Vector and Unit Tangent Vector



The chord PQ

$$\delta\mathbf{r} = \mathbf{r}(t+\delta t) - \mathbf{r}(t)$$

The arc length s

unit tangent vector : direction

magnitude

tangent vector at P

$$\lim_{\delta t \to 0} \frac{\delta \mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt} = \mathbf{T}\left\|\frac{d\mathbf{r}}{dt}\right\| = \dot{\mathbf{r}}$$

unit tangent vector at P

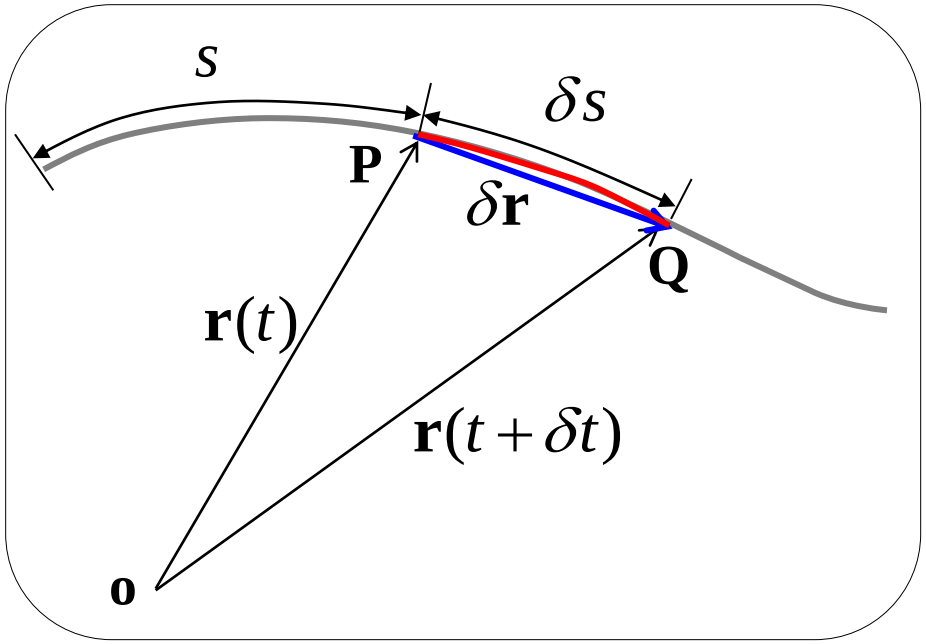
$$\lim_{\delta s \to 0} \frac{\delta \mathbf{r}}{\delta s} = \frac{d\mathbf{r}}{ds} = \mathbf{T}$$

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{T}\frac{ds}{dt}$$

$$\therefore \frac{ds}{dt} = \dot{s} = \left\|\frac{d\mathbf{r}}{dt}\right\| = \|\dot{\mathbf{r}}\|$$

Properties of a Curve

Tangent Vector and Unit Tangent Vector



tangent vector at P

$$\lim_{\delta t \to 0} \frac{\delta \mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt} = \mathbf{T}\left\|\frac{d\mathbf{r}}{dt}\right\| = \dot{\mathbf{r}}$$

unit tangent vector at P

$$\lim_{\delta s \to 0} \frac{\delta \mathbf{r}}{\delta s} = \frac{d\mathbf{r}}{ds} = \mathbf{T}$$

$$\frac{ds}{dt} = \dot{s} = \left\|\frac{d\mathbf{r}}{dt}\right\| = \|\dot{\mathbf{r}}\|$$

The chord PQ

$$\delta \mathbf{r} = \mathbf{r}(t+\delta t) - \mathbf{r}(t)$$

The arc length s

tangent vector $\dot{\mathbf{r}}$ & unit tangent vector $\mathbf{T}$ & parameter S

$$\dot{\mathbf{r}} = \mathbf{T}\dot{s} \qquad \|\dot{\mathbf{r}}\| = \dot{s}$$

Properties of a Curve

The tangent vector at P

$$\lim_{\delta t \to 0} \frac{\delta \mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}$$

The unit tangent vector at P

$$\lim_{\delta s \to 0} \frac{\delta \mathbf{r}}{\delta s} = \frac{d\mathbf{r}}{ds} = \mathbf{T}$$

tangent vector $\dot{\mathbf{r}}$ & unit tangent vector $\mathbf{T}$ & parameter s

$$\dot{\mathbf{r}} = \mathbf{T}\dot{s} \qquad \|\dot{\mathbf{r}}\| = \dot{s}$$

Principal Normal Vector

$$\mathbf{T} \bullet \mathbf{T} = 1$$

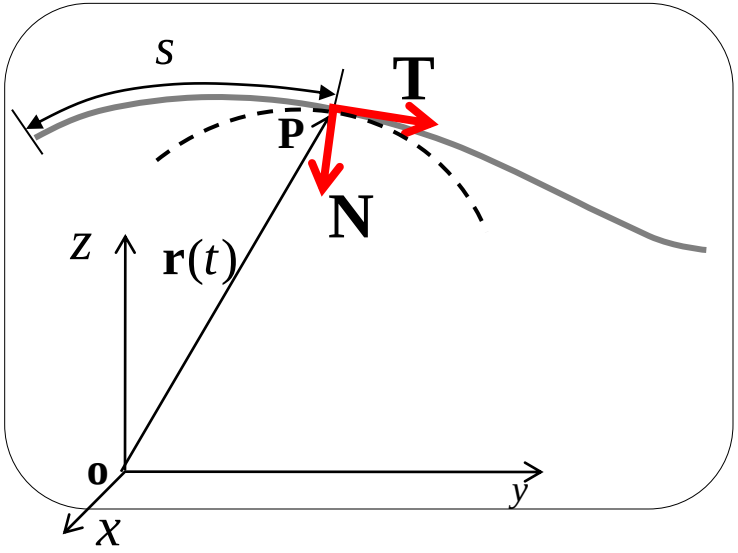
$$\frac{d(\mathbf{T} \bullet \mathbf{T})}{dt} = \mathbf{T} \bullet \frac{d\mathbf{T}}{dt} + \frac{d\mathbf{T}}{dt} \bullet \mathbf{T} = 2\dot{\mathbf{T}} \bullet \mathbf{T} = 0$$

$$\therefore \dot{\mathbf{T}} \perp \mathbf{T}$$

unit vector in the direction of $\dot{\mathbf{T}}$ = Principal Normal Vector $\mathbf{N}$

$$\mathbf{N} = \frac{\dot{\mathbf{T}}}{\|\dot{\mathbf{T}}\|} = \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}$$

$$\rightarrow \dot{\mathbf{T}} = \mathbf{N}\left\|\frac{d\mathbf{T}}{dt}\right\|$$



Properties of a Curve

The tangent vector at P

$$\lim_{\delta t \to 0} \frac{\delta \mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}$$

The unit tangent vector at P

$$\lim_{\delta s \to 0} \frac{\delta \mathbf{r}}{\delta s} = \frac{d\mathbf{r}}{ds} = \mathbf{T}$$

tangent vector $\dot{\mathbf{r}}$ & unit tangent vector $\mathbf{T}$ & parameter s

$$\dot{\mathbf{r}} = \mathbf{T}\dot{s} \qquad \|\dot{\mathbf{r}}\| = \dot{s}$$

unit vector in the direction of $\dot{\mathbf{T}}$ = Principal Normal Vector: $\mathbf{N}$

$$\mathbf{N} = \frac{\dot{\mathbf{T}}}{\|\dot{\mathbf{T}}\|} = \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}$$

$$\rightarrow \dot{\mathbf{T}} = \mathbf{N}\left\|\frac{d\mathbf{T}}{dt}\right\|$$

direction

magnitude

meanwhile,

$$\dot{\mathbf{T}} = \frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} = \frac{d\mathbf{T}}{ds}\dot{s}$$

$$\therefore \frac{d\mathbf{T}}{ds}, \mathbf{N} \text{ : same direction} \Rightarrow \therefore \frac{d\mathbf{T}}{ds} = \mathbf{N}\left\|\frac{d\mathbf{T}}{ds}\right\|$$

Properties of a Curve

$$\frac{d\mathbf{T}}{ds} = \mathbf{N}\left\|\frac{d\mathbf{T}}{ds}\right\|$$

Curvature κ

let $\kappa = \left\|\frac{d\mathbf{T}}{ds}\right\|$ then $\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$

$$\therefore \dot{\mathbf{T}} = \frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} = \frac{d\mathbf{T}}{ds}\dot{s} = \dot{s}\frac{d\mathbf{T}}{ds} = \dot{s}\kappa\mathbf{N}$$

The tangent vector at P

$$\lim_{\delta t \to 0}\frac{\delta \mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}$$

The unit tangent vector at P

$$\lim_{\delta s \to 0}\frac{\delta \mathbf{r}}{\delta s} = \frac{d\mathbf{r}}{ds} = \mathbf{T}$$

tangent vector $\dot{\mathbf{r}}$ & unit tangent vector $\mathbf{T}$ & parameter s

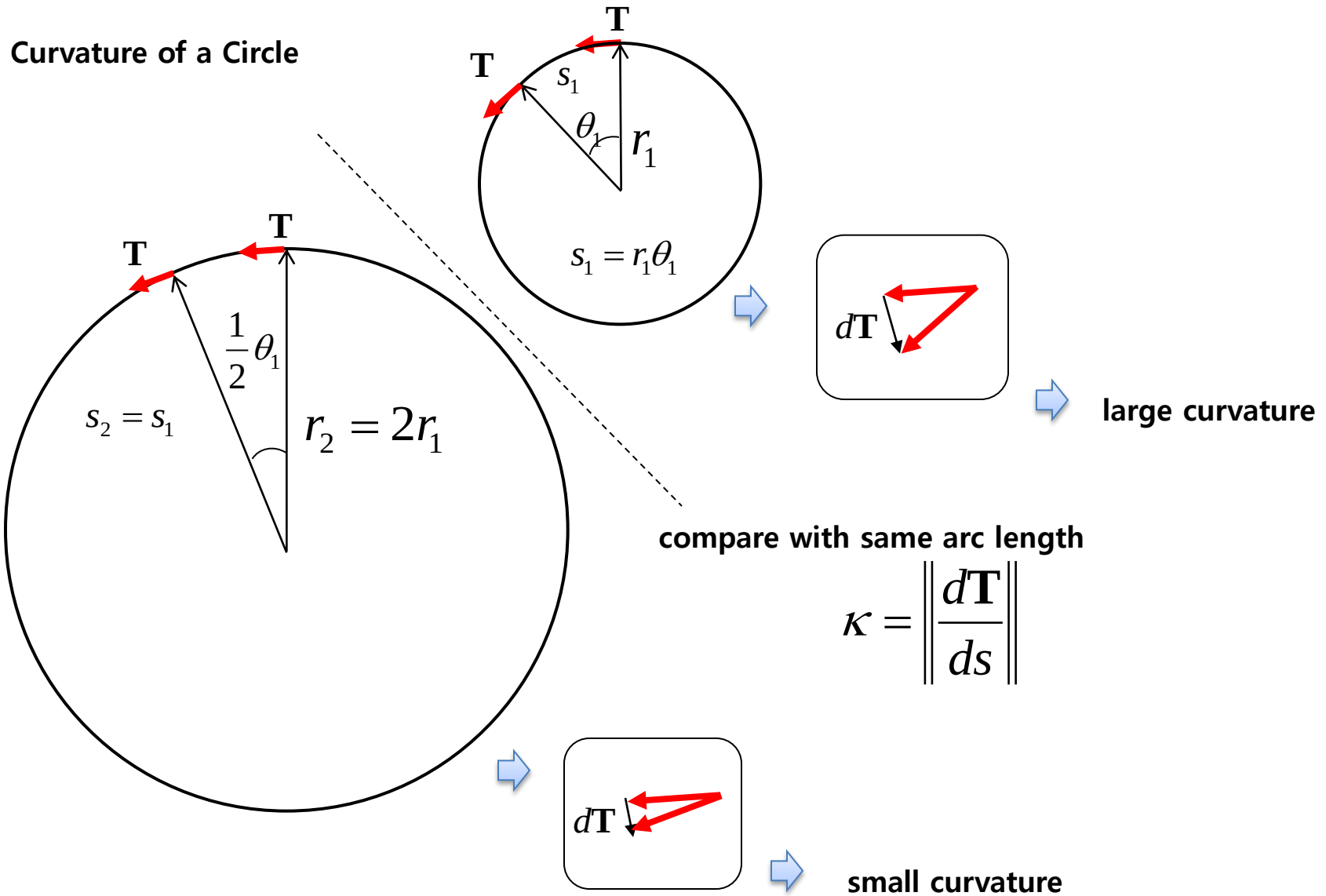
$$\dot{\mathbf{r}} = \mathbf{T}\dot{s} \quad \left\|\dot{\mathbf{r}}\right\| = \dot{s}$$

principal normal vector $\mathbf{N}$

$$\mathbf{N} = \dot{\mathbf{T}} / \left\|\dot{\mathbf{T}}\right\|$$

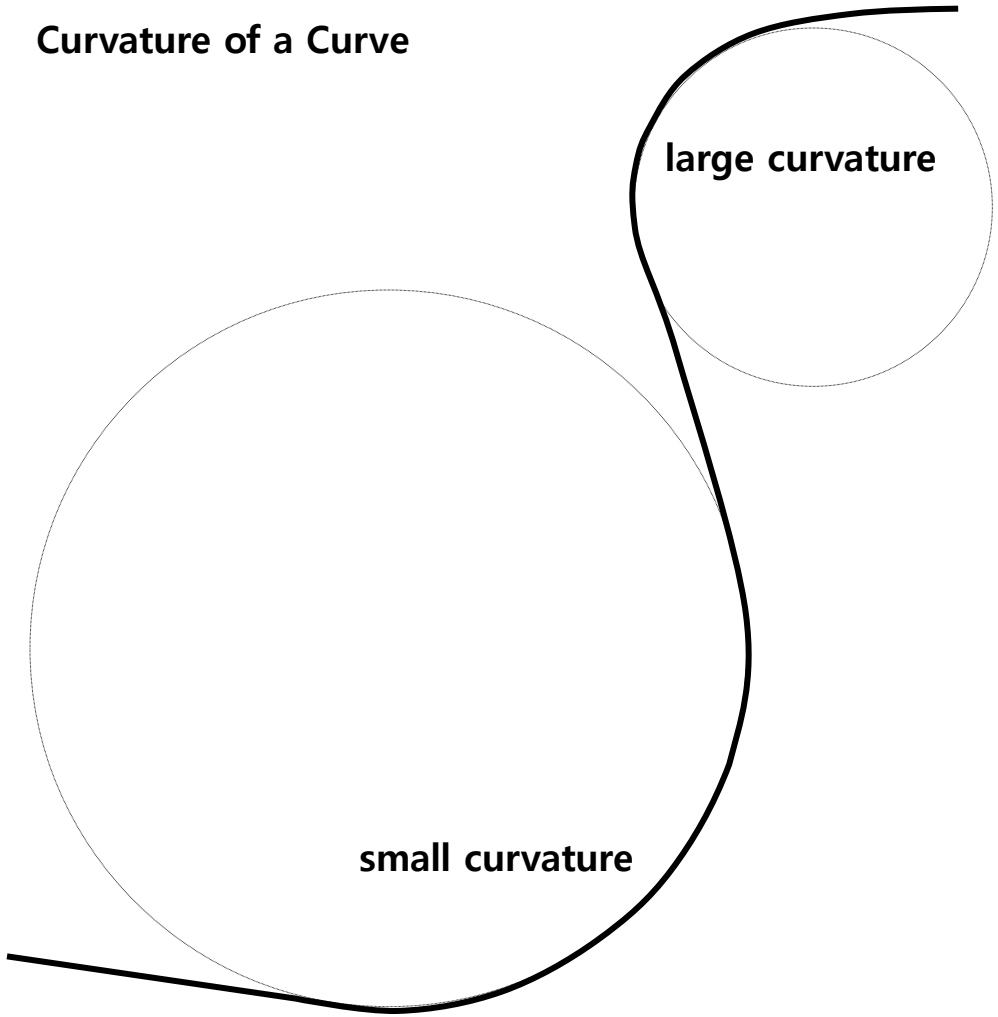
Properties of a Curve

Curvature of a Circle



Properties of a Curve

Curvature of a Curve



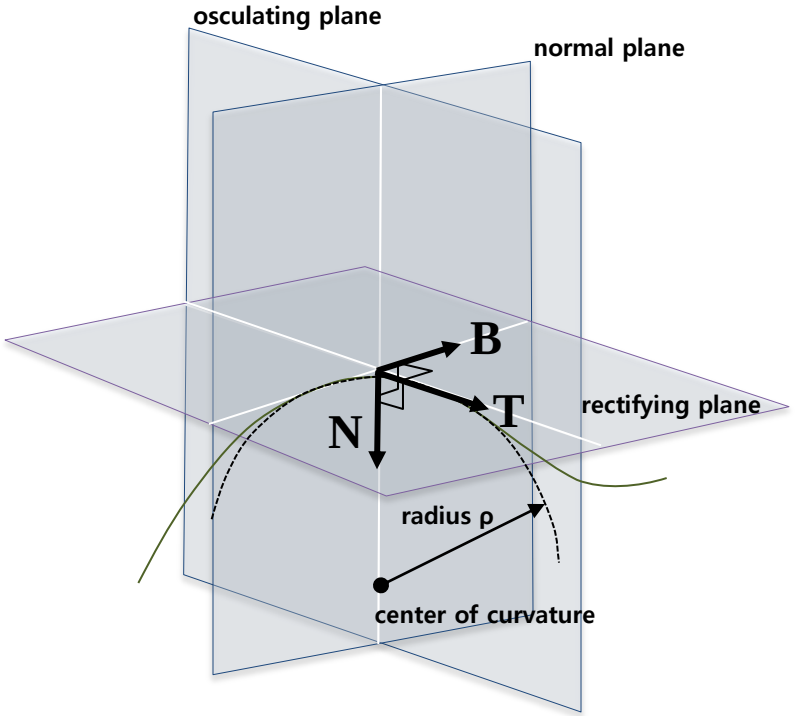
Properties of a Curve

Binormal vector $\mathbf{B}$

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

Three vectors T, N and B form a right-handed set of mutually orthogonal unit vectors

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$
$$\mathbf{T} = \mathbf{N} \times \mathbf{B}$$
$$\mathbf{N} = \mathbf{B} \times \mathbf{T}$$



The tangent vector at P

$$\lim_{\delta t \to 0} \frac{\delta \mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}$$

The unit tangent vector at P

$$\lim_{\delta s \to 0} \frac{\delta \mathbf{r}}{\delta s} = \frac{d\mathbf{r}}{ds} = \mathbf{T}$$

tangent vector $\dot{\mathbf{r}}$ & unit tangent vector $\mathbf{T}$ & parameter s

$$\dot{\mathbf{r}} = \mathbf{T}\dot{s} \quad \|\dot{\mathbf{r}}\| = \dot{s}$$

principal normal vector $\mathbf{N}$

$$\mathbf{N} = \dot{\mathbf{T}} / \|\dot{\mathbf{T}}\|$$

curvature κ

$$\kappa = \|d\mathbf{T} / ds\|$$

unit tangent vector $\mathbf{T}$ principal normal vector $\mathbf{N}$

$$\dot{\mathbf{T}} = \dot{s}\kappa\mathbf{N}$$

Properties of a Curve

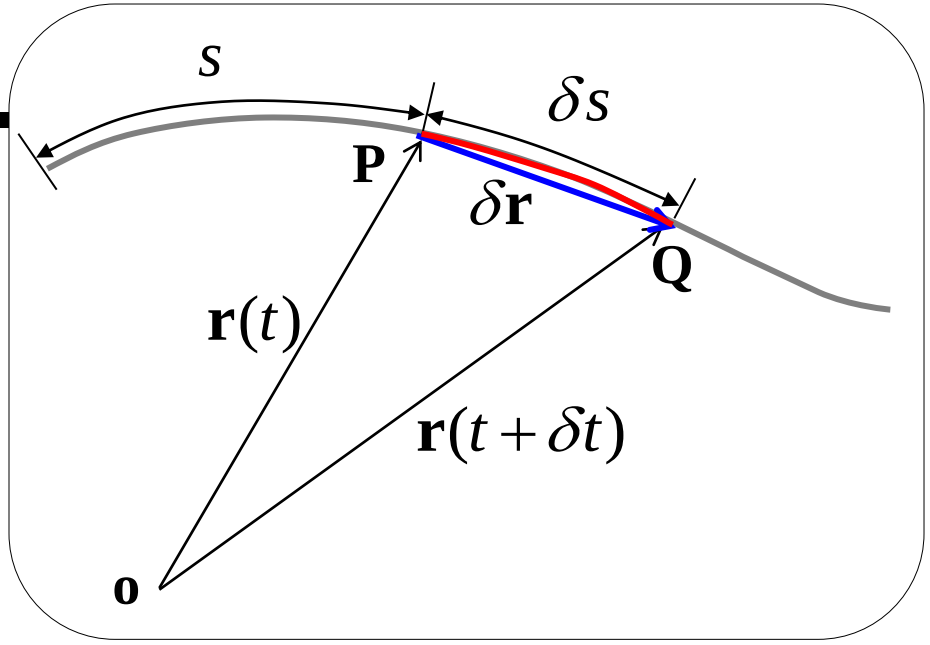
Length of curve

$$\dot{s}(t) = \lim_{\delta t \to 0} \frac{\delta s}{\delta t} = \frac{ds}{dt}$$

$$\therefore ds = \dot{s}(t)dt$$

$$\because \|\dot{\mathbf{r}}\| = \dot{s}$$

$$s = \int_{t_0}^{t_1} ds = \int_{t_0}^{t_1} \dot{s}(t)dt = \int_{t_0}^{t_1} \|\dot{\mathbf{r}}(t)\| dt$$



$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{T}\frac{ds}{dt}$$

$$\frac{d\mathbf{r}}{dt} = \mathbf{T}\left\|\frac{d\mathbf{r}}{dt}\right\|$$

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} : \text{unit tangent vector}$$

$$\therefore \dot{s} = \frac{ds}{dt} = \left\|\frac{d\mathbf{r}}{dt}\right\| = \|\dot{\mathbf{r}}\| = \sqrt{\left(\frac{df(t)}{dt}\right)^2 + \left(\frac{dg(t)}{dt}\right)^2 + \left(\frac{dh(t)}{dt}\right)^2} = \sqrt{\left[\dot{f}(t)\right]^2 + \left[\dot{g}(t)\right]^2 + \left[\dot{h}(t)\right]^2}$$

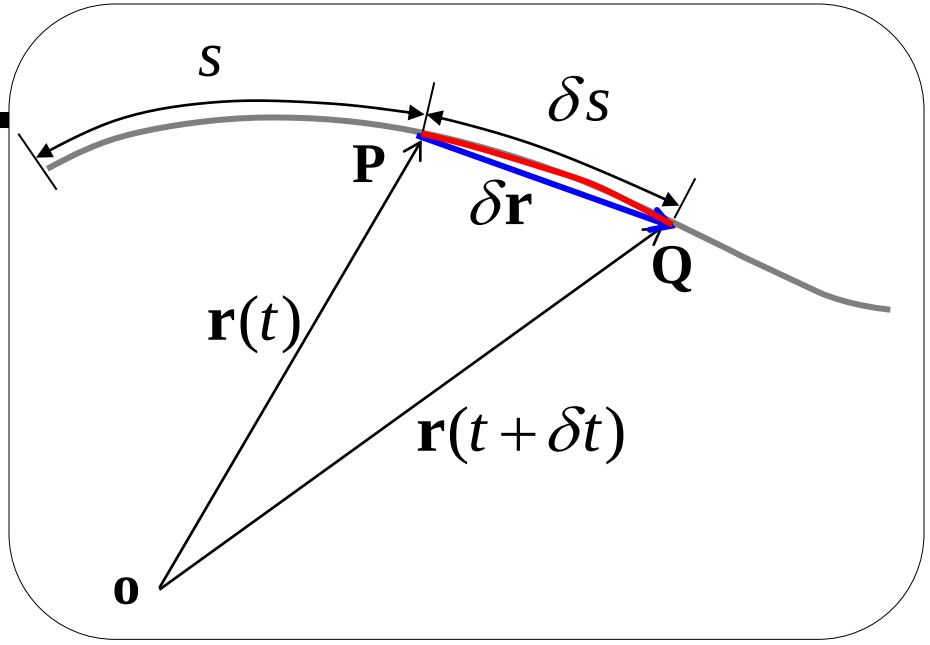
Properties of a Curve

Length of curve

$$\dot{s}(t) = \lim_{\delta t \to 0} \frac{\delta s}{\delta t} = \frac{ds}{dt}$$

$$\therefore ds = \dot{s}(t)dt$$

$$s = \int_{t_0}^{t_1} ds = \int_{t_0}^{t_1} \dot{s}(t)dt = \int_{t_0}^{t_1} \left\| \dot{\mathbf{r}}(t) \right\| dt$$



$$s = \int_{t_0}^{t_1} \left\| \dot{\mathbf{r}}(t) \right\| dt$$

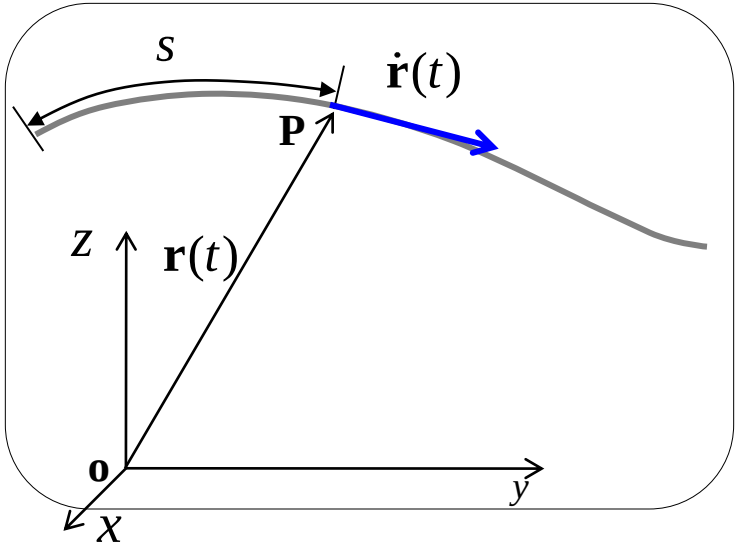
$$= \int_{t_0}^{t_1} \sqrt{\left[\dot{f}(t) \right]^2 + \left[\dot{g}(t) \right]^2 + \left[\dot{h}(t) \right]^2} dt$$

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} : \text{unit tangent vector}$$

Curvilinear Motion of Particles

Tangential and Normal Components of acceleration



The velocity of the particle on C is $\mathbf{v}(t) = \dfrac{d\mathbf{r}(t)}{dt} = \dot{\mathbf{r}}(t)$

Speed is $\dfrac{ds}{dt} = \dot{s} = v = \left\|\mathbf{v}(t)\right\|$

Acceleration $\mathbf{a}(t) = \dot{\mathbf{v}}(t) = \dfrac{d}{dt}(v\mathbf{T}) = v\dfrac{d\mathbf{T}}{dt} + \dfrac{dv}{dt}\mathbf{T}$

The tangent vector at P

$$\lim_{\delta t \to 0} \frac{\delta \mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}$$

The unit tangent vector at P

$$\lim_{\delta s \to 0} \frac{\delta \mathbf{r}}{\delta s} = \frac{d\mathbf{r}}{ds} = \mathbf{T}$$

tangent vector $\dot{\mathbf{r}}$ & unit tangent vector $\mathbf{T}$ & parameter s

$$\dot{\mathbf{r}} = \mathbf{T}\dot{s} \qquad \left\|\dot{\mathbf{r}}\right\| = \dot{s}$$

principal normal vector $\mathbf{N}$

$$\mathbf{N} = \dot{\mathbf{T}} / \left\|\dot{\mathbf{T}}\right\|$$

curvature κ

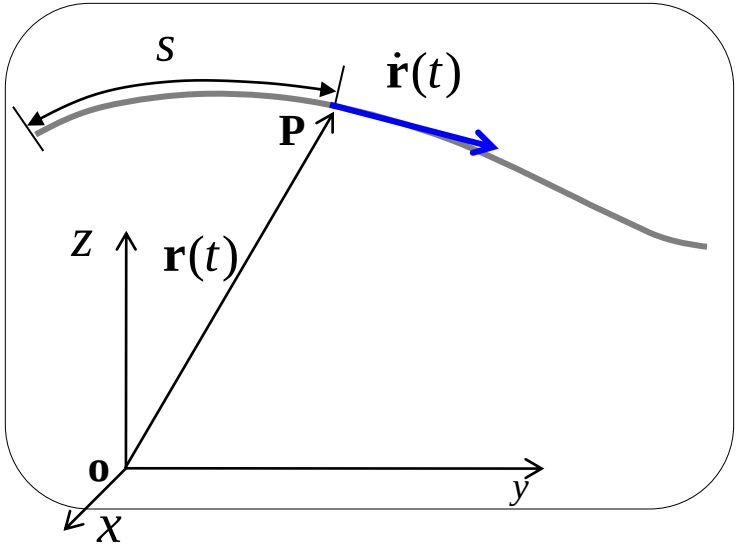
$$\kappa = \left\|d\mathbf{T} / ds\right\|$$

unit tangent vector $\mathbf{T}$ principal normal vector $\mathbf{N}$

$$\dot{\mathbf{T}} = \dot{s}\kappa\mathbf{N}$$

Curvilinear Motion of Particles

Tangential and Normal Components of acceleration



$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \dot{\mathbf{r}}(t)$$

$$\frac{ds}{dt} = \dot{s} = v = \|\mathbf{v}(t)\|$$

Acceleration

$$\mathbf{a}(t) = \dot{\mathbf{v}}(t) = \frac{d}{dt}(v\mathbf{T}) = v\frac{d\mathbf{T}}{dt} + \frac{dv}{dt}\mathbf{T}$$

$$= v\dot{s}\kappa\mathbf{N} + \frac{dv}{dt}\mathbf{T}$$

$$= \kappa v^2\mathbf{N} + \frac{dv}{dt}\mathbf{T}$$

$$\mathbf{a}(t) = a_N\mathbf{N} + a_T\mathbf{T}$$

The tangent vector at P

$$\lim_{\delta t\to 0}\frac{\delta\mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}$$

The unit tangent vector at P

$$\lim_{\delta s\to 0}\frac{\delta\mathbf{r}}{\delta s} = \frac{d\mathbf{r}}{ds} = \mathbf{T}$$

tangent vector $\dot{\mathbf{r}}$ & unit tangent vector $\mathbf{T}$ & parameter s

$$\dot{\mathbf{r}} = \mathbf{T}\dot{s} \qquad \|\dot{\mathbf{r}}\| = \dot{s}$$

principal normal vector $\mathbf{N}$

$$\mathbf{N} = \dot{\mathbf{T}} / \|\dot{\mathbf{T}}\|$$

curvature κ

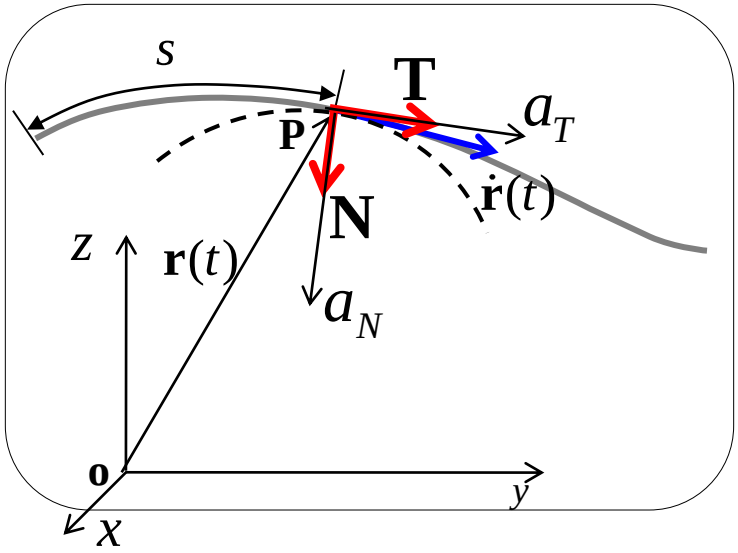
$$\kappa = \|d\mathbf{T} / ds\|$$

unit tangent vector $\mathbf{T}$ principal normal vector $\mathbf{N}$

$$\dot{\mathbf{T}} = \dot{s}\kappa\mathbf{N}$$

Curvilinear Motion of Particles

Tangential and Normal Components of acceleration



$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \dot{\mathbf{r}}(t)$$

$$\mathbf{a}(t) = a_N\mathbf{N} + a_T\mathbf{T}$$

$$= \kappa v^2\mathbf{N} + \frac{dv}{dt}\mathbf{T}$$

Normal Components of acceleration

$$a_N = \kappa v^2$$

Tangential Components of acceleration

$$a_T = \frac{dv}{dt}$$

The tangent vector at P

$$\lim_{\delta t \to 0} \frac{\delta \mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}$$

The unit tangent vector at P

$$\lim_{\delta s \to 0} \frac{\delta \mathbf{r}}{\delta s} = \frac{d\mathbf{r}}{ds} = \mathbf{T}$$

tangent vector $\dot{\mathbf{r}}$ & unit tangent vector $\mathbf{T}$ & parameter s

$$\dot{\mathbf{r}} = \mathbf{T}\dot{s} \quad \|\dot{\mathbf{r}}\| = \dot{s}$$

principal normal vector $\mathbf{N}$

$$\mathbf{N} = \dot{\mathbf{T}} / \|\dot{\mathbf{T}}\|$$

curvature κ

$$\kappa = \|d\mathbf{T} / ds\|$$

unit tangent vector $\mathbf{T}$ principal normal vector $\mathbf{N}$

$$\dot{\mathbf{T}} = \dot{s}\kappa\mathbf{N}$$

Curvilinear Motion of Particles

Normal & Tangential Components of Acceleration

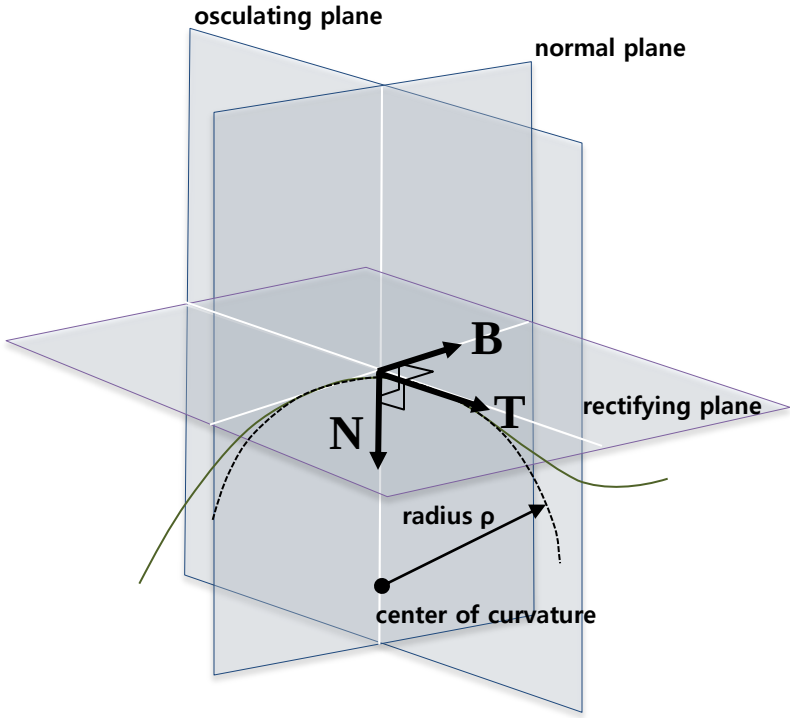
$$\mathbf{a}(t) = a_N\mathbf{N} + a_T\mathbf{T} = \kappa v^2\mathbf{N} + \frac{dv}{dt}\mathbf{T}$$

$$\mathbf{v}(t) = v\mathbf{T}$$

$$\mathbf{v}(t)\cdot\mathbf{a}(t) = a_N v\underbrace{\mathbf{T}\cdot\mathbf{N}}_{0} + a_T v\underbrace{\mathbf{T}\cdot\mathbf{T}}_{1} = a_T v$$

$$\rightarrow a_T = \frac{\mathbf{v}(t)\cdot\mathbf{a}(t)}{v}$$

$$\therefore a_T = \frac{dv}{dt} = \frac{\mathbf{v}(t)\cdot\mathbf{a}(t)}{v} = \frac{\dot{\mathbf{r}}(t)\cdot\ddot{\mathbf{r}}(t)}{\|\dot{\mathbf{r}}(t)\|}$$



Curvilinear Motion of Particles

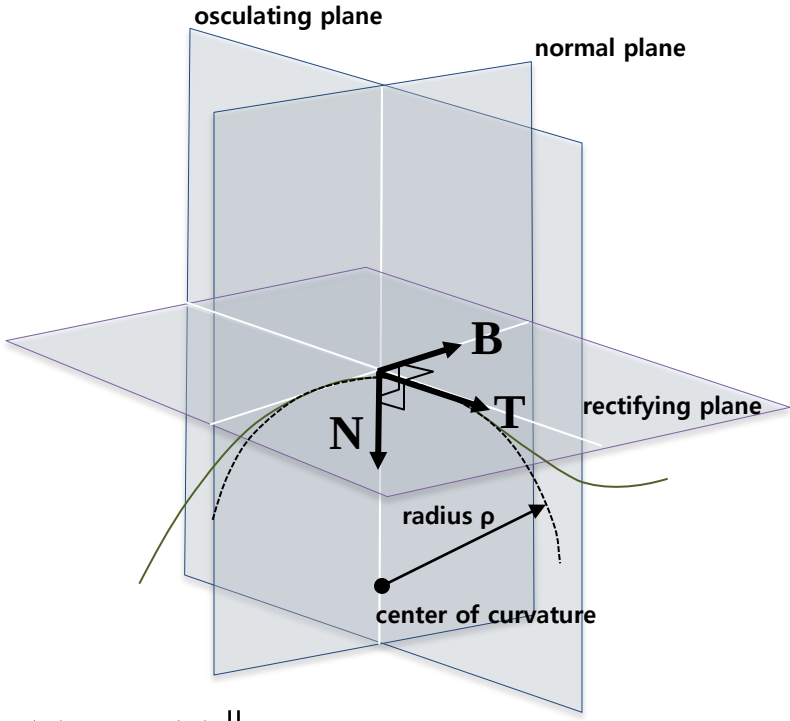
Normal & Tangential Components of Acceleration

$$\mathbf{a}(t) = a_N\mathbf{N} + a_T\mathbf{T} = \kappa v^2\mathbf{N} + \frac{dv}{dt}\mathbf{T}$$

$$\mathbf{v}(t) = v\mathbf{T}$$

$$\mathbf{v}(t)\times\mathbf{a}(t) = a_N v\underbrace{\mathbf{T}\times\mathbf{N}}_{\mathbf{B}} + a_T v\underbrace{\mathbf{T}\times\mathbf{T}}_{0} = a_N v\mathbf{B}$$

$$\rightarrow a_N = \frac{\mathbf{v}(t)\times\mathbf{a}(t)}{v\mathbf{B}}$$



$$\therefore a_N = \kappa v^2 = \left\|\frac{\mathbf{v}(t)\times\mathbf{a}(t)}{v\mathbf{B}}\right\| = \frac{\left\|\dot{\mathbf{r}}(t)\times\ddot{\mathbf{r}}(t)\right\|}{v\left\|\mathbf{B}\right\|} = \frac{\left\|\dot{\mathbf{r}}(t)\times\ddot{\mathbf{r}}(t)\right\|}{\left\|\dot{\mathbf{r}}(t)\right\|}$$

$$\therefore \kappa v^2 = \frac{\left\|\dot{\mathbf{r}}(t)\times\ddot{\mathbf{r}}(t)\right\|}{\left\|\dot{\mathbf{r}}(t)\right\|} \rightarrow \kappa = \frac{\left\|\dot{\mathbf{r}}(t)\times\ddot{\mathbf{r}}(t)\right\|}{\left\|\dot{\mathbf{r}}(t)\right\|^3}$$

Curvilinear Motion of Particles

Normal & Tangential Components of Acceleration

$$\mathbf{a}(t) = a_N\mathbf{N} + a_T\mathbf{T} = \kappa v^2\mathbf{N} + \frac{dv}{dt}\mathbf{T}$$

$$\mathbf{v}(t) = v\mathbf{T}$$

$$a_T = \frac{\dot{\mathbf{r}}(t)\cdot\ddot{\mathbf{r}}(t)}{\left\|\dot{\mathbf{r}}(t)\right\|}$$

Tangential Component of Acceleration

$$a_N = \frac{\left\|\dot{\mathbf{r}}(t)\times\ddot{\mathbf{r}}(t)\right\|}{\left\|\dot{\mathbf{r}}(t)\right\|}$$

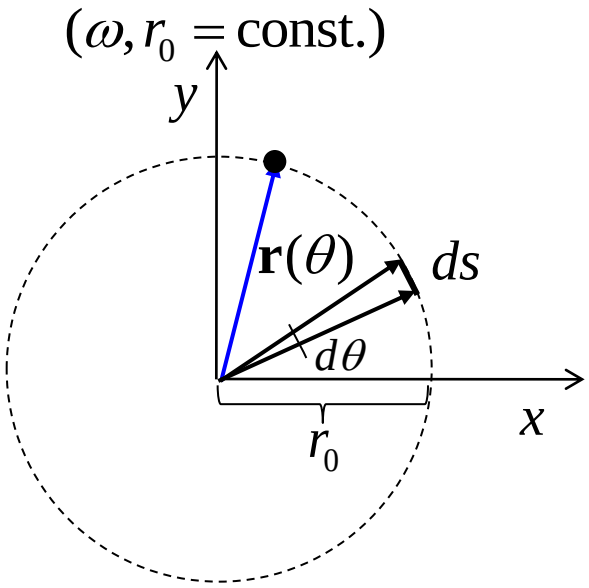
Tangential Component of Acceleration

$$\kappa = \frac{\left\|\dot{\mathbf{r}}(t)\times\ddot{\mathbf{r}}(t)\right\|}{\left\|\dot{\mathbf{r}}(t)\right\|^3}$$

Curvature

Curvilinear Motion of Particles

Ex) For circular motion in the plane, described by $\mathbf{r}(\theta) = r_0 \cos\theta\mathbf{i} + r_0 \sin\theta\mathbf{j}$



$$\mathbf{r}(\theta) = r_0 \cos\theta\mathbf{i} + r_0 \sin\theta\mathbf{j}$$

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{r_0 d\theta} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$$

$$\frac{d\mathbf{T}}{d\theta} = -\cos\theta\mathbf{i} - \sin\theta\mathbf{j}$$

$$\kappa\mathbf{N} = \frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{d\theta}\frac{d\theta}{ds} = \frac{1}{r_0}(-\cos\theta\mathbf{i} - \sin\theta\mathbf{j})$$

$$\mathbf{N} = -\cos\theta\mathbf{i} - \sin\theta\mathbf{j} \qquad \because \kappa = \frac{1}{r_0}$$

Curvilinear Motion of Particles

Normal & Tangential Components of Acceleration

$$\mathbf{a}(t) = a_N \mathbf{N} + a_T \mathbf{T} = \kappa v^2 \mathbf{N} + \frac{dv}{dt}\mathbf{T}$$

$$\mathbf{v}(t) = v\mathbf{T}$$

$$a_T = \frac{\dot{\mathbf{r}}(t) \cdot \ddot{\mathbf{r}}(t)}{\left\|\dot{\mathbf{r}}(t)\right\|}$$

$$a_N = \frac{\left\|\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)\right\|}{\left\|\dot{\mathbf{r}}(t)\right\|}$$

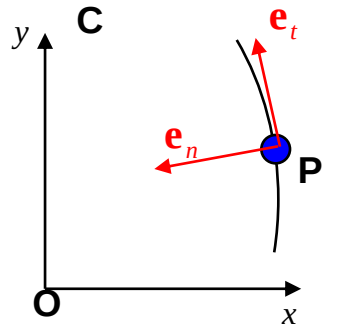
$$\kappa = \frac{\left\|\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)\right\|}{\left\|\dot{\mathbf{r}}(t)\right\|^3}$$

$$\mathbf{T} = \mathbf{e}_t$$

$$\mathbf{N} = \mathbf{e}_n$$

$$\kappa = \frac{1}{\rho}$$

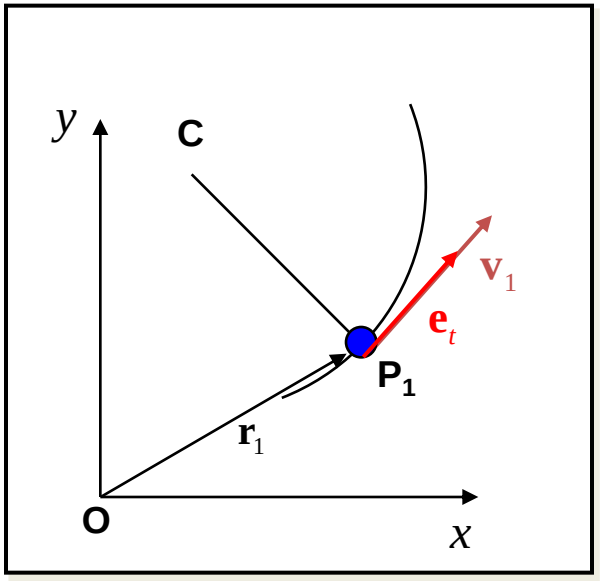
Ref. Dynamics

	Tangential & Normal
Feature	
Velocity	$\mathbf{v} = v \cdot \mathbf{e}_t$
Acceleration	$\mathbf{a} = \frac{dv}{dt}\mathbf{e}_t + \frac{v^2}{\rho}\mathbf{e}_n$

Reference slides

Dynamics

Dynamics - Tangential & Normal component



- the velocity v of the particle is tangent to the path. so,

unit tangent vector

$$\mathbf{v} = v \cdot \mathbf{e}_t$$

magnitude(speed)

- Acceleration :

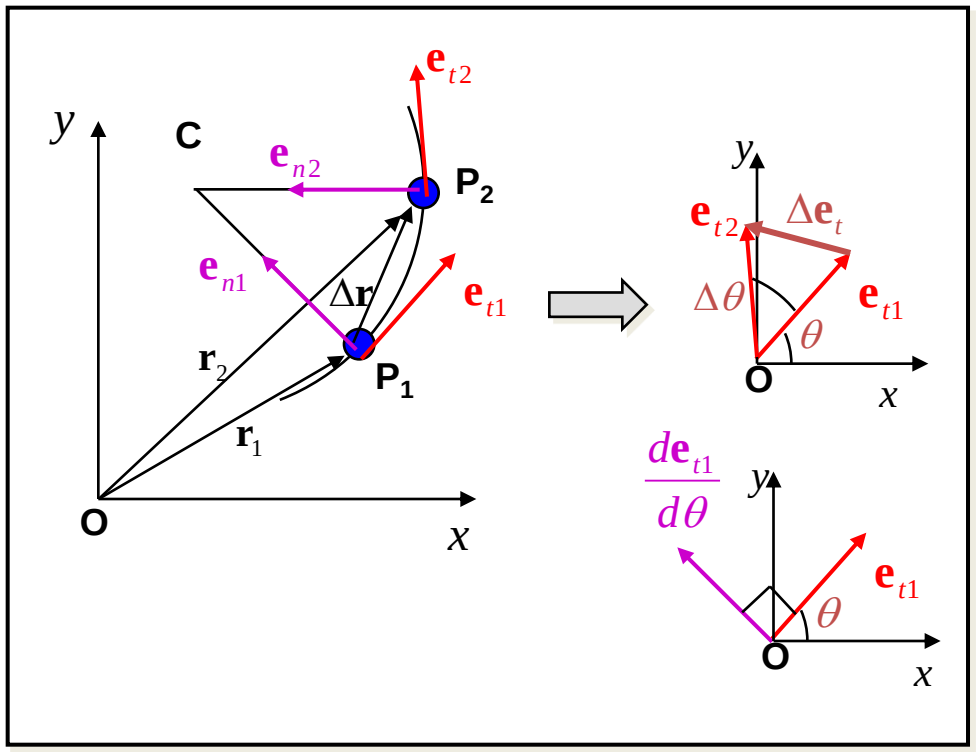
unknown term

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(v\mathbf{e}_t) = \frac{dv}{dt}\mathbf{e}_t + v\frac{d\mathbf{e}_t}{dt}$$

known terms

- To know the unknown term, differentiate the unit tangential vector $\mathbf{e}_t$ at next page.

Dynamics - Tangential & Normal component



- We can express $\mathbf{e}_{t1}, \mathbf{e}_{t2}$ with θ $\quad (\theta = \theta(t))$

$$\mathbf{e}_{t1} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

$$\mathbf{e}_{t2} = \cos(\theta+\Delta\theta)\,\mathbf{i} + \sin(\theta+\Delta\theta)\,\mathbf{j}$$

$$= \{\cos\theta\cos\Delta\theta - \sin\theta\sin\Delta\theta\}\,\mathbf{i} + \{\sin\theta\cos\Delta\theta + \cos\theta\sin\Delta\theta\}\,\mathbf{j}$$

If $\Delta\theta \to 0$, $\sin\Delta\theta \approx \Delta\theta$, $\cos\Delta\theta \approx 1$

$$\mathbf{e}_{t2} = \{\cos\theta - \sin\theta\cdot\Delta\theta\}\,\mathbf{i} + \{\sin\theta + \cos\theta\cdot\Delta\theta\}\,\mathbf{j}$$

$$\Delta\mathbf{e}_t = \mathbf{e}_{t2} - \mathbf{e}_{t1} = (-\sin\theta\cdot\Delta\theta)\,\mathbf{i} + (\cos\theta\cdot\Delta\theta)\,\mathbf{j}$$

$$\frac{\Delta\mathbf{e}_t}{\Delta\theta} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$$

$$\lim_{\Delta\theta\to 0}\frac{\Delta\mathbf{e}_t}{\Delta\theta} = \frac{d\mathbf{e}_t}{d\theta}$$

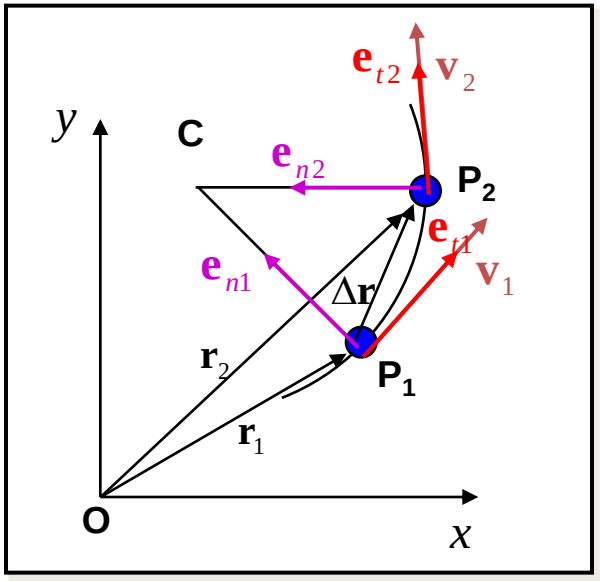
- relation between $\mathbf{e}_t$ and $\frac{d\mathbf{e}_t}{d\theta}$

differential

$$\mathbf{e}_t\cdot\mathbf{e}_t = |\mathbf{e}_t|^2 = 1 \Rightarrow \frac{d(\mathbf{e}_t\cdot\mathbf{e}_t)}{d\theta} = 2\mathbf{e}_t\cdot\frac{d\mathbf{e}_t}{d\theta} = 0 \Rightarrow \therefore \mathbf{e}_t \perp \frac{d\mathbf{e}_t}{d\theta} \quad \left(\mathbf{e}_n = \frac{d\mathbf{e}_t}{d\theta}\right)$$

Unit vector

Dynamics - Tangential & Normal component



- **Acceleration :**

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(v\mathbf{e}_t) = \frac{dv}{dt}\mathbf{e}_t + v\frac{d\mathbf{e}_t}{dt}$$

$$\frac{d\mathbf{e}_t}{dt} = \frac{d\mathbf{e}_t}{d\theta}\cdot\frac{d\theta}{dt} = \mathbf{e}_n\cdot\frac{d\theta}{dt}$$

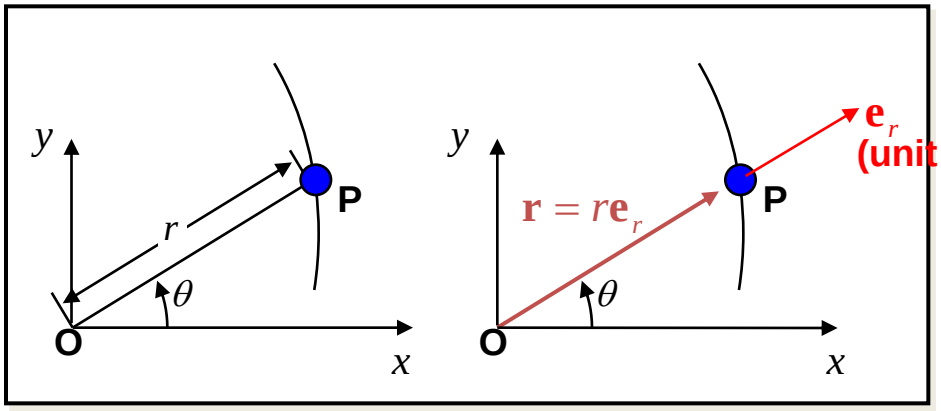
$$= \mathbf{e}_n\cdot\frac{v}{\rho}$$

$$\therefore \mathbf{a} = \frac{dv}{dt}\mathbf{e}_t + \frac{v^2}{\rho}\mathbf{e}_n$$

$\mathbf{a}_n$: **normal component of the acceleration**

$\mathbf{a}_t$: **tangential component of the acceleration**

Dynamics - Radial & Transverse component



- **Position vector r**

$$\mathbf{r} = r\mathbf{e}_r$$

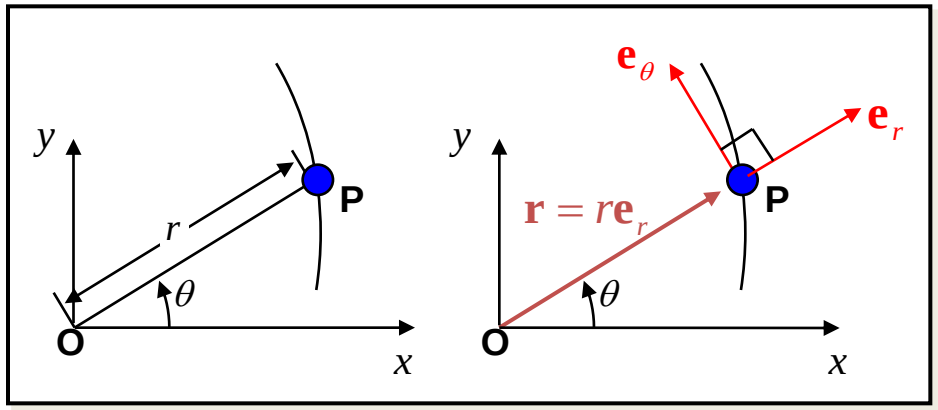
- **Velocity**

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(r\mathbf{e}_r) = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt}$$

unknown terms

known terms

Dynamics - Radial & Transverse component



$$\mathbf{P} = (r\cos\theta, r\sin\theta) \qquad (r = r(t)\ ,\ \theta = \theta(t))$$

$$\mathbf{e}_r = \mathbf{e}_r(\theta) = (\cos\theta, \sin\theta)$$

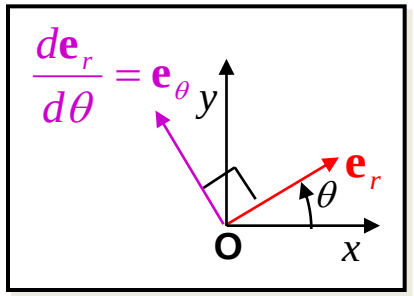
$$\mathbf{e}_\theta = \mathbf{e}_\theta(\theta) = \left(\cos\left(\frac{\pi}{2}+\theta\right), \sin\left(\frac{\pi}{2}+\theta\right)\right)$$

$$= (-\sin\theta, \cos\theta)$$

- time derivatives of $\mathbf{e}_r$

$$\mathbf{e}_r \cdot \mathbf{e}_r = |\mathbf{e}_r|^2 = 1 \Rightarrow \frac{d(\mathbf{e}_r \cdot \mathbf{e}_r)}{d\theta} = 2\mathbf{e}_r \cdot \frac{d\mathbf{e}_r}{d\theta} = 0$$

$$\Rightarrow \therefore \mathbf{e}_r \perp \frac{d\mathbf{e}_r}{d\theta} \Rightarrow \frac{d\mathbf{e}_r}{d\theta} = \mathbf{e}_\theta$$

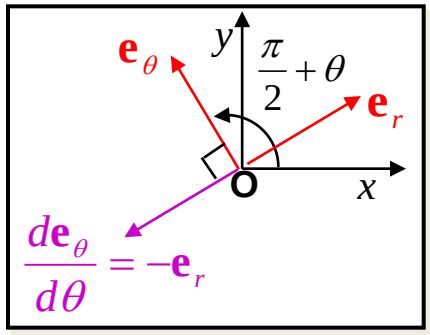


$$\frac{d\mathbf{e}_r}{dt} = \frac{d\mathbf{e}_r}{d\theta}\cdot\frac{d\theta}{dt} = \mathbf{e}_\theta \frac{d\theta}{dt} = \mathbf{e}_\theta \dot{\theta}$$

- time derivatives of $\mathbf{e}_\theta$

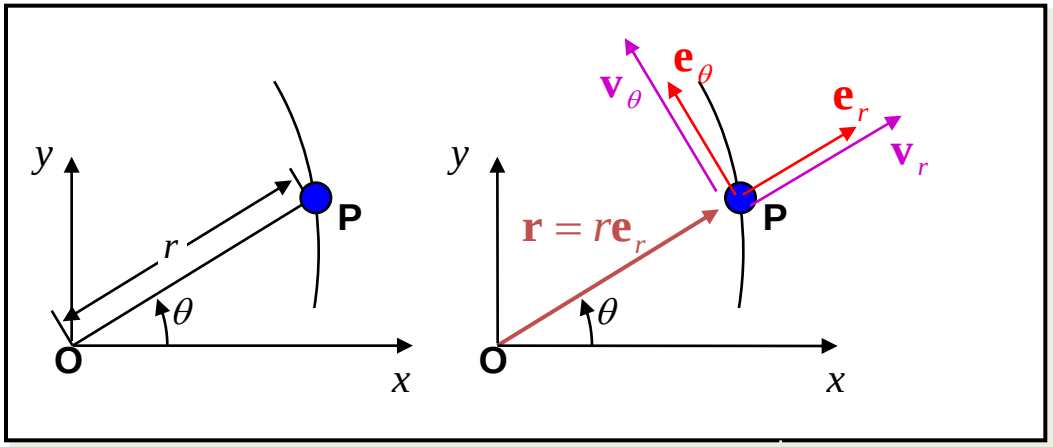
$$\mathbf{e}_\theta \cdot \mathbf{e}_\theta = |\mathbf{e}_\theta|^2 = 1 \Rightarrow \frac{d(\mathbf{e}_\theta \cdot \mathbf{e}_\theta)}{d\theta} = 2\mathbf{e}_\theta \cdot \frac{d\mathbf{e}_\theta}{d\theta} = 0$$

$$\Rightarrow \therefore \mathbf{e}_\theta \perp \frac{d\mathbf{e}_\theta}{d\theta} \Rightarrow \frac{d\mathbf{e}_\theta}{d\theta} = (-\cos\theta, -\sin\theta) = -\mathbf{e}_r$$



$$\frac{d\mathbf{e}_\theta}{dt} = \frac{d\mathbf{e}_\theta}{d\theta}\cdot\frac{d\theta}{dt} = -\mathbf{e}_r \frac{d\theta}{dt} = -\mathbf{e}_r \dot{\theta}$$

Dynamics - Radial & Transverse component



$$\left(\frac{d\mathbf{e}_r}{dt} = \mathbf{e}_\theta \dot{\theta} \, , \frac{d\mathbf{e}_\theta}{dt} = -\mathbf{e}_r \dot{\theta} \right)$$

- **Position Vector**

$$\mathbf{r} = r\,\mathbf{e}_r$$

- **Velocity**

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(r\mathbf{e}_r) = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt}$$

$$= \underbrace{\dot{r}}_{v_r}\mathbf{e}_r + \underbrace{r\dot{\theta}}_{v_\theta}\mathbf{e}_\theta$$

- **Acceleration**

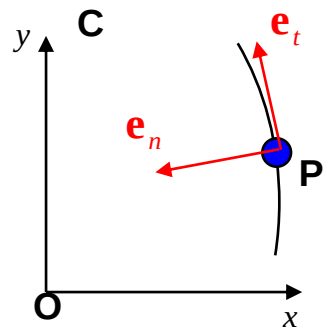
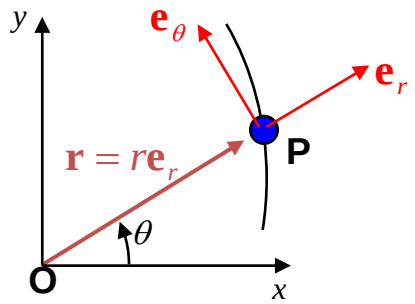
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\dot{r}\,\mathbf{e}_r + r\dot{\theta}\,\mathbf{e}_\theta)$$

$$= \ddot{r}\mathbf{e}_r + \dot{r}\frac{d\mathbf{e}_r}{dt} + \dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta + r\dot{\theta}\frac{d\mathbf{e}_\theta}{dt}$$

$$= \ddot{r}\mathbf{e}_r + \dot{r}(\mathbf{e}_\theta\dot{\theta}) + \dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta + r\dot{\theta}(-\mathbf{e}_r\dot{\theta})$$

$$= \underbrace{(\ddot{r} - r\dot{\theta}^2)}_{a_r}\mathbf{e}_r + \underbrace{(r\ddot{\theta} + 2\dot{r}\dot{\theta})}_{a_\theta}\mathbf{e}_\theta$$

Dynamics - Radial & Transverse component

Summary	Tangential & Normal	Radial & Transverse
Feature		
Velocity	$\mathbf{v} = v \cdot \mathbf{e}_t$	$\mathbf{v} = \dot{r}\ \mathbf{e}_r + r\dot{\theta}\ \mathbf{e}_\theta$
Acceleration	$\mathbf{a} = \frac{dv}{dt}\mathbf{e}_t + \frac{v^2}{\rho}\mathbf{e}_n$	$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\ \mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\ \mathbf{e}_\theta$

angle : θ
angular velocity : $\omega = \dot{\theta}$
angular acceleration :
$\alpha = \dot{\omega} = \ddot{\theta}$

- **Uniform circular motion** $(\frac{d\theta}{dt} = 0)$

$$\mathbf{v} = \dot{r}\ \mathbf{e}_r + r\dot{\theta}\ \mathbf{e}_\theta = \dot{r}\mathbf{e}_r$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\ \mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\ \mathbf{e}_\theta = \ddot{r}\mathbf{e}_r$$

- **Uniformly accelerated circular motion** $(\frac{d\theta}{dt} = C)$

$$\mathbf{v} = \dot{r}\ \mathbf{e}_r + r\dot{\theta}\ \mathbf{e}_\theta$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\ \mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\ \mathbf{e}_\theta = (\ddot{r} - r\dot{\theta}^2)\ \mathbf{e}_r + 2\dot{r}\dot{\theta}\mathbf{e}_\theta$$

09 Vector Calculus (2) : Vector Functions, Vector Fields

Vector Functions

Partial Derivatives

Directional Derivatives

Tangent Planes and Normal Lines

Vector Fields and Fluid Mechanics

Seoul National Univ.

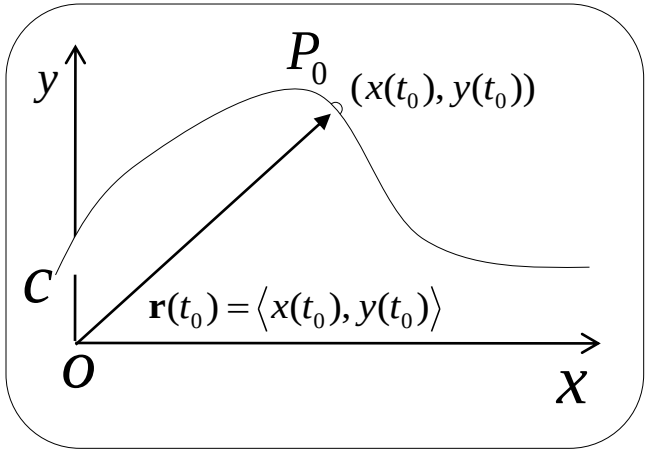
@SDAL
Advanced Ship Design Automation Lab.
http://asdal.snu.ac.kr

Vector Functions

■ Parametric Curve

■ Vector Function

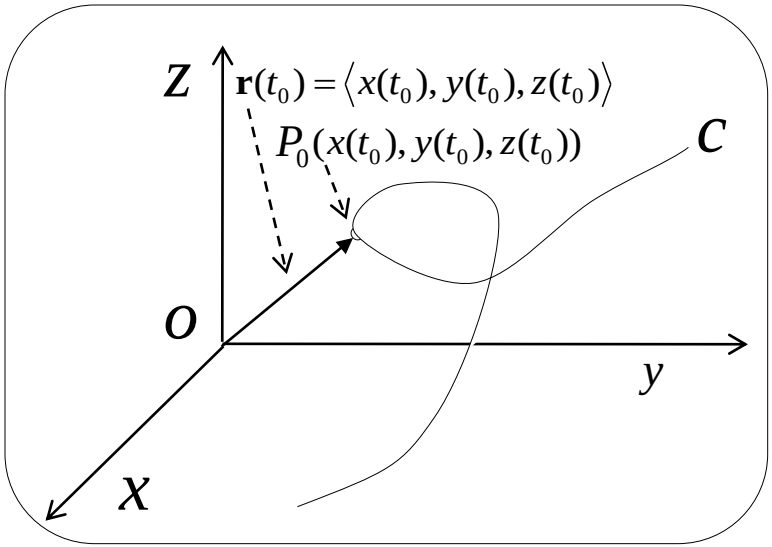
2-D



$$\mathbf{r}(t) = \langle x(t), y(t) \rangle$$
$$= f(t)\mathbf{i} + g(t)\mathbf{j}$$

$$x(t) = f(t)$$
$$y(t) = g(t)$$

3-D



$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$
$$= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

$$x(t) = f(t)$$
$$y(t) = g(t)$$
$$z(t) = h(t)$$

For a given parameter t_0 , $\mathbf{r}(t_0)$ is a position vector of a point P on a curve C

Vector Functions

Definition 9.1 **Limit of a Vector Function**

If $\lim_{t\to a} f(t)$, $\lim_{t\to a} g(t)$, and $\lim_{t\to a} h(t)$ exist, then

$$\lim_{t\to a}\mathbf{r}(t)=\left\langle \lim_{t\to a} f(t),\lim_{t\to a} g(t),\lim_{t\to a} h(t)\right\rangle$$

Theorem 9.1 **Properties of Limits**

If $\lim_{t\to a}\mathbf{r}_1(t)=\mathbf{L}_1$ and $\lim_{t\to a}\mathbf{r}_2(t)=\mathbf{L}_2$, then

(i) $\lim_{t\to a} c\mathbf{r}_1(t)=c\mathbf{L}_1,\quad c$ is a scalar

(ii) $\lim_{t\to a}\left[\mathbf{r}_1(t)+\mathbf{r}_2(t)\right]=\mathbf{L}_1+\mathbf{L}_1$

(iii) $\lim_{t\to a}\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)=\mathbf{L}_1\cdot\mathbf{L}_1$

Vector Functions

Definition 9.2 **Continuity**

A vector function $\mathbf{r}$ is said to be continuous at $t = a$ if

$(i)\ \mathbf{r}(a)$ is defined

$(ii)\ \mathbf{r}(t)$ exists, and

$(iii)\ \lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$

Definition 9.3 **Derivatives of Vector Function**

The derivative of a vector $\mathbf{r}$ is $\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\mathbf{r}(t + \Delta t) - \mathbf{r}(t)\right]$

for all t for which the limit exists

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt}$$

Vector Functions

Theorem 9.2 **Differentiation of Components**

If $\mathbf{r}(t) = \langle f(x), g(x), h(x) \rangle$ where f, g and h are differentiable, then

$$\mathbf{r}'(t) = \langle f'(x), g'(x), h'(x) \rangle$$

■ Proof

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} [\mathbf{r}(t + \Delta t) - \mathbf{r}(t)]$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \langle f(t + \Delta t), g(t + \Delta t), h(t + \Delta t) \rangle - \langle f(t), g(t), h(t) \rangle$$

$$= \lim_{\Delta t \to 0} \left\langle \frac{f(t + \Delta t) - f(t)}{\Delta t}, \frac{g(t + \Delta t) - g(t)}{\Delta t}, h\frac{f(t + \Delta t) - h(t)}{\Delta t} \right\rangle$$

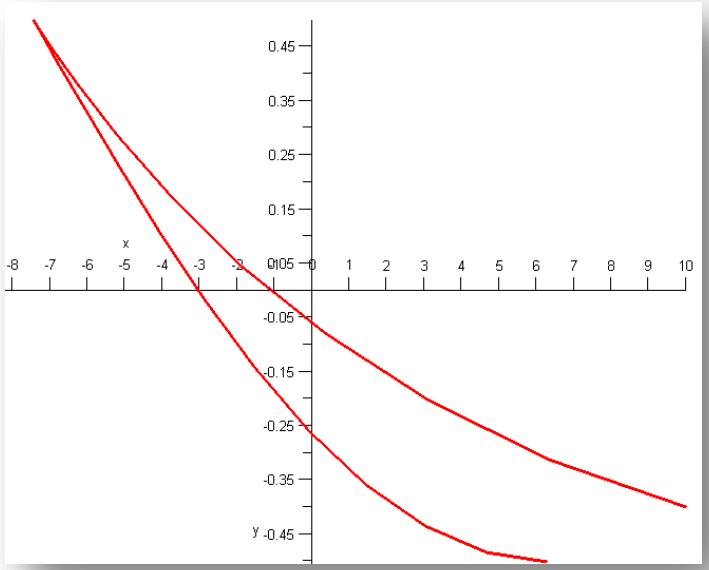
Vector Functions

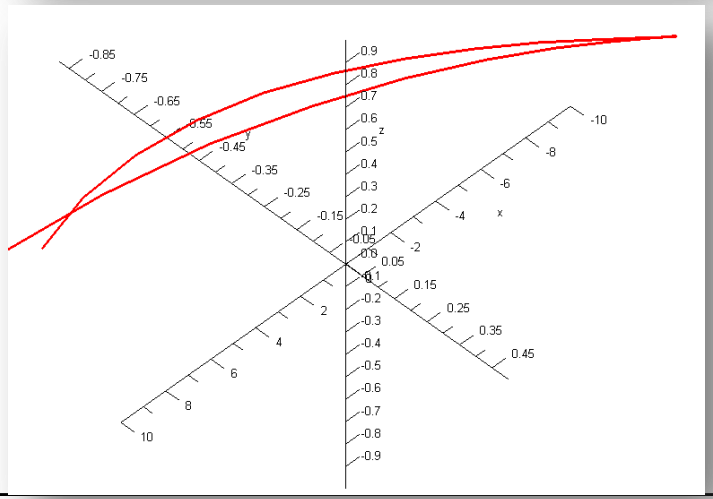
■ Smooth curve

When the component function of a vector function $\mathbf{r}$ have continuous first derivatives and $\mathbf{r}'(t) \neq 0$ for all t in the open interval (a, b), then $\mathbf{r}$ is said to be a smooth function and the curve C traced by $\mathbf{r}$ is called a smooth curve

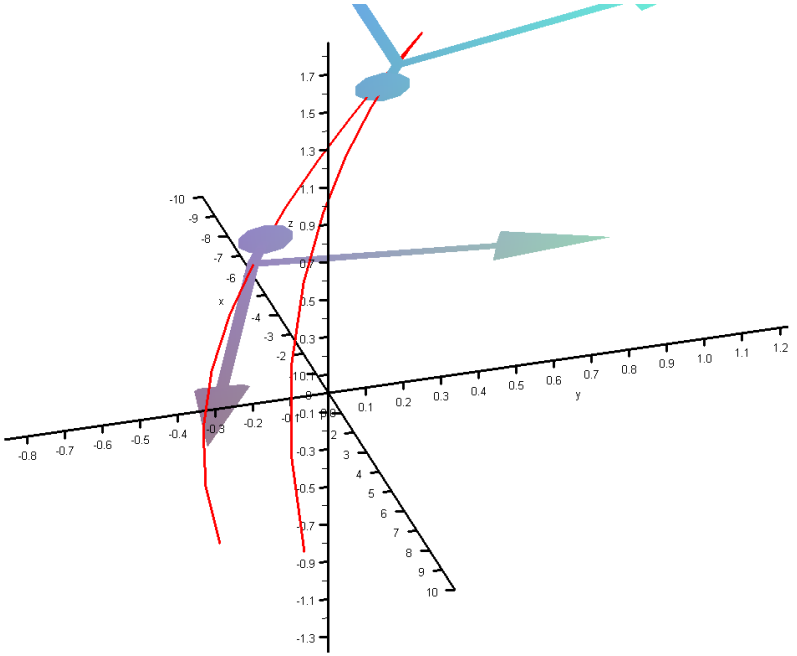
Vector Functions

■ Not Smooth curve ex.)





$$F(t) = \left(2t^3 + (4 - \frac{3}{2}\pi)t^2 - 4\pi t + 2\pi\right)\mathbf{i} + \left(-\frac{1}{2}\cos 2t\right)\mathbf{j} + (\sin t)\mathbf{k}$$



Vector Functions

Theorem 9.2 **Differentiation of Components**

If $\mathbf{r}(t) = \langle f(x), g(x), h(x) \rangle$ where f, g and h are differentiable, then

$$\mathbf{r}'(t) = \langle f'(x), g'(x), h'(x) \rangle$$

■ High-Order Derivatives

Also obtained by differentiating its components.

Ex) Second-order

$$\mathbf{r}''(t) = \langle f''(x), g''(x), h''(x) \rangle$$

Vector Functions

☑ Example 4
Tangent Vectors

Graph the curve C that is traced by a point P whose position is given by $\mathbf{r}(t)=\cos 2t\mathbf{i}+\sin t\mathbf{j}$, $0\le t\le 2\pi$. Graph $\mathbf{r}'(0)$ and $\mathbf{r}'(\pi/6)$.

Solution)

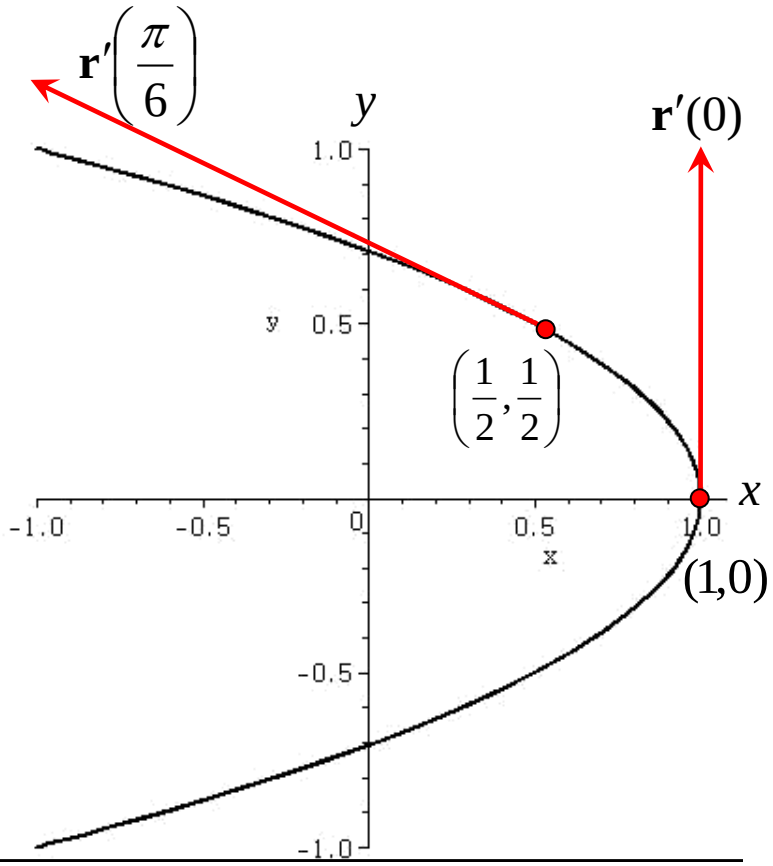
$$\begin{cases} x=\cos 2t=\cos^2 t-\sin^2 t \quad (0\le t\le 2\pi) \\ \quad =(1-\sin^2 t)-\sin^2 t \\ \quad =1-2\sin^2 t \\ y=\sin t \end{cases}$$

$$x=1-2y^2,\ -1\le x\le 1$$

$$\mathbf{r}'(t)=-2\sin 2t\mathbf{i}+\cos t\mathbf{j}$$

$$\mathbf{r}'(0)=\mathbf{j}$$

$$\mathbf{r}'\left(\frac{\pi}{6}\right)=-\sqrt{3}\mathbf{i}+\frac{\sqrt{3}}{2}\mathbf{j}$$



Vector Functions

☑ Example 5 Tangent Line

Find parametric equations of the tangent line to the graph of the curve C whose parametric equations are $x=t^2$, $y=t^2-t$, $z=-7t$ at $t=3$.

Solution)

$$\mathbf{r}(t) = t^2\mathbf{i} + (t^2 - t)\mathbf{j} - 7t\mathbf{k}$$

$$\mathbf{r}(3) = 9\mathbf{i} + 6\mathbf{j} - 21\mathbf{k}$$

$$\mathbf{r}'(t) = 2t\mathbf{i} + (2t - 1)\mathbf{j} - 7\mathbf{k}$$

$$\mathbf{r}'(3) = 6\mathbf{i} + 5\mathbf{j} - 7\mathbf{k}$$

$$\mathbf{r}(3) + \mathbf{r}'(3)\cdot t = (9\mathbf{i} + 6\mathbf{j} - 21\mathbf{k}) + (6\mathbf{i} + 5\mathbf{j} - 7\mathbf{k})\cdot t$$
$$= (9 + 6t)\mathbf{i} + (6 + 5t)\mathbf{j} + (-21 - 7t)\mathbf{k}$$

$$x = 9 + 6t,\ y = 6 + 5t,\ z = -21 - 7t$$

Vector Functions

☑ Example 6
Derivative of a Vector Function

If $\mathbf{r}(t)=(t^3-2t^2)\mathbf{i}+4t\mathbf{j}+e^{-t}\mathbf{k}$,

then $\mathbf{r}'(t)=(3t^2-4t)\mathbf{i}+4\mathbf{j}-e^{-t}\mathbf{k}$,

and $\mathbf{r}''(t)=(6t-4)\mathbf{i}+e^{-t}\mathbf{k}$.

Vector Functions

Theorem 9.3 **Chain Rule**

If $\mathbf{r}$ is a differentiable vector function and $s = u(t)$ is a differentiable scalar function, then, the derivatives of $\mathbf{r}(s)$ with respect to t is

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{r}'(s)u'(t)$$

Vector Functions

☑ Example 7
Chain Rule

If $\mathbf{r}(s)=\cos 2s\mathbf{i}+\sin 2s\mathbf{j}+e^{-3s}\mathbf{k}$, where $s=t^4$, then

$$\frac{d\mathbf{r}}{dt}=\left[-2\sin 2s\mathbf{i}+2\cos 2s\mathbf{j}-3e^{-3s}\mathbf{k}\right]4t^3$$

$$=-8t^3\sin(2t^4)\mathbf{i}+8t^3\cos(2t^4)\,j-12t^3e^{-3t^4}\mathbf{k}.$$

Vector Functions

Theorem 9.4

Rules of Differentiation

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be differentiable vector functions $u(t)$ a differentiable scalar function.

$$(i)\ \frac{d}{dt}[\mathbf{r}_1(t)+\mathbf{r}_2(t)]=\mathbf{r}_1'(t)+\mathbf{r}_2'(t)$$

$$(ii)\ \frac{d}{dt}[u(t)\mathbf{r}_1(t)]=u(t)\mathbf{r}_1'(t)+u'(t)\mathbf{r}_1(t)$$

$$(iii)\ \frac{d}{dt}[\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)]=\mathbf{r}_1'(t)\cdot\mathbf{r}_2(t)+\mathbf{r}_1(t)\cdot\mathbf{r}_2'(t)$$

$$(iv)\ \frac{d}{dt}[\mathbf{r}_1(t)\times\mathbf{r}_2(t)]=\mathbf{r}_1'(t)\times\mathbf{r}_2(t)+\mathbf{r}_1(t)\times\mathbf{r}_2'(t)$$

Differentiation of Cross Product

Vector Functions

■ Integral of Vector Functions

If f, g and h are integrable, then the indefinite and definite integrals of a vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ are defined, respectively, by

$$\int \mathbf{r}(t)dt = \left(\int f(t)dt\right)\mathbf{i} + \left(\int g(t)dt\right)\mathbf{j} + \left(\int h(t)dt\right)\mathbf{k}$$

$$\int_a^b \mathbf{r}(t)dt = \left(\int_a^b f(t)dt\right)\mathbf{i} + \left(\int_a^b g(t)dt\right)\mathbf{j} + \left(\int_a^b h(t)dt\right)\mathbf{k}$$

$$\int \mathbf{r}(t)dt = \mathbf{R} + \mathbf{c}, \quad where\ \mathbf{R}'(t) = \mathbf{r}(t)$$

☑ Example 8

Integral of a Vector Function

If $\mathbf{r}(t)=6t^2\mathbf{i}+4e^{-2t}\mathbf{j}+8\cos 4t\mathbf{k}$ then

$$\int r(t)dt = \left|\int 6t^2 dt\right|\mathbf{i} + \left|\int 4e^{-2t}dt\right|\mathbf{j} + \left|\int 8\cos 4t dt\right|\mathbf{k}$$

$$= \left[2t^3 + c_1\right]\mathbf{i} + \left[-2e^{-2t} + c_2\right]\mathbf{j} + \left[2\sin 4t + c_3\right]\mathbf{k}$$

$$= 2t^3\mathbf{i} + -2e^{-2t}\mathbf{j} + 2\sin 4t\mathbf{k} + \mathbf{c},$$

where $\mathbf{c}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$.

Vector Functions

☑ Example 9

Consider the helix of Example 1. Find the length of the curve from $\mathbf{r}(0)$ to an arbitrary point $\mathbf{r}(t)$. Find the parametric equations of the helix.

Solution)

$$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}$$

$$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$$

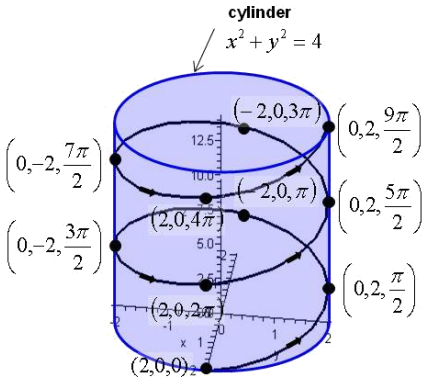
$$\|\mathbf{r}'(t)\| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 1^2}$$

$$= \sqrt{4\sin^2 t + 4\cos^2 t + 1} = \sqrt{5}$$

$$\|\mathbf{r}'(t)\| = \sqrt{5}$$

9.1 Example 1.

$$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}, \quad t \geq 0.$$



$$s = \int_0^t \|\mathbf{r}'(u)\| du = \int_0^t \sqrt{5} du = \sqrt{5}t$$

$$\mathbf{r}(s) = 2\cos\frac{s}{\sqrt{5}}\mathbf{i} + 2\sin\frac{s}{\sqrt{5}}\mathbf{j} + \frac{s}{\sqrt{5}}\mathbf{k}$$

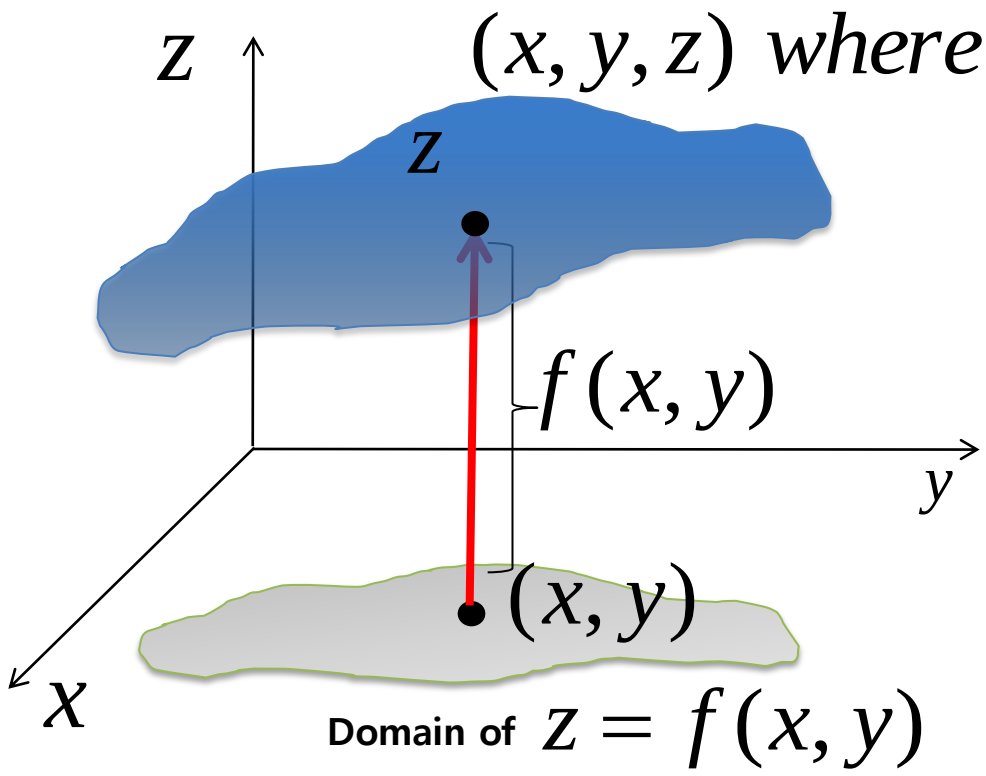
$$f(s) = 2\cos\frac{s}{\sqrt{5}}$$

$$g(s) = 2\sin\frac{s}{\sqrt{5}}$$

$$h(s) = \frac{s}{\sqrt{5}}$$

Partial Derivatives

Function of two variables $z = f(x, y)$



x, y : independent variables

z : dependent variable

Partial Derivatives

Ordinary Derivatives

$$y = f(x)$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Partial Derivatives

$$z = f(x, y)$$

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x} = \frac{\partial f}{\partial x} = z_x = f_x$$

partial derivative with respect to x

treating y **as a constant**

$$\frac{\partial z}{\partial y} = \lim_{\Delta x \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y} = \frac{\partial f}{\partial y} = z_y = f_y$$

partial derivative with respect to y

treating x **as a constant**

Partial Derivatives

☑ Example 3

Partial Derivatives

If $z=4x^3y^2-4x^2+y^6+1$, find $\partial z/\partial x$ and $\partial z/\partial y$.

$$\frac{\partial z}{\partial x} = 12x^2y^2 - 8x$$

$$\frac{\partial z}{\partial y} = 8x^3y^2 + 6y^5$$

Partial Derivatives

☑ Example 4

Partial Derivatives

If $F(x,y,t)=e^{-3\pi t}\cos 4x \sin 6y$, then the partial derivatives with respect to x, y, and t are, in turn,

$$F_x(x,y,t) = -4e^{-3\pi t}\sin 4x \sin 6y$$

$$F_y(x,y,t) = 6e^{-3\pi t}\cos 4x \cos 6y$$

$$F_t(x,y,t) = -3\pi e^{-3\pi t}\cos 4x \sin 6y.$$

Partial Derivatives

Chain Rule

$$y = f(u)\ , u = g(x)$$

$$\rightarrow y = f(g(x))$$

$$\frac{dy}{dx} = \frac{\partial y}{\partial u}\frac{du}{dx}$$

Theorem 9.5 **Chain Rule**

If $z = f(u,v)$ **is differentiable and** $u = g(x,y)$ **and** $v = h(x,y)$ **have continuous first partial derivatives, then**

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$z = f(u,v) = f(g(x,y), h(x,y))$$

Partial Derivatives

☑ Example 5

Chain Rule

If $z=u^2-v^3$ and $u=e^{2x-3y}$, $v=\sin(x^2-y^2)$, find $\partial z/\partial x$ and $\partial z/\partial y$.

Solution)

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$=2u\bullet(2e^{2x-3y})-3v^2\left[2x\cos(x^2-y^2)\right]$$

$$=4u2e^{2x-3y}-6v^2x\cos(x^2-y^2)$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$=2u\bullet(-3e^{2x-3y})-3v^2\left[(-2y)\cos(x^2-y^2)\right]$$

$$=-6u2e^{2x-3y}+6yv^2\cos(x^2-y^2)$$

Partial Derivatives

Chain Rule

$$z = f(u,v)\ ,\ u = g(x,y)\ ,\ v = h(x,y)$$

$$z = f(u,v) = f(g(x,y), h(x,y))$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

---

special case

$$z = f(u,v)\ ,\ u = g(t)\ ,v = h(t)$$

$$z = f(u,v) = f(g(t), h(t))$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt}$$

Partial Derivatives

Chain Rule

example)

If $r(x,y,z) = x^2+y^5z^3$
and $x(u,v)=uve^{2s}$, $y(u,v)=u^2-v^2s$, $z(u,v)=\sin(uvs^2)$,

find $\partial r/\partial s$.

Chain Rule

solution)

$$\frac{\partial r}{\partial s} = \frac{\partial r}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial r}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial r}{\partial z}\frac{\partial z}{\partial s}$$

$$= 2x(2uve^{2s}) + 5y^{4z^3}(-v^2) + 3y^5z^2\left(2uvs\cos(uvs^2)\right)$$

Partial Derivatives

special case

$$z = f(u,v)\ ,\ u = g(t)\ \ ,v = h(t)$$

$$z = f(u,v) = f(g(t),h(t))$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt}$$

---

generalization

$$z = f(u_1,u_2,...,u_n) \qquad u_1 = g_1(t)$$

$$u_2 = g_2(t)$$

$$\vdots$$

$$u_n = g_n(t)$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial u_1}\frac{du_1}{dt} + \frac{\partial z}{\partial u_2}\frac{du_2}{dt} + \cdots + \frac{\partial z}{\partial u_n}\frac{du_n}{dt}$$

Partial Derivatives

Chain Rule $z = f(u,v)\ ,\ u = g(x,y)\ ,\ v = h(x,y)$

$$z = f(u,v) = f(g(t),h(t))$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

---

generalization

$$z = f(u_1, u_2, \ldots, u_n)$$

$$u_1 = g_1(x_1, x_2, \ldots, x_k)$$
$$u_2 = g_2(x_1, x_2, \ldots, x_k)$$
$$\vdots$$
$$u_n = g_n(x_1, x_2, \ldots, x_k)$$

$$\frac{dz}{dx_i} = \frac{\partial z}{\partial u_1}\frac{\partial u_1}{\partial x_i} + \frac{\partial z}{\partial u_2}\frac{\partial u_2}{\partial x_i} + \cdots + \frac{\partial z}{\partial u_n}\frac{\partial u_n}{\partial x_i}$$

Gradient of a Function

Introduce a new vector based on partial differentiation

Differentiable function

$$z = f(x, y)$$

$$w = F(x, y, z)$$

Apply Vector differential operator

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}$$

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

∇ : "del" or "nabla"

∇f : "grad f "

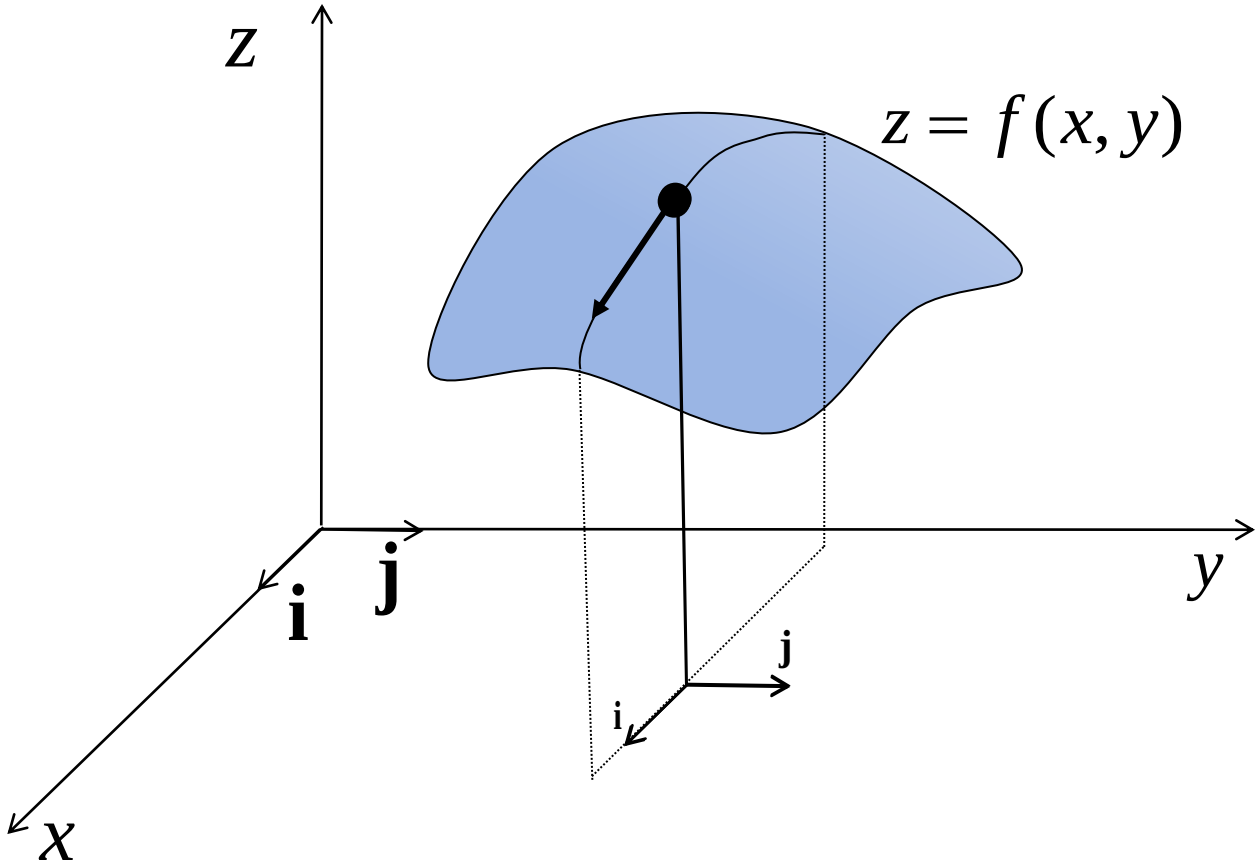
Gradients of function

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

$$\nabla F(x, y, z) = \frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k}$$

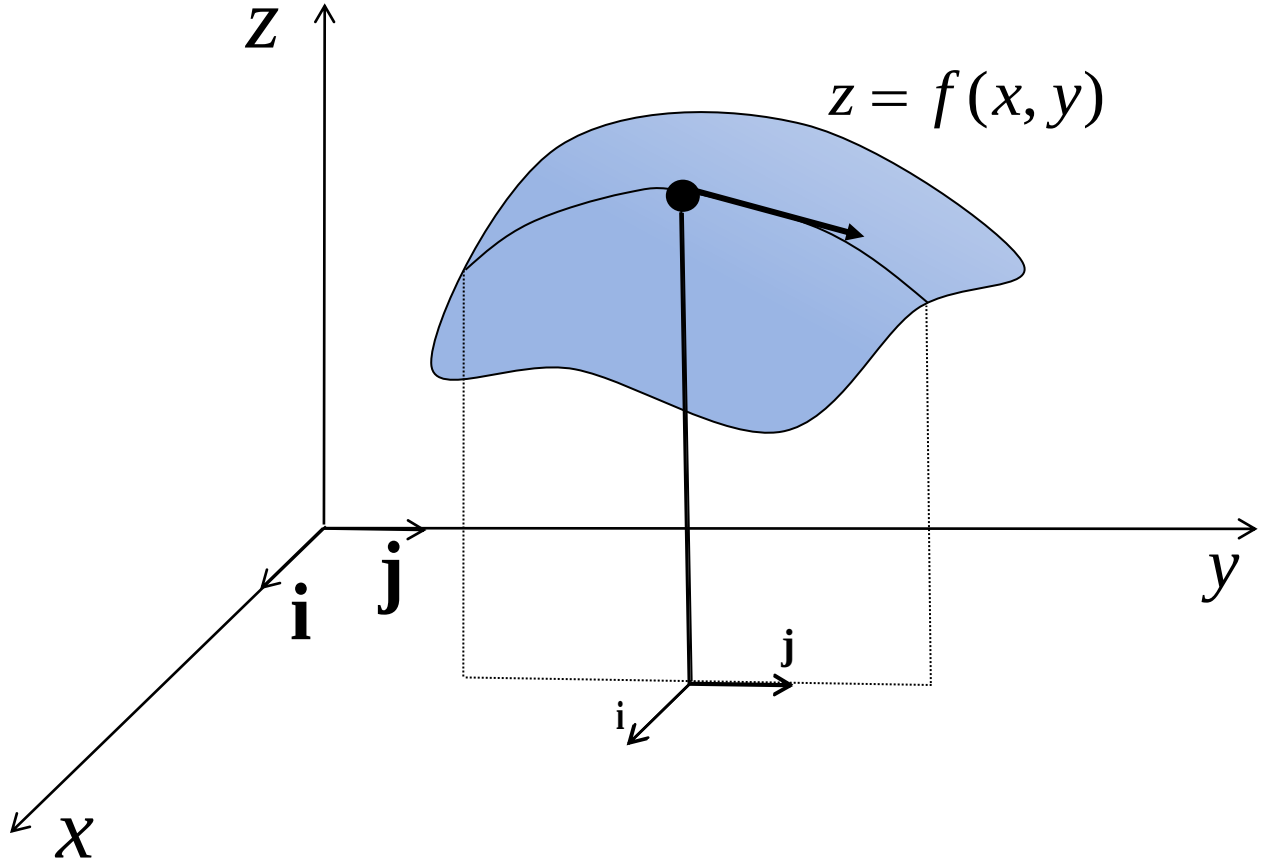
Directional Derivatives

$\frac{\partial f}{\partial x}$:Rate of change of f in the $\mathbf{i}$ -direction



Directional Derivatives

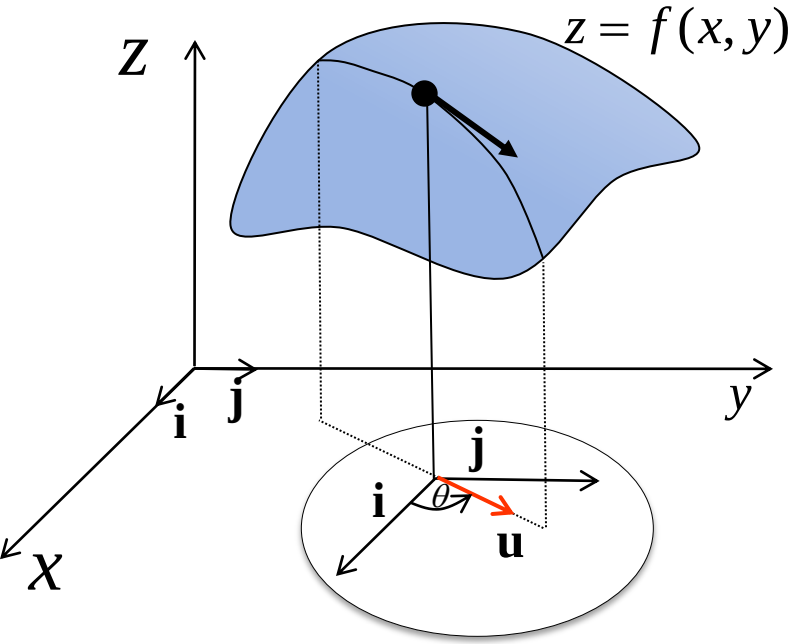
$\frac{\partial f}{\partial y}$:Rate of change of f in the $\mathbf{j}$ -direction



Directional Derivatives

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

The rate of change of f in the direction given by the vector $\mathbf{u}$: $D_{\mathbf{u}} f$



The rate of change of f in the direction of $\mathbf{x}$: $\frac{\partial f}{\partial x}$

The component of $\frac{\partial f}{\partial x}$ in the direction of $\mathbf{u}$: $\frac{\partial f}{\partial x}\cos\theta$

The rate of change of f in the direction of $\mathbf{y}$: $\frac{\partial f}{\partial y}$

The component of $\frac{\partial f}{\partial y}$ in the direction of $\mathbf{u}$: $\frac{\partial f}{\partial y}\cos(\frac{\pi}{2}-\theta)$

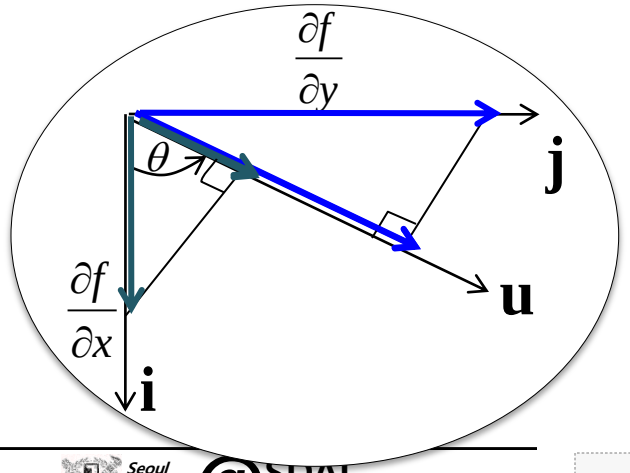
$= \frac{\partial f}{\partial y}\sin\theta$

The rate of change of f in the direction given by the vector $\mathbf{u}$: $D_{\mathbf{u}} f$

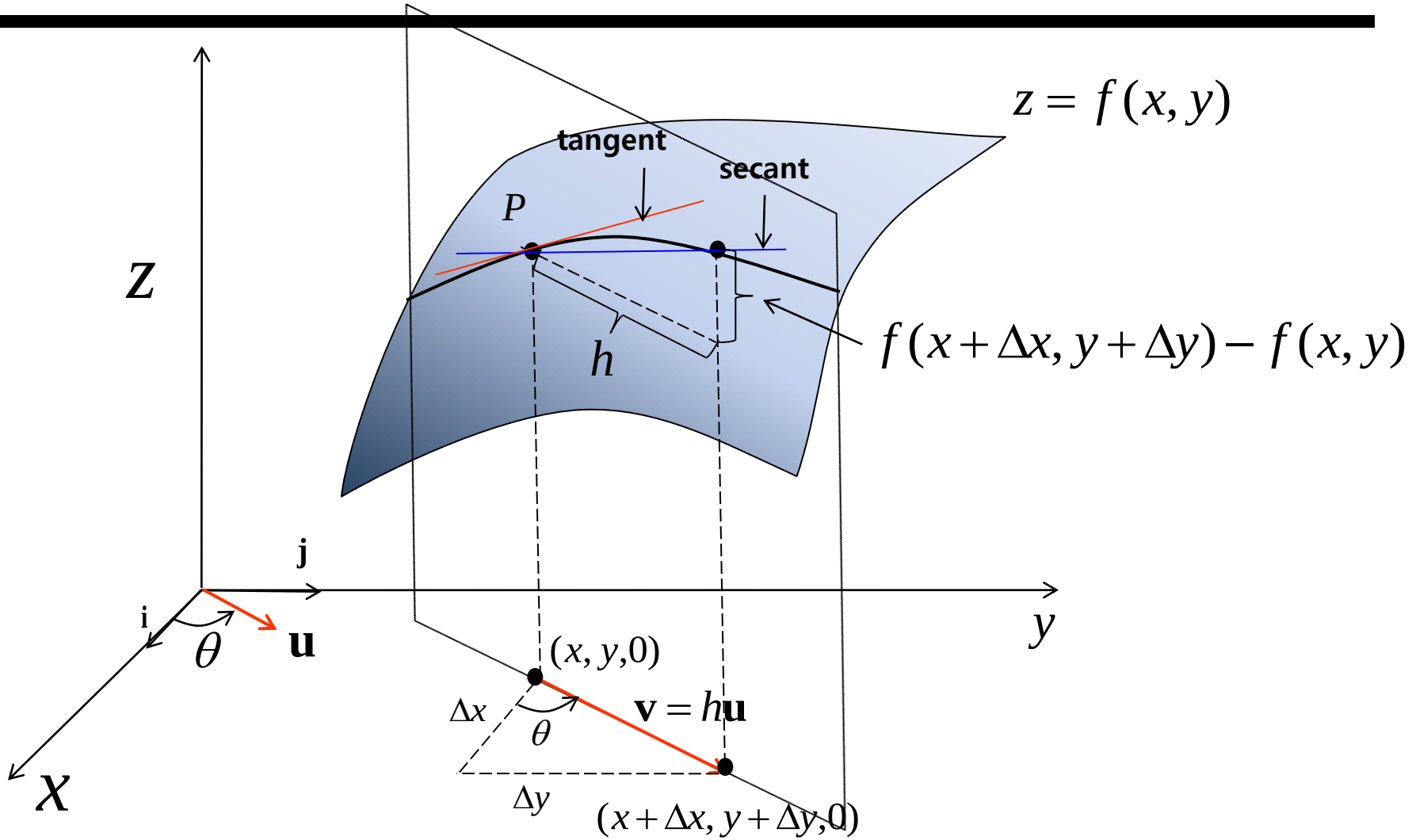
= +

= $\frac{\partial f}{\partial x}\cos\theta$ + $\frac{\partial f}{\partial y}\sin\theta$

$= (\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}) \bullet (\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) = \nabla f \cdot \mathbf{u}$



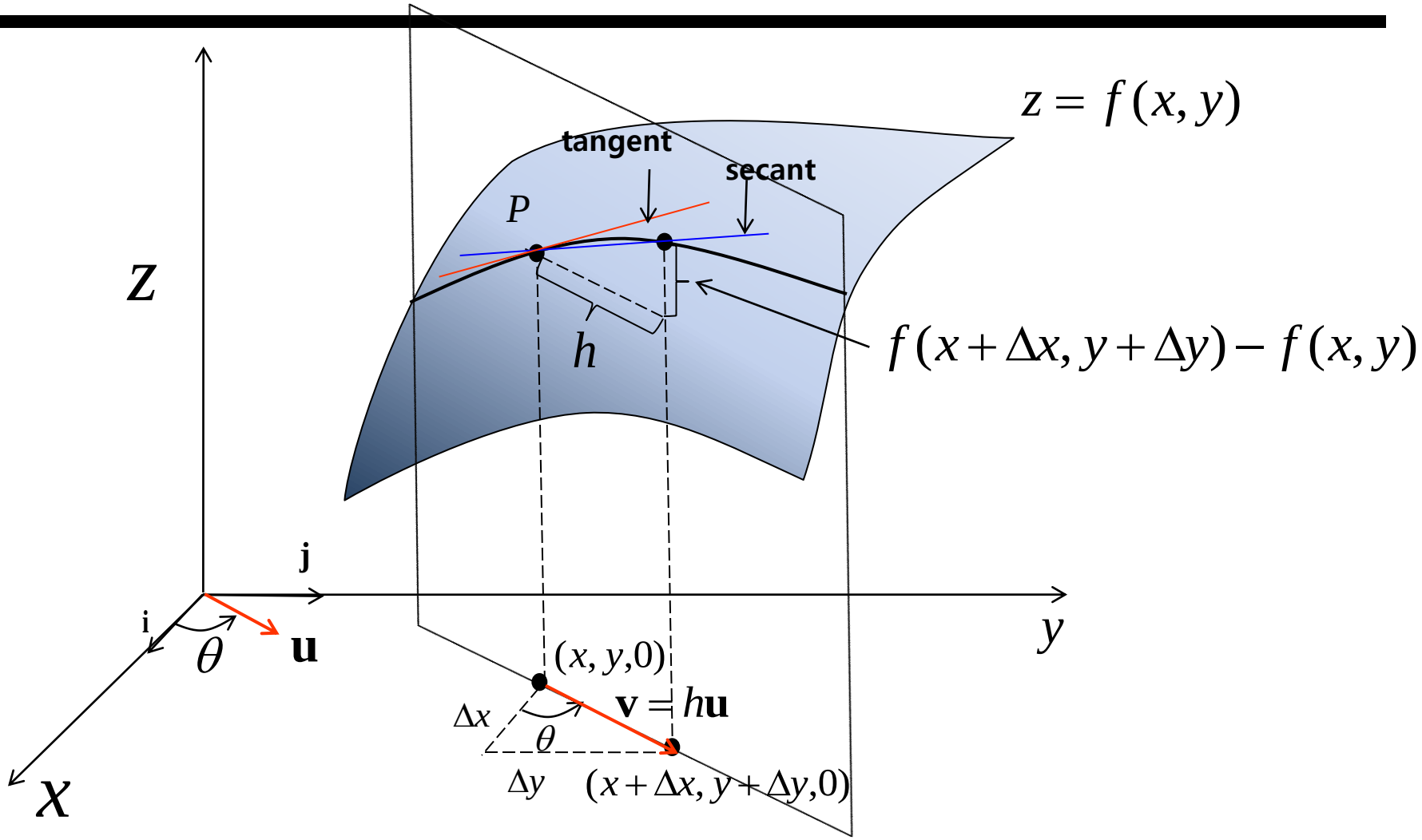
Directional Derivatives



Slope of indicated secant line is

$$\frac{f(x+\Delta x, y+\Delta y) - f(x, y)}{h} = \frac{f(x+h\cos\theta, y+h\sin\theta) - f(x, y)}{h}$$

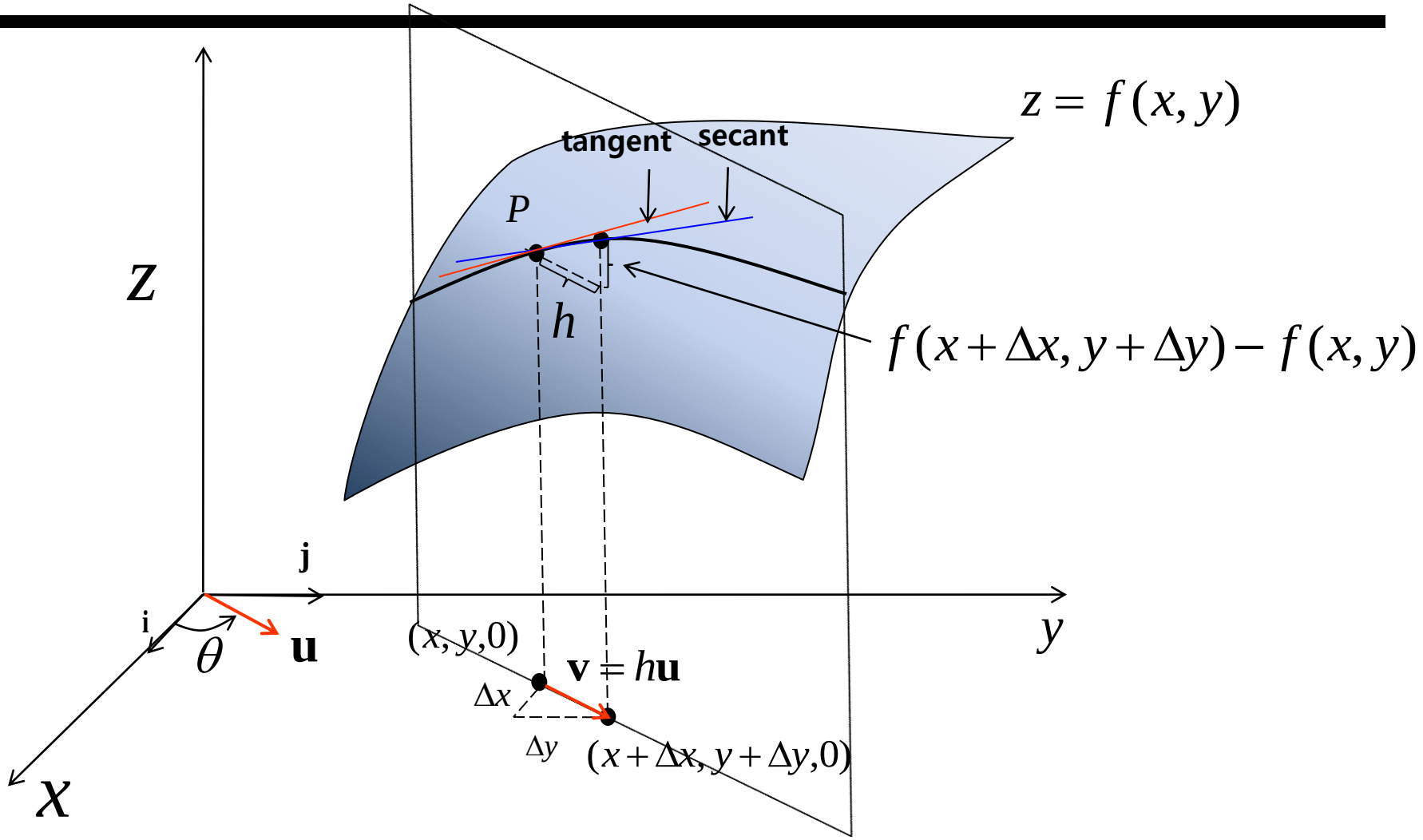
Directional Derivatives



Slope of indicated secant line is

$$\frac{f(x+\Delta x, y+\Delta y) - f(x, y)}{h} = \frac{f(x+h\cos\theta, y+h\sin\theta) - f(x, y)}{h}$$

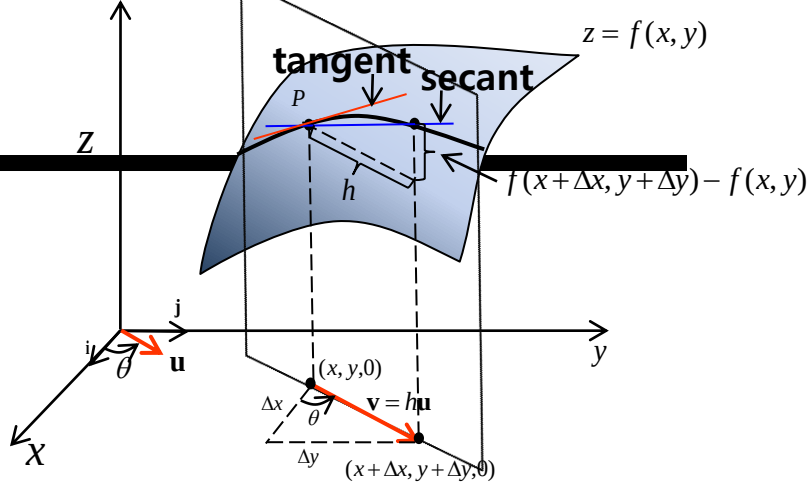
Directional Derivatives



As $h \to 0$ we expect the slope of secant line would be tangent line

$$\frac{f(x+\Delta x, y+\Delta y) - f(x, y)}{h} = \frac{f(x+h\cos\theta, y+h\sin\theta) - f(x, y)}{h}$$

Directional Derivatives



Slope of secant line

$$\frac{f(x+\Delta x, y+\Delta y)-f(x,y)}{h} = \frac{f(x+h\cos\theta, y+h\sin\theta)-f(x,y)}{h}$$

Definition 9.5 **Directional Derivative**

The directional derivative of $z = f(x,y)$ **in the direction of a unit vector** $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ **is**

$$D_{\mathbf{u}} f(x,y) = \lim_{h\to 0} \frac{f(x+h\cos\theta, y+h\sin\theta)-f(x,y)}{h}$$

provided the limit exists.

$$\theta = 0 \text{ implies } D_{\mathbf{i}} f(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} = \frac{\partial z}{\partial x}$$

$$\theta = \frac{\pi}{2} \text{ implies } D_{\mathbf{j}} f(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} = \frac{\partial z}{\partial y}$$

Directional Derivatives

Theorem 9.6 **Computing a Directional Derivative**

If $z = f(x, y)$ is differentiable function of x and y and $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ then,

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

Proof)

$$g(t) = f(x + t\cos\theta, y + t\sin\theta)$$

by definition of a derivative

$$g'(0) = \lim_{h\to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0} \frac{f(x + h\cos\theta, y + h\sin\theta) - f(x, y)}{h} = D_{\mathbf{u}} f(x, y)$$

by the Chain Rule,

$$g'(t) = f_1(x + t\cos\theta, y + t\sin\theta)\frac{d}{dt}(x + t\cos\theta) + f_2(x + t\cos\theta, y + t\sin\theta)\frac{d}{dt}(y + t\sin\theta)$$

$$= f_1(x + t\cos\theta, y + t\sin\theta)\cos\theta + f_2(x + t\cos\theta, y + t\sin\theta)\sin\theta$$

$$\therefore g'(0) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

Directional Derivatives

Theorem 9.6 **Computing a Directional Derivative**

If $z = f(x, y)$ is differentiable function of x and y and $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ then,

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

Proof)

$$g(t) = f(x + t\cos\theta, y + t\sin\theta)$$

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

$$\begin{aligned} D_{\mathbf{u}} f(x, y) = g'(0) &= f_x(x, y)\cos\theta + f_y(x, y)\sin\theta \\ &= [f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}] \cdot (\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) \\ &= \nabla f(x, y) \cdot \mathbf{u} \end{aligned}$$

Directional Derivatives

☑ Example 2 Gradient at a Point

If $F(x,y,z)=xy^2+3x^2-z^3$, find $\nabla F(x,y,z)$ at $(2,-1,4)$.

Solution)

$$\nabla F(x,y,z) = \frac{\partial}{\partial x}(xy^2+3x^2-z^3)\mathbf{i} + \frac{\partial}{dy}(xy^2+3x^2-z^3)\mathbf{j} + \frac{\partial}{dz}(xy^2+3x^2-z^3)\mathbf{k} = (y^2+6x)\mathbf{i}+2xy\mathbf{j}-3z^2\mathbf{k}$$

$$\nabla F(2,-1,4) = 13\mathbf{i}-4\mathbf{j}-48\mathbf{k}$$

Directional Derivatives

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

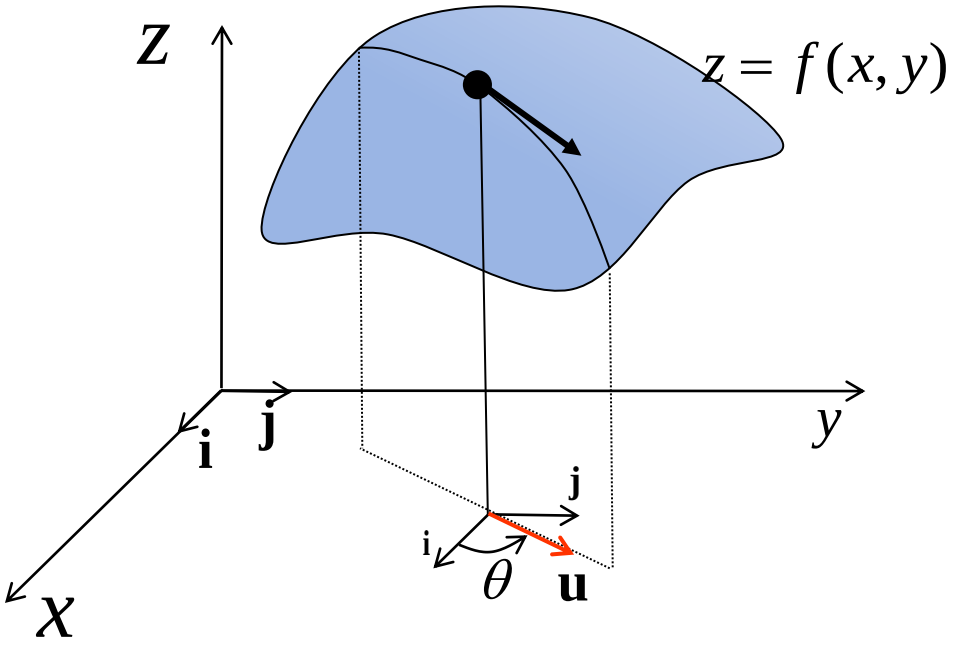
Theorem 9.6 — **Computing a Directional Derivative**

If $z = f(x,y)$ **is differentiable function of x and y and** $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ **then,**

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u}$$

$$D_{\mathbf{u}} f(x,y) = [\,f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}] \cdot (\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$$

$$= \nabla f(x,y) \cdot \mathbf{u}$$



Directional Derivatives

☑ Example 3
Directional Derivative

Find the directional derivative of $f(x,y)=2x^2y^3+6xy$ at $(1,1)$ in the direction of a unit vector whose angle with the positive x-axis is $\pi/6$.

Solution)

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

$$= (4xy^3 + 6y)\mathbf{i} + (6x^2y^2 + 6x)\mathbf{j}$$

$$\nabla f(1,1) = 10\mathbf{i} + 12\mathbf{j}$$

$$\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} = \cos\frac{\pi}{6}\mathbf{i} + \sin\frac{\pi}{6}\mathbf{j}$$

$$D_{\mathbf{u}} f(1,1) = \nabla f(1,1)\cdot\mathbf{u}$$

$$= (10\mathbf{i} + 12\mathbf{j})\cdot\left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right)$$

$$= 5\sqrt{3} + 6$$

Directional Derivatives

☑ Example 4

Directional Derivative

Consider the plane that is perpendicular to the xy-plane and passes through the points $P(2,1)$ and $Q(3,2)$. What is the slope of the tangent line to the curve on intersection of this plane with the surface $f(x,y)=4x^2+y^2$ at $(2,1,17)$ in the direction of Q?

Solution)

$$f(x,y) = 4x^2 + y^2$$

$$\nabla f(x,y) = 8x\mathbf{i} + 2y\mathbf{j}$$

$$\nabla f(2,1) = 16\mathbf{i} + 2\mathbf{j}$$

$$\overrightarrow{PQ} = \mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$$D_{\mathbf{u}} f(2,1) = \nabla f(2,1) \bullet \mathbf{u}$$

$$= (16\mathbf{i} + 2\mathbf{j}) \bullet \left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \right)$$

$$= 9\sqrt{2}$$

Directional Derivatives

☑ Example 5
Directional Derivative

Find the directional derivative of $F(x,y,z)=xy^2-4x^2y+z^2$ at (1,-1,2) in the direction of $6\mathbf{i}+2\mathbf{j}+3\mathbf{k}$.

Solution)

$$f(x,y,z)=xy^2-4x^2y+z^2$$

$$\nabla f(x,y,z)=\frac{\partial}{\partial x}(xy^2-4x^2y+z)\mathbf{i}+\frac{\partial}{\partial y}(xy^2-4x^2y+z)\mathbf{j}+\frac{\partial}{\partial z}(xy^2-4x^2y+z)\mathbf{k}$$

$$=(y^2-8xy)\mathbf{i}+(2xy-4x^2)\mathbf{j}+2z\mathbf{k}$$

$$\nabla f(1,-1,2)=9\mathbf{i}-6\mathbf{j}+4\mathbf{k}$$

$$\left\|6\mathbf{i}+2\mathbf{j}+3\mathbf{k}\right\|=7$$

$$\mathbf{u}=\frac{1}{7}\cdot(6\mathbf{i}+2\mathbf{j}+3\mathbf{k})=\frac{6}{7}\mathbf{i}+\frac{2}{7}\mathbf{j}+\frac{3}{7}\mathbf{k}$$

$$D_{\mathbf{u}}F(1,-1,2)=(9\mathbf{i}-6\mathbf{j}+4\mathbf{k})\bullet\left(\frac{6}{7}\mathbf{i}+\frac{2}{7}\mathbf{j}+\frac{3}{7}\mathbf{k}\right)$$

$$=\frac{54}{7}$$

Directional Derivatives

∇f **Points in the direction of maximum increase of** f **at P**

The rate of change of f **in the direction given by the vector** $\mathbf{u}$ **:**

$$D_{\mathbf{u}} f(x,y) = [f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}] \cdot (\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) = \nabla f(x,y) \cdot \mathbf{u}$$

$$D_{\mathbf{u}} f = |\nabla f||\mathbf{u}|\cos\phi = |\nabla f|\cos\phi, \quad \phi: \text{angle between } \nabla f \text{ and } \mathbf{u}$$

$$-1 \le \cos\phi \le 1$$

The maximum value of $D_{\mathbf{u}} f$ ⇒ $D_{\mathbf{u}} f = |\nabla f|$ **, When** $\cos\phi = 1,\ \phi = 0$

$\mathbf{u}$ **has the same direction of** ∇f

⇒ ∇f **is the direction of maximum increase of** f **at P**

$-\nabla f$ **is the direction of maximum decrease of** f **at P**

Directional Derivatives

☑ Example 6

Max/Min of Directional Derivative

In Example 5 the maximum value of the directional derivative at F at $(1,-1,2)$ is

$$\|\nabla F(1,-1,2)\| = \sqrt{133}.$$

The minimum value of $D_{\mathbf{u}}F(1,-1,2)$ is then $-\sqrt{133}$.

Directional Derivatives

☑ Example 7 Direction of Steepest Ascent

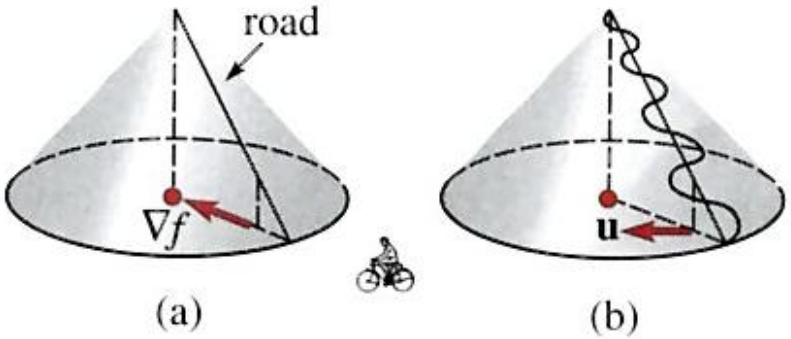
Each year in Los Angeles there is a bicycle race up to the top of a hill by a road known to be the steepest in the city. To understand why a bicyclist with a modicum of sanity will zigzag up the road, let us suppose the graph of

$$f(x,y)=4-\frac{2}{3}\sqrt{x^2+y^2},\ 0\le z\le 4,$$

shown in Figure (a) is a mathematical model of the hill. The gradient of f is

$$\nabla f(x,y)=\frac{2}{3}\left[\frac{-x}{\sqrt{x^2+y^2}}\mathbf{i}+\frac{-y}{\sqrt{x^2+y^2}}\mathbf{j}\right]=\frac{2/3}{\sqrt{x^2+y^2}}\mathbf{r}$$

where $\mathbf{r}=-x\mathbf{i}-y\mathbf{j}$ is a vector pointing to the center of the circular base.



Thus the steepest ascent up the hill is a straight road whose projection in the xy-plane is a radius of the circular base. Since $D_{\mathbf{u}}f=\text{comp}_{\mathbf{u}}\nabla f$, a bicyclist will zigzag, or seek a direction $\mathbf{u}$ other than ∇f, in order to reduce this component.

Directional Derivatives

☑ Example 8 Direction to Cool Off Fastest

The temperature in a rectangular box is approximated by

$$T(x,y,z) = xyz(1-x)(2-y)(3-z)$$

$$0 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 3.$$

If a mosquito is located at (½,1,1), in which direction should it fly to cool off as rapidly as possible?

Solution)

$$\nabla T(x,y,z) = \frac{\partial T(x,y,z)}{\partial x}\mathbf{i} + \frac{\partial T(x,y,z)}{\partial y}\mathbf{j} + \frac{\partial T(x,y,z)}{\partial x}\mathbf{k}$$

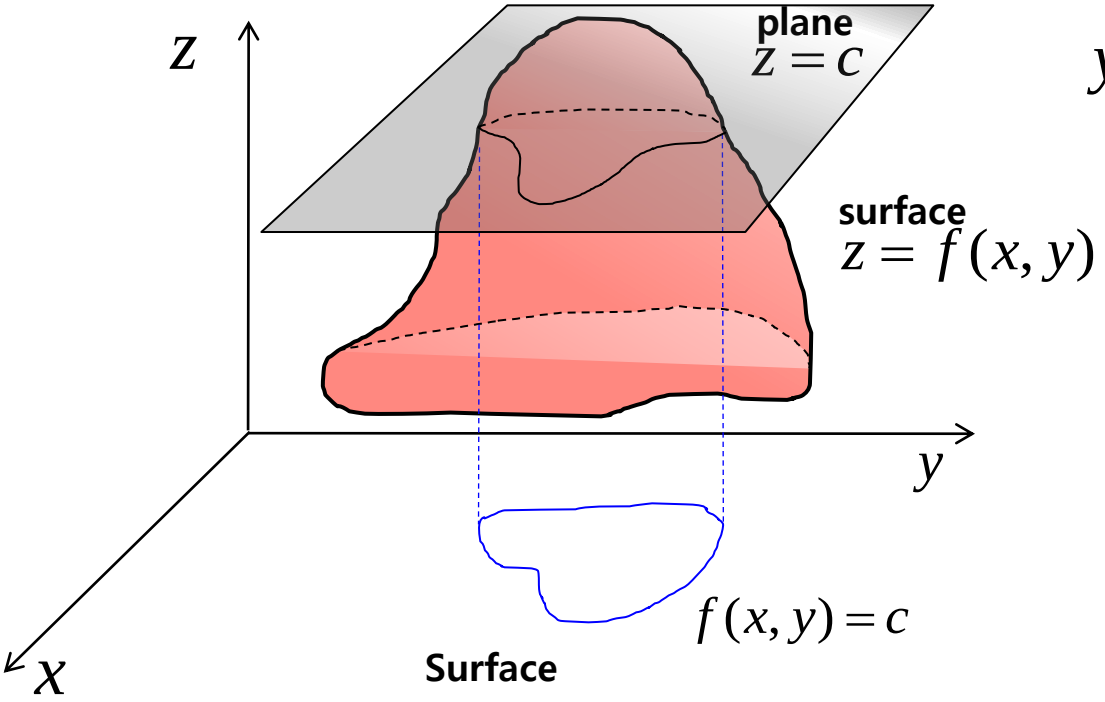
$$= -yz(2-y)(3-z)(1-2x)\mathbf{i}$$
$$+ xz(1-x)(3-z)(2-2y)\mathbf{j}$$
$$+ xy(1-x)(2-y)(3-2z)\mathbf{k}$$

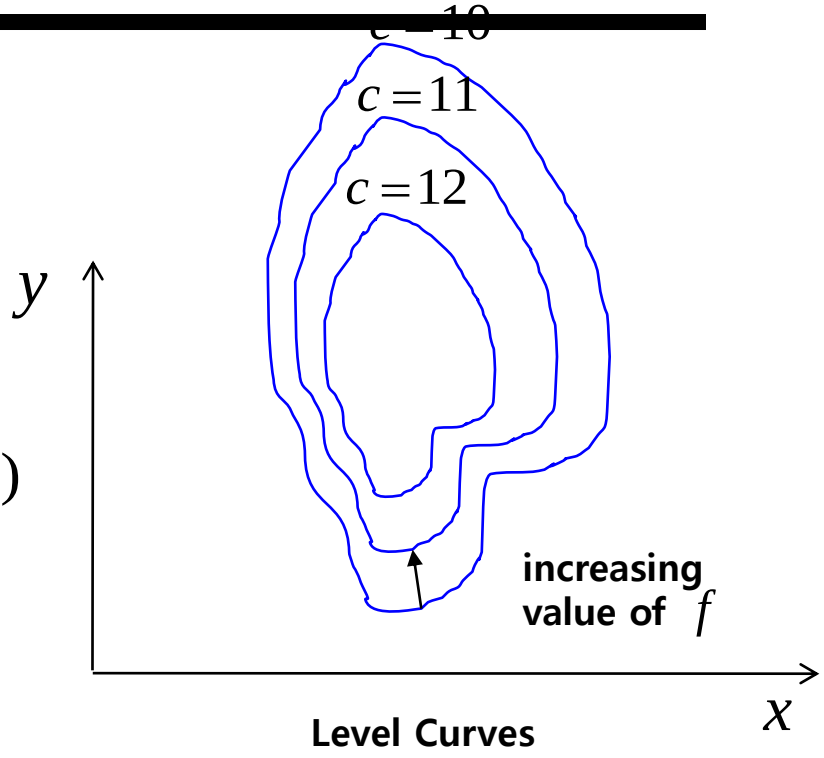
$$\nabla T\left(\frac{1}{2},1,1\right) = \frac{1}{4}\mathbf{k}$$

To cool off most rapidly, the mosquito should fly in the direction of -1/4k; that is, it should dive for the floor of the box, where the temperature is $T(x,y,0)=0$.

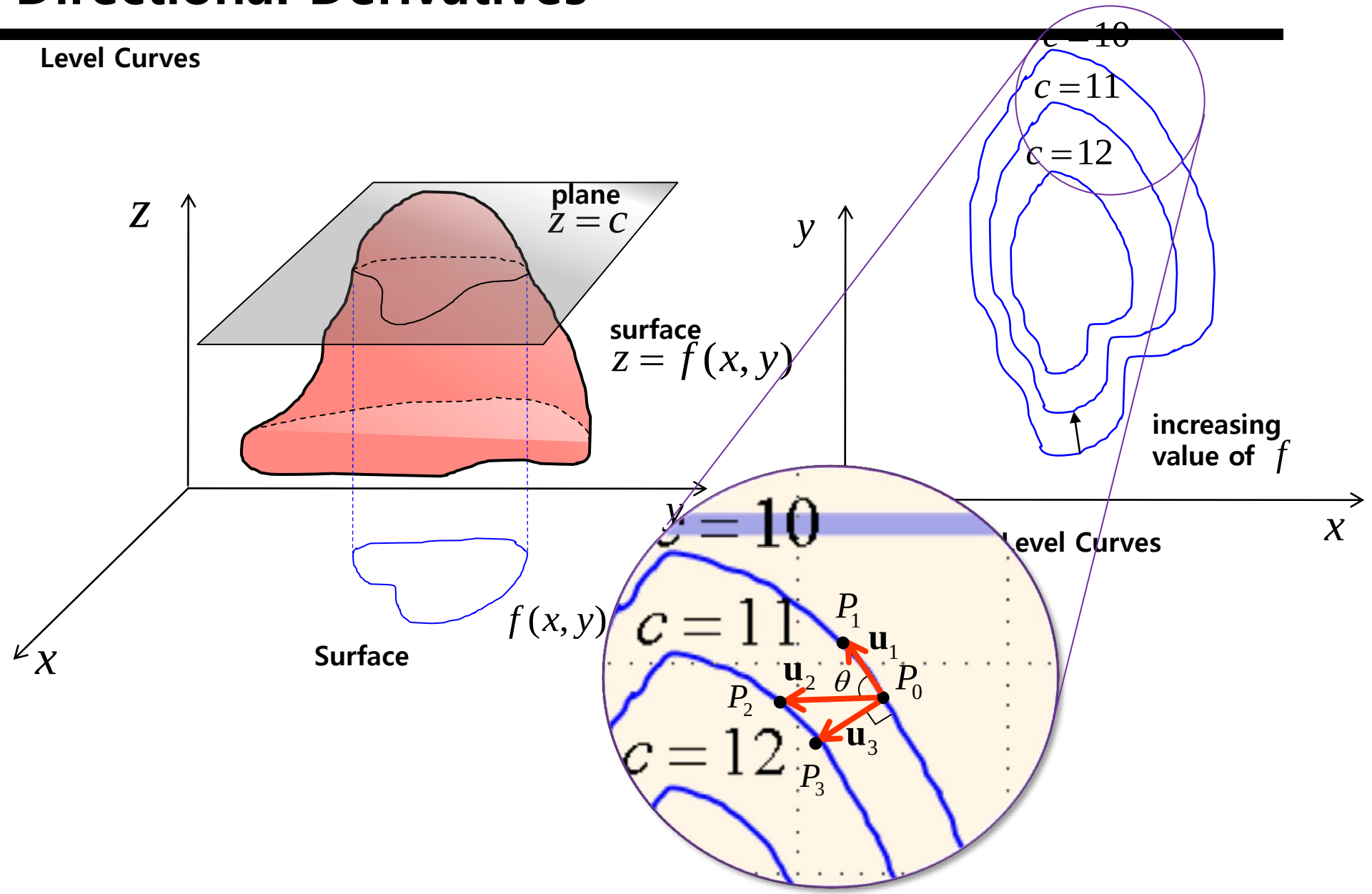
Directional Derivatives

Level Curves



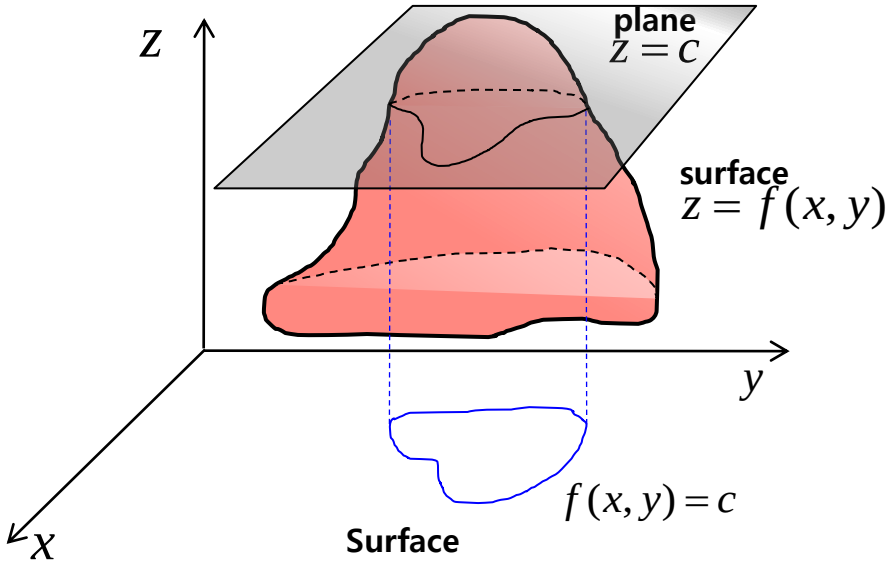


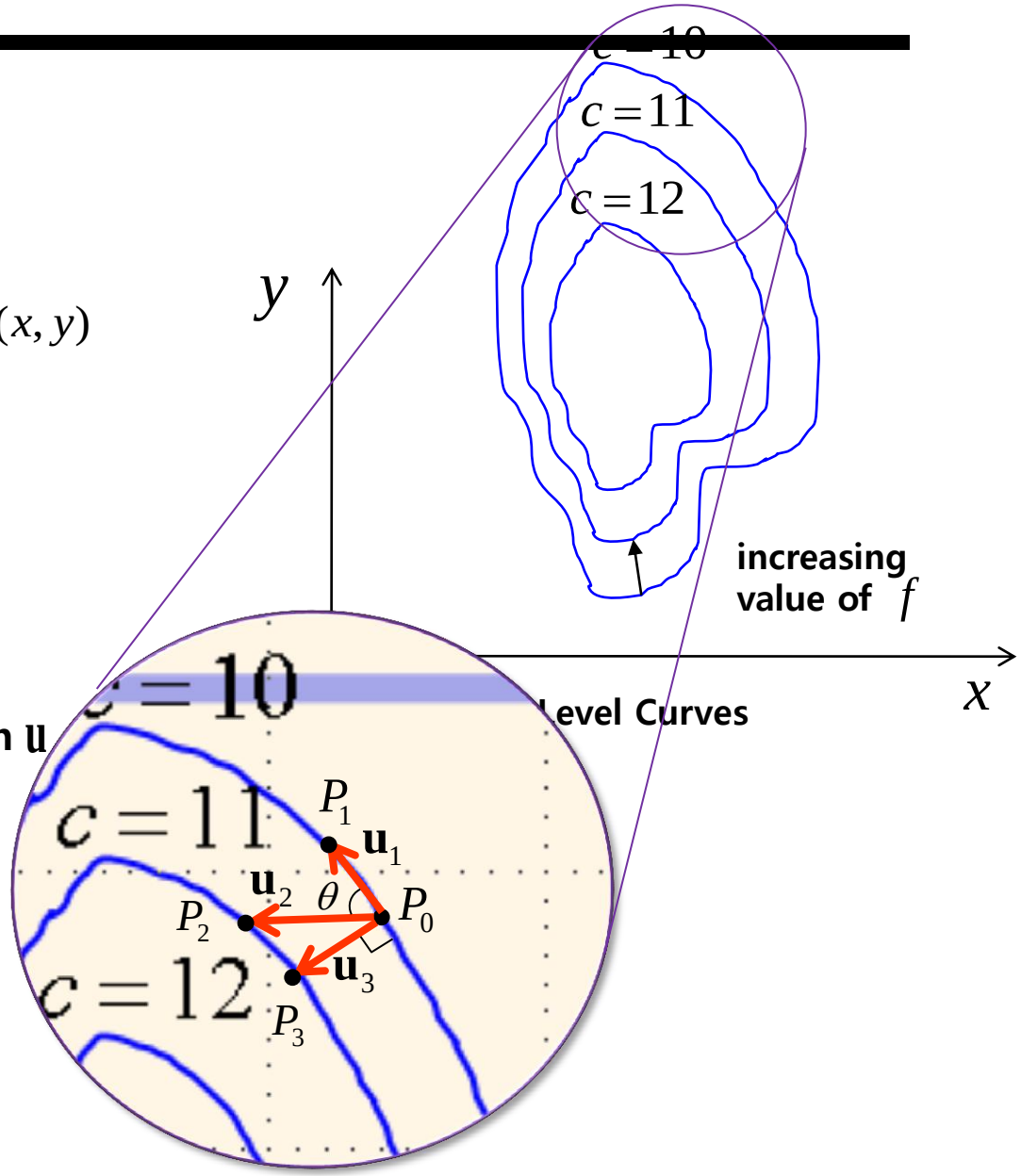
Directional Derivatives



Directional Derivatives

Level Curves





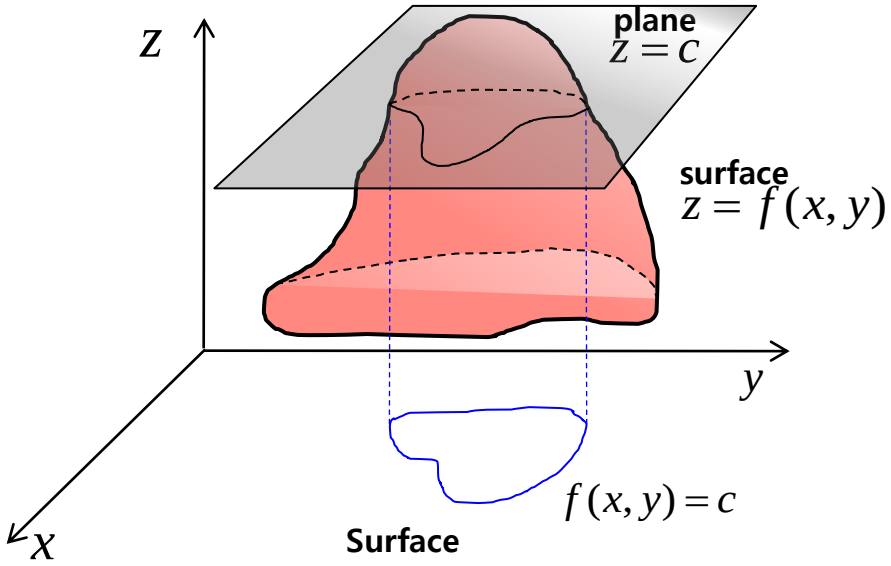
The rate of change of f in the direction $\mathbf{u}$ given by the vector : $D_{\mathbf{u}} f(x, y)$

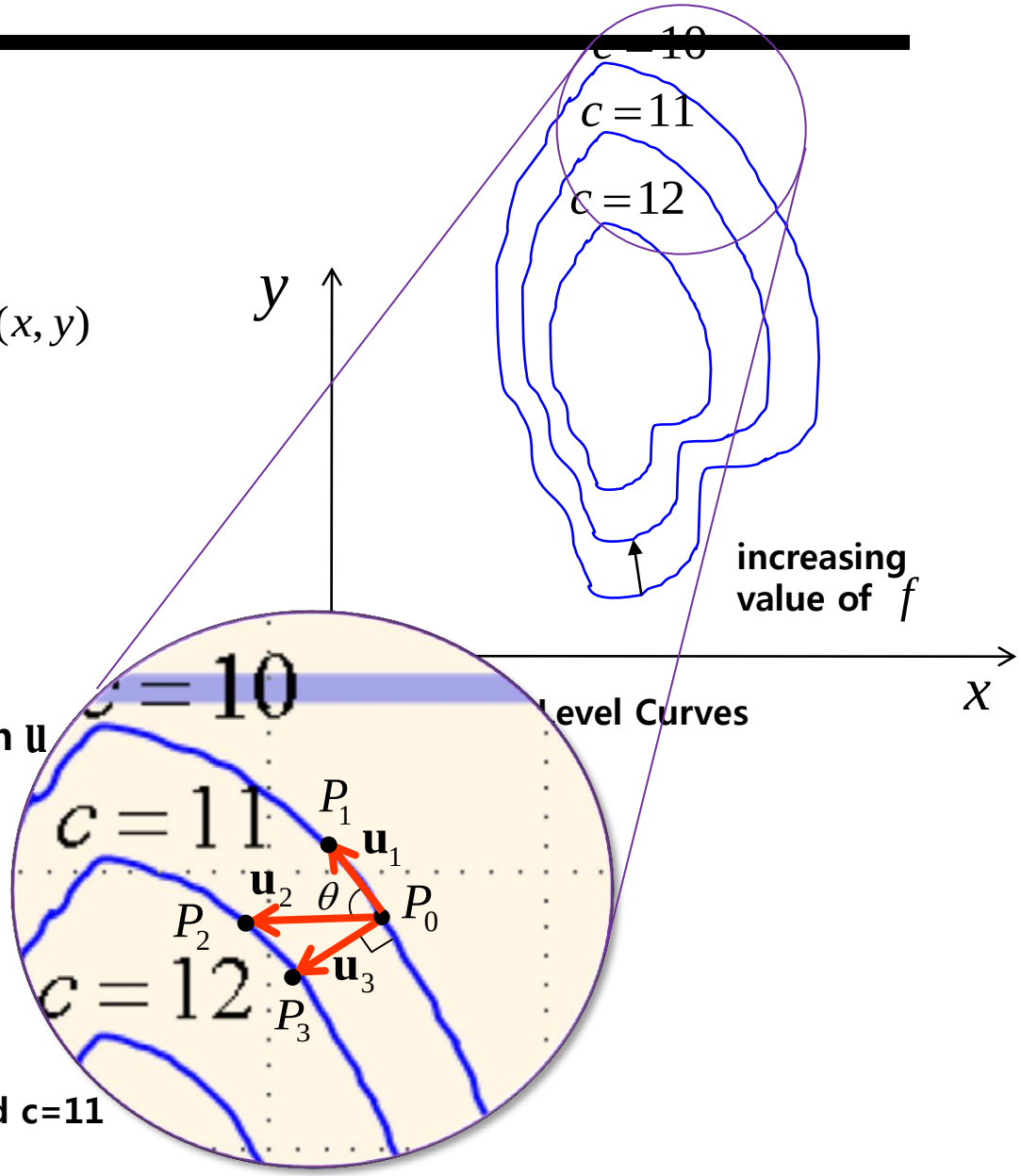
$$D_{\mathbf{u}_1} f(x, y) = 0$$

$$\because \frac{f(P_1) - f(P_0)}{\overline{P_1P_0}} = \frac{10 - 10}{\overline{P_1P_0}} = 0$$

Directional Derivatives

Level Curves





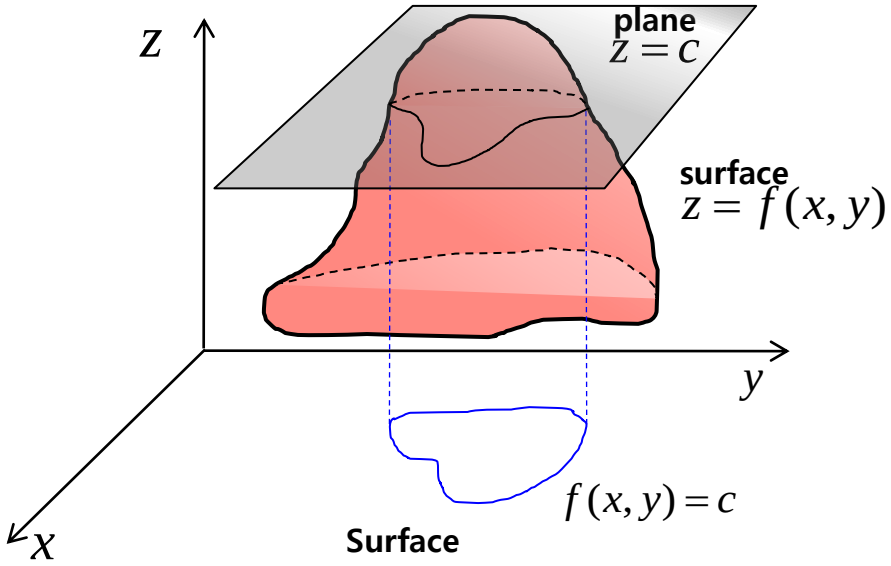
The rate of change of f in the direction $\mathbf{u}$ given by the vector : $D_{\mathbf{u}} f(x, y)$

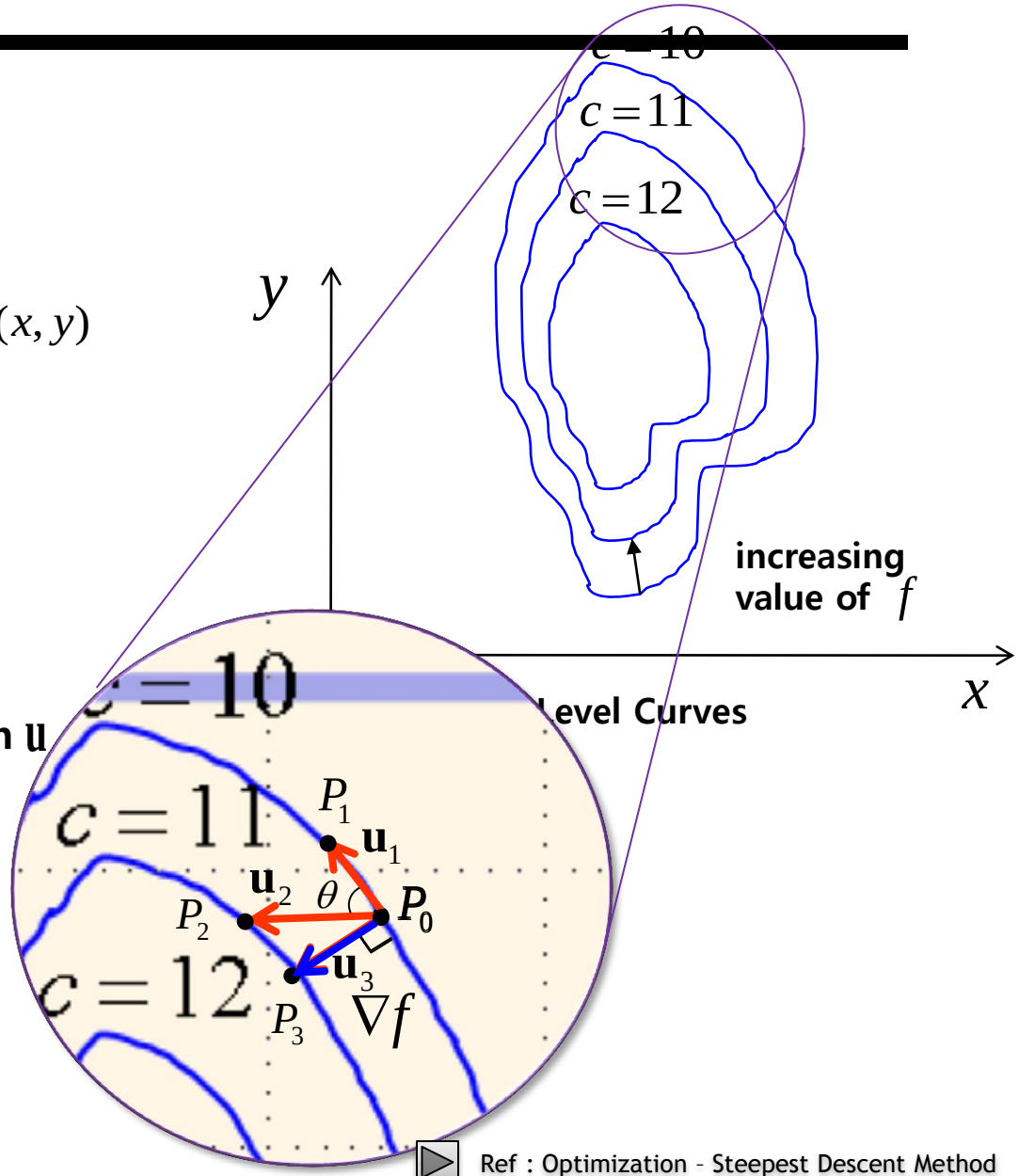
$$\frac{f(P_2) - f(P_0)}{\overline{P_2P_0}} < \frac{f(P_3) - f(P_0)}{\overline{P_3P_0}}$$

$\because \overline{P_3P_0}$ is the shortest path between c=10 and c=11

Directional Derivatives

Level Curves





The rate of change of f in the direction $\mathbf{u}$ given by the vector : $D_{\mathbf{u}} f(x, y)$

∇f is the direction of maximum increase of f at P_0

Ref : Optimization - Steepest Descent Method

Gradient as Curve Normal Vector

$f\big(x(t), y(t)\big) = c$: **A curve passes through a specified point** $P(x_0, y_0)$

$\mathbf{r} = \big(x(t), y(t)\big)$: **A position vector of a point on the curve.**

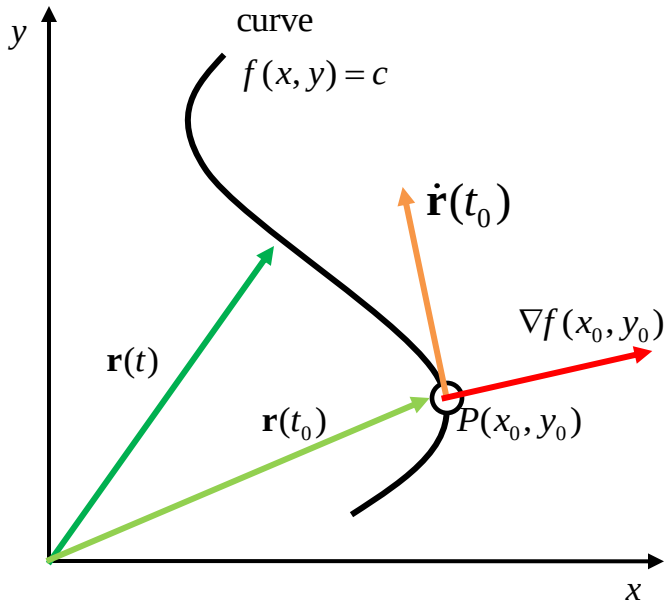
$$f\big(x(t), y(t)\big) = c \quad \Rightarrow \frac{df\big(x(t), y(t)\big)}{dt} = 0$$

$$\text{LHS}: \frac{df\big(x(t), y(t)\big)}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= \underbrace{\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right)}_{= \nabla f(x, y)} \bullet \underbrace{\left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}\right)}_{= \dot{\mathbf{r}}(t)}$$

Tangent vector

$$\therefore \nabla f(x, y) \bullet \dot{\mathbf{r}}(\mathbf{t}) = 0$$



$\nabla f(x, y)$ **is normal to the curve at the point P.**

Gradient as Curve Normal Vector

☑ Example 1

Gradient at a Point

Find the level curve of $f(x,y)=-x^2+y^2$ passing through (2,3). Graph the gradient at the point.

Solution)

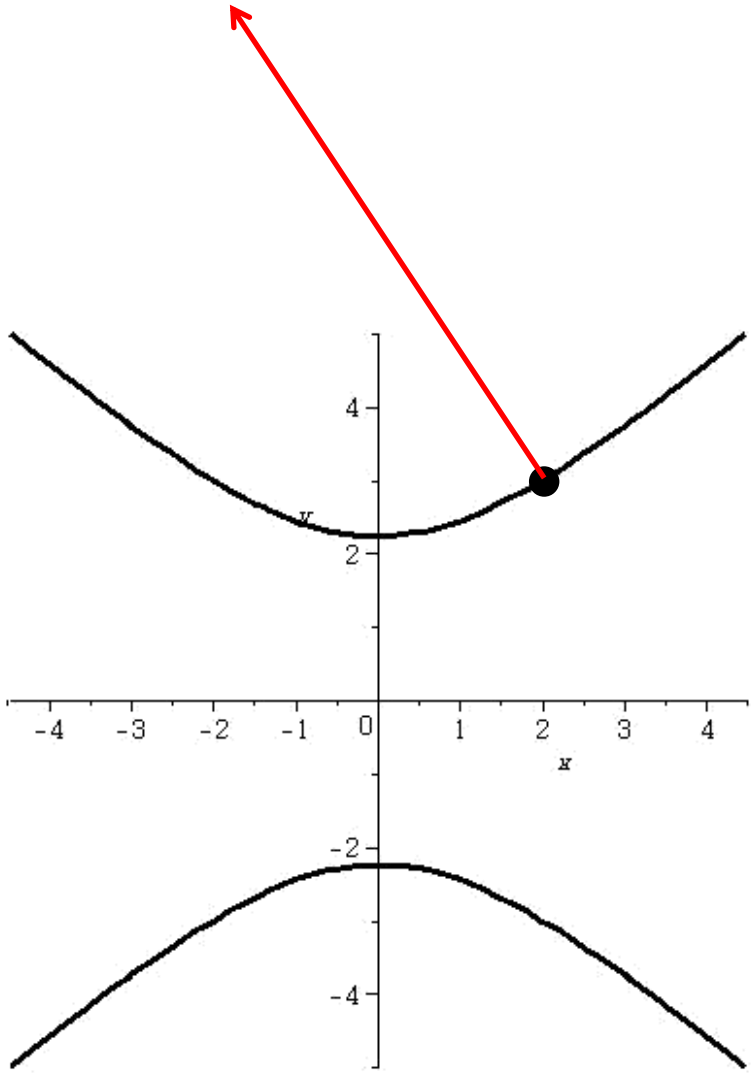
$$f(2,3) = -2^2 + 3^2 = 5$$

Level curve : $-x^2 + y^2 = 5$

$$\nabla f(x,y) = -2x\mathbf{i} + 2y\mathbf{j}$$

$$\nabla f(2,3) = -2\cdot 2\mathbf{i} + 2\cdot 3\mathbf{j}$$

$$= -4\mathbf{i} + 6\mathbf{j}$$



Gradient as Surface Normal Vector

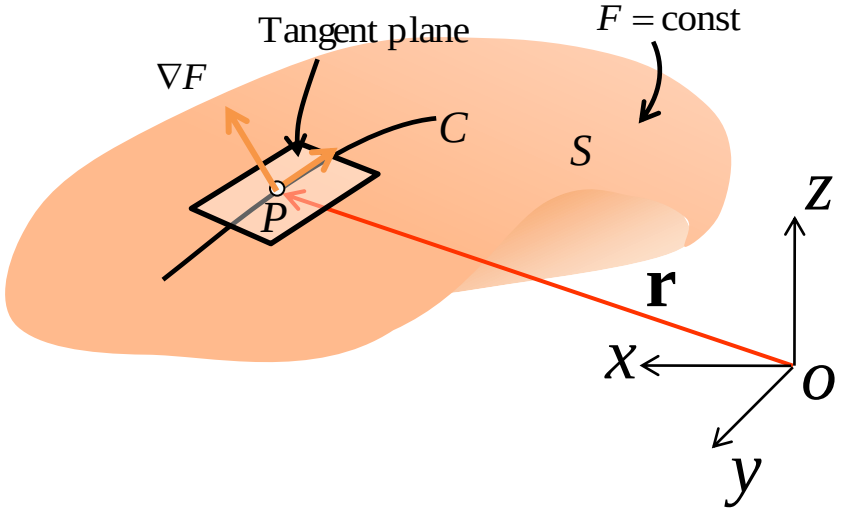
surface $S: F(x,y,z)=c$

curve C on the surface:

$$\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k}$$

$$F\left(x(t),y(t),z(t)\right)=c$$

$$\frac{dF\left(x(t),y(t),z(t)\right)}{dt}=0$$



$$\text{LHS}: \frac{\partial F}{\partial x}\frac{dx}{dt}+\frac{\partial F}{\partial y}\frac{dy}{dt}+\frac{\partial F}{\partial z}\frac{dz}{dt} = \underbrace{\left(\frac{\partial F}{\partial x}\mathbf{i}+\frac{\partial F}{\partial y}\mathbf{j}+\frac{\partial F}{\partial z}\mathbf{k}\right)}_{=\nabla F\left(x(t),y(t),z(t)\right)} \bullet \underbrace{\left(\dot{x}\mathbf{i}+\dot{y}\mathbf{j}+\dot{z}\mathbf{k}\right)}_{=\dot{\mathbf{r}}(t)}$$

$$\text{R.H.S.}: 0$$

Tangent vector to curve C at the point P on the surface
The tangent vectors of all curves on S passing through P will generally form a plane, called tangent plane of S at P

$$\therefore \nabla F \bullet \dot{\mathbf{r}} = 0$$

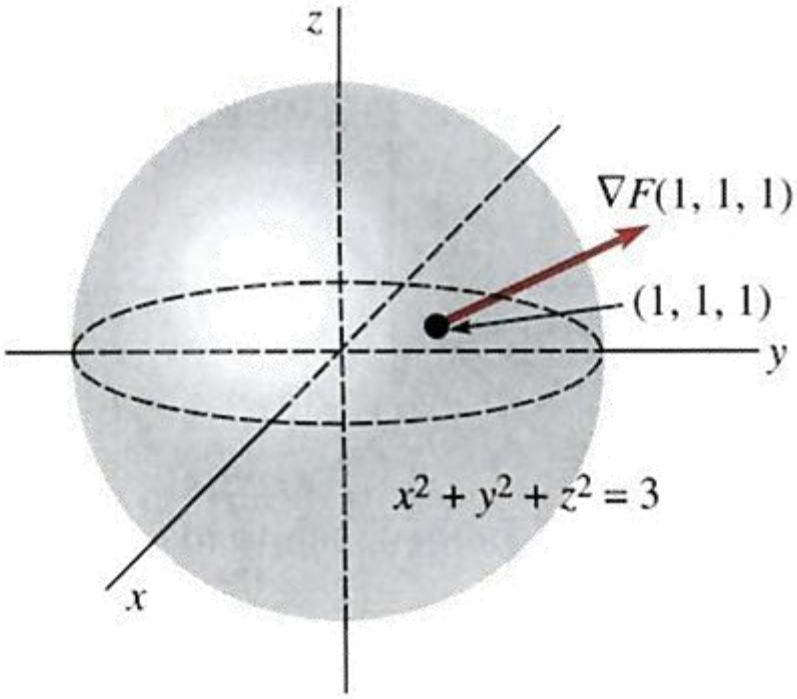
∇F is normal to the Surface at the point P.

Gradient as Surface Normal Vector

☑ Example 2

Gradient at a Point

Find the level surface of $F(x,y,z)=x^2+y^2+z^2$ passing through (1,1,1). Graph the gradient at the point.



Solution)

$$F(1,1,1)=1^2+1^2+1^2=3$$

Level surface : $x^2+y^2+z^2=3$

$$\nabla F(x,y,z)=2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}$$

$$\nabla F(1,1,1)=2\cdot 1\mathbf{i}+2\cdot 1\mathbf{j}+2\cdot 1\mathbf{k}$$

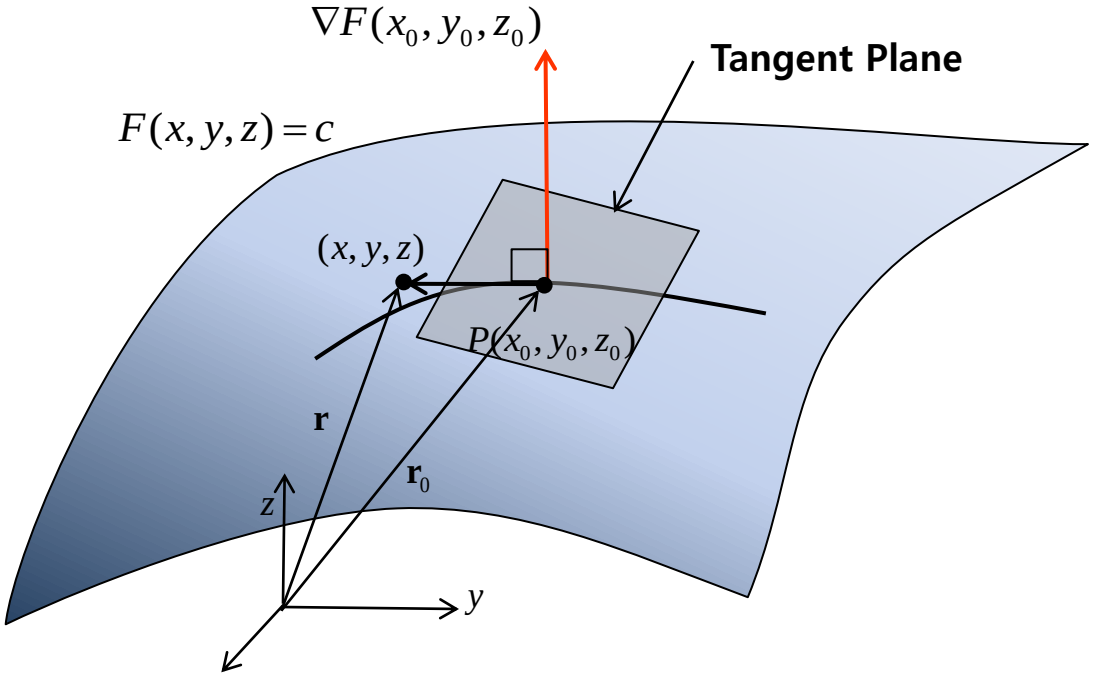
Tangent Planes and Normal Lines

Tangent Plane

Definition 9.6 **Tangent Plane**

Let $P(x_0, y_0, z_0)$ be a point on the curve of $F(x, y, z) = c$ where ∇F is not 0.

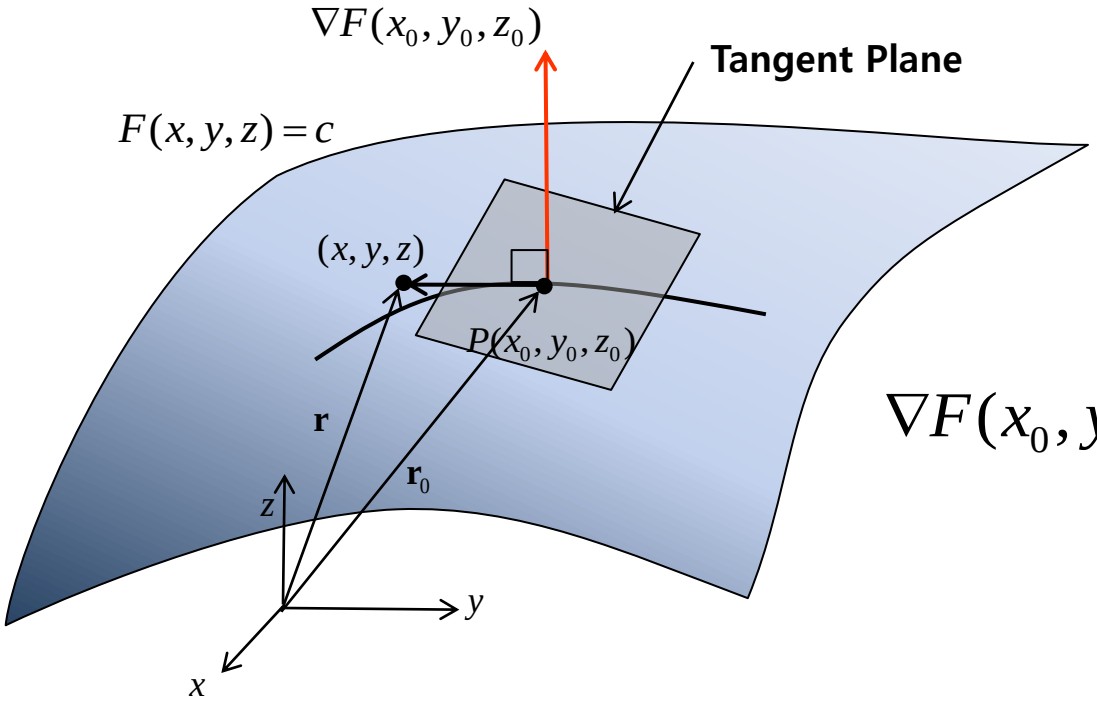
The tangent plane at P is that plane through P that is perpendicular to ∇F evaluated at P



Normal Line

The line containing $P(x_0, y_0, z_0)$ that is parallel to $\nabla F(x_0, y_0, z_0)$ is called normal line

Tangent Planes and Normal Lines



$$\nabla F(x_0, y_0, z_0) \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

Theorem 9.7 — **Equation of Tangent Plane**

Let $P(x_0, y_0, z_0)$ be a point on the graph of $F(x, y, z) = c$ where ∇F is not 0. Then an equation of the tangent plane at P is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Tangent Planes and Normal Lines

☑ Example 3 Equation of Tangent Plane

Find an equation of the tangent plane to the graph of $x^2-4y^2+z^2=16$ at $(2,1,4)$.

Solution)

$$F(x,y,z)=x^2-4y^2+z^2$$

$$F(2,1,4)=2^2-4\cdot 1^2+4^2=16$$

$$F_x(x,y,z)=2x,\ F_x(2,1,4)=2$$

$$F_y(x,y,z)=-8y,\ F_y(2,1,4)=-8$$

$$F_z(x,y,z)=2z,\ F_z(2,1,4)=8$$

equation of the tangent plane :

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0$$

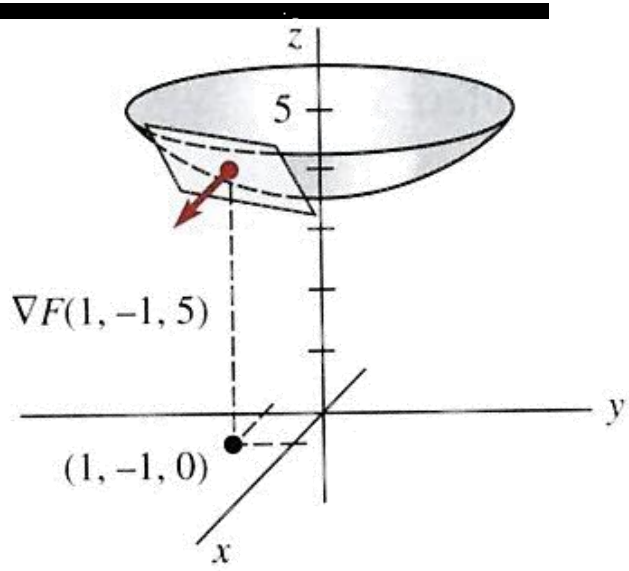
$$\therefore 4(x-2)-8(y-1)+8(z-4)=0$$

$$x-2y+2z=8$$

Tangent Planes and Normal Lines

☑ Example 4
Equation of Tangent Plane

Find an equation of the tangent plane to the graph of $z = \frac{1}{2}x^2 + \frac{1}{2}y^2 + 4$ at (1,-1,5).



Solution)

$$F(x,y,z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 - z + 4$$

$$F(1,-1,5) = \frac{1}{2}\cdot 1^2 + \frac{1}{2}\cdot(-1)^2 - 5 + 4 = 0$$

$$\nabla F(x,y,z) = x\mathbf{i} + y\mathbf{j} - \mathbf{k}$$

$$\nabla F(1,-1,5) = \mathbf{i} - \mathbf{j} - \mathbf{k}$$

equation of the tangent plane :

$$F_x(x_0,y_0,z_0)(x-x_0)$$
$$+ F_y(x_0,y_0,z_0)(y-y_0)$$
$$+ F_z(x_0,y_0,z_0)(z-z_0) = 0$$

$$(x+1) - (y-1) - (z-5) = 0$$

$$\therefore -x + y + z = 7$$

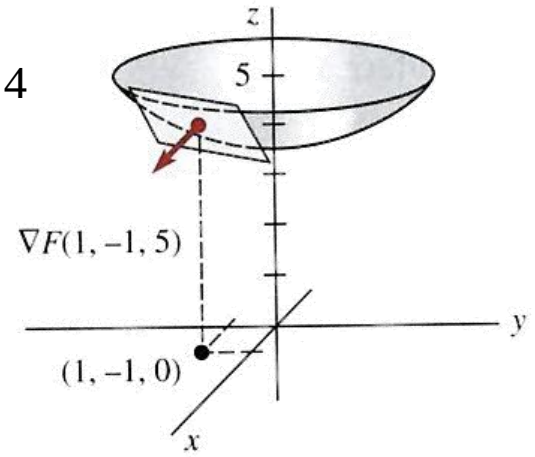
Tangent Planes and Normal Lines

☑ Example 5
Normal Line to a Surface

Find parametric equation for the normal line to the surface in Example 4 at (1,-1,5).

Example 4.

$$z = \frac{1}{2}x^2 + \frac{1}{2}y^2 + 4$$



Solution)

$$F(x, y, z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 - z + 4$$

$$\nabla F(x, y, z) = x\mathbf{i} + y\mathbf{j} - \mathbf{k}$$

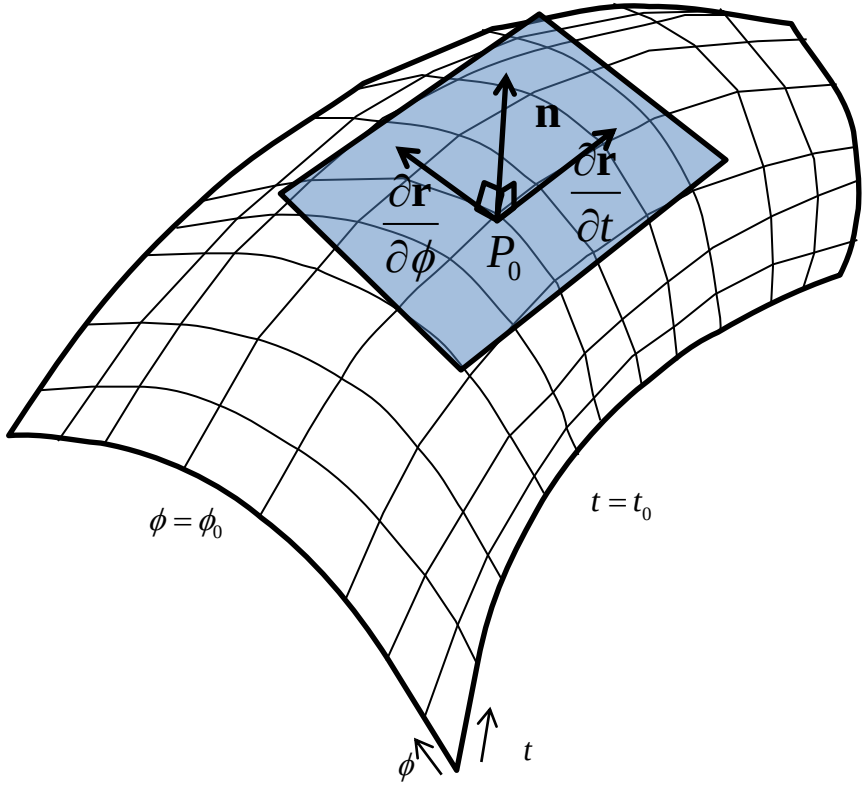
$$\begin{cases} x = 1 + t \\ y = -1 - t \\ z = 5 - t \end{cases}$$

Tangent Planes and Normal Lines

The tangent plane at $\mathbf{r} = \mathbf{r}(u_0, v_0)$ **contains two tangent vectors** $\dfrac{\partial \mathbf{r}}{\partial u}, \dfrac{\partial \mathbf{r}}{\partial v}$

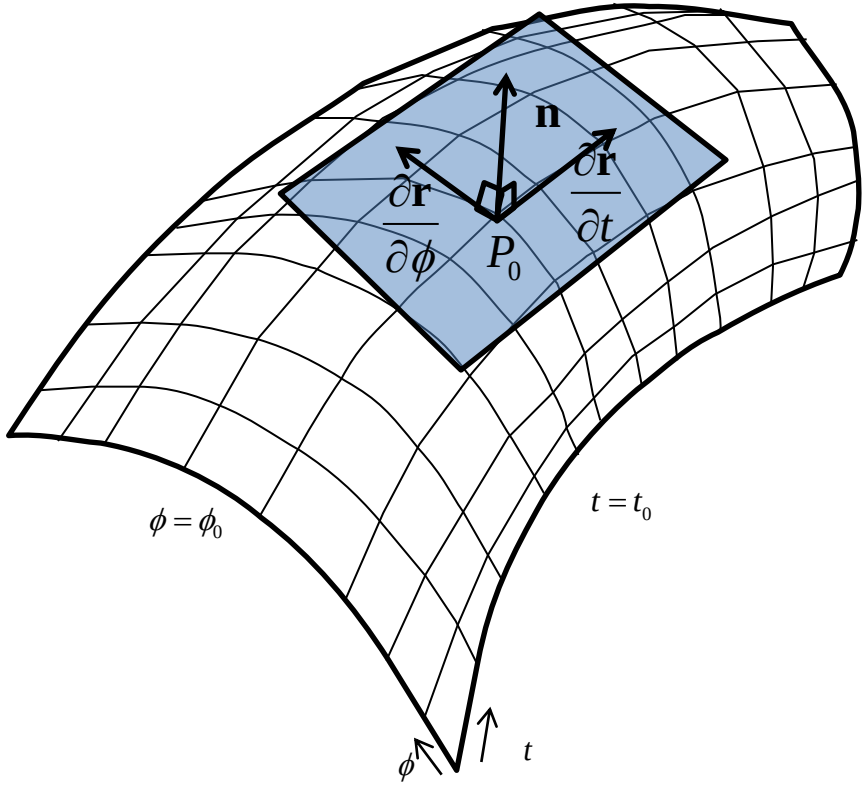
So the normal to the surface is a multiple of their vector product.

$$\mathbf{n} = \pm\left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) \Big/ \left|\frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial v}\right|$$



Tangent Planes and Normal Lines

$$\mathbf{n} = \pm\left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) \Big/ \left|\frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial v}\right|$$



Ex) On the paraboloid of revolution given by

$$\mathbf{r} = \mathbf{r}(t, \phi) = 2at\cos\phi\mathbf{i} + 2at\sin\phi\mathbf{j} + at^2\mathbf{k}$$

We have

$$\frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial \phi} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2a\cos\phi & 2a\sin\phi & 2at \\ -2at\sin\phi & 2at\cos\phi & 0 \end{vmatrix}$$

$$= -4a^2t^2\cos\phi\mathbf{i} - 4a^2t^2\sin\phi\mathbf{j} + 4a^2t\mathbf{k}$$

Then

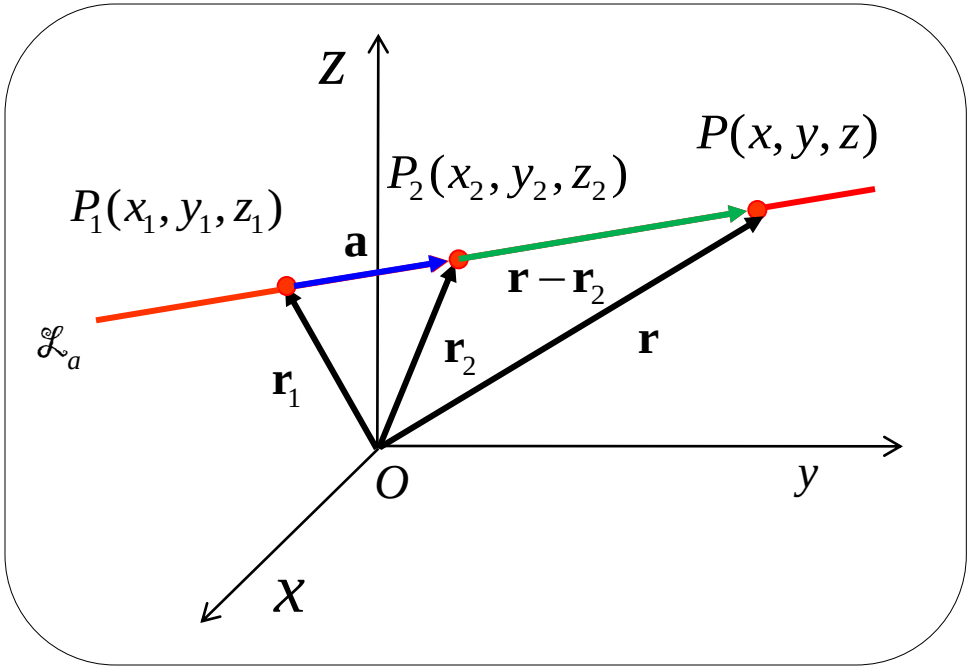
$$\left|\frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial \phi}\right|^2 = 16a^4t^2(1+t^2)$$

Hence

$$\mathbf{n} = \frac{1}{\sqrt{1+t^2}}(-t\cos\phi, -t\sin\phi, 1)$$

Lines and Planes in 3-Space

■ Lines: Vector Equation



To find an equation of the line through P_1 and P_2, assume P is any point on the line.

$$\mathbf{r}-\mathbf{r}_2 = t(\mathbf{r}_2-\mathbf{r}_1)$$

$$\mathbf{a} = \mathbf{r}_2 - \mathbf{r}_1$$

Vector equation for the line $\mathscr{L}_a$ is

$$\mathbf{r} = \mathbf{r}_2 + t\mathbf{a}$$

direction vector

$\mathbf{r} = r_1 + t\mathbf{a}$, $\mathbf{r} = r_1 + t(-\mathbf{a})$, $\mathbf{r} = r_1 + t(k\mathbf{a})$, Are also equation of $\mathscr{L}_a$

Lines and Planes in 3-Space

■ Example 1 Vector Equation of a Line

Find a vector equation for the line through (2,-1,8) and (5,6,-3)

Sol) **Let** $\mathbf{a} = \langle 2-5, -1-6, 8-(-3) \rangle = \langle -3, 7, 11 \rangle$

Then, The three possible vector equations for the line :

$$\langle x, y, z \rangle = \langle 2, -1, 8 \rangle + t \langle -3, -7, 11 \rangle$$

$$\langle x, y, z \rangle = \langle 5, 6, -3 \rangle + t \langle -3, -7, 11 \rangle$$

$$\langle x, y, z \rangle = \langle 5, 6, -3 \rangle + t \langle 3, 7, -11 \rangle$$

Lines and Planes in 3-Space

■ Parametric Equations

$$\mathbf{r} = \mathbf{r}_2 + t\mathbf{a}$$

Write this equation by component,

$$\begin{aligned}\langle x, y, z\rangle &= \langle x_2 + t(x_2 - x_1), y_2 + t(y_2 - y_1), z_2 + t(z_2 - z_1)\rangle \\ &= \langle x_2 + a_1 t, y_2 + a_2 t, z_2 + a_3 t\rangle\end{aligned}$$

$$x = x_2 + a_1 t, \quad y = y_2 + a_2 t, \quad z = z_2 + a_3 t$$

This is called parametric equations

■ Example 2 Parametric Equations of a Line

Find a parametric equations for the line in Example 1

$\langle x, y, z \rangle = \langle 2, -1, 8 \rangle + t \langle -3, -7, 11 \rangle$: Line in Example 1

Sol)

$$\langle x, y, z \rangle = \langle 2 - 3t, -1 - 7t, 8 + 11t \rangle$$

$$x = 2 - 3t,\ y = -1 - 7t,\ z = 8 + 11t$$

■ Example 3 Vector Parallel to a Line

Find a vector **a** that is parallel to the line $\mathscr{L}_a$ whose parametric equations are

$$x = 4 + 9t,\ y = -14 + 5t,\ z = 1 - 3t$$

Sol) The coefficients of the parameter in each equation are the components of a vector that is parallel to the line.

$$\therefore \mathbf{a} = 9\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$$

Lines and Planes in 3-Space

■ Symmetric Equations

From $x = x_2 + a_1 t, \;\; y = y_2 + a_2 t, \;\; z = z_2 + a_3 t$

$$t = \frac{x - x_2}{a_1} = \frac{y - y_2}{a_2} = \frac{z - z_2}{a_3}$$

$$\frac{x - x_2}{a_1} = \frac{y - y_2}{a_2} = \frac{z - z_2}{a_3}$$

Are said to be **symmetric equations** for the line through P_1 and P_2

■ Example 4 Symmetric Equations of a Line

Find a symmetric equations for the line through (4,10,-6) and (7,9,2)

Sol)

$$\frac{x-x_2}{a_1}=\frac{y-y_2}{a_2}=\frac{z-z_2}{a_3}$$ **Symmetric equation**

$$a_1=4-7=-3,\quad a_2=10-9=1,\quad a_3=-6-2=-8$$

$$\frac{x-7}{3}=\frac{y-9}{-1}=\frac{z-2}{8}$$

Lines and Planes in 3-Space

■ Example 5 Symmetric Equations of a Line

Find a symmetric equations for the line through (5,3,1) and (2,1,1)

Sol) $\dfrac{x-x_2}{a_1}=\dfrac{y-y_2}{a_2}=\dfrac{z-z_2}{a_3}$ **Symmetric equation**

Comes from $x=x_2+a_1t,\ y=y_2+a_2t,\ z=z_2+a_3t$

$$a_1=5-2=3,\ a_2=3-1=2,\ a_3=1-1=0$$

$$x=x_2+a_1t,\ y=y_2+a_2t,\ z=z_2+a_3t \Rightarrow \frac{x-5}{3}=t,\ \frac{y-3}{2}=t,\ z=1$$

$$a_1=3,\ a_2=2,\ a_3=0$$

$$(x_2,y_2,z_2)=(5,3,1)$$

$$\frac{x-5}{3}=\frac{y-3}{2},\ z=1$$

The symmetric equations describe a line in the plane z=1

Lines and Planes in 3-Space

■ Example 6 Line Parallel to a Vector

Write vector, parametric, and symmetric equations for the line through (4,6,-3) and parallel to $\mathbf{a} = 5\mathbf{i} - 10\mathbf{j} + 2\mathbf{k}$

Sol) $\langle x, y, z \rangle = \langle x_2 + a_1 t, y_2 + a_2 t, z_2 + a_3 t \rangle$ **Vector equation**

$x = x_2 + a_1 t, \;\; y = y_2 + a_2 t, \;\; z = z_2 + a_3 t$ **Parametric equation**

$\dfrac{x - x_2}{a_1} = \dfrac{y - y_2}{a_2} = \dfrac{z - z_2}{a_3}$ **Symmetric equation**

$a_1 = 5,\ a_2 = -10,\ a_3 = 2$

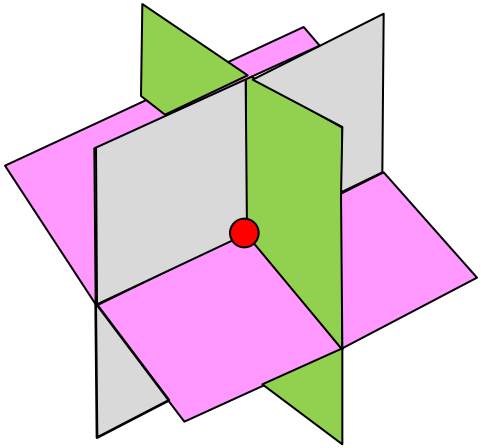
$Vector : \langle x, y, z \rangle = \langle 4, 6, -3 \rangle + t \langle 5, -10, 2 \rangle$

$Parametric : x = 4 + 5t,\ y = 6 - 10t,\ z = -3 + 2t$

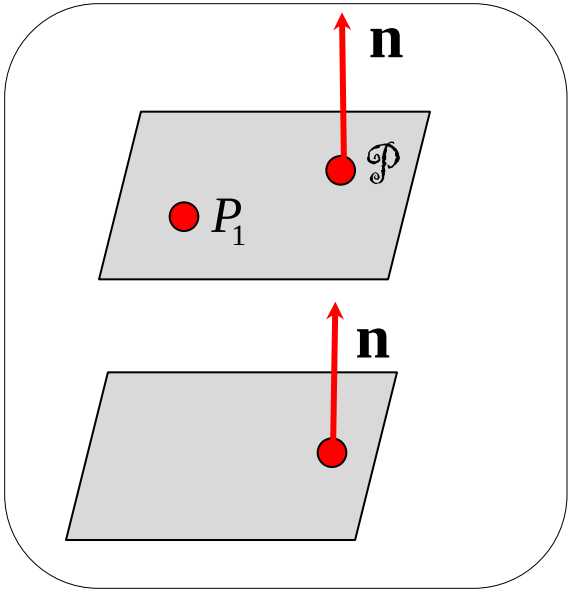
$Symmetric : \dfrac{x - x_2}{a_1} = \dfrac{y - y_2}{a_2} = \dfrac{z - z_2}{a_3}$

Lines and Planes in 3-Space

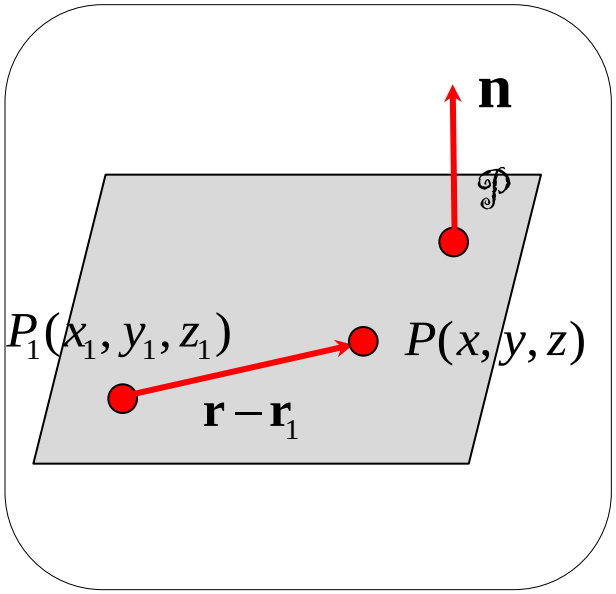
■ Planes: Vector Equation



Through a given point there pass an infinite number of planes



If point and normal are specified, only one plane is determined.

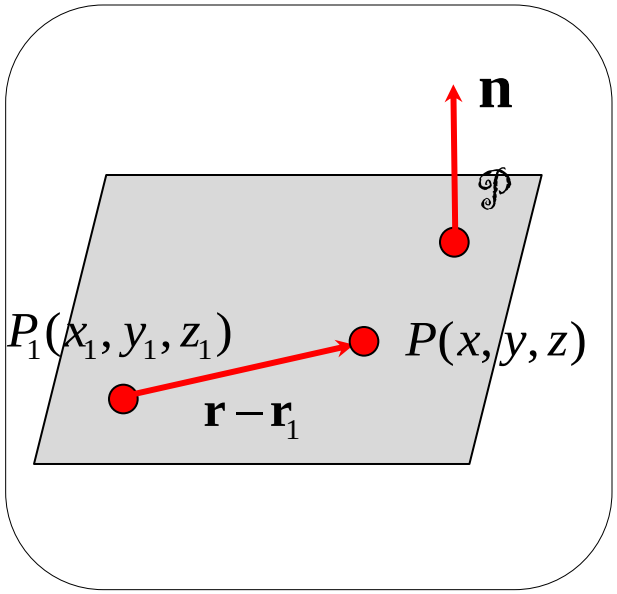


$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_1) = 0$$

"Vector equation of plane"

Lines and Planes in 3-Space

■ Cartesian Equation



$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_1) = 0$$

If normal Vector is $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

"Cartesian Equation" of plane containing $P_1(x_1, y_1, z_1)$

■ Example 7 Plane Perpendicular to a Vector

Find an equation of the plane that contains the point (4,-1,3) and is perpendicular to the vector $\mathbf{n} = 2\mathbf{i} + 8\mathbf{j} - 5\mathbf{k}$

Sol)

$$2(x-4)+8(y+1)-5(z-3)=0$$

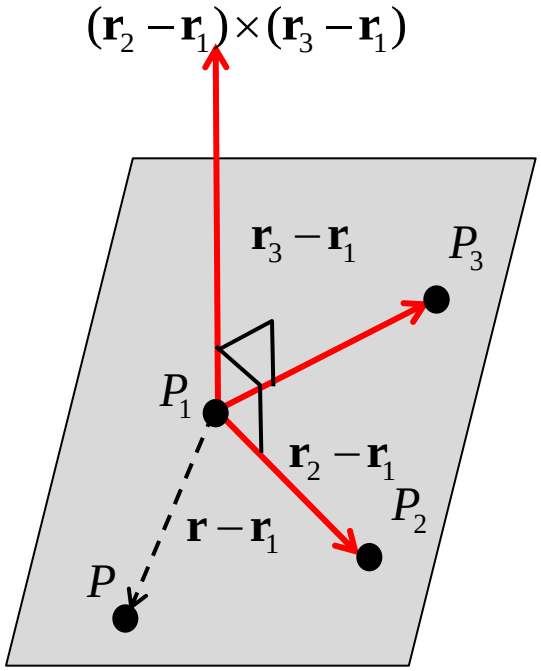
$$or \quad 2x+8y-5z+15=0$$

Theorem 7.3 **Plane with Normal Vector**

The graph of any equation $ax+by+cz+d=0, a, b, c$ **not all zero, is a plane with the normal vector** $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

Lines and Planes in 3-Space

■ Three noncollinear points



$$\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}_1)=0 \quad \text{“Vector Equation”}$$

$$\mathbf{n}=(\mathbf{r}_2-\mathbf{r}_1)\times(\mathbf{r}_3-\mathbf{r}_1)$$

$$[(\mathbf{r}_2-\mathbf{r}_1)\times(\mathbf{r}_3-\mathbf{r}_1)]\cdot(\mathbf{r}-\mathbf{r}_1)=0$$

Three noncollinear points P_1, P_2 *and* P_3 also determine a plane

■ Example 9 Three Points That Determine a Plane

Find a equation of the plane that contains (1,0,-1),(3,1,4) and (2,-2,0)

Sol) $\left.\begin{matrix}(1,0,-1)\\(3,1,4)\end{matrix}\right\}\mathbf{u}=2\mathbf{i}+\mathbf{j}+5\mathbf{k}$ $\left.\begin{matrix}(3,1,4)\\(2,-2,0)\end{matrix}\right\}\mathbf{v}=\mathbf{i}+3\mathbf{j}+4\mathbf{k}$

$$\left.\begin{matrix}(x,y,z)\\(2,-2,0)\end{matrix}\right\}\mathbf{w}=(x-2)\mathbf{i}+(y+2)\mathbf{j}+z\mathbf{k}$$

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&1&5\\1&3&4\end{vmatrix}=-11\mathbf{i}-3\mathbf{j}+5\mathbf{k}$$

$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}=0$ yields

$$-11(x-2)-3(y+2)+5z=0 \quad or \quad -11x-3y+5z+16=0$$

Lines and Planes in 3-Space

■ Graphs

To graph a plane, we should try to find

(i) the x-,y-, and z-intercept

If necessary,

(ii) The trace of the plane in each coordinate plane

Lines and Planes in 3-Space

■ Example 10 Graph of a Plane

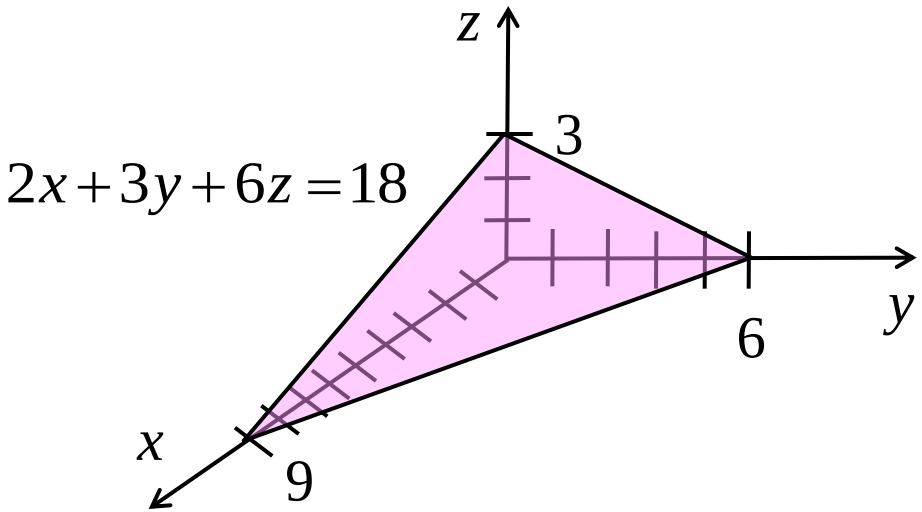
Graph the equation $2x+3y+6z=18$

Sol) $y=0,\ z=0\ gives\ x=9$

$x=0,\ z=0\ gives\ y=6$

$x=0,\ y=0\ gives\ z=3$

$\Rightarrow\ x-\text{intercept}=9,\ y-\text{intercept}=6,\ z-\text{intercept}=3$



■ Example 11 Graph of a Plane

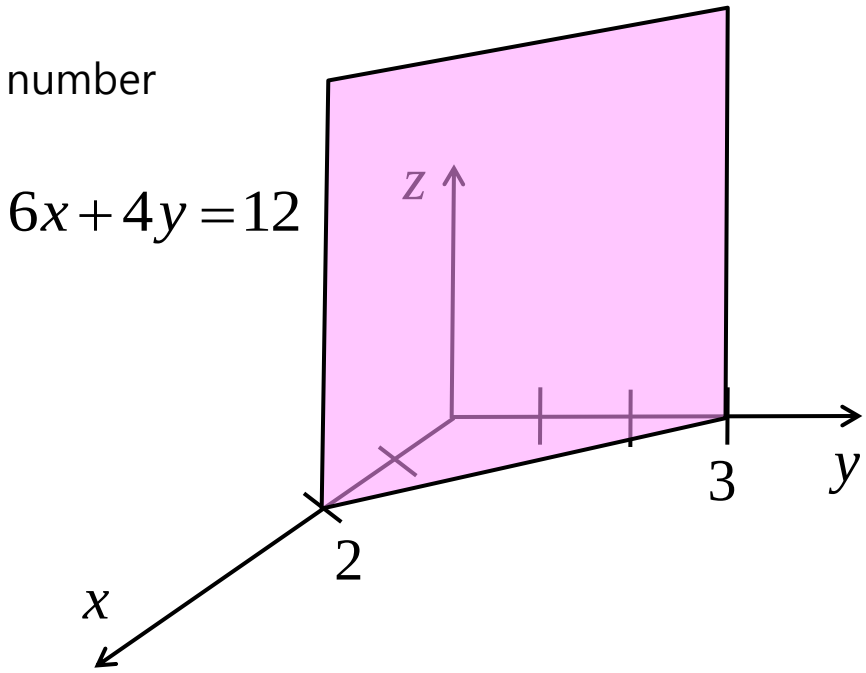
Graph the equation $6x+4y=12$

Sol) $x-\text{intercept}=2$

$y-\text{intercept}=3$

In three dimensions, it is the trace of a plane in the xy-coordinate plane.

Since z is not specified, it can be any real number



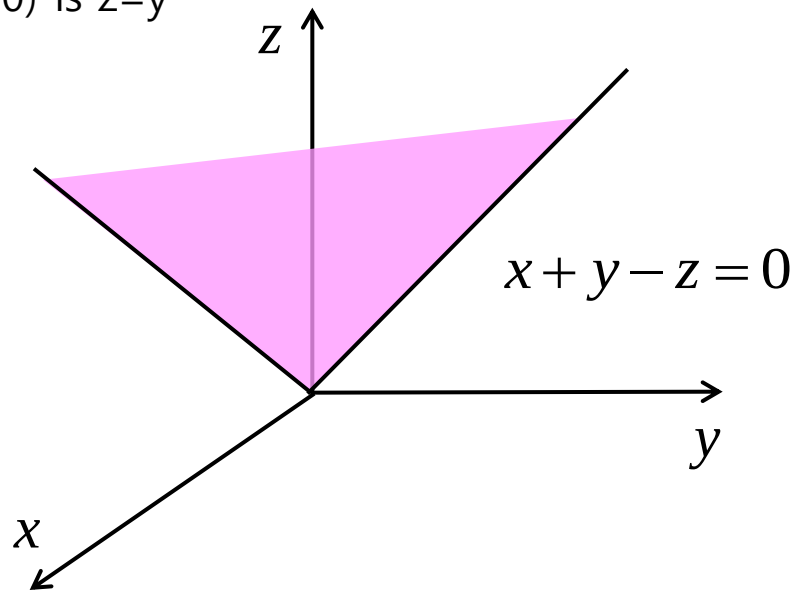
Lines and Planes in 3-Space

■ Example 12 Graph of a Plane

Graph the equation $x + y - z = 0$

Sol) Trace in xz-plane(y=0) is z=x

Trace in yz-plane(x=0) is z=y



■ Example 13 Line of Intersection of Two Planes

Find parametric equations for the line of intersection of

$$2x-3y+4z=1$$

$$x-y-z=5$$

Sol) In a system of 2 equations and 3 unknowns, we choose one variable arbitrary, $z=t$

Then,

$$x=14+7t,$$

$$y=9+6t$$

$$z=t$$

Lines and Planes in 3-Space

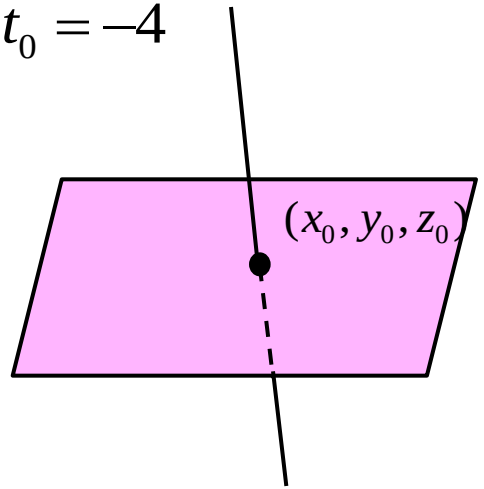
■ Example 14 Point of Intersection of a Line and Plane

Find the point of intersection of the plane $3x-2y+z=-5$
and the line $x=1+t, y=-2+2t, z=4t$

Sol) For some number t_0, plane and line are intersect in a point (x_0, y_0, z_0)

So, $2(1+t_0)-2(-2+2t_0)+4t_0=-5 \quad or \quad t_0=-4$

$$x_0=-3,\ y_0=-10,\ z_0=-16$$

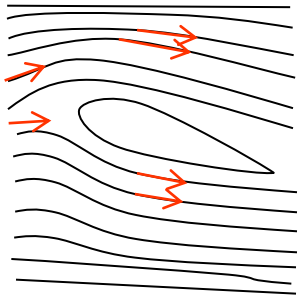


Vector Fields

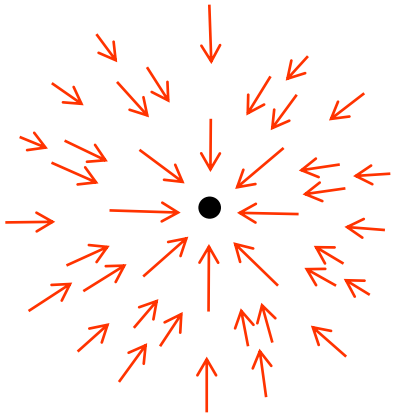
Vector Fields : Vector Functions of two or three variables

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$

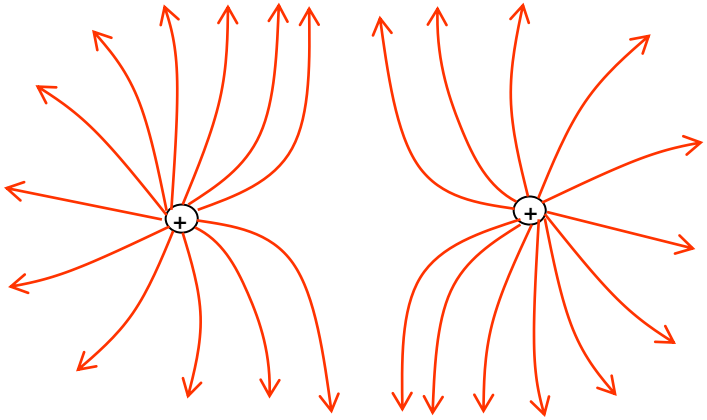
$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$



Air flow around
an airplane wing



Magnitude of
attractive force



Force around two
equal positive charge

Vector Fields

Example
Two-dimensional Vector Field

Graph the two-dimensional vector field

$$\mathbf{F}(x,y)=-y\mathbf{i}+x\mathbf{j}.$$

Solution)

$$\mathbf{F}(1,1) = -\mathbf{i} + \mathbf{j}$$

$$\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$$

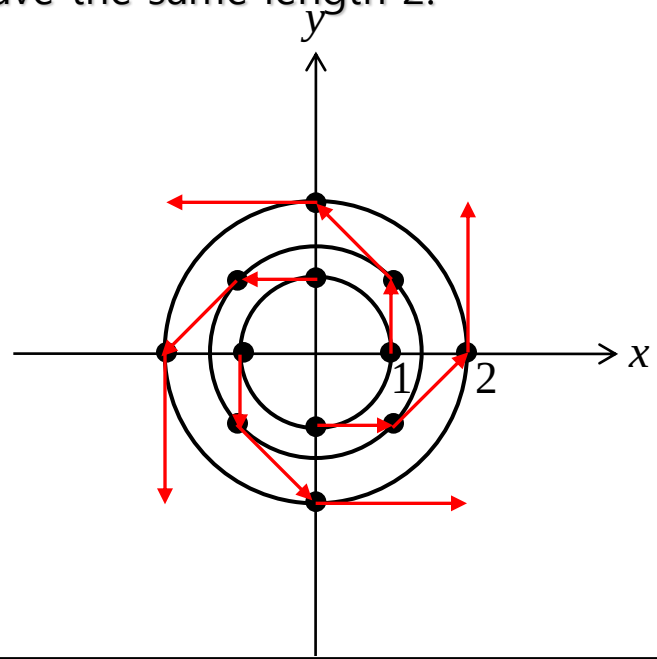
$$\|\mathbf{F}(x, y)\| = \sqrt{y^2 + x^2} = \sqrt{x^2 + y^2}$$

Vectors of the same length k must lie along the curve defined by $\sqrt{x^2 + y^2} = k.$

$\sqrt{x^2 + y^2} = 1$: At the points (1,0), (0,1), (-1,0), (0,-1), the corresponding vectors **j, -i, -j, i** have the same length 1.

$\sqrt{x^2 + y^2} = 2$: At the points (1,1), (-1,1), (-1,-1), (1,-1), the corresponding vectors –i+**j, -i-j, i-j, i+j** have the same length $\sqrt{2}$.

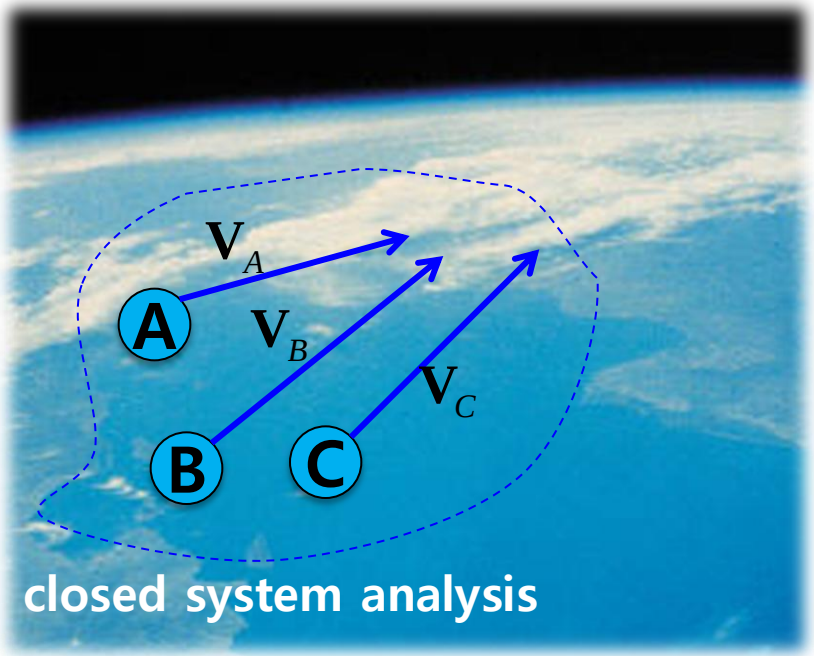
$\sqrt{x^2 + y^2} = 4$: At the points (2,0), (0,2), (-2,0), (0,-2), the corresponding vectors 2**j, -2i, -2j, 2 i** have the same length 2.



Vector Fields and Fluid Mechanics

Fluid Kinematics* : how fluids flow and how to describe fluid motion

•Lagrangian Description



Keeping track of

the position vector $\mathbf{x}_A(t), \mathbf{x}_B(t), \mathbf{x}_C(t), \ldots$

and the velocity vector $\mathbf{V}_A(t), \mathbf{V}_B(t), \mathbf{V}_C(t), \ldots$

of each fluid particle, A B C

*Cimbala J.M., Cengel Y.A., Essential of Fluid Mechanics : Fundamental and Application, McGraw Hill, 2008, p90~95

Vector Fields and Fluid Mechanics

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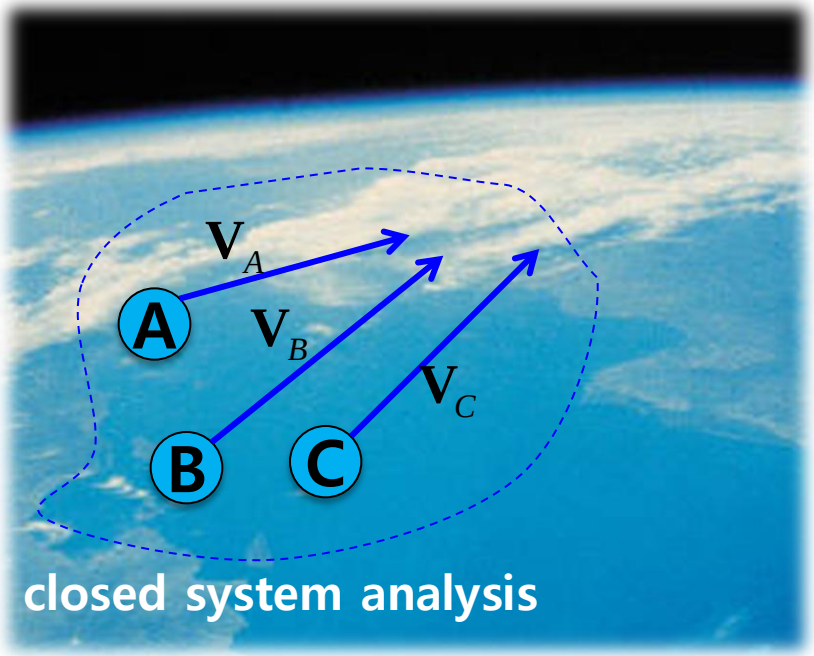
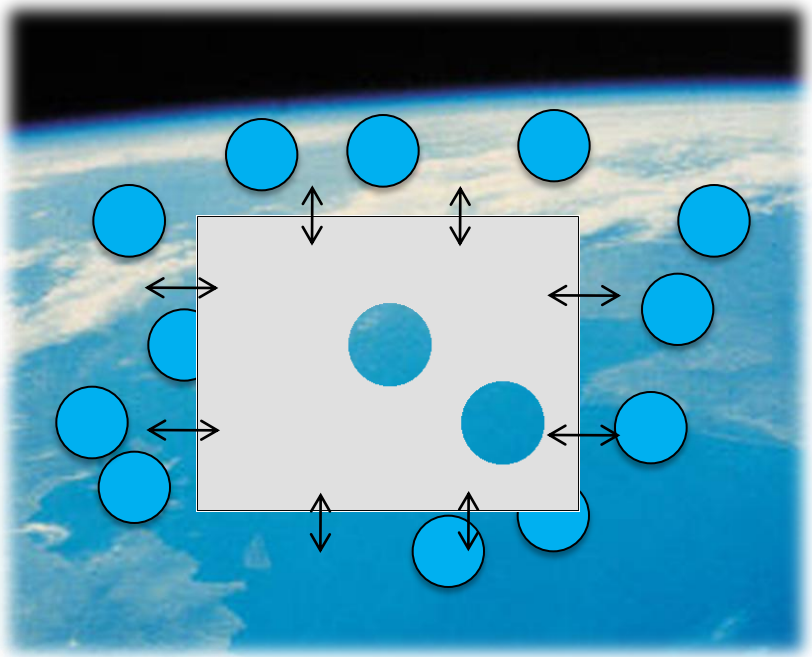


•Eulerian Description



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Vector Fields and Fluid Mechanics

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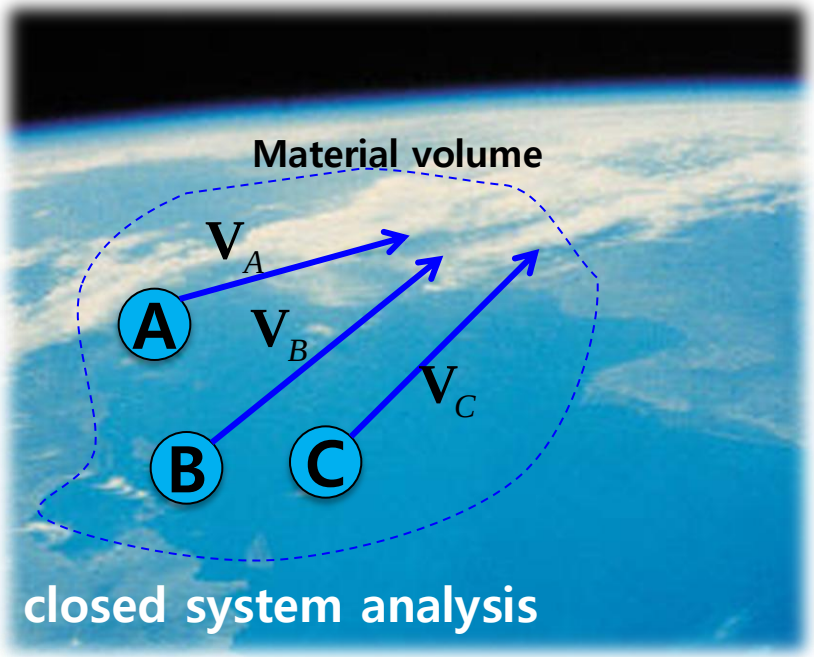

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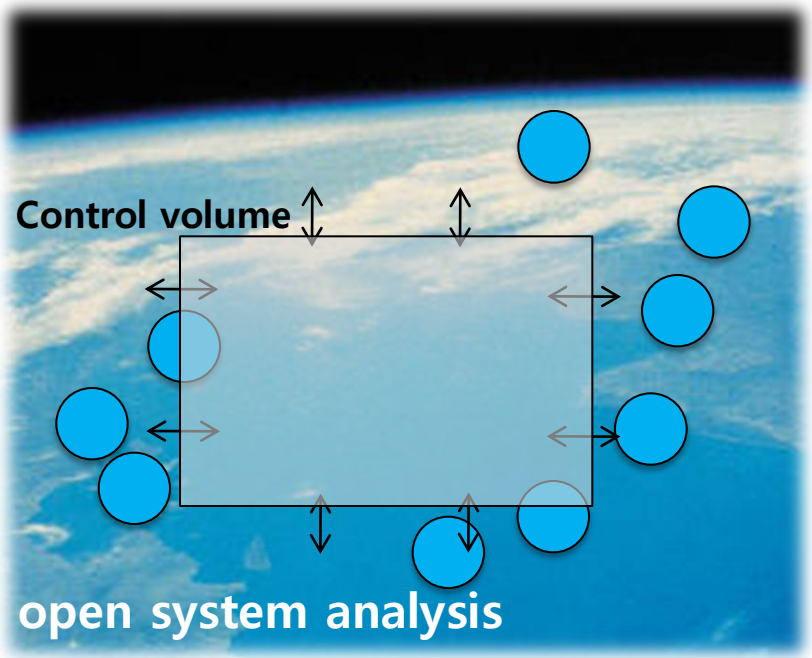

Define Field Variables

The field variables at a particular location and

A particular time is...

the value of the variable for whichever

fluid particle happen to occupy that location

at that time

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Vector Fields and Fluid Mechanics

Fluid Kinematics* : how fluids flow and how to describe fluid motion

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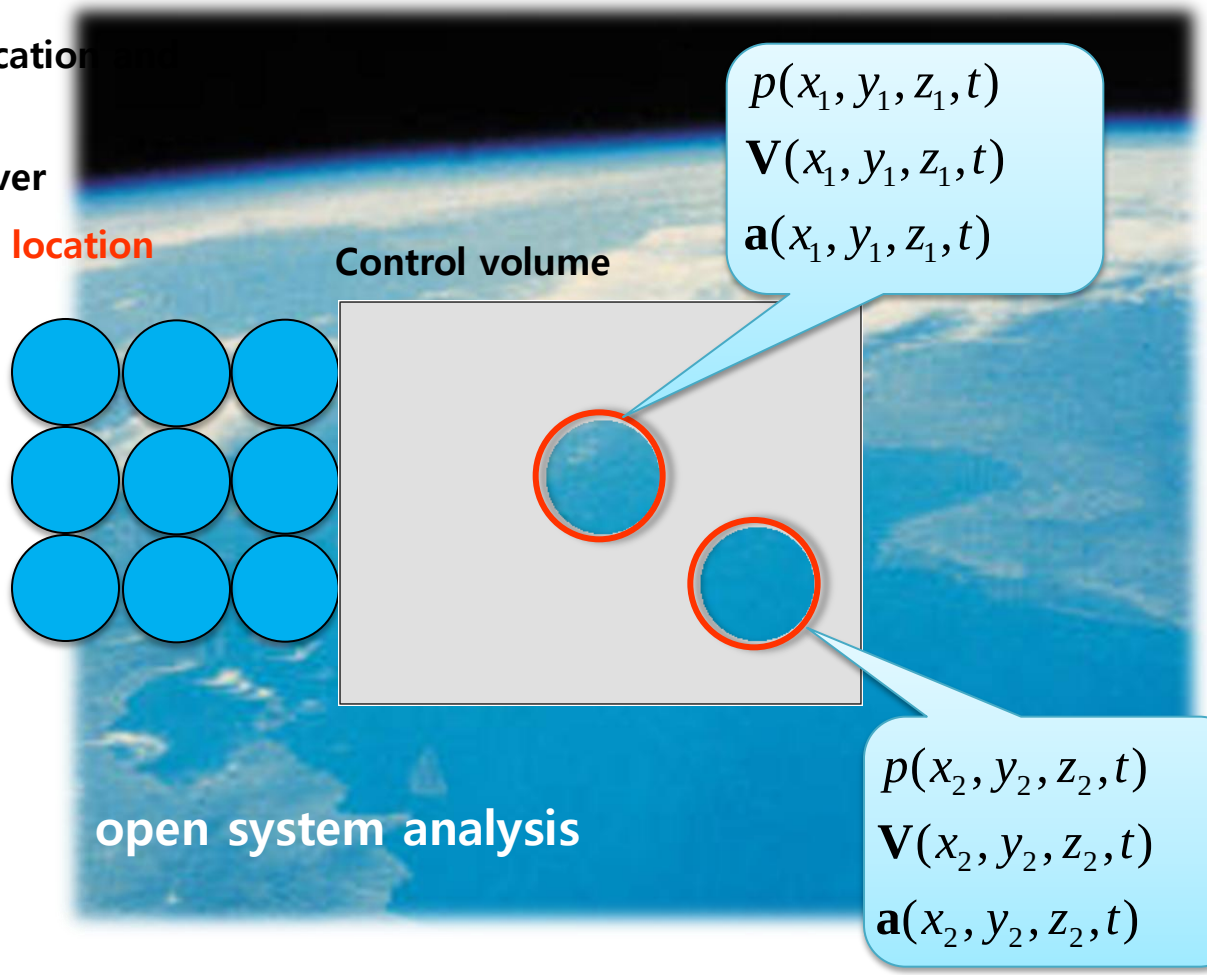

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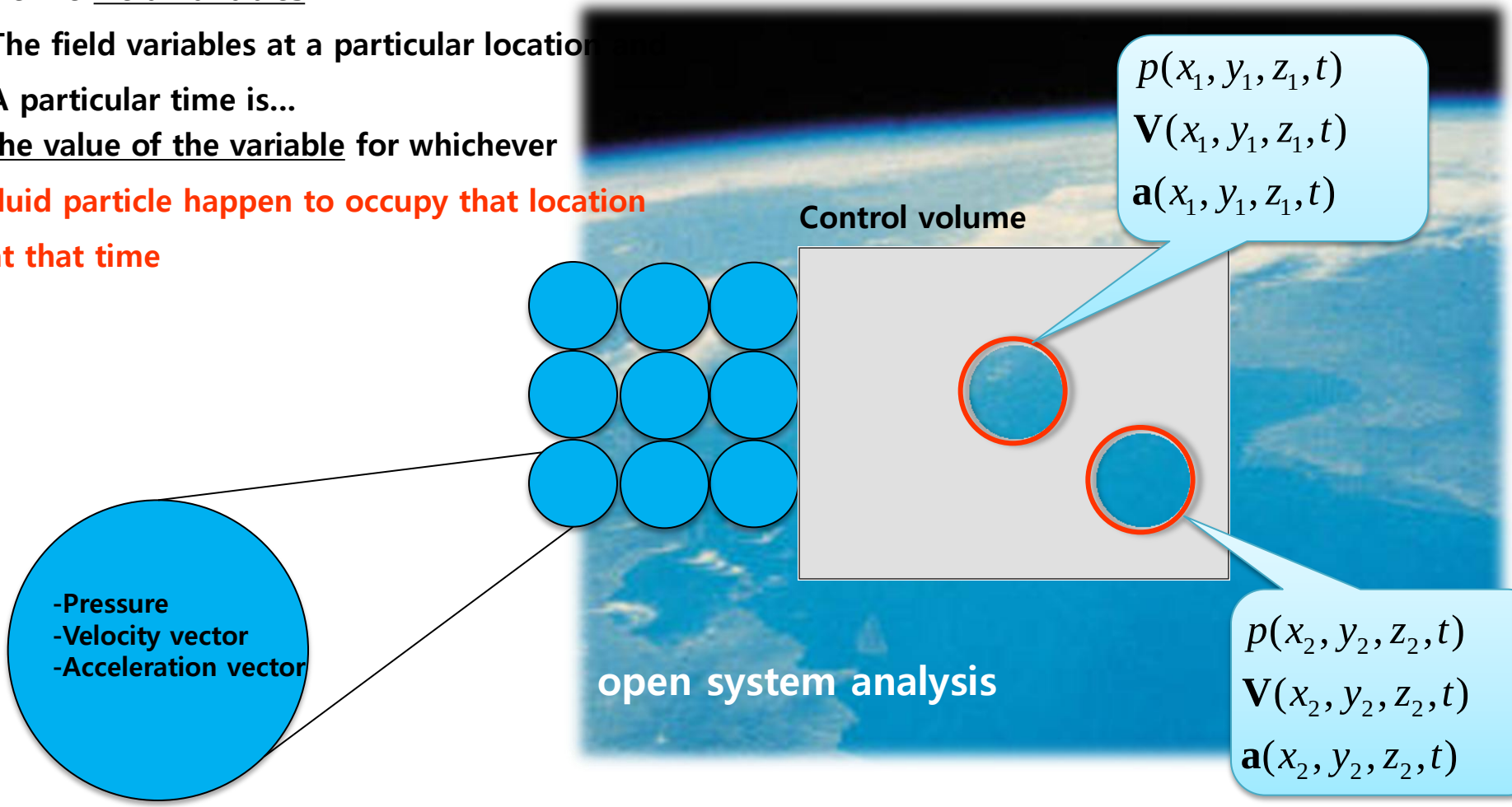

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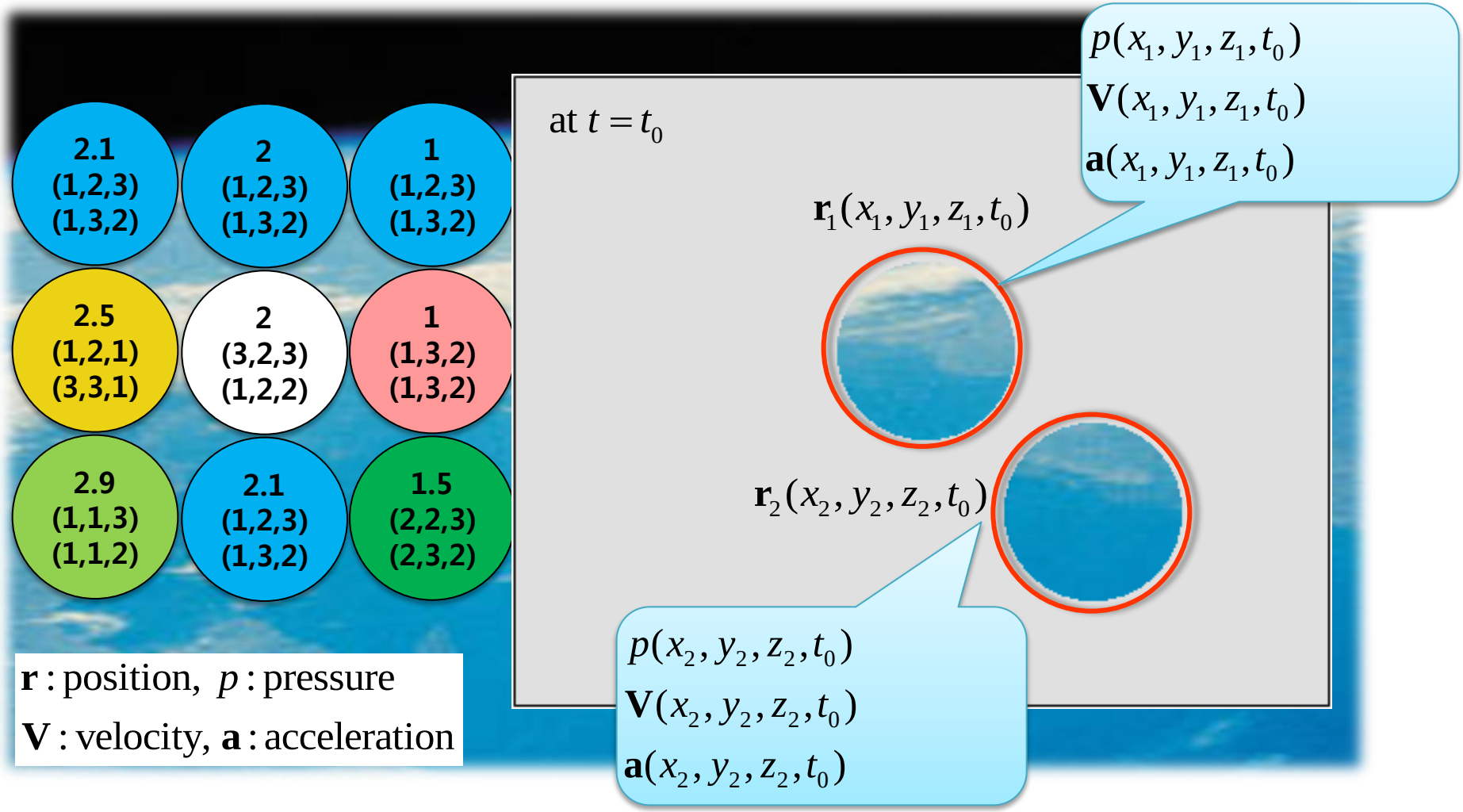
- **Eulerian Description**



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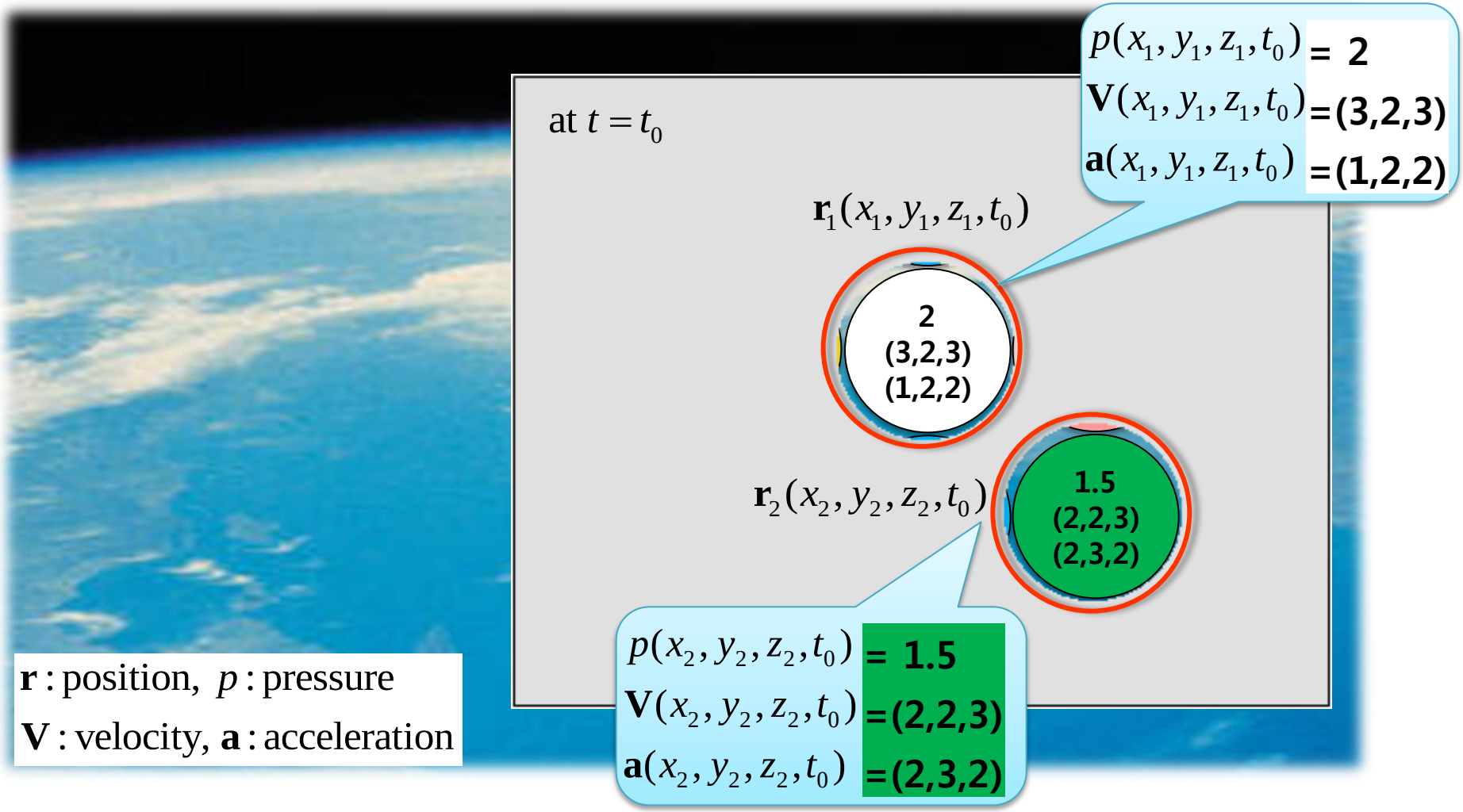

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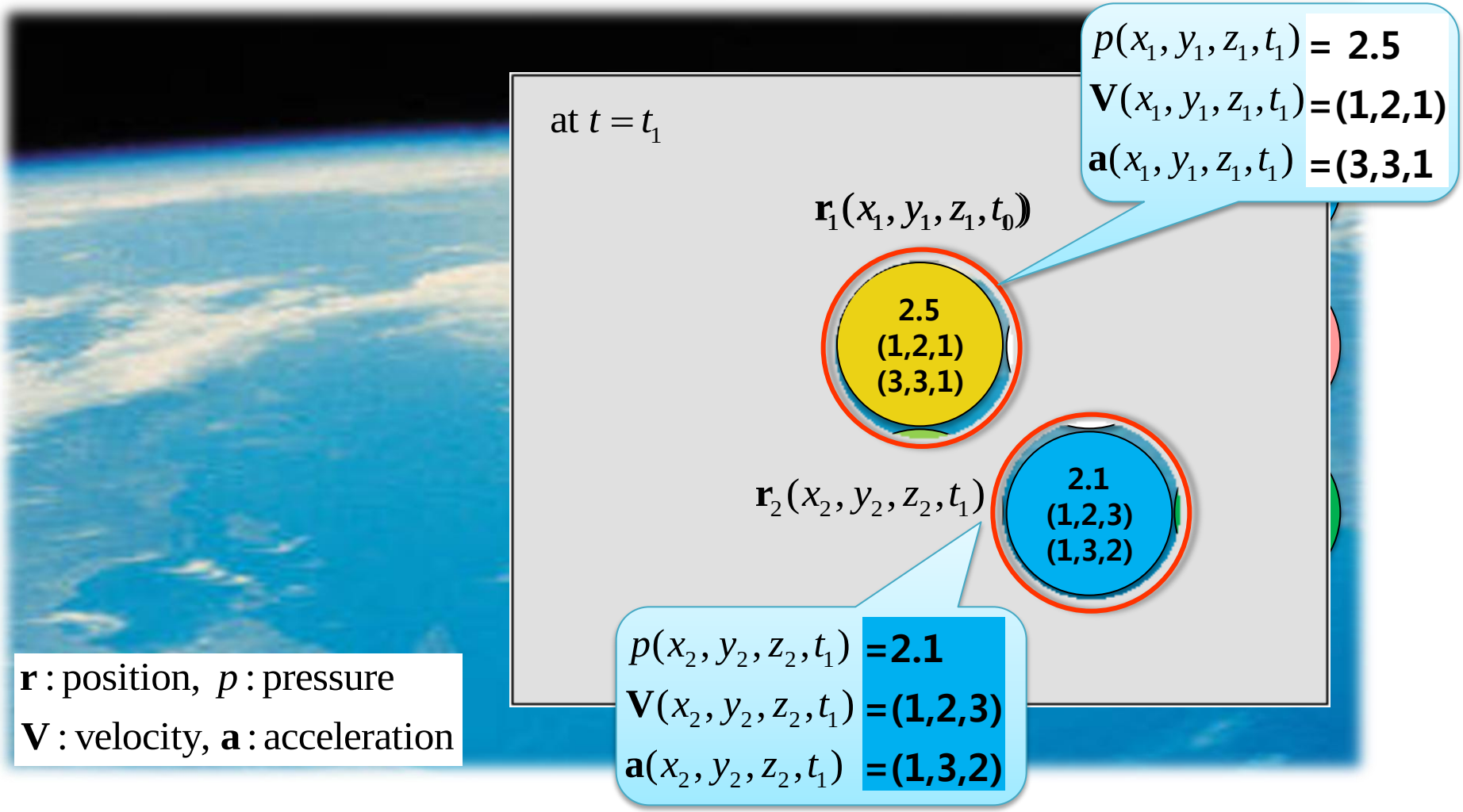
Vector Fields and Fluid Mechanics

Fluid Kinematics* : how fluids flow and how to describe fluid motion

- **Eulerian Description**

Ref. Lagrangian & Eulerian Description



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Vector Fields and Fluid Mechanics

■ Example 4-1* A Steady Two-Dimensional Velocity Field

A steady, incompressible, two-dimensional velocity filed is given by

$$\mathbf{V} = (u, v) = (0.5 + 0.8x)\mathbf{i} + (1.5 - 0.8y)\mathbf{j}$$

A stagnation point is defined as a point in the flow field where the velocity is identically zero.

(a) Determine if there are any stagnation points in this flow filed and, if so, where?

(b) Sketch velocity vectors at several locations in the domain between x=-2 to 2 and y=0 to 5

Sol) (a)

$$\left.\begin{aligned} u = 0.5 + 0.8x = 0 \;&\rightarrow\; x = -0.625 \\ v = 1.5 - 0.8y = 0 \;&\rightarrow\; y = 1.875 \end{aligned}\right\} \text{Stagnation point}$$

*Cimbala J.M., Cengel Y.A., Essential of Fluid Mechanics : Fundamental and Application, McGraw Hill, 2008, p91

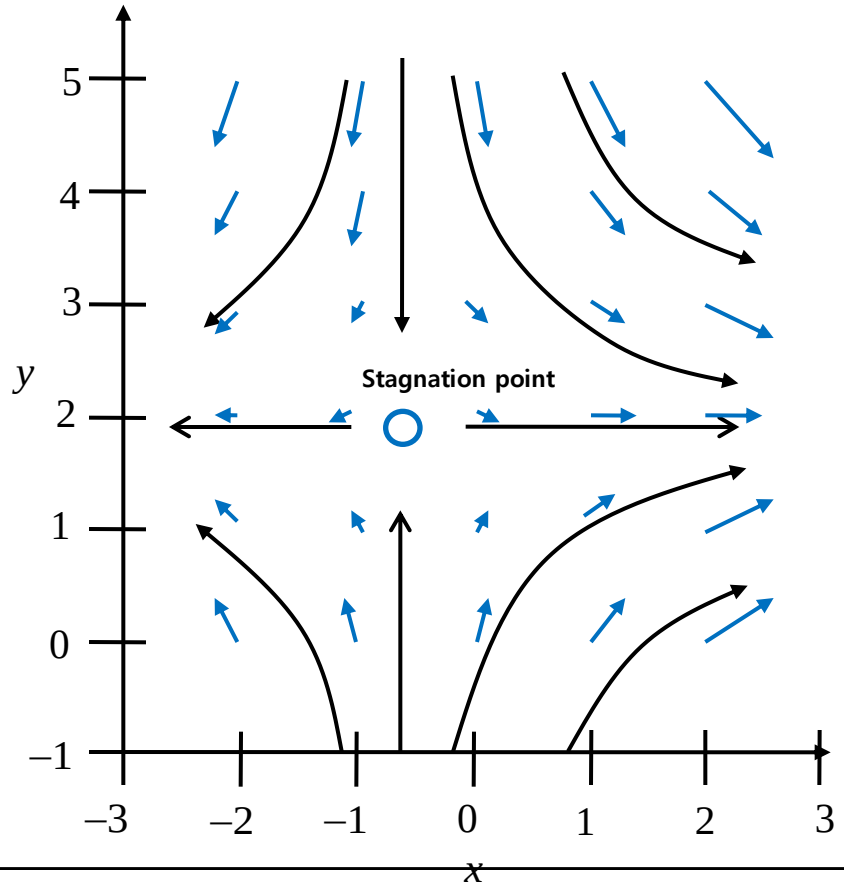
Vector Fields and Fluid Mechanics

■ Example 4-1* A Steady Two-Dimensional Velocity Field

$$\mathbf{V} = (u, v) = (0.5 + 0.8x)\mathbf{i} + (1.5 - 0.8y)\mathbf{j}$$

(b) Sketch velocity vectors at several locations in the domain between x=-2 to 2 and y=0 to 5

Sol) (b)



*Cimbala J.M., Cengel Y.A., Essential of Fluid Mechanics : Fundamental and Application, McGraw Hill, 2008, p91

Curl of a Vector Fields

Definition. Curl (introduced by Maxwell*)

Let $\mathbf{v}(x,y,z)=[v_1,v_2,v_3]=v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$ be a differentiable vector function of Cartesian coordinates x, y, z.

Then the curl of the vector function $\mathbf{v}$ or of the vector field given by $\mathbf{v}$ is defined by the "symbolic" determinant

$$\operatorname{curl}\mathbf{v}=\nabla\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ v_1 & v_2 & v_3\end{vmatrix}$$

$$=\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)\mathbf{k}.$$

Curl of a Vector Fields

$$\operatorname{curl}\mathbf{v}=\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)\mathbf{k}$$

Let v=[yz,3zx,z]=yzi+3zxi+zk with right-handed x,y,z.

Find *curl* v.

Solution)

$$\operatorname{curl}\mathbf{v}=\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)\mathbf{k}$$

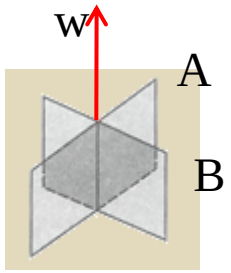
$$=\left(\frac{\partial z}{\partial y}-\frac{\partial 3zx}{\partial z}\right)\mathbf{i}+\left(\frac{\partial yz}{\partial z}-\frac{\partial z}{\partial x}\right)\mathbf{j}+\left(\frac{\partial 3zx}{\partial x}-\frac{\partial yz}{\partial y}\right)\mathbf{k}$$

$$=-3x\mathbf{i}+y\mathbf{j}+2z\mathbf{k}$$

Curl of a Vector Fields

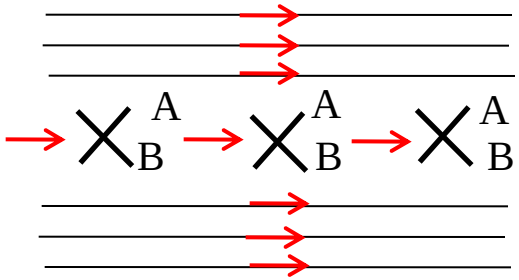
Paddle device

If a paddle device is inserted in a flowing fluid, then the curl of the velocity field **v** is a measure of the tendency of the fluid to turn the device about its vertical axis *w*.



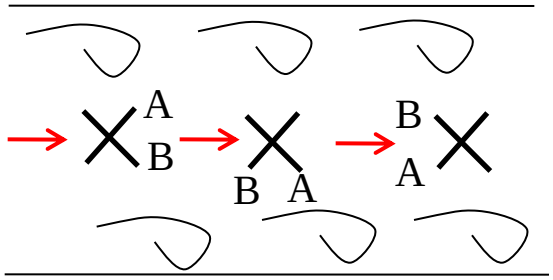
Paddle device

Physical Interpretations



Irrotational flow

$(\text{curl}\ \mathbf{F} = 0)$



Rotational flow

$(\text{curl}\ \mathbf{F} \neq 0)$

Curl of a Vector Fields

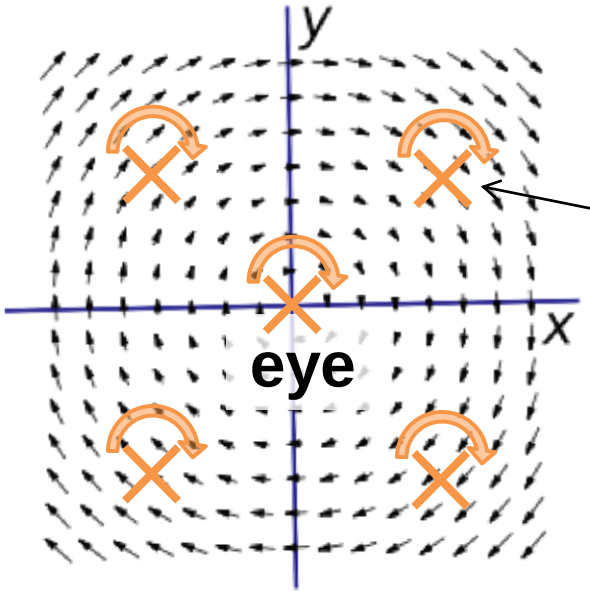
revolution

In a tornado the winds are rotating about the eye,
and a vector field showing wind velocities would have a non-zero curl at the eye, and possibly elsewhere.

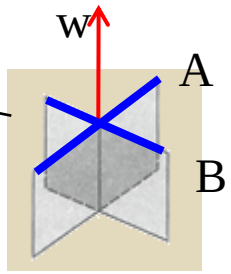
rotation

$$\mathbf{V}(x,y,z) = y\mathbf{i} + x\mathbf{j}$$

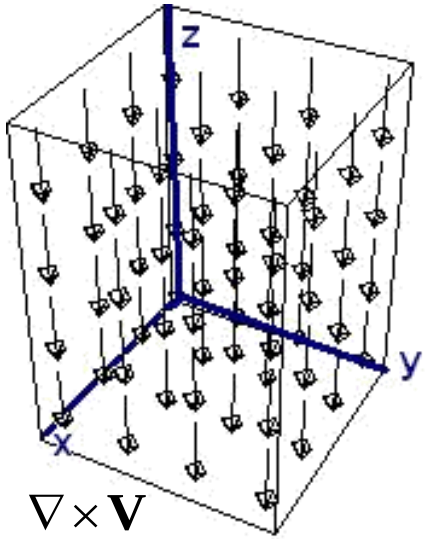
$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} - 2\mathbf{k} = -2\mathbf{k}$$



Velocity Vector

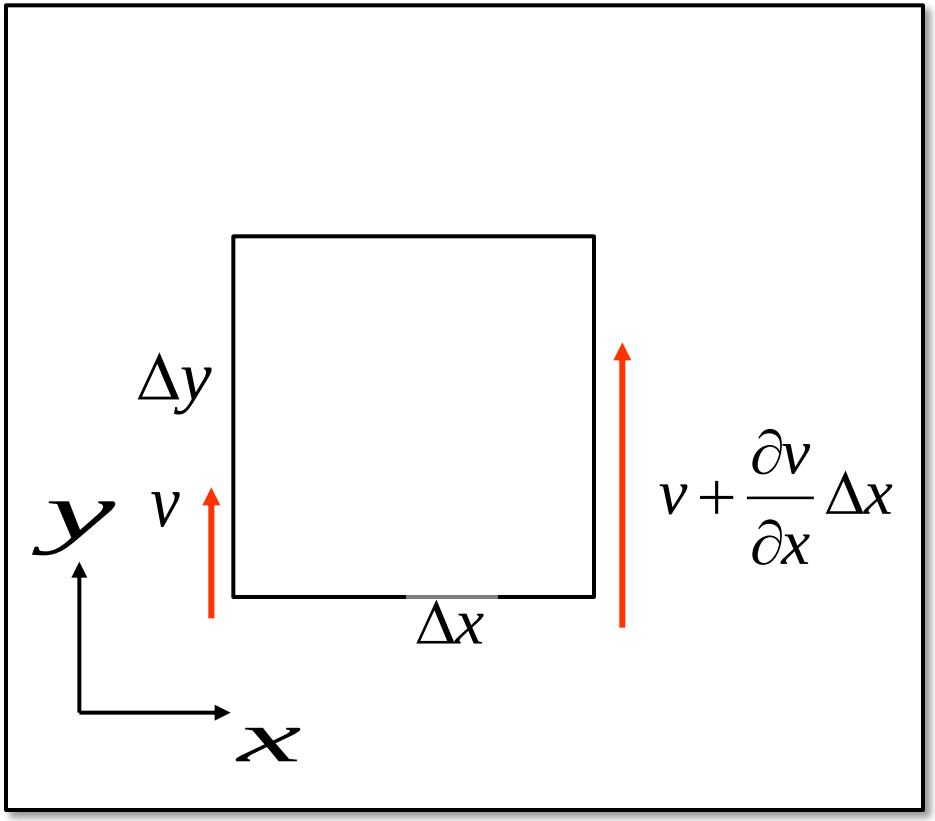


Paddle device



$\nabla \times \mathbf{V}$

Curl and Rotation in Fluids



✓ 유체 입자의 왼쪽 아래의 *y*방향 속도를 *v*라고 할 때, *x*축으로 △*x* 만큼 떨어진 지점에서의 *y*방향 속도
(Tayler Series Expansion)

$$v = v(x, y, t)$$

↓ *y,t* 가 고정이라면,

$$v = v(x)$$

↓ Tayler Series Expansion

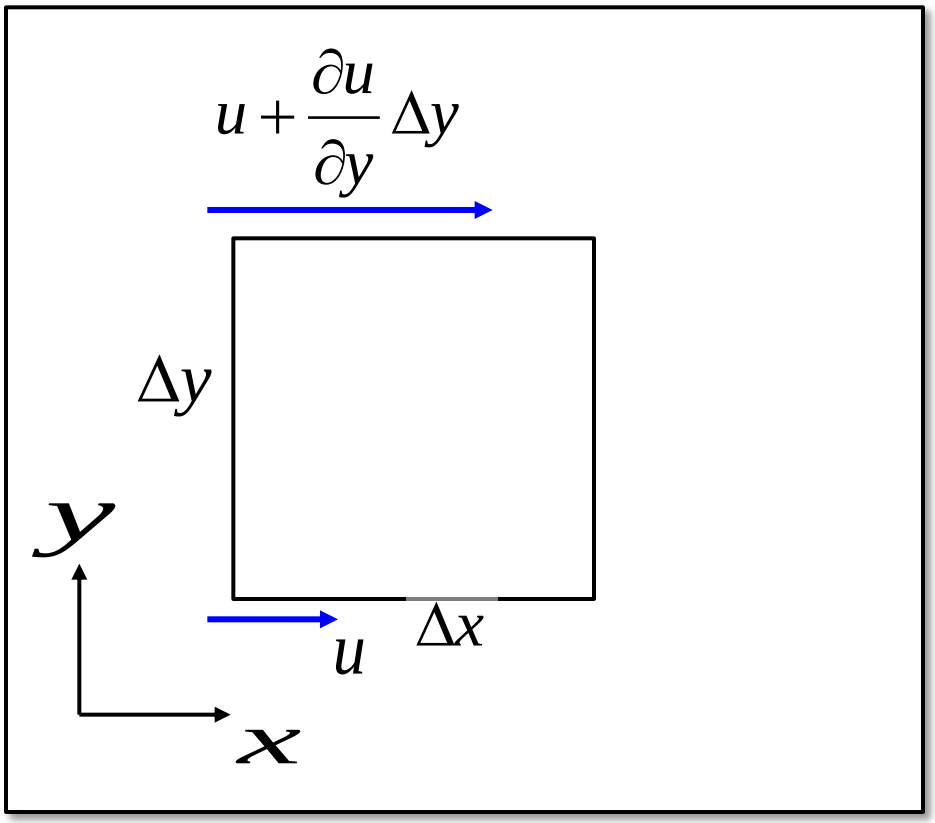
$$v(x + \Delta x) = v(x) + \frac{\partial v}{\partial x}\Delta x + \frac{\partial^2 v}{\partial x^2}(\Delta x)^2 + \ldots$$

↓ △*x*가 작다고 가정하면,

$$v(x + \Delta x) = v(x) + \frac{\partial v}{\partial x}\Delta x$$

(반시계 방향)

Curl and Rotation in Fluids



✓ 유체 입자의 왼쪽 아래의 *x*방향 속도를 *u*라고 할 때, *y*축으로 △*y* 만큼 떨어진 지점에서의 *x*방향 속도
(Tayler Series Expansion)

$$u = u(x, y, t)$$

↓ *x,t* 가 고정이라면,

$$u = u(y)$$

↓ Tayler Series Expansion

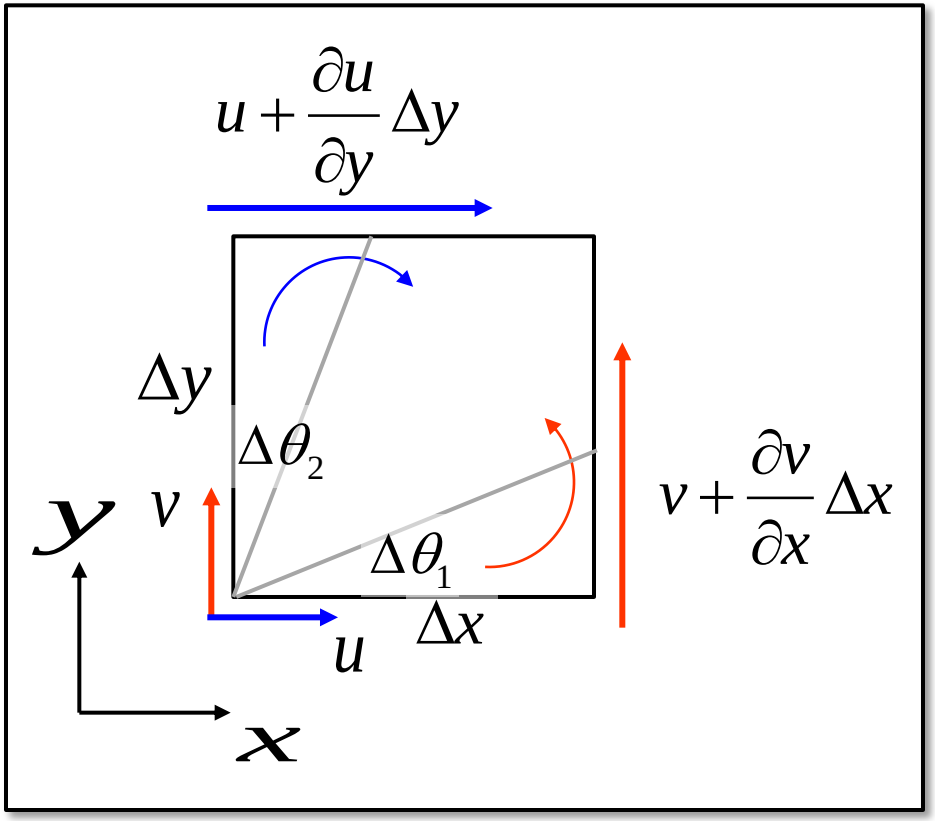
$$u(y + \Delta y) = u(y) + \frac{\partial u}{\partial y}\Delta y + \frac{\partial^2 u}{\partial y^2}(\Delta y)^2 + \ldots$$

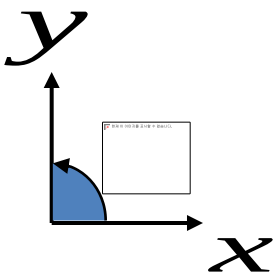
↓ △*y*가 작다고 가정하면,

$$u(y + \Delta y) = u(y) + \frac{\partial u}{\partial y}\Delta y$$

(시계 방향)

Curl and Rotation in Fluids





✓ 각속도

(속도 차만큼 변위 발생)

$$\omega_{z1}=\frac{\Delta\theta_1}{\Delta t}\approx\frac{\tan\Delta\theta_1}{\Delta t}=\frac{\Delta v\Delta t}{\Delta x}\cdot\frac{1}{\Delta t}$$

$$=\frac{v+\frac{\partial v}{\partial x}\Delta x-v}{\Delta x}=\frac{\partial v}{\partial x}$$

(반시계 방향)

$$\omega_{z2}=-\frac{\Delta\theta_2}{\Delta t}\approx-\frac{\tan\Delta\theta_2}{\Delta t}=-\frac{\Delta u\Delta t}{\Delta y}\cdot\frac{1}{\Delta t}$$

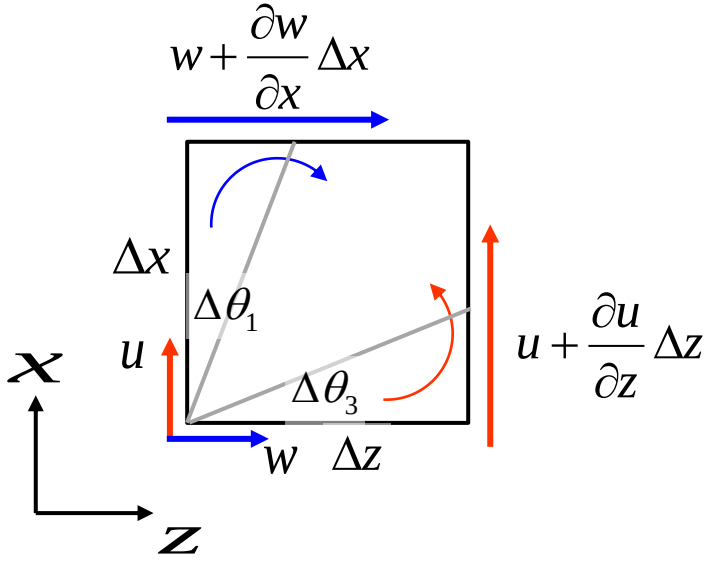
$$=-\frac{u+\frac{\partial u}{\partial y}\Delta y-u}{\Delta y}=-\frac{\partial u}{\partial y}$$

(시계 방향)

✓ z축에 대한 각속도는 두 각속도의 평균으로 정의하면

$$\omega_z=\frac{1}{2}\left(\omega_{z1}+\omega_{z2}\right)=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

Curl and Rotation in Fluids

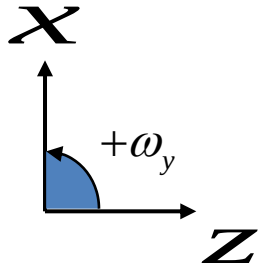


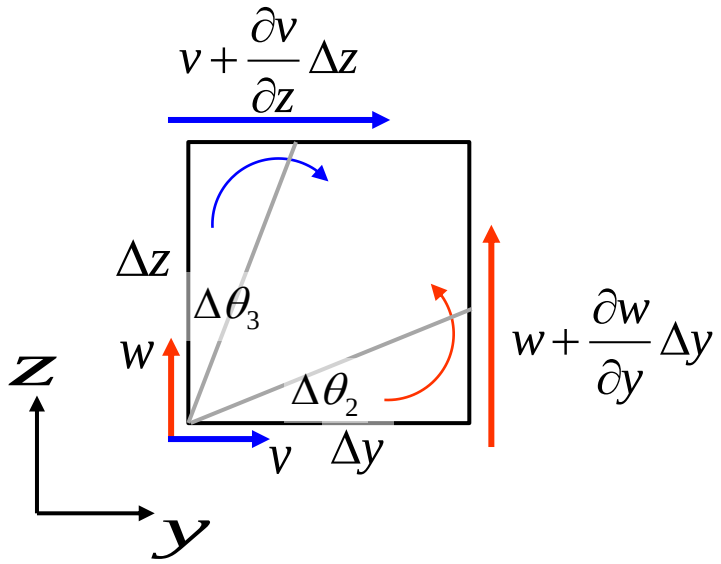
✓ y축에 대한 각속도

(반시계 방향) $\omega_{y1}=\frac{\partial u}{\partial z}$

(시계 방향) $\omega_{y2}=-\frac{\partial w}{\partial x}$

$$\therefore \omega_y=\frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)$$



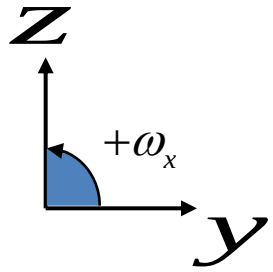


✓ x축에 대한 각속도

(반시계 방향) $\omega_{x1}=\frac{\partial w}{\partial y}$

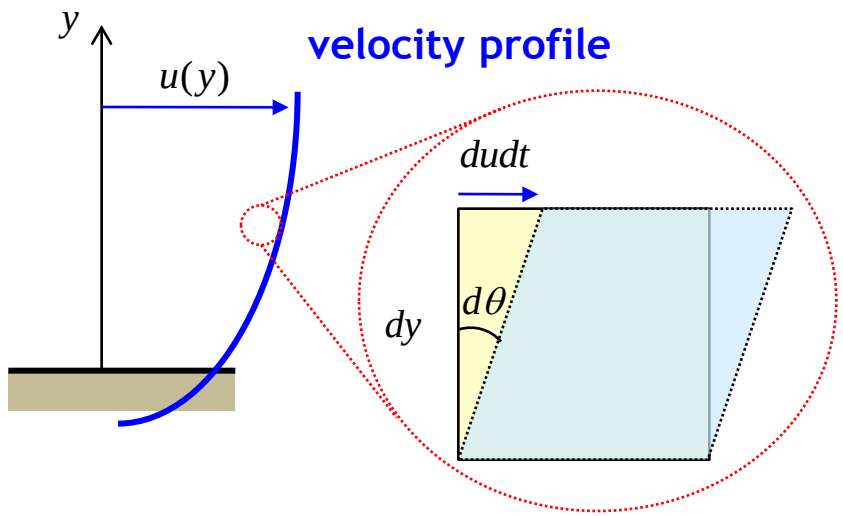
(시계 방향) $\omega_{x2}=-\frac{\partial v}{\partial z}$

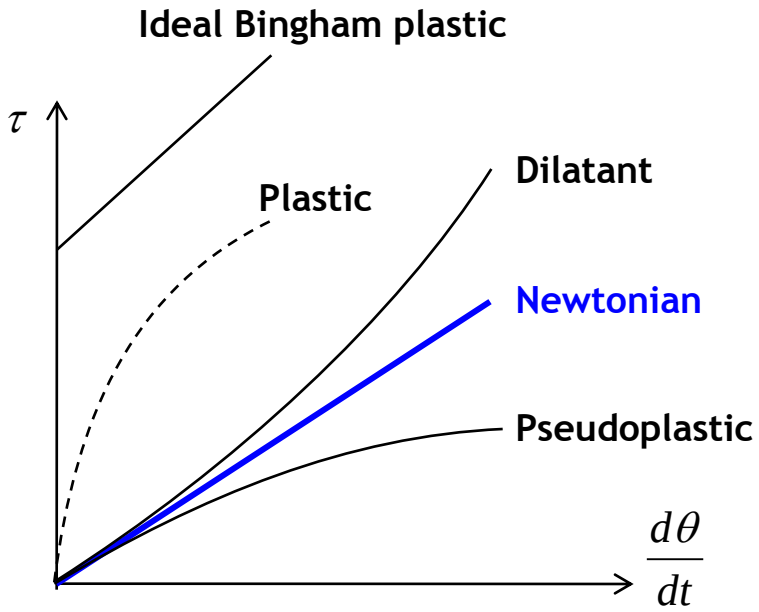
$$\therefore \omega_x=\frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)$$



1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, pp46~50,pp426-431
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.100~105
3) 김찬중, "공학도를 위한 길잡이 유체공학입문" , 문운당, 2002, pp258-264

뉴턴 유체1) (Newtonian Fluid)





✓ 미소 구간에서의 전단변형율

$$d\theta \approx \tan d\theta = \frac{dudt}{dy} = \frac{du}{dy}dt$$

✓ 전단변형율의 시간변화율은 속도 구배와 같음

$$\frac{d\theta}{dt} = \frac{du}{dy} \quad \text{---①}$$

✓ 뉴턴 유체(Newtonian fluid)
: 전단응력이 전단변형율의 시간변화율에 비례

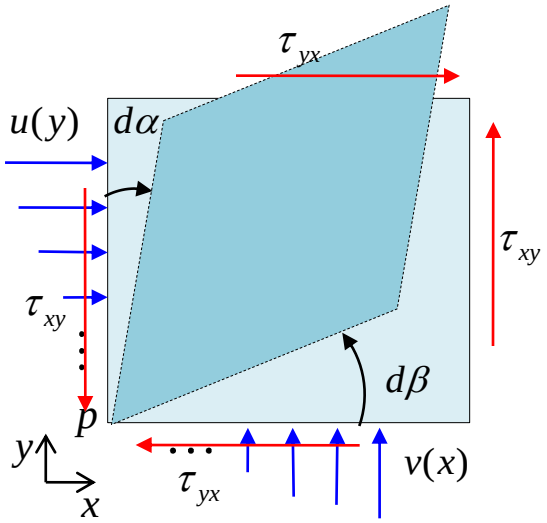
$$\tau \propto \frac{d\theta}{dt} \quad \text{---②}$$

✓ 뉴턴 유체(Newtonian fluid)의 특징
: ①,②에 의해, 전단응력은 속도구배에 비례함
(비례 상수 μ : 점성 계수)

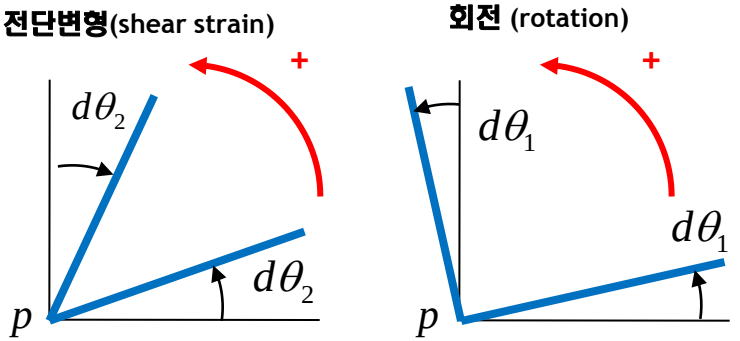
$$\tau \propto \frac{du}{dy} \Rightarrow \tau = \mu \frac{du}{dy}$$

Curl and Rotation in Fluids
- 다른 유도 방법

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, pp46~50,pp426-431
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.100~105
3) 김찬중, "공학도를 위한 길잡이 유체공학입문" , 문운당, 2002, pp258-264



회전과 전단변형으로 나타낼 수 있음



$$-d\alpha = -d\theta_2 + d\theta_1 \quad \cdots ①$$

$$d\beta = d\theta_2 + d\theta_1 \quad \cdots ②$$

Given : $d\alpha, d\beta$

Find : 회전각속도 ω_z

(1) ②와 ①을 더하면, $d\beta - d\alpha = 2d\theta_1$

$$d\theta_1 = \frac{1}{2}(d\beta - d\alpha)$$

➡ z축에 대한 회전 각도

(2) *dt*로 나누면,

$$\frac{d\theta_1}{dt} = \frac{1}{2}\left(\frac{d\beta}{dt} - \frac{d\alpha}{dt}\right)$$

➡ z축에 대한 회전 각속도

(3) 전단변형율의 시간 변화율은 속도 구배와 같음

$$\omega_z = \frac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy}\right)$$

(5) 동일한 방법에 의해 다음도 성립함

➡ *y*축에 대한 회전각속도 : $\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$

➡ *x*축에 대한 회전각속도 : $\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$

Curl and Rotation in Fluids

✓ curl의 정의

$$\text{curl}\,\mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k}$$

$$= 2\left[\frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k}\right]$$

$$= 2\omega_x\mathbf{i} + 2\omega_y\mathbf{j} + 2\omega_z\mathbf{k} = 2\boldsymbol{\omega}$$

=> curl v 크기는 는 유체 입자의 회전 각속도x2에 해당

✓ x축에 대한 각속도

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

✓ y축에 대한 각속도

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

✓ z축에 대한 각속도

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Curl and Rotation in Fluids

✓ curl의 정의

$$\operatorname{curl}\mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k}$$

✓ Irrotational Flow : $\nabla \times \mathbf{v} = 0 \left(\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}\right)$

See also
Streeter V.L., Fluid Mechanics, McGraw-Hill, 1948,
p19, '9. Irrotational Flow, Velocity Potential'

✓ $\nabla \times \mathbf{v} = 0$ 일 때, $\mathbf{v} = \nabla\Phi$ 인 Scalar Function Φ 가 존재함 ⇒ Velocity Potential

Curl and Rotation in Fluids

For any scalar function $\phi = \phi(x, y, z)$, $\operatorname{curl}(\operatorname{grad}\phi) = \mathbf{0}$ **is always true.**

$$\operatorname{curl}(\operatorname{grad}\phi) = \operatorname{curl}\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)$$

$$= \left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right)\mathbf{i} - \left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right)\mathbf{j} + \left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right)\mathbf{k} = 0$$

Velocity Vector : $\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ **and** $\operatorname{curl}\mathbf{V} = \mathbf{0}$

$$\left.\begin{array}{l}\operatorname{curl}(\operatorname{grad}\phi) = \mathbf{0} \\ \operatorname{curl}\mathbf{V} = \mathbf{0}\end{array}\right\} \mathbf{V} = \operatorname{grad}\phi$$

$\phi = \phi(x, y, z)$: Velocity Potential

$\mathbf{V}$ ➡ ϕ

(Vector, 3 unknown components) **(1 unknown)**

We can reduce the unknowns from 3 to 1.

Divergence of a Vector Fields

Definition. **Divergence**

Let $\mathbf{v}(x,y,z)$ be a differentiable vector function, where x,y,z are Cartesian coordinates. And let v_1, v_2, v_3 be components of $\mathbf{v}$. Then the function

$$\operatorname{div}\mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \quad \cdots(1)$$

is called the divergence of $\mathbf{v}$ or the divergence of the vector field defined by $\mathbf{v}$.

$$\operatorname{div}\mathbf{v} = \nabla \bullet \mathbf{v} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \bullet [v_1, v_2, v_3]$$

$$= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \bullet (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Divergence of a Vector Fields

divergence of v : $\operatorname{div}\mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \quad \cdots(1)$

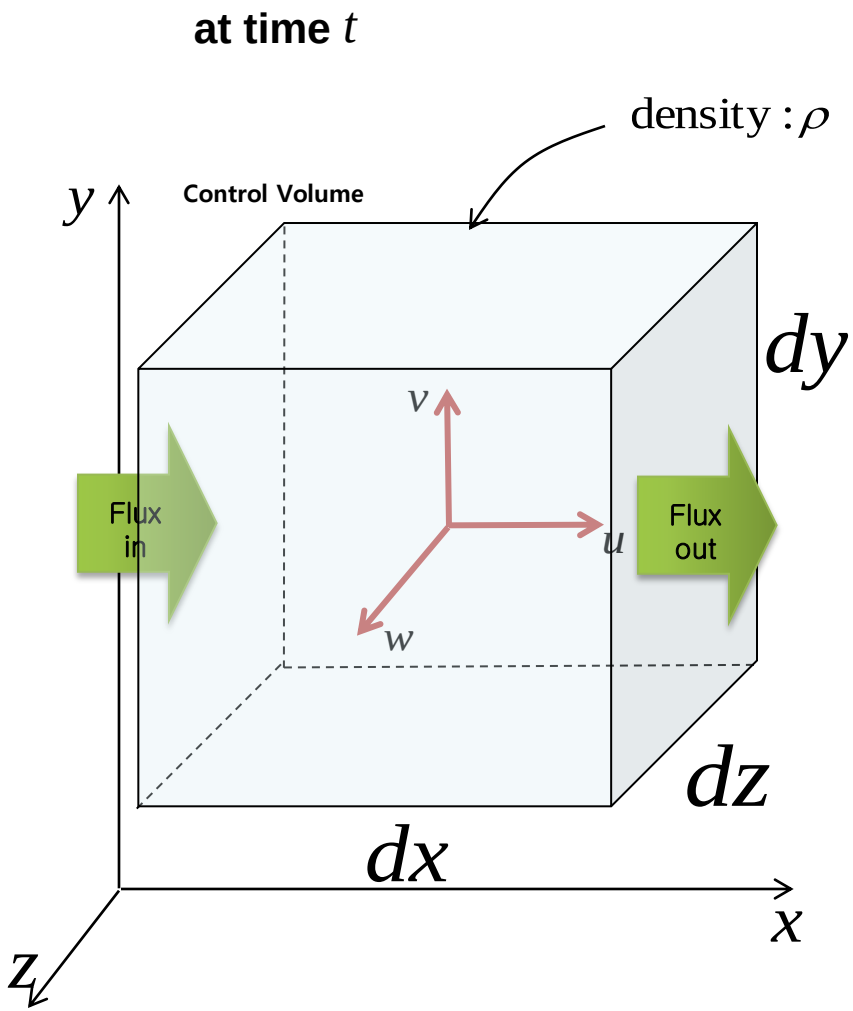
$$\text{If } \mathbf{v} = \operatorname{grad} f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k},$$

$$\operatorname{div}\mathbf{v} = \operatorname{div}(\operatorname{grad} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f.$$

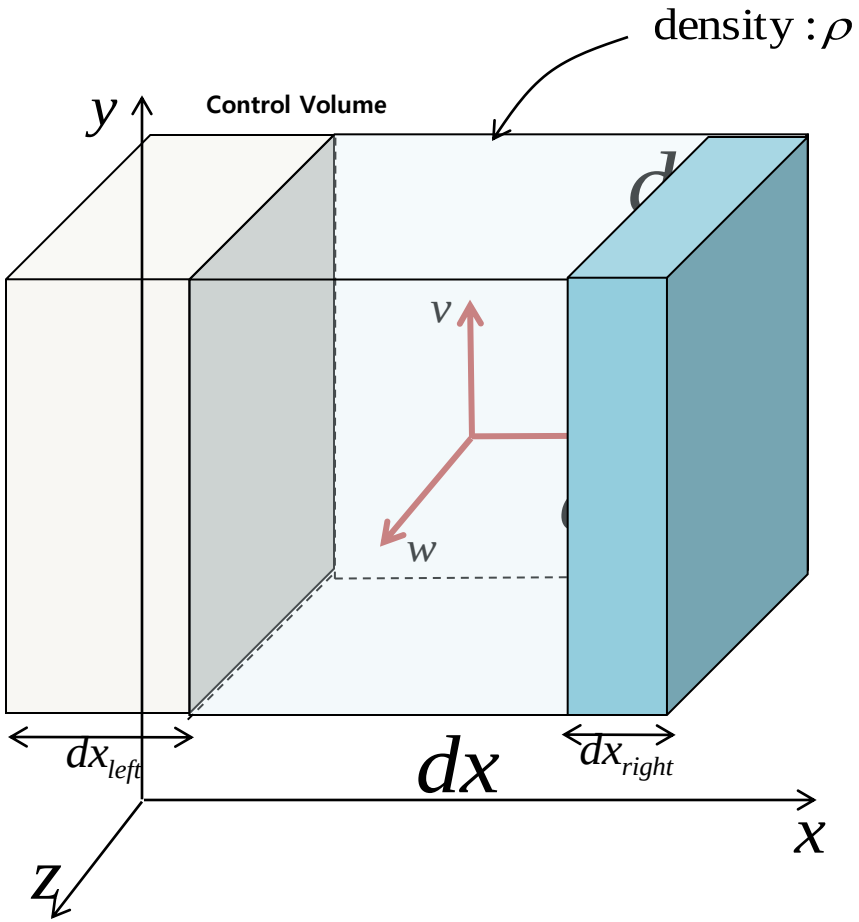
See also
Streeter V.L., Fluid Mechanics, McGraw-Hill, 1948,
p26, '11. The Laplace Equation'

Laplace operator

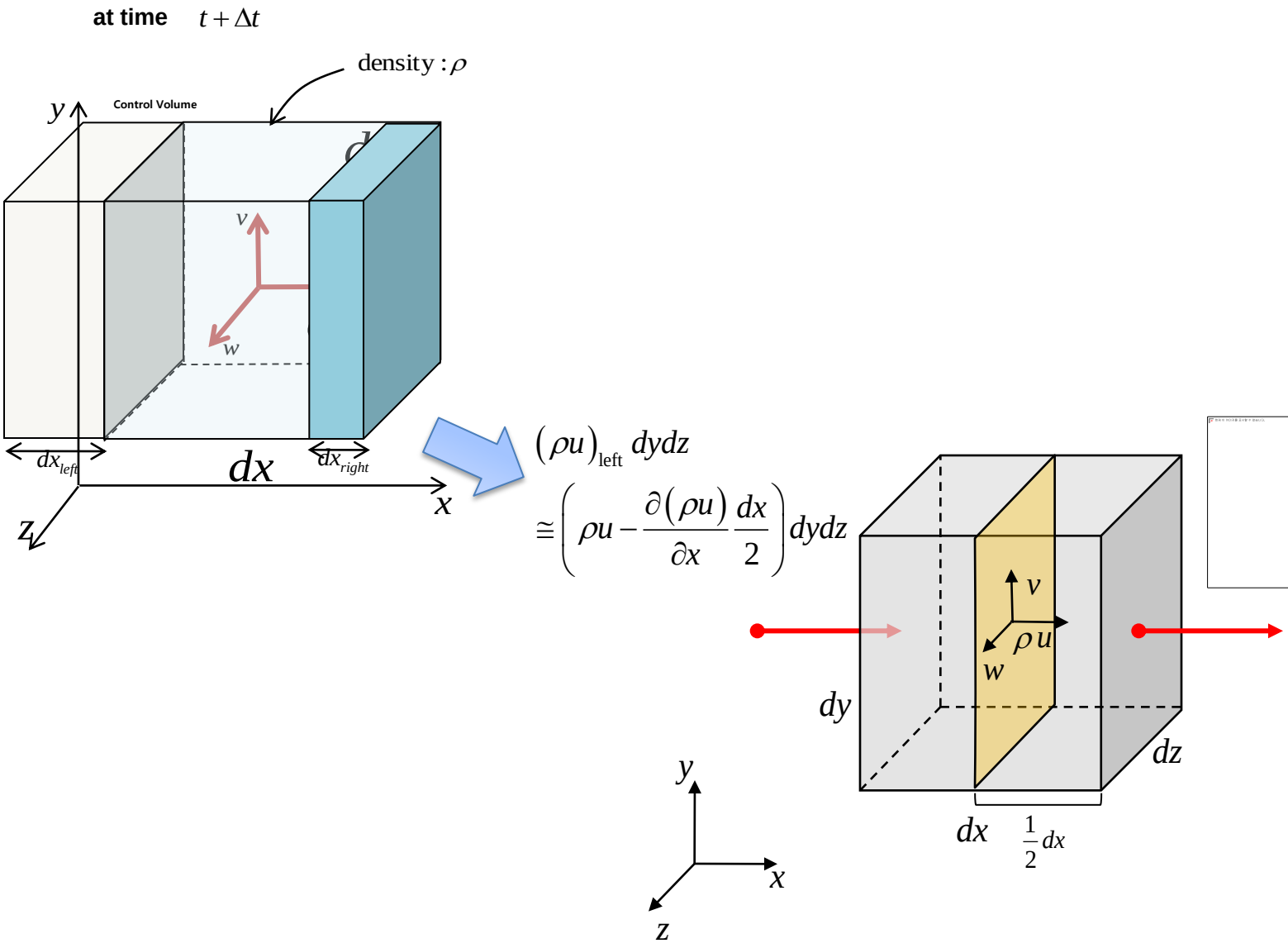
Divergence of a Vector Fields : Continuity Eqn.



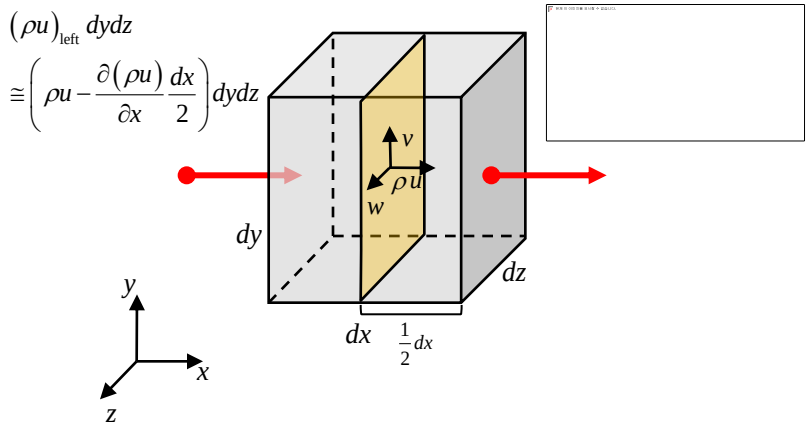
Divergence of a Vector Fields : Continuity Eqn.



Divergence of a Vector Fields : Continuity Eqn.



Divergence of a Vector Fields : Continuity Eqn.



✓ 오른쪽 면을 통해 검사체적으로부터 빠져나간 유체의 부피

$$dV_{right} = (dydz)dx_{right} = (dydz)u_{right}dt$$

✓ 오른쪽 면을 통해 검사체적으로부터 빠져나간 유체의 질량

$$\rho_{\text{right}} dV_{\text{right}} = \rho_{\text{right}} (dydz) u_{\text{right}} dt = (\rho u)_{\text{right}} dydz \cdot dt$$

✓ 단위 시간당 오른쪽 면을 통해 빠져나간 유체의 질량

$$\rho_{\text{right}} \frac{dV_{\text{right}}}{dt} = \rho_{right} (dydz) u_{\text{right}} = (\rho u)_{\text{right}} dydz$$

✓ $(\rho u)_{right}$ 를 Tayler Series로 전개하면,

$$(\rho u)_{right} = \rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2} + \frac{1}{2!}\frac{\partial^2(\rho u)}{\partial x^2}\left(\frac{dx}{2}\right)^2 + \cdots$$

$$\cong \rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}$$ (1차항까지만 선택)

✓ 단위 시간당 오른쪽 면을 통해 빠져나간 유체의 질량

$$(\rho u)_{\text{right}} dydz = \left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right) dydz$$

Divergence of a Vector Fields : Continuity Eqn.

Flow of a Compressible Fluid. Physical meaning of Divergence

✓ 단위 시간당 각 면을 통해 빠져나간 유체의 질량

Right(+x) : $(\rho u)_{\text{right}}\, dydz \cong \left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right)dydz$

Top(+y) : $(\rho v)_{\text{top}}\, dxdz \cong \left(\rho v + \frac{\partial(\rho v)}{\partial z}\frac{dy}{2}\right)dxdz$

Front(+z) : $(\rho w)_{\text{front}}\, dxdy \cong \left(\rho w + \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}\right)dxdy$

✓ 단위 시간당 각 면을 통해 들어온 유체의 질량

left(-x) : $(\rho u)_{\text{left}}\, dydz \cong \left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right)dydz$

Bottom(-y) : $(\rho v)_{\text{bottom}} dxdz \cong \left(\rho v - \frac{\partial(\rho v)}{\partial z}\frac{dy}{2}\right)dxdz$

Rear(-z) : $(\rho w)_{\text{rear}}\, dxdy \cong \left(\rho w - \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}\right)dxdy$

✓ 단위 시간당 검사 체적을 통과한 유체의 유입량 (들어온 양을 +로 봄)

$$\sum \dot{m} = \sum_{in} \dot{m} - \sum_{out} \dot{m} = \left[\left((\rho u) - \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right)dydz + \left((\rho v) - \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\right)dxdz + \left((\rho w) - \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}\right)dxdy\right]$$

$$-\left[\left((\rho u) + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right)dydz + \left((\rho v) + \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\right)dxdz + \left((\rho w) + \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}\right)dxdy\right]$$

$$= -\frac{\partial(\rho u)}{\partial x}dxdydz - \frac{\partial(\rho v)}{\partial y}dxdydz - \frac{\partial(\rho w)}{\partial z}dxdydz$$

Divergence of a Vector Fields : Continuity Eqn.

Mass conservation

✓ 단위 시간당 검사 체적을 통과한 유체의 유입량 (들어온 양을 +로 봄)

$$= -\frac{\partial(\rho u)}{\partial x}dxdydz - \frac{\partial(\rho v)}{\partial y}dxdydz - \frac{\partial(\rho w)}{\partial z}dxdydz \quad \text{---------①}$$

✓ (검사체적 내부의 질량 변화율)$= \frac{\partial \rho}{\partial t}dxdydz$ ---------②

① = ②

$$\frac{\partial \rho}{\partial t}dxdydz = -\frac{\partial(\rho u)}{\partial x}dxdydz - \frac{\partial(\rho v)}{\partial y}dxdydz - \frac{\partial(\rho w)}{\partial z}dxdydz$$

$dxdydz$ 으로 양변을 나누면,

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x} - \frac{\partial(\rho v)}{\partial y} - \frac{\partial(\rho w)}{\partial z}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{V}) = 0$$

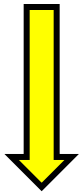
=> Continuity Equation

See also
Streeter V.L., Fluid Mechanics, McGraw-Hill, 1948,
p14, '7. Equation of Continuity'

Mass Conservation

Continuity Equation

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{V}) = 0$$

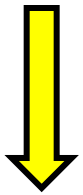


비압축성 유체(Incompressible fluid)라고 가정하면,

$$(\rho = const)$$

$$\nabla \bullet \mathbf{V} = 0 \quad \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \right)$$

⇨ **Divergence**



비회전 유동(Irrotational flow)이라고 가정하면,

$$(\mathbf{V} = \nabla \Phi)$$

⇨ **Gradient**

$$\nabla \bullet \nabla \Phi = \nabla^2 \Phi = 0$$

$$\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \right)$$

⇨ **Laplace Equation**

Further Applications of the Divergence Theorem :Potential Theory, Harmonic Function

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F}$$

$$\frac{\partial}{\partial x}P(x,y,z) + \frac{\partial}{\partial y}Q(x,y,z) + \frac{\partial}{\partial z}R(x,y,z)$$

- **Potential theory : the theory of solutions of Laplace's equation**

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

- **Harmonic function : A solution of Laplace's equation with continuous 2nd order partial derivatives**

A basic property of solutions of Laplace's equation ⇐

$$\iiint_T \operatorname{div}\mathbf{F}\,dV = \iint_S \mathbf{F}\cdot\mathbf{n}\,dA$$

If $\mathbf{F} = \operatorname{grad} f$, $\operatorname{div}\mathbf{F} = \operatorname{div}(\operatorname{grad} f) = \nabla^2 f$

And, $\mathbf{F}\bullet\mathbf{n} = \mathbf{n}\bullet\mathbf{F} = \mathbf{n}\bullet(\operatorname{grad} f) = \dfrac{\partial f}{\partial n}$

Thus the formula in the divergence theorem becomes ⇒

$$\iiint_T \nabla^2 f\,dV = \iint_S \frac{\partial f}{\partial n}\,dA$$

Further Applications of the Divergence Theorem :Potential Theory, Harmonic Function

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F}$$

$$\frac{\partial}{\partial x}P(x,y,z) + \frac{\partial}{\partial y}Q(x,y,z) + \frac{\partial}{\partial z}R(x,y,z)$$

- **Potential theory : the theory of solutions of Laplace's equation**

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

- **Harmonic function : A solution of Laplace's equation with continuous 2nd order partial derivatives**

$$\iiint_T \nabla^2 f dV = \iint_S \frac{\partial f}{\partial n} dA$$

$$if,\ \nabla^2 f = 0 \ \ then \ \ \iiint_T \nabla^2 f dV = 0$$

$$\because \iint_S \frac{\partial f}{\partial n} dA = 0$$

(Theorem 1) A Basic Property of harmonic Functions

Let $f(x,y,z)$ be a harmonic function in some domain D in space. Let S be any piecewise smooth closed orientable surface in D whose entire region it encloses belongs to D. Then the integral of the normal derivatives of f taken over S is zero.

Further Applications of the Divergence Theorem :Potential Theory, Harmonic Function

- **Potential theory : the theory of solutions of Laplace's equation**

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

- **Harmonic function : A solution of Laplace's equation with continuous 2nd order partial derivatives**

$$\iiint_T \nabla^2 f \, dV = \iint_S \boxed{\frac{\partial f}{\partial n}} dA$$

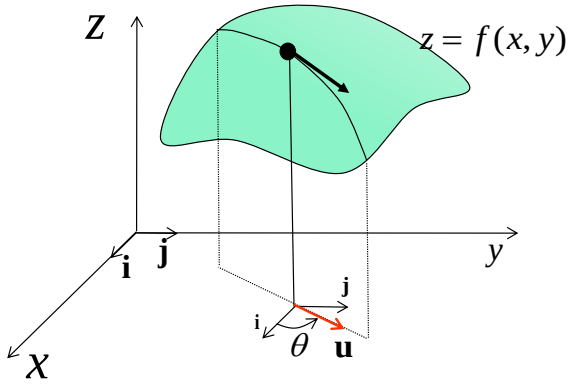
Theorem 9.6 **Computing a Directional Derivative**

If $z = f(x, y)$ is differentiable function of x and y and $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ then,

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

Recall, Directional Derivative

$$\boxed{D_{\mathbf{u}} f(x, y)} = [f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}] \cdot (\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$$
$$= \nabla f(x, y) \cdot \mathbf{u}$$



Further Applications of the Divergence Theorem
: Green's Theorems

$$\mathbf{a}\bullet\mathbf{b}=(a_1b_1+a_2b_2+a_3b_3)=\mathbf{b}\bullet\mathbf{a}$$

Let $\mathbf{F}=f\ \text{grad}\ g$

$$\text{div}\,\mathbf{F}=\text{div}(f\ \text{grad}\ g)=\text{div}\left(\left[f\frac{\partial g}{\partial x},f\frac{\partial g}{\partial y},f\frac{\partial g}{\partial z}\right]\right)$$

$$=\frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial x}\right)+\frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial y}\right)+\frac{\partial}{\partial z}\left(f\frac{\partial g}{\partial z}\right)$$

$$=\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}+f\frac{\partial^2 g}{\partial x^2}\right)+\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}+f\frac{\partial^2 g}{\partial y^2}\right)+\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial z}+f\frac{\partial^2 g}{\partial z^2}\right)$$

$$=f\nabla^2 g+\text{grad}\,f\bullet\text{grad}\,g$$

$$\mathbf{F}\bullet\mathbf{n}=\mathbf{n}\bullet\mathbf{F}=\mathbf{n}\bullet(f\ \text{grad}\ \text{g})=f(\mathbf{n}\bullet\text{grad}\ \text{g})=f\frac{\partial g}{\partial n}$$

(1) Green's first formula

$$\iiint_T\left(f\nabla^2 g+\text{grad}\ f\bullet\text{grad}\ g\right)dV=\iint_S f\frac{\partial g}{\partial n}\,dA$$

Further Applications of the Divergence Theorem : Green's Theorems

Let $\mathbf{F} = f \operatorname{grad} g$

$$\iiint_T \left(f\nabla^2 g + \operatorname{grad} f \bullet \operatorname{grad} g \right) dV = \iint_S f \frac{\partial g}{\partial n} dA \text{ --> ①}$$

Let $\mathbf{F} = g \operatorname{grad} f$

$$\iiint_T \left(g\nabla^2 f + \operatorname{grad} g \bullet \operatorname{grad} f \right) dV = \iint_S g \frac{\partial f}{\partial n} dA \text{ --> ②}$$

① - ② :

(2) Green's second formula

$$\iiint_T \left(f\nabla^2 g - g\nabla^2 f \right) dV = \iint_S \left(f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} \right) dA$$

Further Applications of the Divergence Theorem : Green's Theorems

(2) Green's second formula

$$\iiint_T \left(f\nabla^2 g - g\nabla^2 f \right) dV = \iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) dA$$

If g, f are solution of Laplace equation, then

$$\iiint_T \left(f\nabla^2 g - g\nabla^2 f \right) dV = 0$$

$$\therefore \iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) dA = 0$$

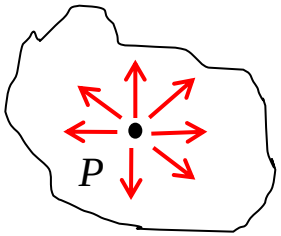
$$\therefore f\frac{\partial g}{\partial n} = g\frac{\partial f}{\partial n}$$

Ref. Froude-Krylov Force & Diffraction Force

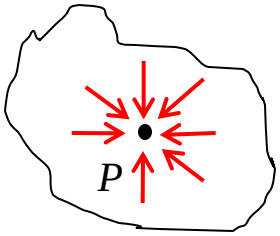
Source & Sink

$\nabla \cdot \mathbf{F} = 0$:incompressible flow

$\nabla \cdot \mathbf{F} \neq 0$:compressible flow



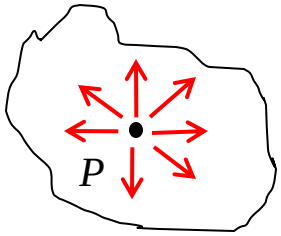
Source
:net outward flow
$(\text{div } \mathbf{F}(P) > 0)$



Sink
:net inward flow
$(\text{div } \mathbf{F}(P) < 0)$

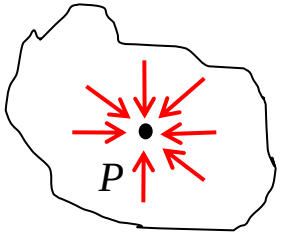
Generate a body shape by using Source and Sink

Source & Sink



Source
:net outward flow

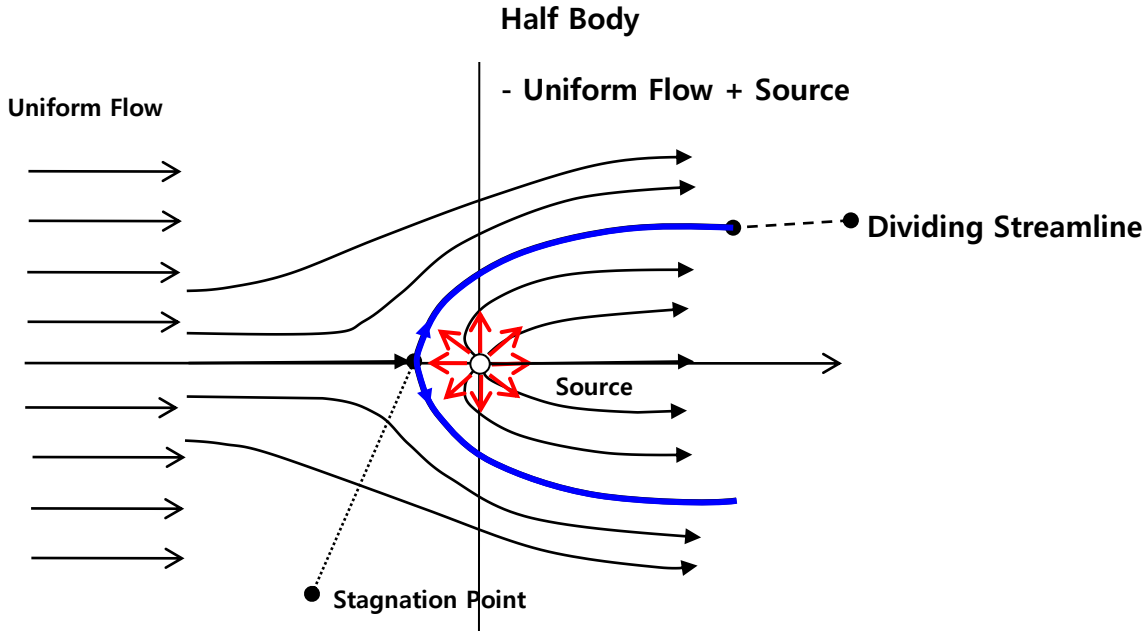
$(\text{div } \mathbf{F}(P) > 0)$



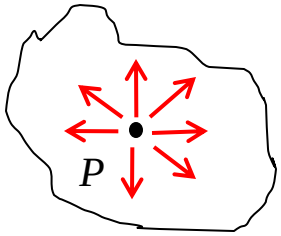
Sink
:net inward flow

$(\text{div } \mathbf{F}(P) < 0)$

Generate a body-like shape by using Source and Sink

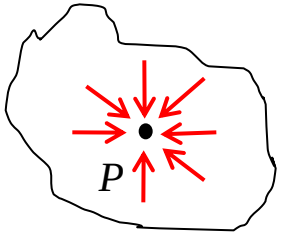


Source & Sink



Source
:net outward flow

$(\text{div } \mathbf{F}(P) > 0)$



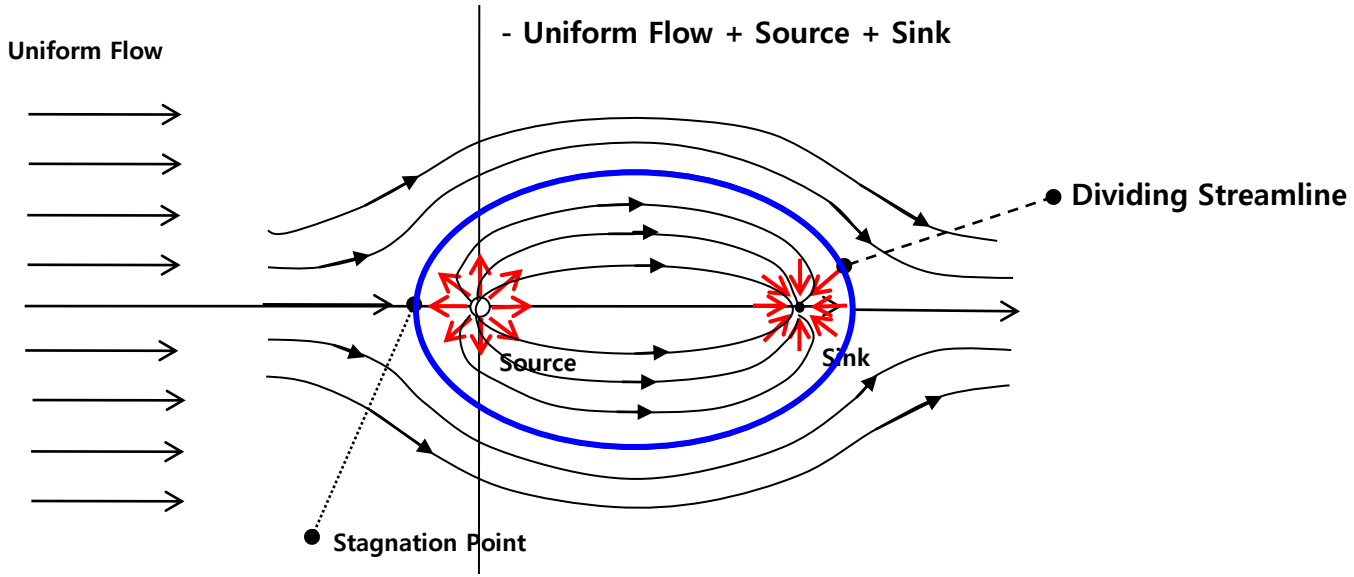
Sink
:net inward flow

$(\text{div } \mathbf{F}(P) < 0)$

Generate a body-like shape by using Source and Sink

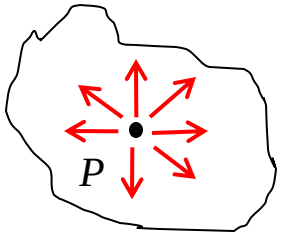
Rankine Ovoid

- Uniform Flow + Source + Sink



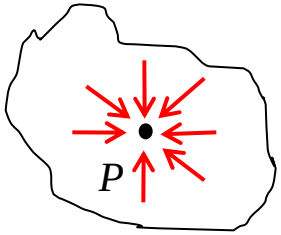
Source & Sink

Ref. Radiation Velocity Potential



Source
:net outward flow

$(\text{div } \mathbf{F}(P) > 0)$



Sink
:net inward flow

$(\text{div } \mathbf{F}(P) < 0)$

Generate a body-like shape by using Source and Sink

✓ Singularity Distribution Method[2] (2-D)

→ Laplace Equation을 만족함 $\nabla^2\Phi = 0$

물체 표면에 특이점 (source, doublet, vortex)을 분포시켜 수학적으로 물체 경계면을 생성시키고, 이들 특이점들의 강도(Strength)를 구하여 전체 유장의 velocity potential을 구하는 방법

Velocity Potential → Pressure → Surface Force to Hull

$$\rho\frac{\partial\Phi}{\partial t} + P + \frac{1}{2}\rho|\nabla\Phi|^2 + \rho g\, z = \mathrm{C} \qquad \mathbf{F}_{Fluid} = \iint_{S_B} P\mathbf{n}dS$$

Reference slides

Differentiation of Cross Product

증명

$$\mathbf{r}_1(t) = \langle x_1(t), y_1(t), z_1(t) \rangle$$
$$\mathbf{r}_1'(t) = \langle x_1'(t), y_1'(t), z_1'(t) \rangle$$
$$\mathbf{r}_2(t) = \langle x_2(t), y_2(t), z_2(t) \rangle$$
$$\mathbf{r}_2(t) = \langle x_2'(t), y_2'(t), z_2'(t) \rangle$$

$$\frac{d}{dt}[\mathbf{r}_1(t) \times \mathbf{r}_2(t)] = \mathbf{r}_1'(t) \times \mathbf{r}_2(t) + \mathbf{r}_1(t) \times \mathbf{r}_2'(t)$$

$L.H.S.:$

$$\frac{d}{dt}[\mathbf{r}_1(t) \times \mathbf{r}_2(t)] = \frac{d}{dt}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1(t) & y_1(t) & z_1(t) \\ x_2(t) & y_2(t) & z_2(t) \end{vmatrix}$$

$$= \frac{d}{dt}\left\{\left[y_1(t)z_2(t) - y_2(t)z_1(t)\right]\mathbf{i} + \left[z_1(t)x_2(t) - z_2(t)x_1(t)\right]\mathbf{j} + \left[x_1(t)y_2(t) - x_2(t)y_1(t)\right]\mathbf{k}\right\}$$

$$= \left\{\frac{d}{dt}\left[y_1(t)z_2(t) - y_2(t)z_1(t)\right]\mathbf{i} + \frac{d}{dt}\left[z_1(t)x_2(t) - z_2(t)x_1(t)\right]\mathbf{j} + \frac{d}{dt}\left[x_1(t)y_2(t) - x_2(t)y_1(t)\right]\mathbf{k}\right\}$$

$$= \left[y_1'(t)z_2(t) + y_1(t)z_2'(t) - y_2'(t)z_1(t) - y_2(t)z_1'(t)\right]\mathbf{i}$$
$$+ \left[z_1'(t)x_2(t) + z_1(t)x_2'(t) - z_2'(t)x_1(t) - z_2(t)x_1'(t)\right]\mathbf{j}$$
$$+ \left[x_1'(t)y_2(t) + x_1(t)y_2'(t) - x_2(t)y_1'(t) - x_2(t)y_1'(t)\right]\mathbf{k}$$

증명

$$\mathbf{r}_1(t) = \langle x_1(t), y_1(t), z_1(t) \rangle$$
$$\mathbf{r}_1'(t) = \langle x_1'(t), y_1'(t), z_1'(t) \rangle$$
$$\mathbf{r}_2(t) = \langle x_2(t), y_2(t), z_2(t) \rangle$$
$$\mathbf{r}_2(t) = \langle x_2'(t), y_2'(t), z_2'(t) \rangle$$

$$\frac{d}{dt}[\mathbf{r}_1(t) \times \mathbf{r}_2(t)] = \mathbf{r}_1'(t) \times \mathbf{r}_2(t) + \mathbf{r}_1(t) \times \mathbf{r}_2'(t)$$

$R.H.S.:$

$$\mathbf{r}_1'(t) \times \mathbf{r}_2(t) + \mathbf{r}_1(t) \times \mathbf{r}_2'(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1'(t) & y_1'(t) & z_1'(t) \\ x_2(t) & y_2(t) & z_2(t) \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1(t) & y_1(t) & z_1(t) \\ x_2'(t) & y_2'(t) & z_2'(t) \end{vmatrix}$$

$$= \left\{ \left[y_1'(t) z_2(t) - y_2(t) z_1'(t) \right] \mathbf{i} + \left[z_1'(t) x_2(t) - z_2(t) x_1'(t) \right] \mathbf{j} + \left[x_1'(t) y_2(t) - x_2(t) y_1'(t) \right] \mathbf{k} \right\}$$
$$+ \left\{ \left[y_1(t) z_2'(t) - y_2'(t) z_1(t) \right] \mathbf{i} + \left[z_1(t) x_2'(t) - z_2'(t) x_1(t) \right] \mathbf{j} + \left[x_1(t) y_2'(t) - x_2'(t) y_1(t) \right] \mathbf{k} \right\}$$

$$= \left[y_1'(t) z_2(t) + y_1(t) z_2'(t) - y_2'(t) z_1(t) - y_2(t) z_1'(t) \right] \mathbf{i}$$
$$+ \left[z_1'(t) x_2(t) + z_1(t) x_2'(t) - z_2'(t) x_1(t) - z_2(t) x_1'(t) \right] \mathbf{j}$$
$$+ \left[x_1'(t) y_2(t) + x_1(t) y_2'(t) - x_2(t) y_1'(t) - x_2(t) y_1'(t) \right] \mathbf{k}$$

증명

$$\frac{d}{dt}[\mathbf{r}_1(t)\times\mathbf{r}_2(t)]=\mathbf{r}_1'(t)\times\mathbf{r}_2(t)+\mathbf{r}_1(t)\times\mathbf{r}_2'(t)$$

$L.H.S:$

$$\begin{aligned}\frac{d}{dt}[\mathbf{r}_1(t)\times\mathbf{r}_2(t)]=&\left[y_1'(t)z_2(t)+y_1(t)z_2'(t)-y_2'(t)z_1(t)-y_2(t)z_1'(t)\right]\mathbf{i}\\&+\left[z_1'(t)x_2(t)+z_1(t)x_2'(t)-z_2'(t)x_1(t)-z_2(t)x_1'(t)\right]\mathbf{j}\\&+\left[x_1'(t)y_2(t)+x_1(t)y_2'(t)-x_2(t)y_1'(t)-x_2(t)y_1'(t)\right]\mathbf{k}\end{aligned}$$

$R.H.S:$

$$\begin{aligned}\mathbf{r}_1'(t)\times\mathbf{r}_2(t)+\mathbf{r}_1(t)\times\mathbf{r}_2'(t)=&\left[y_1'(t)z_2(t)+y_1(t)z_2'(t)-y_2'(t)z_1(t)-y_2(t)z_1'(t)\right]\mathbf{i}\\&+\left[z_1'(t)x_2(t)+z_1(t)x_2'(t)-z_2'(t)x_1(t)-z_2(t)x_1'(t)\right]\mathbf{j}\\&+\left[x_1'(t)y_2(t)+x_1(t)y_2'(t)-x_2(t)y_1'(t)-x_2(t)y_1'(t)\right]\mathbf{k}\end{aligned}$$

$$\therefore L.H.S.=R.H.S$$

09. Vector Calculus (3) : Line, Double and Triple Integrals

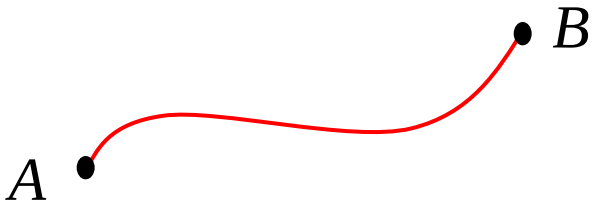
Line Integrals
Independence Path
Double Integrals
Surface Integrals
Triple Integrals

Seoul National Univ.
@SDAL Advanced Ship Design Automation Lab.
http://asdal.snu.ac.kr

Line Integrals

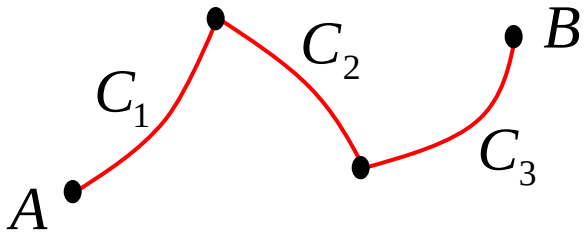
Terminology

(i) C is smooth curve if f' and g' are continuous on the closed interval [a,b] and not simultaneously zero on the open interval (a,b)



(ii) C is piecewise smooth if it consists of a finite number of smooth curves $C_1, C_2, ..., C_n$ joined end to end – that is,

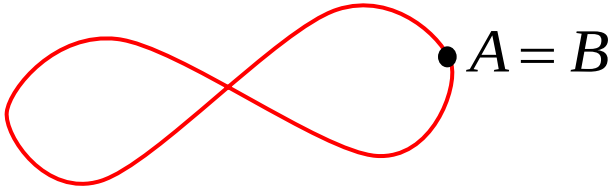
$$C = C_1 \cup C_2 \cup ... \cup C_n$$



Line Integrals

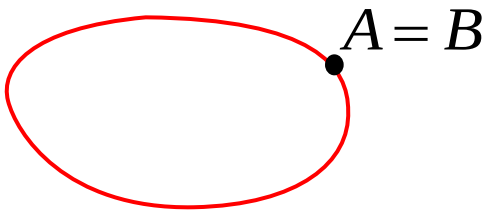
Terminology

(iii) C is closed curve if A=B.



Closed but not simple

(iv) C is simple closed curve if A=B and the curve does not cross itself.

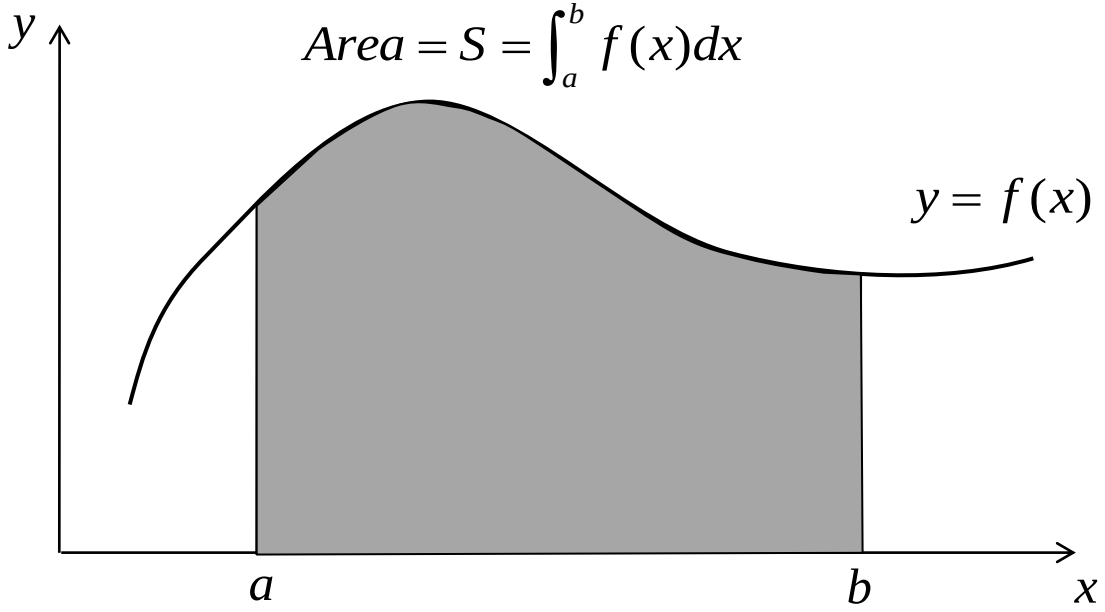


Closed and simple

(v) If C is not a closed curve, then the positive direction on C is the direction corresponding to increasing values of t.

Line Integrals

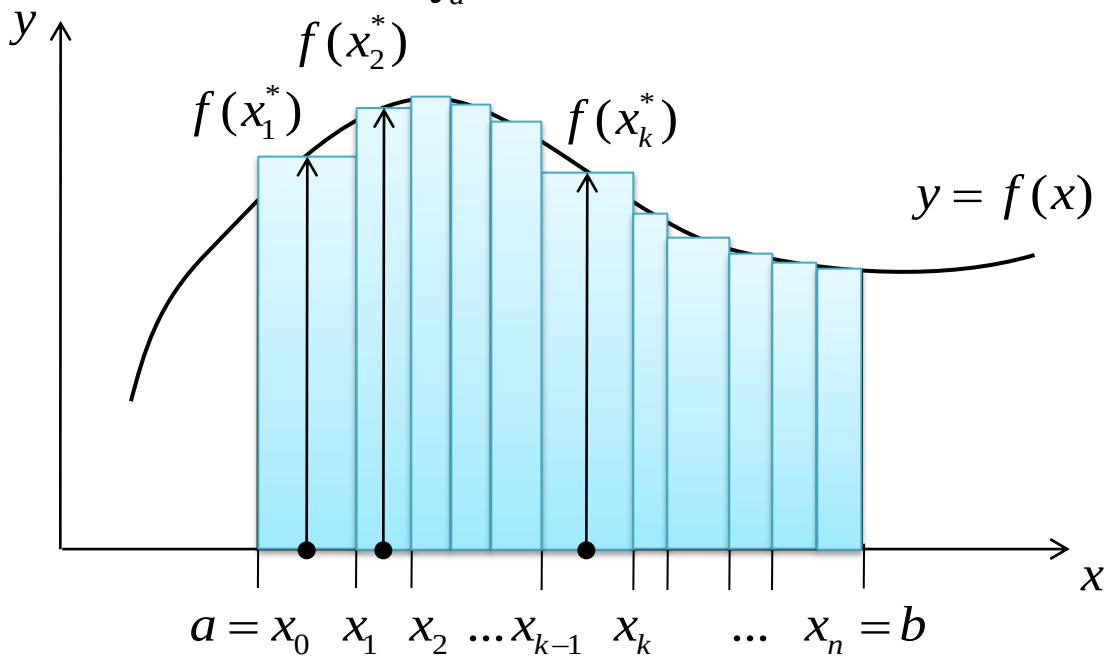
Definite Integral



Line Integrals

Definite Integral

$$S = \int_a^b f(x)dx$$



subinterval

$$x_1^* = \frac{x_0 + x_1}{2}, \Delta x_1 = x_1 - x_0$$

$$x_2^* = \frac{x_1 + x_2}{2}, \Delta x_2 = x_2 - x_1$$

$$\vdots \qquad \vdots$$

$$x_k^* = \frac{x_{k-1} + x_k}{2}, \Delta x_2 = x_2 - x_1$$

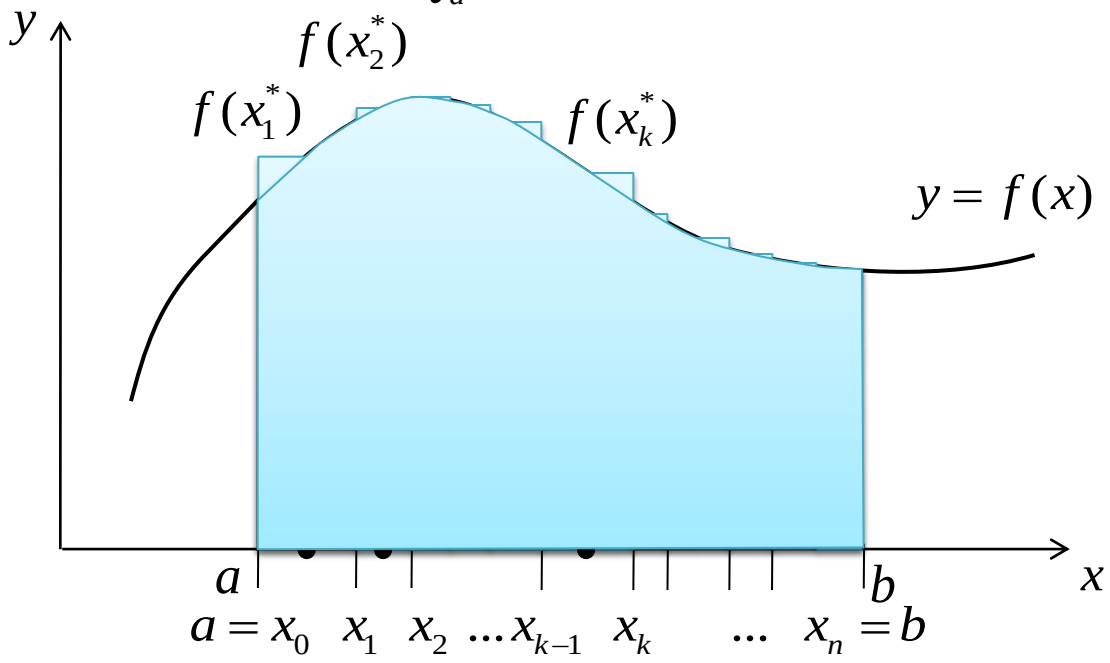
$$f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_k^*)\Delta x_k + \cdots + f(x_b^*)\Delta x_b$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

Line Integrals

Definite Integral

$$S = \int_a^b f(x)dx$$



subinterval

$$x_0 \le x_1^* \le x_1 \quad , \Delta x_1 = x_1 - x_0$$

$$x_1 \le x_2^* \le x_2 \quad , \Delta x_2 = x_2 - x_1$$

$$\vdots \qquad\qquad \vdots$$

$$x_{k-1} \le x_k^* \le x_k , \Delta x_k = x_k - x_{k-1}$$

$$f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_k^*)\Delta x_k + \cdots + f(x_b^*)\Delta x_b$$

The definite integral of a function of a single variable is given by the limit of a sum

$$\int_a^b f(x)dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

$\|P\|$: length of the longest subinterval

Line Integrals

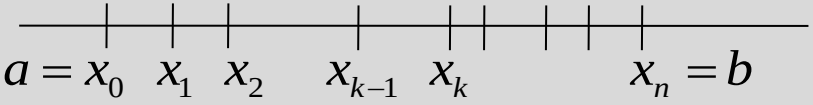
Definite Integral

y=*f* (x)

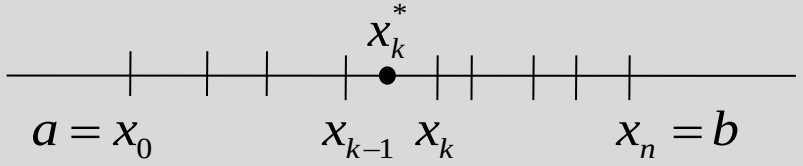
1. Let f be defined on a closed interval [a,b].
2. Partition the interval [a,b] into n subintervals $[x_{k-1}, x_k]$ of length $\Delta x_k = x_k - x_{k-1}$

   Let *P* denote the partition

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$



3. Let $\| P \|$ be the length of the longest subinterval. The number $\| P \|$ is called the **norm** of the partition *P*
4. Choose a number x_k^* in each subinterval.
5. Form the sum $\sum_{k=1}^{n} f(x_k^*)\Delta x_k$

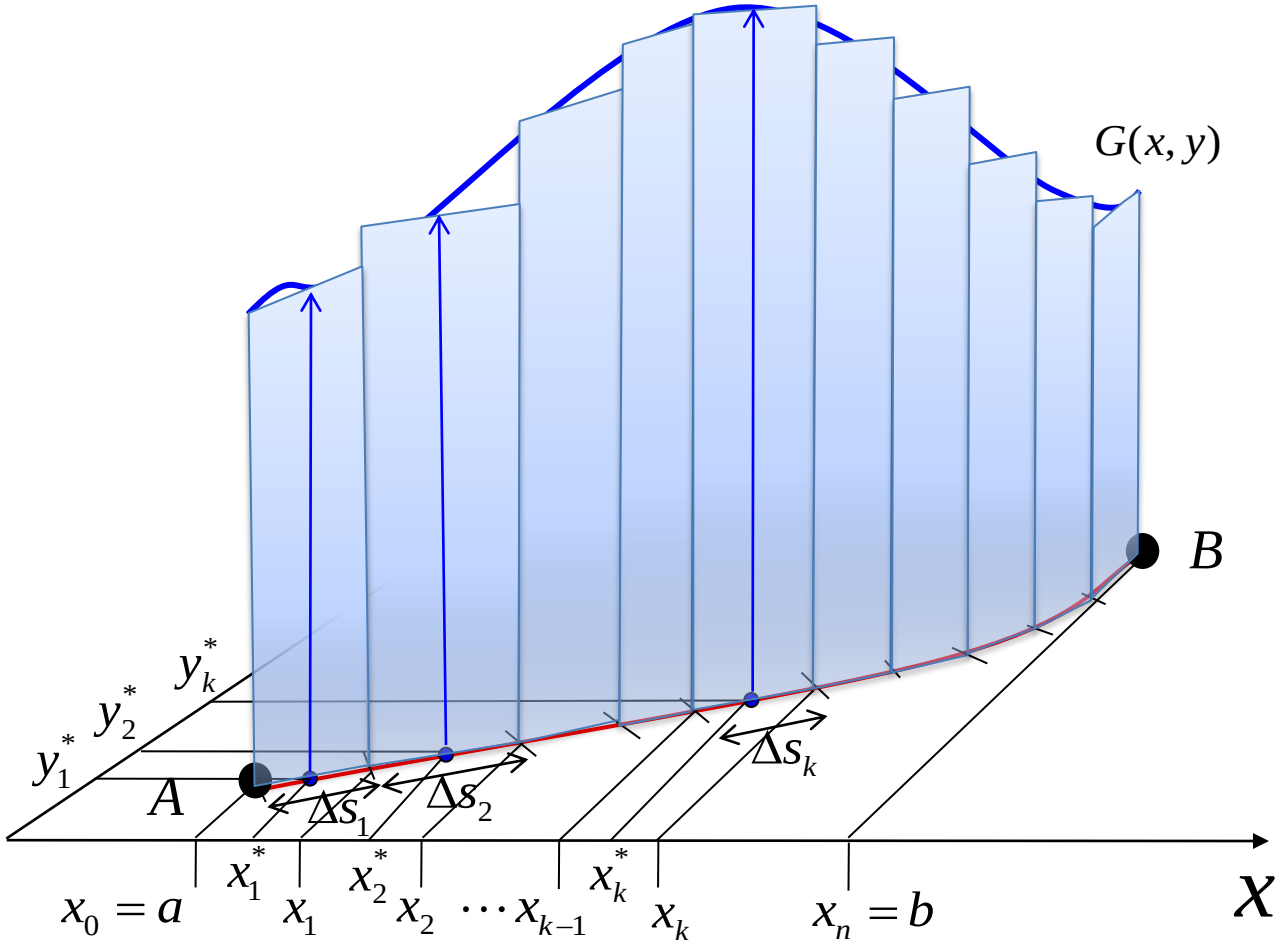


The definite integral of a function of a single variable is given by the limit of a sum :

$$\int_a^b f(x)dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

Line Integrals

Line Integral in the Plane



$$x_0 \le x_1^* \le x_1 \qquad y_0 \le y_1^* \le y_1 \qquad \text{subinterval } \Delta s_1$$

$$x_1 \le x_2^* \le x_2 \qquad y_1 \le y_2^* \le y_2 \qquad \Delta s_2$$

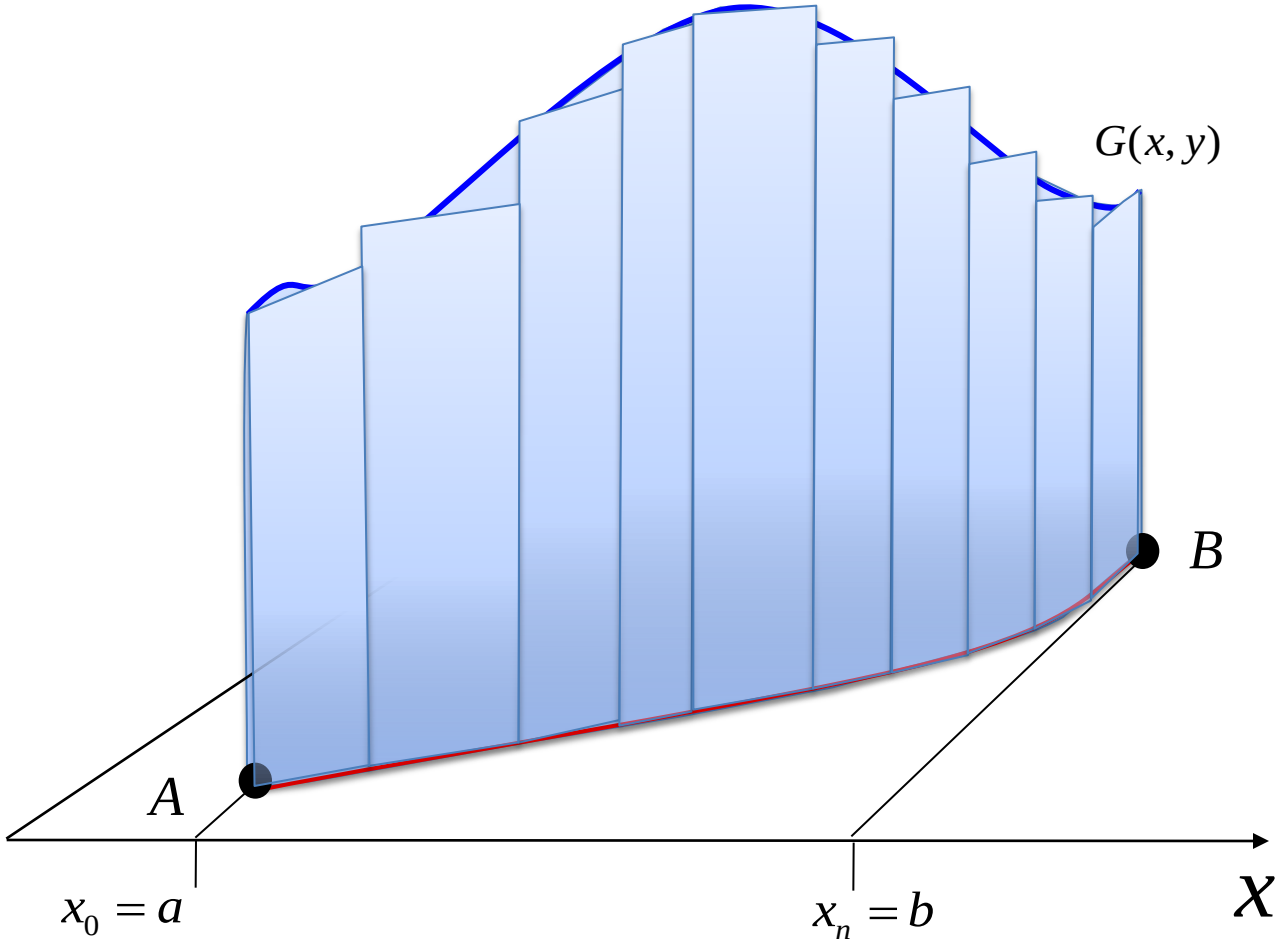
$$\vdots$$

$$x_{k-1} \le x_k^* \le x_k \qquad y_{k-1} \le y_k^* \le y_k \qquad \Delta s_k$$

$$G(x_1^*, y_1^*)\Delta s_1 + G(x_2^*, y_2^*)\Delta s_2 + \cdots + G(x_k^*, y_k^*)\Delta s_k + \cdots + G(x_b^*, y_b^*)\Delta s_b$$

Line Integrals

Line Integral in the Plane



$$x_0 \le x_1^* \le x_1 \qquad y_0 \le y_1^* \le y_1 \qquad \text{subinterval } \Delta s_1$$

$$x_1 \le x_2^* \le x_2 \qquad y_1 \le y_2^* \le y_2 \qquad \Delta s_2$$

$$\vdots$$

$$x_{k-1} \le x_k^* \le x_k \qquad y_{k-1} \le y_k^* \le y_k \qquad \Delta s_k$$

$$G(x_1^*, y_1^*)\Delta s_1 + G(x_2^*, y_2^*)\Delta s_2 + \cdots + G(x_k^*, y_k^*)\Delta s_k + \cdots + G(x_b^*, y_b^*)\Delta s_b \Rightarrow \int_C G(x,y)ds = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta s_k$$

$\|P\|$: length of the longest subinterval

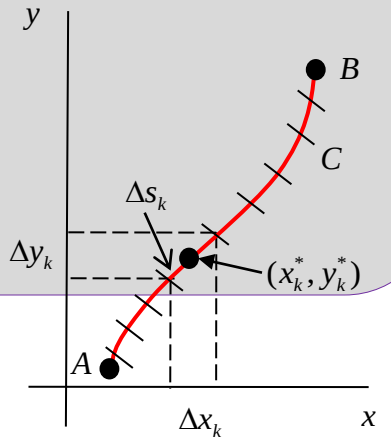
Line Integrals

Line Integral in the Plane

z=G (x,y)

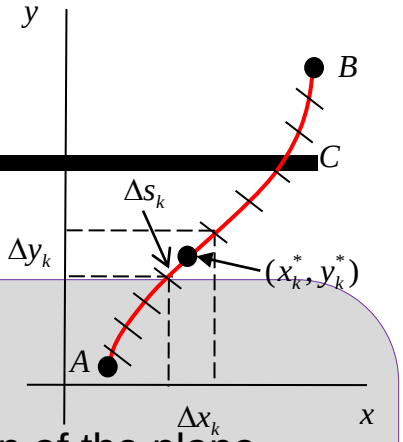
1. Let G be defined in some region that contains the smooth curve C defined by
   $x = f(t), y = g(t), a \le t \le b$
2. Divide C into n subarcs of length Δs_k according to the partition
   $a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$ of $[a,b]$. Let the projection of each subarc onto the x- and y=axes have length Δx_k and Δy_k,respectively.
3. Let $\| P \|$ be the **norm** of the partition or the length of the longest subarc.
4. Choose a point (x_k^*, y_k^*) in each subarc.
5. Form the sum

$$\sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta x_k, \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta y_k, \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta s_k$$



Line Integrals

Line Integral in the Plane



Definition 9.9

Line Integrals in the Plane

Let G be a function of two variables x and y defined on a region of the plane containing a smooth curve *C*.

(i) The line integral of *G* along *C* from *A* to *B* with respect to *x* is

$$\int_C G(x,y)dx = \lim_{\|P\|\to 0}\sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta x_k$$

(ii) The line integral of *G* along *C* from *A* to *B* with respect to y is

$$\int_C G(x,y)dy = \lim_{\|P\|\to 0}\sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta y_k$$

(iii) The line integral of *G* along *C* from *A* to *B* with respect arc length is

$$\int_C G(x,y)ds = \lim_{\|P\|\to 0}\sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta s_k$$

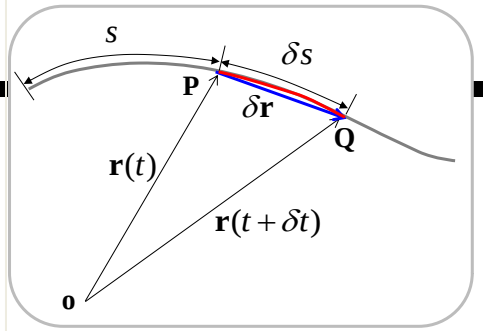
Line Integrals

Method of Evaluation
- Curve Defined Parametrically

If C is a smooth curve parameterized by

$$x = f(t),\ y = g(t), a \le t \le b$$

Tangent Vector and Unit Tangent Vector



The chord PQ

$$\delta\mathbf{r} = \mathbf{r}(t+\delta t) - \mathbf{r}(t)$$

The arc length s

unit tangent vector : direction

magnitude

tangent vector at P

$$\lim_{\delta t \to 0} \frac{\delta \mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt} = \mathbf{T}\left|\frac{d\mathbf{r}}{dt}\right| = \dot{\mathbf{r}}$$

unit tangent vector at P

$$\lim_{\delta s \to 0} \frac{\delta \mathbf{r}}{\delta s} = \frac{d\mathbf{r}}{ds} = \mathbf{T}$$

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{T}\left|\frac{ds}{dt}\right|$$

$$\therefore \frac{ds}{dt} = \dot{s} = \left|\frac{d\mathbf{r}}{dt}\right| = |\dot{\mathbf{r}}|$$

$$dx = f'(t)dt,\ dy = g'(t)dt,\ ds = \sqrt{[f'(t)]^2 + [g'(t)]^2}\ dt$$

$$\int_C G(x,y)dx = \int_a^b G(f(t), g(t))\, f'(t)dt$$

$$\int_C G(x,y)dy = \int_a^b G(f(t), g(t))\, g'(t)dt$$

$$\int_C G(x,y)ds = \int_a^b G(f(t), g(t))\sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

Convert the line integral to a definite integral in a single variable

Line Integrals

Notation

$$\int_C P(x,y)dx + \int_C Q(x,y)dy$$

In practice, It can be written as

$$\int_C P(x,y)dx + Q(x,y)dy \quad \text{or simply} \quad \int_C Pdx + Qdy$$

A line integral along a closed curve C is very often denoted by

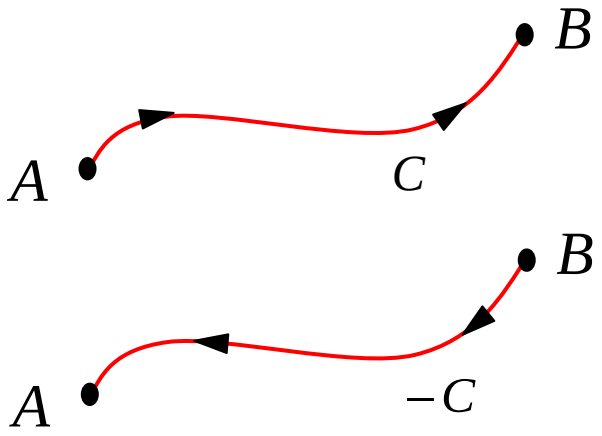
$$\oint_C Pdx + Qdy$$

Line Integrals

Curves with opposite orientation

$$\int_{-C} P\,dx + Q\,dy = -\int_{C} P\,dx + Q\,dy$$

$$\int_{-C} P\,dx + Q\,dy + \int_{C} P\,dx + Q\,dy = 0$$



Line integrals in space

$$\int_{C} G(x,y,z)\,dz = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*, z_k^*)\,\Delta z_k$$

Line integral along a space curve C with respect to z

$\|P\|$: length of the longest subinterval

Line Integrals

Method of Evaluation

If C is a smooth curve in 3-space defined parametric equation

$$x = f(t),\ y = g(t),\ z = h(t), a \le t \le b$$

$$\int_C G(x,y,z)dz = \int_a^b G(f(t),g(t),h(t))h'(t)dt$$

$$\int_C G(x,y,z)ds = \int_a^b G(f(t),g(t),h(t))\sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt$$

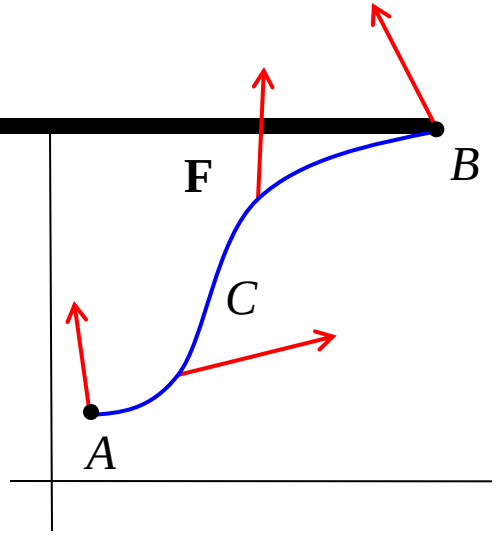
Convert the line integral to a definite integral in a single variable

Line Integrals

$F(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is defined along a curve

$C : x = f(t),\ y = g(t),\ a \le t \le b$

$\mathbf{r} = f(t)\mathbf{i} + g(t)\mathbf{j}$: position vector of points on *C*



$$\frac{d\mathbf{r}}{dt} = f'(t)\mathbf{i} + g'(t)\mathbf{j} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} \qquad dr = \frac{d\mathbf{r}}{dt}dt = dx\mathbf{i} + dy\mathbf{j}$$

$$F(x,y) \bullet d\mathbf{r} = P(x,y)dx + Q(x,y)dy$$

Ref. Line Integrals

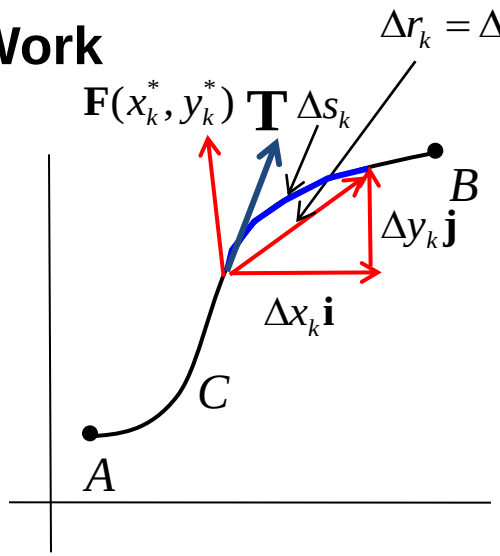
$$\int_C P(x,y)dx + Q(x,y)dy = \int_C \mathbf{F} \cdot d\mathbf{r}$$

Similarly, for a line integral on a space curve,

$$\int_C P(x,y)dx + Q(x,y)dy + R(x,y)dz = \int_C \mathbf{F} \cdot d\mathbf{r}$$

Line Integrals

Work



If Δs_k is small, $\mathbf{F}(x_k^*, y_k^*)$ is constant force, and

$$\Delta s_k \approx \Delta r_k$$

Approximate work done by F over the subarc is

$$(\| \mathbf{F}(x_k^*, y_k^*) \| \cos\theta) \| \Delta r_k \| = \mathbf{F}(x_k^*, y_k^*) \cdot \Delta r_k$$
$$= P(x_k^*, y_k^*)\Delta x_k + Q(x_k^*, y_k^*)\Delta y_k$$

By summing the elements of work and passing to limit,

$$W = \int_C P(x,y)dx + Q(x,y)dy \quad or \quad W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} \qquad d\mathbf{r} = \mathbf{T}ds \ (\mathbf{T} = d\mathbf{r}/ds)$$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds$$

The work done by a force *F* along a curve *C* is due entirely to the tangential component of *F*

Line Integrals

Circulation

$$\text{Circulation} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot \mathbf{T} ds$$

Circulation is a measure of the amount by which the fluid tends to turn the curve *C* by rotating around it.

$\int_C \mathbf{F} \cdot \mathbf{T} ds = 0$: Fluid does not Circulate in curve C

$\int_C \mathbf{F} \cdot \mathbf{T} ds > 0$: Fluid tend to rotate C in counterclockwise

$\int_C \mathbf{F} \cdot \mathbf{T} ds < 0$: Fluid tend to rotate C in clockwise

Line Integrals

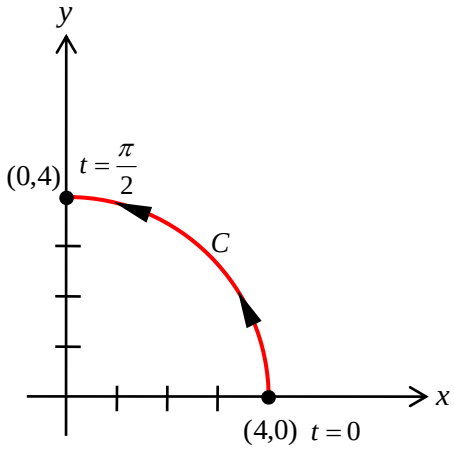
☑ Example 1 Evaluation of Line Integrals

Evaluate

(a) $\int_C xy^2 dx$,

(b) $\int_C xy^2 dy$,

(c) $\int_C xy^2 ds$



on the quarter-circle C defined by $x=4\cos t$, $y=4\sin t$, $0\le t\le \pi/2$. See 9.47.

Solution)

$$(a)\ \int_C xy^2 dx$$

$$= \int_0^{\pi} \overbrace{(4\cos t)}^{x}\overbrace{(16\sin^2 t)}^{y^2}\overbrace{(-4\sin t)dt}^{dx}$$

$$= -256\int_0^{\pi} \sin^3 t)\cos t dt$$

$$= -256\left[\frac{1}{4}\sin^4 t\right]_0^{\pi/2}$$

$$= -64$$

Line Integrals

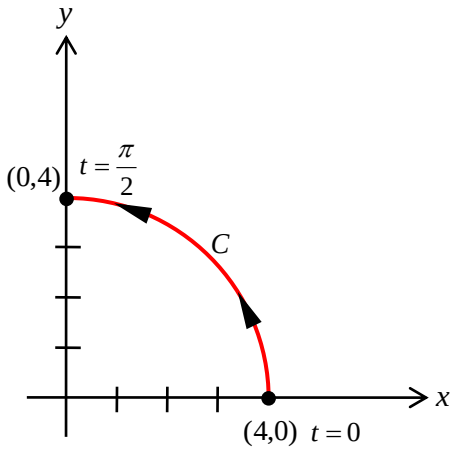
☑ Example 1 Evaluation of Line Integrals

Evaluate

(a) $\int_C xy^2 dx$,

(b) $\int_C xy^2 dy$,

(c) $\int_C xy^2 ds$



on the quarter-circle C defined by $x=4\cos t$, $y=4\sin t$, $0\le t\le \pi/2$. See Figure 9.47.

Solution)

(b) $\int_C xy^2 dy$

$$= \int_0^{\pi/2} \overbrace{(4\cos t)}^{x} \overbrace{(16\sin^2 t)}^{y^2} \overbrace{(4\cos t)dt}^{dx}$$

$$= 256\int_0^{\pi/2} (\sin^2 t)(\cos^2 t)dt$$

$$= 64\int_0^{\pi/2} (2\sin t\cos t)^2 dt$$

$$= 32\int_0^{\pi/2} (\sin 2t)^2 dt$$

$$= 32\int_0^{\pi/2} 1-(\cos^2 2t-\sin^2 2t)dt$$

$$= 32\int_0^{\pi/2} (1-\cos 4t)dt$$

$$= 32\left[t-\frac{1}{4}\sin 4t\right]_0^{\pi/2} = 16\pi$$

Line Integrals

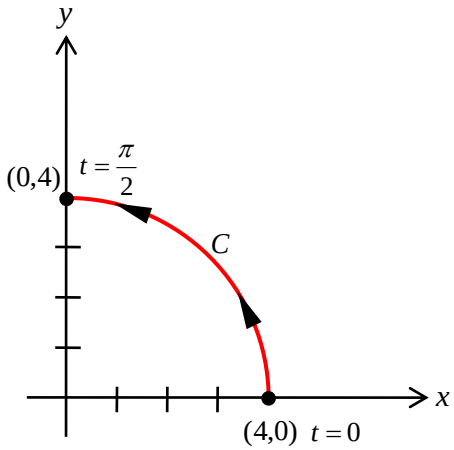
☑ Example 1 Evaluation of Line Integrals

Evaluate

(a) $\int_C xy^2 dx$,

(b) $\int_C xy^2 dy$,

(c) $\int_C xy^2 ds$



on the quarter-circle C defined by $x=4\cos t$, $y=4\sin t$, $0\le t\le \pi/2$. See 9.47.

Solution)

$$(c)\ \int_C xy^2 ds$$

$$= \int_0^{\pi/2} \overbrace{(4\cos t)}^{x} \overbrace{(16\sin^2 t)}^{y^2} \overbrace{\sqrt{(4\cos t)^2 + (4\sin t)^2}\, dt}^{ds}$$

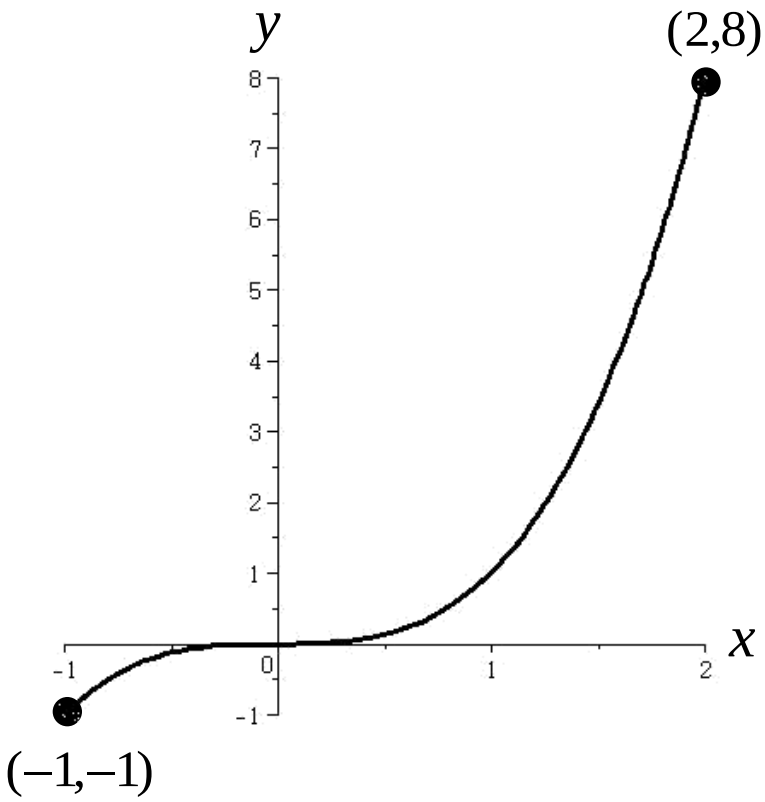
$$= 256\int_0^{\pi/2} (\sin^2 t \cos t)\, dt$$

$$= 256\left[\frac{1}{3}\sin^3 t\right]_0^{\pi/2} = \frac{256}{3}$$

Line Integrals

☑ Example 2
Curve Defined by an Explicit Function

Evaluate $\int_C xydx + x^2dy$, where C is given by $y=x^3$, $-1 \le x \le 2$.



Solution)

$$\int_C xydx + x^2dy$$

$$= \int_C xydx + \int_C x^2dy$$

$$= \int_{-1}^{2} x(x^3)dx + \int_{-1}^{2} x^2(3x^2dx)$$

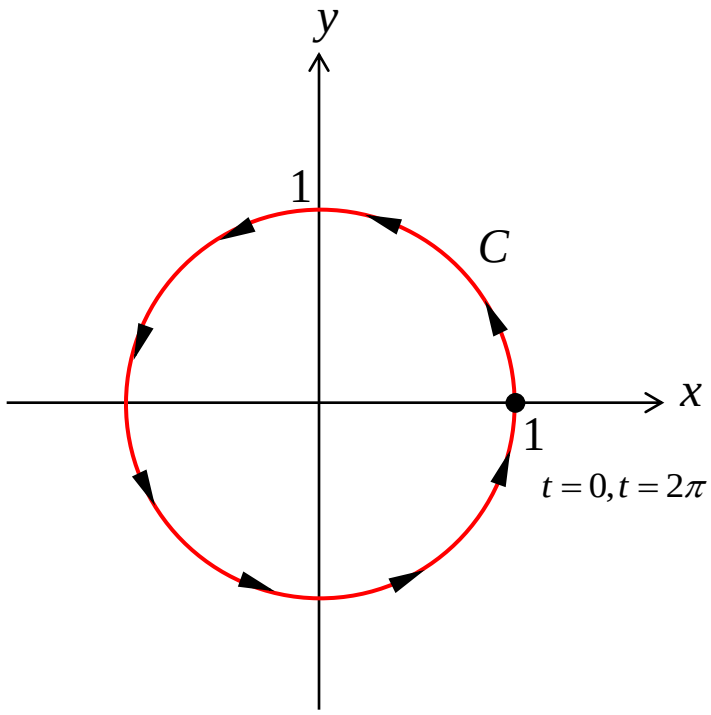
$$= \int_{-1}^{2} x^4 + 3x^4dx = \int_{-1}^{2} 4x^4dx$$

$$= \left[\frac{4}{5}x^5\right]_{-1}^{2} = \frac{132}{5}$$

Line Integrals

☑ Example 3
Curve Defined Parametrically

Evaluate $\oint_C x dx$, where C is the circle x=cost, y=sint, 0≤t≤2π.



Solution)

$$\oint_C x dx = \int_0^{2\pi} \cos t(-\sin t dt)$$

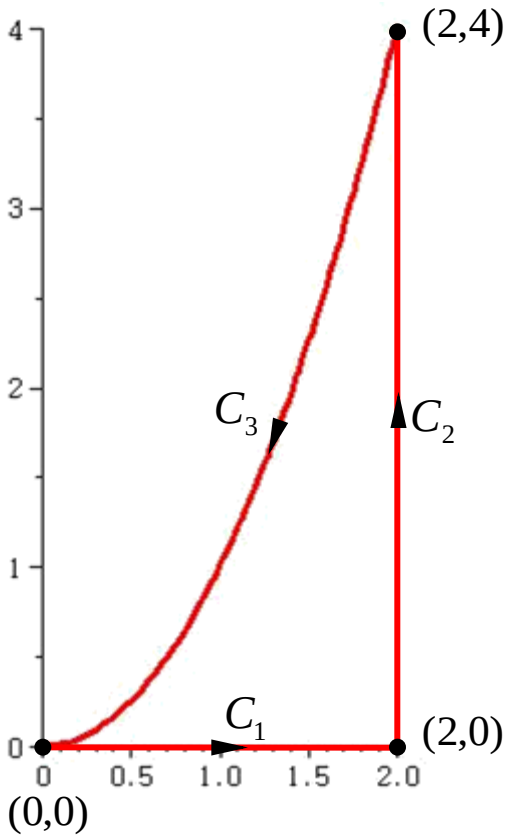
$$= \int_0^{2\pi} -\cos t \sin t dt$$

$$= \left[\frac{1}{2}\cos^2 t\right]_0^{2\pi} = 0$$

Line Integrals

☑ Example 4
Closed Curve

Evaluate $\oint_C y^2dx - x^2dy$ on the closed curve C that is shown in Figure 9.49(a).



Solution)

$$\oint_C y^2dx - x^2dy$$

$$= \int_{C_1} y^2dx - x^2dy + \int_{C_2} y^2dx - x^2dy + \int_{C_3} y^2dx - x^2dy$$

$$= \left[\int_0^2 0dx - x^2(0)\right] + \left[\int_0^4 y^2(0) - 4dy\right] + \left[\int_2^0 x^4dx - x^2(2xdx)\right]$$

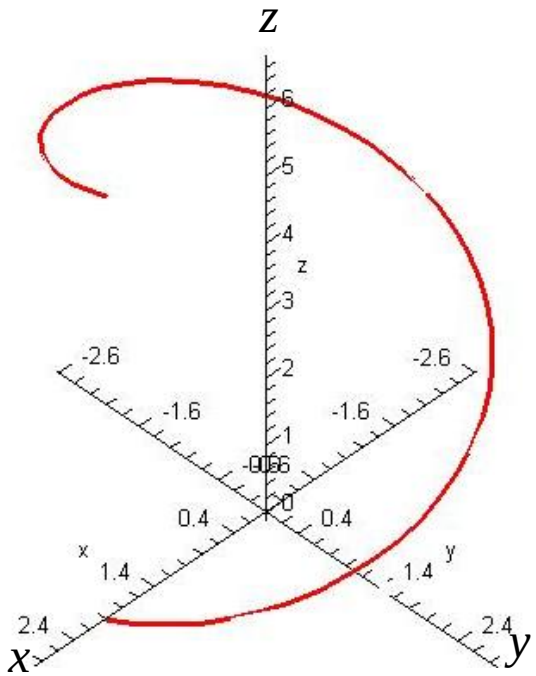
$$= [0] - [4y]_{y=0}^{y=4} + \left[\frac{1}{5}x^5 - \frac{1}{2}x^4\right]_{x=2}^{x=0}$$

$$= 0 - 16 + \frac{8}{5} = -\frac{72}{5}$$

Line Integrals

☑ Example 5
Line Integral on a Curve in 3-Space

Evaluate $\int_C ydx + xdy + zdz$, where C is the helix $x=2\cos t$, $y=2\sin t$, $z=t$, $0 \le t \le 2\pi$.



Solution)

$$x = 2\cos t \quad \Rightarrow \quad dx = -2\sin t \cdot dt$$
$$y = 2\sin t \quad \Rightarrow \quad dy = 2\cos t \cdot dt$$
$$z = t \quad \Rightarrow \quad dz = 1 \cdot dt = dt$$

$$\int_C ydx + xdy + zdz$$
$$= \int_0^{2\pi} 2\sin t \cdot (-2\sin t \cdot dt) + 2\cos t \cdot (2\cos t \cdot dt) + t \cdot (dt)$$
$$= \int_0^{2\pi} (-4\sin^2 t + 4\cos^2 t + t)dt$$
$$= \int_0^{2\pi} (4\cos 2t + t)dt$$
$$= \left[2\sin 2t + \frac{t^2}{2} \right]_0^{2\pi} = 2\pi^2$$

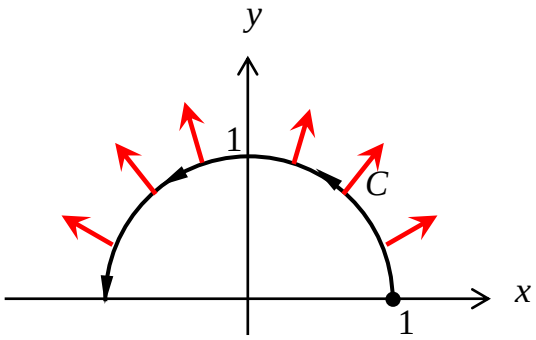
Line Integrals

☑ Example 6
Work Done by a Force

Find the work done by (a) $\mathbf{F}=x\mathbf{i}+y\mathbf{j}$ and (b) $\mathbf{F}=\frac{3}{4}\mathbf{i}+\frac{1}{2}\mathbf{j}$ along the curve C traced by $\mathbf{r}(t)=\cos t\mathbf{i}+\sin t\mathbf{j}$ from $t=0$ to $t=\pi$.

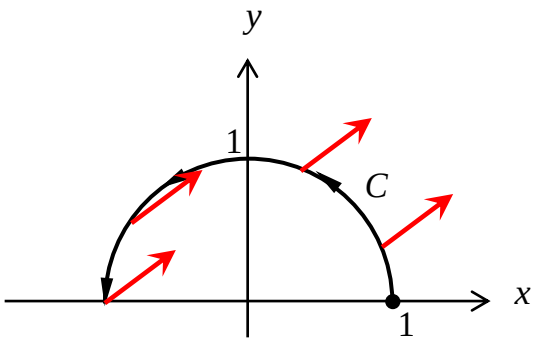
Solution)

$(a)\ \mathbf{F} = x\mathbf{i} + y\mathbf{j}$



$$W = \int_C \mathbf{F}\bullet d\mathbf{r} = \int_C (x\mathbf{i}+y\mathbf{j})\bullet d\mathbf{r}$$
$$= \int_0^{\pi} (\cos t\mathbf{i}+\sin t\mathbf{j})\bullet(-\sin t\mathbf{i}+\cos t\mathbf{j})dt$$
$$= \int_0^{\pi} (-\cos t\sin t+\sin t\cos t)dt = 0$$

$(b)\ \mathbf{F} = \frac{3}{4}\mathbf{i} + \frac{1}{2}\mathbf{j}$



$$W = \int_C \mathbf{F}\bullet d\mathbf{r} = \int_C \left(\frac{3}{4}\mathbf{i}+\frac{1}{2}\mathbf{j}\right)\bullet d\mathbf{r}$$
$$= \int_0^{\pi} \left(\frac{3}{4}\mathbf{i}+\frac{1}{2}\mathbf{j}\right)\bullet(-\sin t\mathbf{i}+\cos t\mathbf{j})dt$$
$$= \int_0^{\pi} \left(-\frac{3}{4}\sin t+\frac{1}{2}\cos t\right)dt$$
$$= \left[\frac{3}{4}\cos t+\frac{1}{2}\sin t\right]_0^{\pi} = -\frac{3}{2}$$

Independence of path

Differential – Functions of Two Variables

The differential of a function two variables $\phi(x, y)$ is

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$$

It is said to be exact differential equation if there exists a function $\phi(x, y)$ such that

$$d\phi = P(x, y)dx + Q(x, y)dy$$

$d\phi = x^2 y^3 dx + x^3 y^2 dy$ Is differential of $\phi(x, y) = \frac{1}{3} x^3 y^3$ => Exact differential

$d\phi = (2y^2 - 2y)dx + (2xy - x)dy$ There is no function ϕ satisfying this equation .

=> Not an exact differential

Independence of path

Path Independence

A line integral whose value is the same for ***every*** curve or path connection A and B is said to be **independent of the path**.

Theorem 9.8 **Fundamental Theorem for Line Integrals**

Suppose there exists a function $\phi(x, y)$ such that $d\phi = Pdx + Qdy$; that is, $Pdx + Qdy$ is an exact differential. Then $\int_C Pdx + Qdy$ depends on only the endpoints A and B of the path C and

$$\int_C Pdx + Qdy = \phi(B) - \phi(A).$$

=> If $\phi(x, y)$ is exact, line integral of $\phi(x, y)$ is said to be path independent.

Independence of path

Theorem 9.8 **Fundamental Theorem for Line Integrals**

Suppose there exists a function $\phi(x, y)$ such that $d\phi = Pdx + Qdy$; that is, $Pdx + Qdy$ is an exact differential. Then $\int_C Pdx + Qdy$ depends on only the endpoints A and B of the path C and

$$\int_C Pdx + Qdy = \phi(B) - \phi(A).$$

Proof)

By chain rule, $\int_C Pdx + Qdy = \int_a^b \left(\frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} \right) dt$

$$= \int_a^b \frac{d\phi}{dt} dt = \phi(f(t), g(t))\Big]_a^b$$

$$= \phi(f(b), g(b)) - \phi(f(a), g(a))$$

$$= \phi(B) - \phi(A).$$

Independence of path

Ref. Simply Connected

Test for Path Independence in plane

Theorem 9.9 — **Test for Path Independence**

Let P and Q have continuous first partial derivatives in an open simply connected region. Then $\int_C Pdx + Qdy$ is independent of path C if and only if

For all (x,y) in the region

Independence of path

Conservative Vector Fields

Ref. Conservative Force and Mechanical Energy Conservation

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = Pdx + Qdy$$

$$= (P\mathbf{i} + Q\mathbf{j}) \cdot (dx\mathbf{i} + dy\mathbf{j}) = \mathbf{F} \cdot d\mathbf{r}$$

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} \quad (P = \frac{\partial \phi}{\partial x}, \quad Q = \frac{\partial \phi}{\partial y})$$

$\Rightarrow$ Vector field **F** is a gradient of the function ϕ

$\mathbf{F}$ is said to be **gradient field** and ϕ is said to be a **potential function**.

Gradient force field **F** is path independent and the work done by the force along a closed path is *zero*. For this reason, such a force field is also said to be **conservative**. In a conservative field **F** the *law of conservation* of mechanical energy holds.

Independence of path

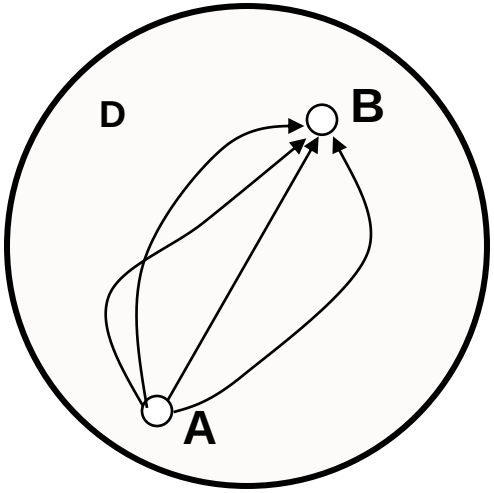
Test for Path Independence in space

Theorem 9.10 — **Test for Path Independence**

Let P , Q and R have continuous first partial derivatives in an open simply connected region of space. Then $\int_C Pdx + Qdy + Rdz$ is independent of path C if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

Independence of path



path independent in a domain D in space

$$\text{(1)}\quad \int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_C \left(F_1 dx + F_2 dy + F_3 dz\right)$$

The integral (1) has the same value for all paths in D that begin at A and end at B.

Next 3 ideas give path independence of (1) in a domain D if and only if :

$$\mathbf{F} = \text{grad } f$$

Integration around closed curves C in D always gives 0.

$$\text{curl}\,\mathbf{F} = 0$$

Independence of path

$$\text{(1)}\quad \int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_C \left(F_1 dx + F_2 dy + F_3 dz\right)$$

(Theorem 1)

A line integral (1) with continuous F_1, F_2, F_3 in a domain D in space is path independent in D if and only if $\mathbf{F} = \left[F_1, F_2, F_3\right]$ is the gradient of some function f in D

(2) $\mathbf{F} = \text{grad } f$ **, thus,** $F_1 = \dfrac{\partial f}{\partial x}, F_2 = \dfrac{\partial f}{\partial y}, F_3 = \dfrac{\partial f}{\partial z}$

Independence of path

(proof) $\mathbf{F} = \text{grad } f \quad \left([F_1, F_2, F_3] = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] \right)$

$$\int_C \left(F_1 dx + F_2 dy + F_3 dz \right) = \int_C \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \right)$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt$$

$$= \int_a^b \frac{df}{dt} dt = f\left[x(t), y(t), z(t) \right]_{t=a}^{t=b}$$

$$= f\left(x(b), y(b), z(b) \right) - f\left(x(a), y(a), z(a) \right)$$

$$= f(B) - f(A)$$

$$\therefore \int_C \left(F_1 dx + F_2 dy + F_3 dz \right) = f(B) - f(A)$$

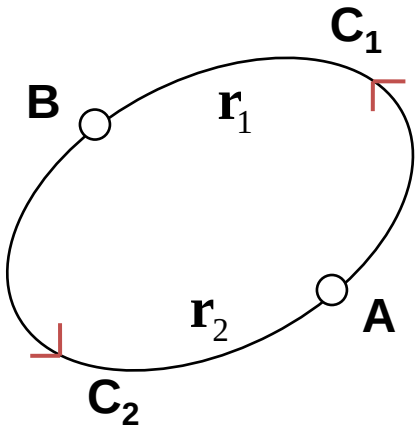
a line integral is independent of path

Independence of path

(Theorem 2)

Integration around closed curves C in D always gives 0.

(proof)



$$\int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_{C_1} \mathbf{F}(\mathbf{r}_1) \bullet d\mathbf{r}_1 + \int_{C_2} \mathbf{F}(\mathbf{r}_2) \bullet d\mathbf{r}_2 = 0$$

$$C_1 : A \to B, C_2 : B \to A$$

$$\int_A^B \mathbf{F}(\mathbf{r}_1) \bullet d\mathbf{r}_1 + \int_B^A \mathbf{F}(\mathbf{r}_2) \bullet d\mathbf{r}_2 = 0$$

Move 2nd term to the right.

$$\int_A^B \mathbf{F}(\mathbf{r}_1) \bullet d\mathbf{r}_1 = -\int_B^A \mathbf{F}(\mathbf{r}_2) \bullet d\mathbf{r}_2 = \int_A^B \mathbf{F}(\mathbf{r}_2) \bullet d\mathbf{r}_2$$

In conclusion, a line integral is path independent

Independence of path

(Theorem 3)

(4) $\mathbf{F} \bullet d\mathbf{r} = F_1 dx + F_2 dy + F_3 dz$

Let F_1, F_2, F_3 in the line integral (1) $\int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_C \left(F_1 dx + F_2 dy + F_3 dz \right)$

be continuous and have continuous first partial derivatives in a domain D in space. Then :

(a) If the differential form (4) is exact in D - and thus (1) is path independent by theorem (3*)- then in D,

(6) $\operatorname{curl} \mathbf{F} = 0$

$$\left(\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right] \right)$$

in components

(6') $$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

Independence of path

(4) $\mathbf{F} \bullet d\mathbf{r} = F_1 dx + F_2 dy + F_3 dz$

(Theorem 3)

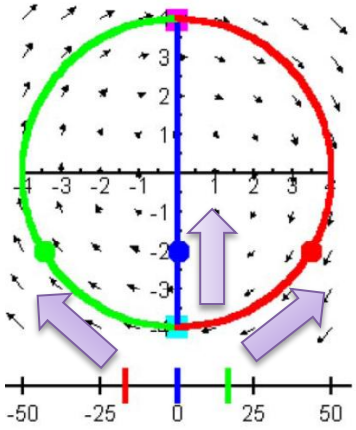
Let F_1, F_2, F_3 in the line integral (1) $\int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_C \left(F_1 dx + F_2 dy + F_3 dz \right)$

(6) $\operatorname{curl} \mathbf{F} = 0$

$$\left(\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right] \right)$$

in components

(6') $$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$



If it is rotational field,

$\int \mathbf{F} \bullet d\mathbf{r} > 0$

$\int \mathbf{F} \bullet d\mathbf{r} = 0$

$\int \mathbf{F} \bullet d\mathbf{r} < 0$

Path dependent → not a conservative field

Independence of path

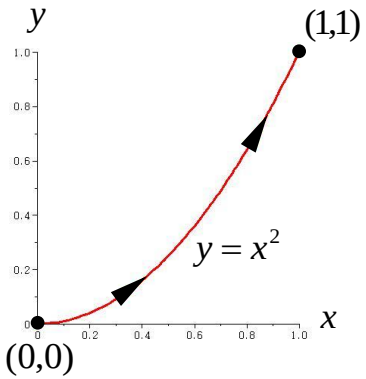
☑ Example 1

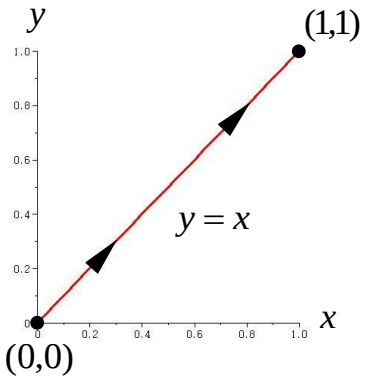
An Integral Independent of the Path

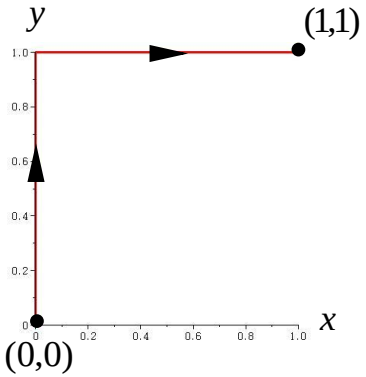
The integral $\int_C ydx + xdy$ has the same value on each path C between (0,0) and (1,1) shown in Figure 9.65. You may recall from Problems 11-14 of Exercises 9.8 that on these paths

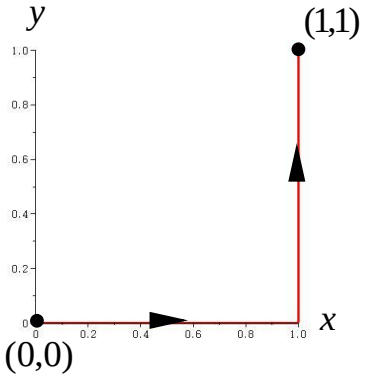
$$\int_C ydx + xdy = 1$$

In example 2 we shall prove that the given integral is independent of the path.









Independence of path

☑ Example 2
Using Theorem 9.8

In Example 1 note that $d(xy)=y\,dx + x\,dy$; that is, $y\,dx + x\,dy$ is an exact differential. Hence, $\int_C y\,dx + x\,dy$ is independent of the path between any two points A and B. Specifically, if A and B are, respectively, $(0,0)$, and $(1,1)$, we then have, from theorem 9.8,

$$\int_{(0,0)}^{(1,1)} ydx + xdy = \int_{(0,0)}^{(1,1)} d(xy) = xy\Big]_{(0,0)}^{(1,1)} = 1.$$

9.9 Example 1.

$$\int_C ydx + xdy = 1$$

Independence of path

☑ Example 3

A Path-Dependent Integral

Show that the integral $\int_C (x^2 - 2y^3)dx + (x + 5y)dy$ is not independent of the path C.

Solution)

$$P = x^2 - 2y^3, \; Q = x + 5y$$

$$\frac{\partial P}{\partial y} = -6y^2, \; \frac{\partial Q}{\partial x} = 1$$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}.$$

$\therefore$ Integral is not independent of the path.

Independence of path

☑ Example 4

An Integral Independent of the Path

Show that $\int_C (y^2 - 6xy + 6)dx + (2xy - 3x^2)dy$ is independent of any path C between (-1,0) and (3,4). Evaluate.

Solution)

$$\int_C (y^2 - 6xy + 6)dx + (2xy - 3x^2)dy$$

$$= \int_C P(x,y)dx + Q(x,y)dy$$

$$P = y^2 - 6xy + 6 \quad \Rightarrow \quad \frac{\partial P}{\partial y} = 2y - 6x$$

$$Q = 2xy - 3x^2 \quad \Rightarrow \quad \frac{\partial Q}{\partial x} = 2y - 6x$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

$\therefore$ Integral is independent of the path.

$$P = \frac{\partial \phi}{\partial x} = y^2 - 6xy + 6$$

$$Q = \frac{\partial \phi}{\partial y} = 2xy - 3x^2$$

$$\phi = \int y^2 - 6xy + 6dx = xy^2 - 3x^2 y + 6x + g(y)$$

$$\frac{\partial \phi}{\partial y} = 2xy - 3x^2 + g'(y)$$

$$\therefore g'(y) = 0,\ g(y) = C$$

$$\int_{(-1,0)}^{(3,4)} (y^2 - 6xy + 6)dx + (2xy - 3x^2)dy$$

$$= \int_{(-1,0)}^{(3,4)} d(xy^2 - 3x^2 y + 6x)$$

$$= \left[xy^2 - 3x^2 y + 6x\right]_{(-1,0)}^{(3,4)}$$

$$= (48 - 108 + 18) - (-6) = -36$$

Independence of path

☑ Example 5 Gradient Field

Show that the vector field $\mathbf{F}=(y^2+5)\mathbf{i}+(2xy-8)\mathbf{j}$ is a gradient field. Find a potential function for $\mathbf{F}$.

Solution)

$$\mathbf{F} = (y^2+5)\mathbf{i}+(2xy-8)\mathbf{j}$$
$$= P(x,y)\mathbf{i}+Q(x,y)\mathbf{j}$$

$$\frac{\partial P}{\partial y}=2y,\ \frac{\partial Q}{\partial x}=2y$$

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$$

∴Vector field is a gradient field.

$$P=\frac{\partial \phi}{\partial x}=y^2+5$$

$$Q=\frac{\partial \phi}{\partial y}=2xy-8$$

$$\phi=\int y^2+5dx=y^2x+5x+g(y)$$

$$\frac{\partial \phi}{\partial y}=2xy+g'(y)$$

$$\therefore g'(y)=-8,\ g(y)=-8y+C$$

$$\phi=y^2x+5x-8y+C$$

$$\nabla\phi=\frac{\partial \phi}{\partial x}\mathbf{i}+\frac{\partial \phi}{\partial y}\mathbf{i}=(y^2+5)\mathbf{i}+(2xy-8)\mathbf{j}$$

Independence of path

☑ Example 6
An Integral Independent of the Path

Show that $\int_C (y+yz)dx+(x+3z^3+xz)dy+(9yz^2+xy-1)dz$ is independent of any path C between (1,1,1) and (2,1,4). Evaluate

Solution)

$$\int_C (y+yz)dx+(x+3z^3+xz)dy+(9yz^2+xy-1)dz$$

$$=\int_C P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz$$

$$\frac{\partial P}{\partial y}=1+z=\frac{\partial Q}{\partial x},\ \frac{\partial P}{\partial z}=y=\frac{\partial R}{\partial x},\ \frac{\partial Q}{\partial z}=9z^2+x=\frac{\partial R}{\partial y}$$

∴Integral is independent of the path.

$$P=\frac{\partial Q}{\partial x}=y+yz$$

$$\phi=\int y+yzdx=yx+xyz+g(y,z)$$

$$\frac{\partial \phi}{\partial y}=x+xz+\frac{\partial g(y,z)}{\partial y}=x+3z^3+xz$$

$$\phi=xy+xyz+3yz^3+h(z)$$

$$\frac{\partial \phi}{\partial z}=xy+9yz^2+\frac{dh(z)}{dz}=9yz^2+xy-1$$

$$\therefore \phi=xy+xyz+3yz^3-z$$

$$\int_C (y+yz)dx+(x+3z^3+xz)dy+(9yz^2+xy-1)dz$$

$$=\int_{(1,1,1)}^{(2,1,4)} (y+yz)dx+(x+3z^3+xz)dy+(9yz^2+xy-1)dz$$

$$=\int_{(1,1,1)}^{(2,1,4)} d(xy+xyz+3yz^3)$$

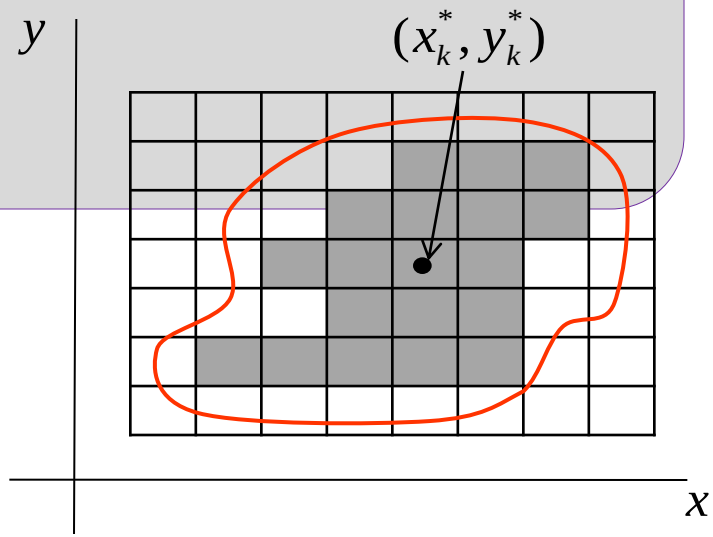
$$=(xy+xyz+3yz^3-z)\Big|_{(1,1,1)}^{(2,1,4)}$$

$$=198-4-194$$

Double Integrals

z=f (x,y)

1. Let f be defined in closed and bounded region R
2. By means of a grid of vertical and horizontal lines parallel to the coordinate axes, form a partition P of R into n rectangular subregions of areas ΔA_k that lie entirely in R.
3. Let $\| P \|$ be the **norm** of the partition or the length of the longest diagonal of the R_k.
4. Choose a point (x_k^*, y_k^*) in each subregion R_k.
5. Form the sum $\sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$



Double Integrals

Definition 9.10 **The Double Integral**

Let f be a function of two variables defined on a closed region R. Then the **double integral of f over *R*** is given by

$$\iint_R f(x,y)dA = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k \cdots (1)$$

Integrability

If the limit in integral exists, we say that f is **integrable** over R and that R is the region of integration. When f is continuous on R, then f is necessarily integrable over R

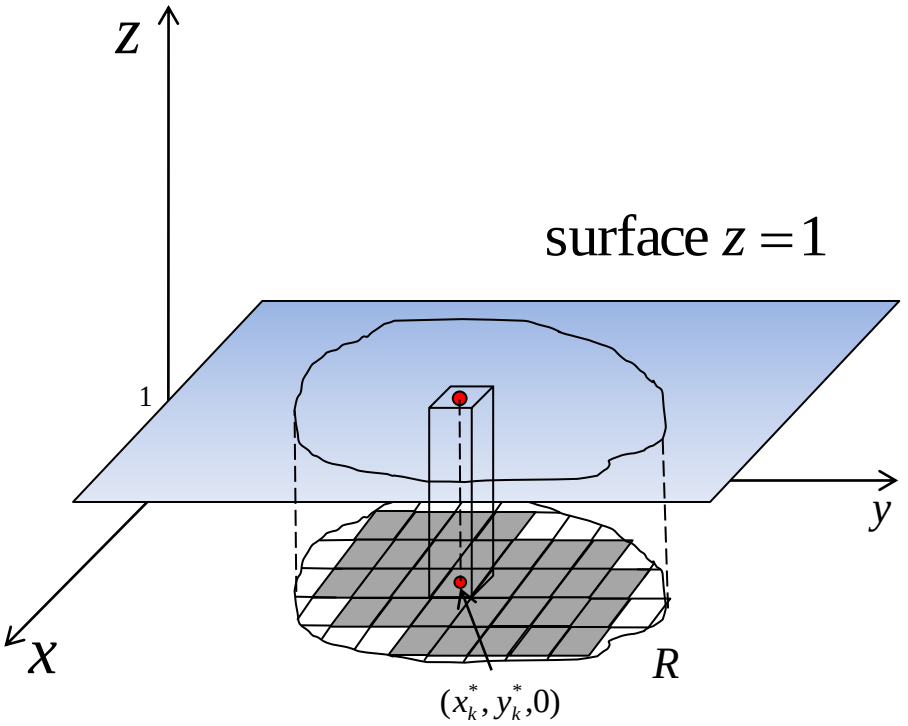
Double Integrals

$$\iint_R f(x,y)dA = \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k \cdots (1)$$

Area

when $f(x,y)=1$ on R, then $\lim_{\|P\|\to 0}\sum_{k=1}^{n}\Delta A$ simply give the **area** A of the region

$$A = \iint_R dA$$

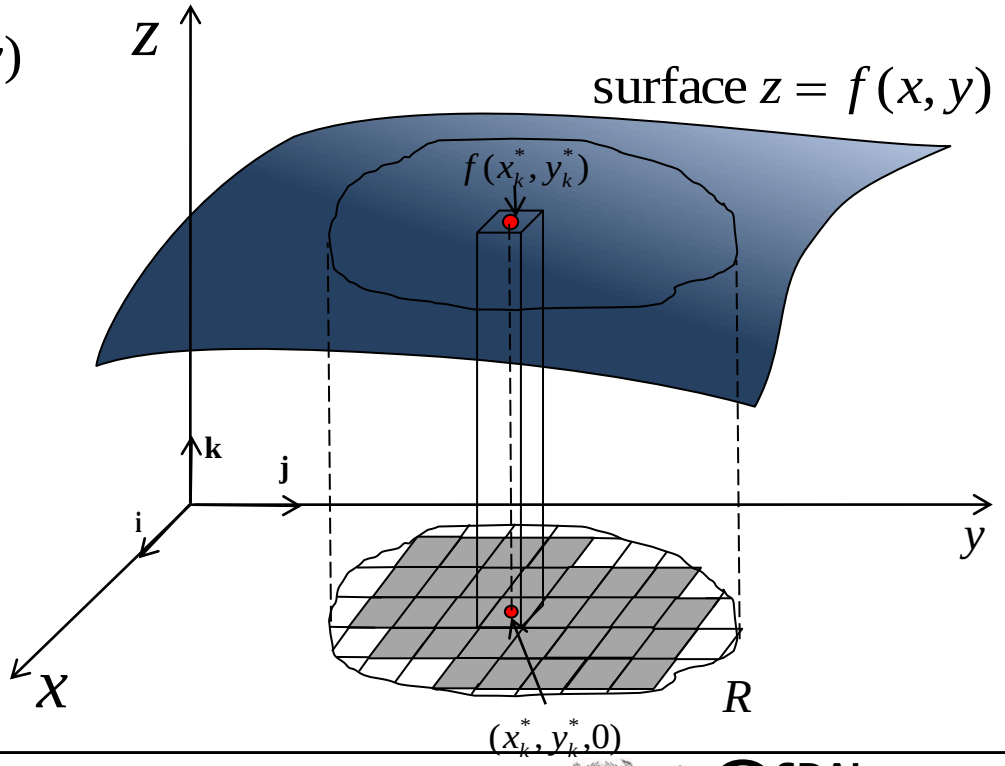


Double Integrals

$$\iint_R f(x,y)dA = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k \cdots (1)$$

Volume

If $f(x,y) > 0$ on *R*, then the product $f(x_k^*, y_k^*)\Delta A_k$ give the volume of rectangular prism. The summation of volume $\sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$ is approximation to the **volume** *V,* of the solid *above* the region R and *below* the surface $z = f(x,y)$



The limit of this sum as $\| P \| \to 0$

$$V = \iint_R f(x,y)dA$$

Double Integrals

Theorem 9.11 **Properties of Double Integrals**

Let f and g be functions of two variables that are integrable over a region *R*, Then

$$(ii)\iint_R [f(x,y) \pm g(x,y)]dA = \iint_R f(x,y)dA + \iint_R g(x,y)dA$$

$$(iii)\iint_R f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R_2} g(x,y)dA, \text{ where } R_1 \text{ and } R_2$$

$$\text{are subregions of } R \text{ that do not overlap and } R = R_1 \cup R_2$$

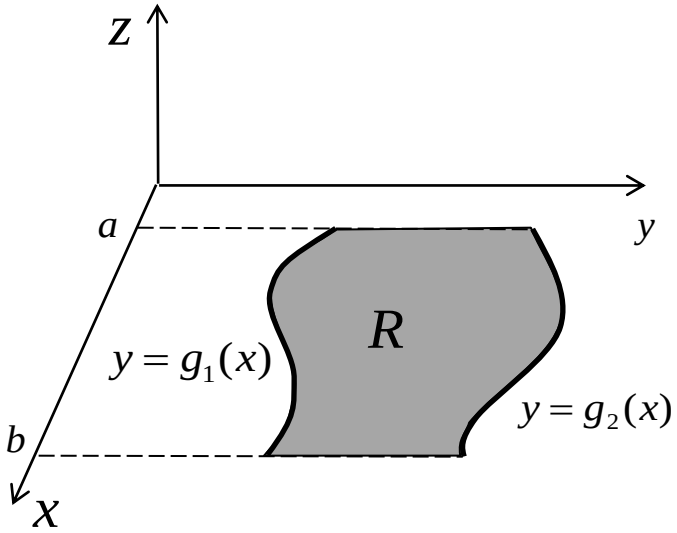
Double Integrals

Regions of Type I and Type II

Regions of Type I

$$R: a \le x \le b, \quad g_1(x) \le y \le g_2(x)$$

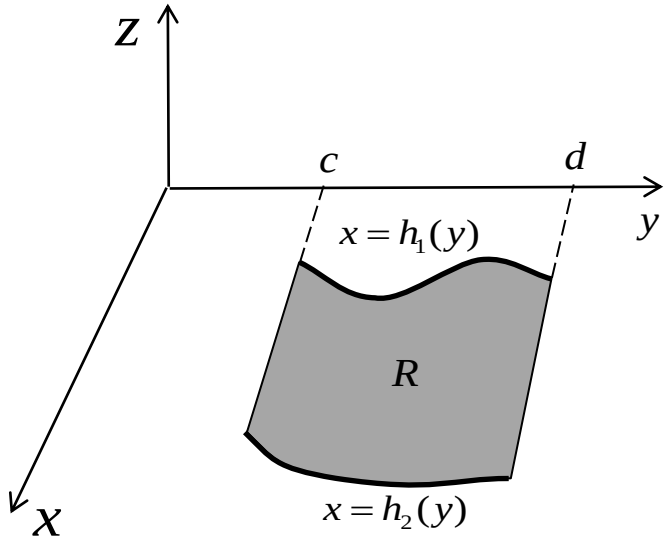
Where g_1 and g_2 are continuous.



Regions of Type II

$$R: c \le y \le d, \quad h_1(y) \le x \le h_2(y)$$

Where h_1 and h_2 are continuous.



Double Integrals

Type I:

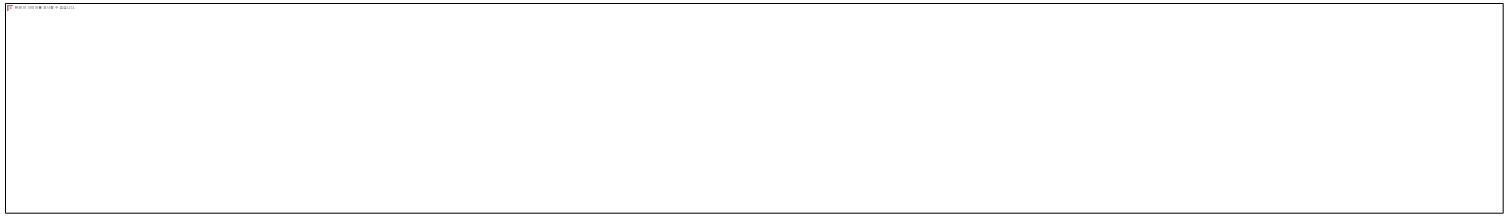
$$R: a \le x \le b, \quad g_1(x) \le y \le g_2(x)$$

Type II:

$$R: c \le y \le d, \quad h_1(y) \le x \le h_2(y)$$

Iterated Integrals

For type I



For type II

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right] dy$$

Double Integrals

Type I:

$R: a \le x \le b, \quad g_1(x) \le y \le g_2(x)$

Type II:

$R: c \le y \le d, \quad h_1(y) \le x \le h_2(y)$

Theorem 9.12 **Evaluation of Double Integrals**

Let f be continuous on region *R*

(i) If R is Type I, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

(ii) If R is Type II, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Double Integrals

Laminas with Variable Density – Center of Mass

If density is constant,

$$m = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \rho f(x_k^*)\Delta x_k = \int_a^b \rho f(x)dx$$

If density is not constant,

$$m = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \rho f(x_k^*)\Delta x_k = \iint_R \rho(x,y)dA$$

Center of mass $\bar{x} = \dfrac{M_y}{m},\ \bar{y} = \dfrac{M_x}{m},$

Where,

$$M_y = \iint_R x\rho(x,y)dA \qquad M_x = \iint_R y\rho(x,y)dA$$

Moment of lamina about y- and x- axis

Double Integrals

Moments of Inertia

Moments of inertia about the x-axis

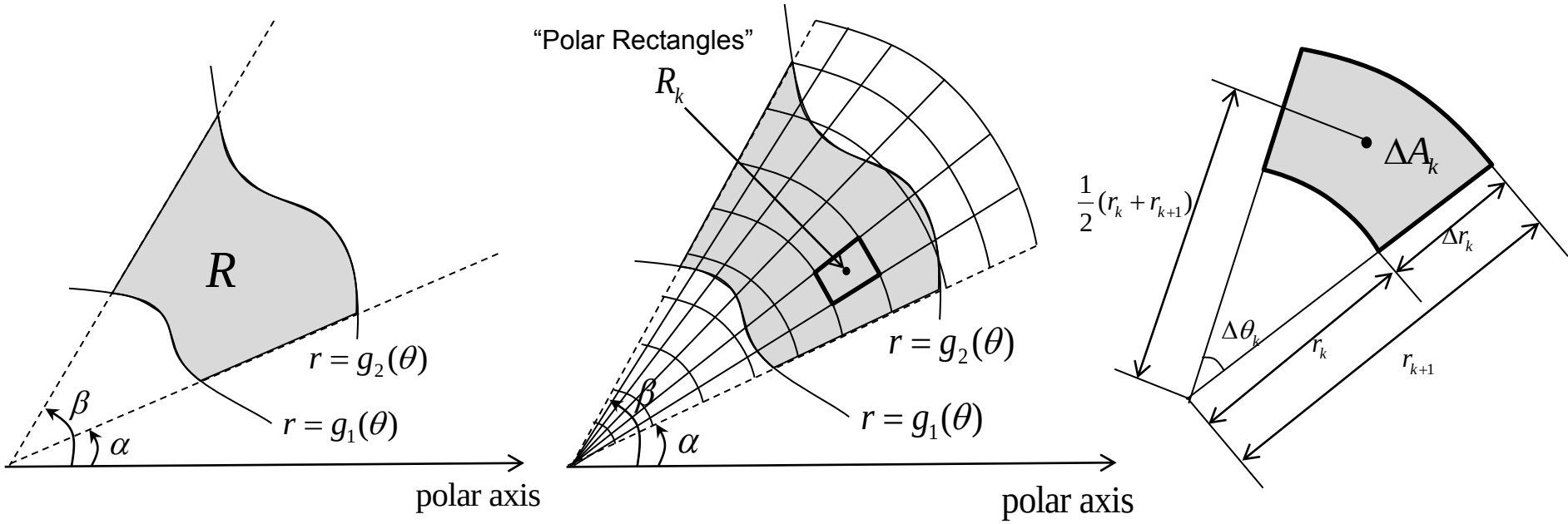
$$I_x = \iint_R y^2 \rho(x, y) dA$$

Moments of inertia about the y-axis

$$I_y = \iint_R x^2 \rho(x, y) dA$$

Double Integrals in Polar Coordinates

Polar Rectangles



$$\Delta A_k = \frac{1}{2} r_{k+1}^2 \Delta\theta_k - \frac{1}{2} r_k^2 \Delta\theta_k$$

Ref. Area of Polar Rectangles

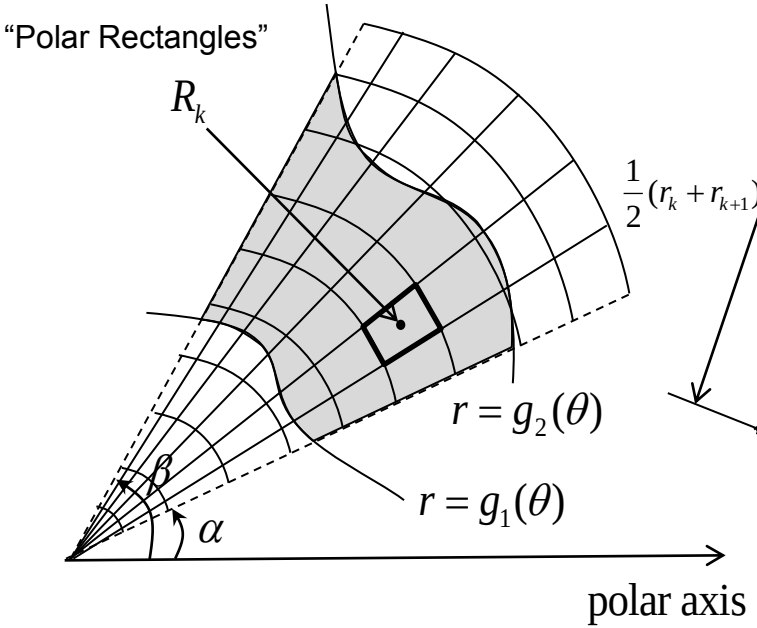
$$= \frac{1}{2}\left(r_{k+1}^2 - r_k^2\right)\Delta\theta_k = \frac{1}{2}(r_{k+1} + r_k)(r_{k+1} - r_k)\Delta\theta_k = r_k^* \Delta r_k \Delta\theta_k$$

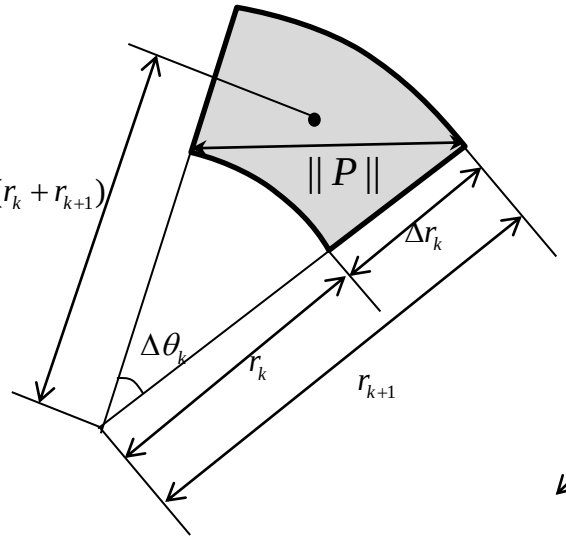
$$\Delta r_k = (r_{k+1} - r_k)$$

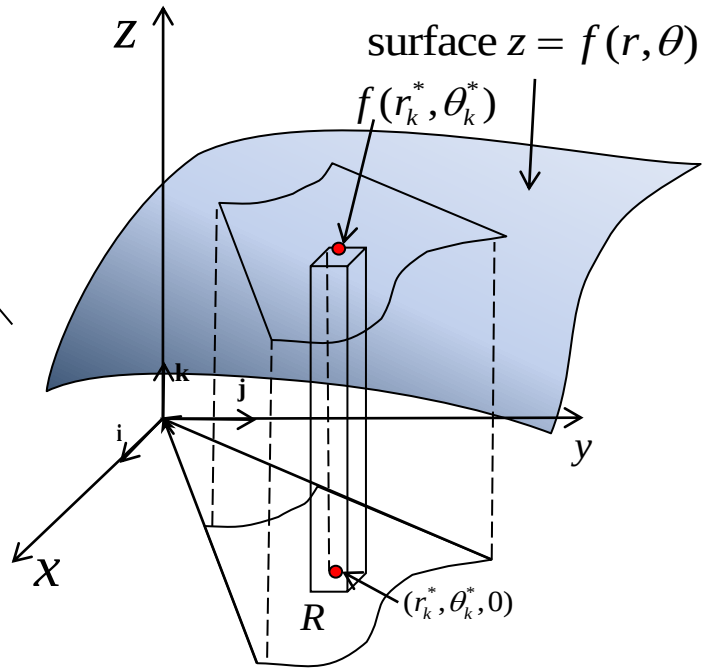
$$r_k^* = \frac{1}{2}(r_{k+1} + r_k)$$

Double Integrals in Polar Coordinates

Polar Rectangles







$$\Delta A_k = r_k^* \Delta r_k \Delta\theta_k$$

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(r_k^*, \theta_k^*)\Delta A_k = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(r_k^*, \theta_k^*) r_k^* \Delta r_k \Delta\theta_k = \iint_R f(r,\theta)\,dA$$

$$\iint_R f(r,\theta)\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

Double Integrals in Polar Coordinates

Change of Variables : Rectangular to Polar Coordinates

If R is describes in polar coordinates as

$$0 \le g_1(\theta) \le r \le g_2(\theta)$$

$$\alpha \le \theta \le \beta$$

$$0 \le \beta - \alpha \le 2\pi$$

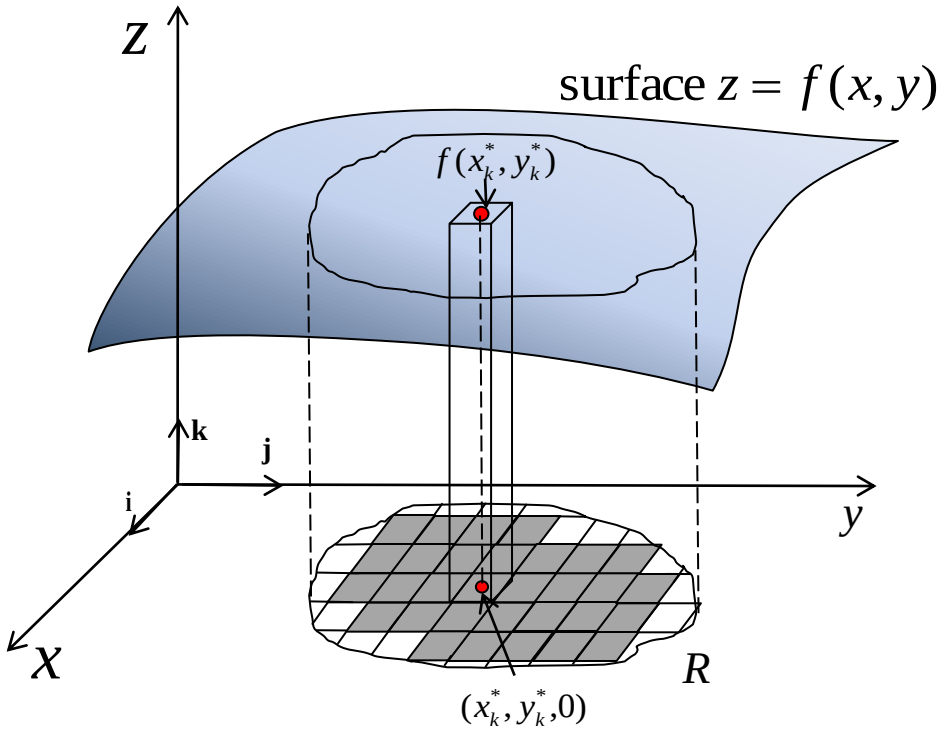
Then, $$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Above equation is particularly useful when f contains the expression $x^2 + y^2$ Since, in polar coordinates, we can write

$x^2 + y^2 = r^2$ *and* $\sqrt{x^2 + y^2} = r$

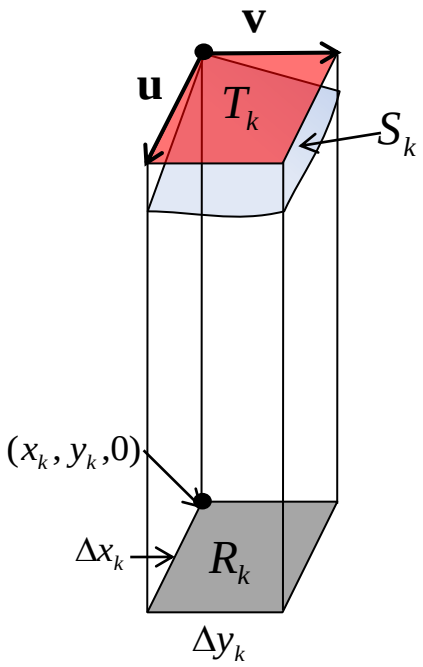
Surface Integrals

Surface Area



$$\mathbf{u} = \Delta x_k \mathbf{i} + f_x(x_k, y_k)\Delta x_k \mathbf{k}$$

$$\mathbf{v} = \Delta y_k \mathbf{j} + f_y(x_k, y_k)\Delta y_k \mathbf{k}$$

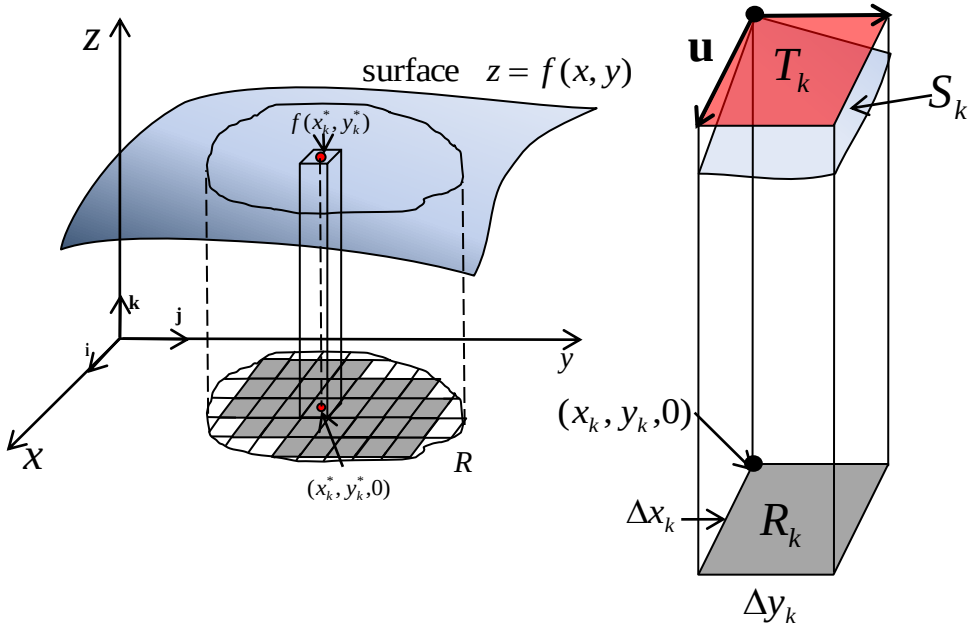


If R_k is small, $\Delta T_k \approx \Delta S_k$ $(\Delta T_k, \Delta S_k : \text{Area of } T_k, S_k)$

$$\Delta T_k = \| \mathbf{u} \times \mathbf{v} \| \quad , \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x_k & 0 & f_x(x_k, y_k)\Delta x_k \\ 0 & \Delta y_k & f_y(x_k, y_k)\Delta y_k \end{vmatrix} = [-f_x(x_k, y_k)\mathbf{i} - f_y(x_k, y_k)\mathbf{j} + \mathbf{k}]\Delta x_k \Delta y_k$$

Surface Integrals

Surface Area



$$\mathbf{u} = \Delta x_k \mathbf{i} + f_x(x_k, y_k)\Delta x_k \mathbf{k}$$
$$\mathbf{v} = \Delta y_k \mathbf{j} + f_y(x_k, y_k)\Delta y_k \mathbf{k}$$

If R_k is small, $\Delta T_k \approx \Delta S_k$

($\Delta T_k, \Delta S_k$: Area of T_k, S_k)

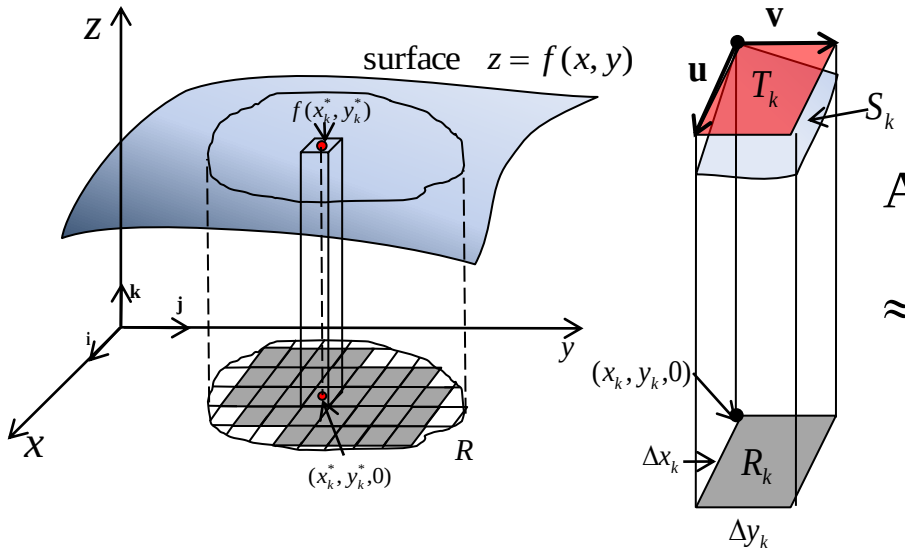
$$\Delta T_k = \| \mathbf{u} \times \mathbf{v} \|$$

$$\mathbf{u} \times \mathbf{v} = [-f_x(x_k, y_k)\mathbf{i} - f_y(x_k, y_k)\mathbf{j} + \mathbf{k}]\Delta x_k \Delta y_k$$

$$\Delta T_k = \sqrt{[f_x(x_k, y_k)]^2 + [f_y(x_k, y_k)]^2 + 1}\ \Delta x_k \Delta y_k$$

$$\text{Area of surface} \approx \sum_{k=1}^{n} \sqrt{[f_x(x_k, y_k)]^2 + [f_y(x_k, y_k)]^2 + 1}\ \Delta x_k \Delta y_k$$

Surface Integrals



Area of surface

$$\approx \sum_{k=1}^{n} \sqrt{[f_x(x_k, y_k)]^2 + [f_y(x_k, y_k)]^2 + 1}\ \Delta x_k \Delta y_k$$

Definition 9.11

Surface Area

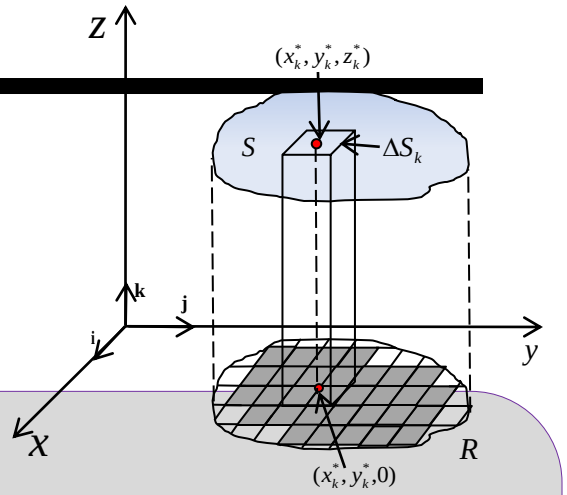
Let f be a function for the first partial derivatives f_x and f_y are continuous on a closed region *R*. Then the area of the surface over ***R*** is given by

$$A(s) = \iint_R \sqrt{[f_x(x_k, y_k)]^2 + [f_y(x_k, y_k)]^2 + 1}\ dA \cdots (2)$$

Surface Integrals

Differential of Surface Area

$$dS = \sqrt{1+[f_x(x_k, y_k)]^2 + [f_y(x_k, y_k)]^2}\ dA$$



w =G (x, y, z)

1. Let G be defined in a region of 3-space that contains a surface *S*, which is the graph of a function $z = f(x, y)$. Let the projection *R* of the surface onto the xy-plane be either a Type I or Type II region
2. Divide the surface into n pieces of areas ΔS_k corresponding to a partition *P* of *R* into n rectangles R_k of area ΔA_k
3. Let $\| P \|$ be the **norm** of the partition or the length of the longest diagonal of the R_k
4. Choose a point (x_k^*, y_k^*, z_k^*) in each element of surface area
5. Form the sum $\sum_{k=1}^{n} G(x_k^*, y_k^*, z_k^*)\Delta S_k$

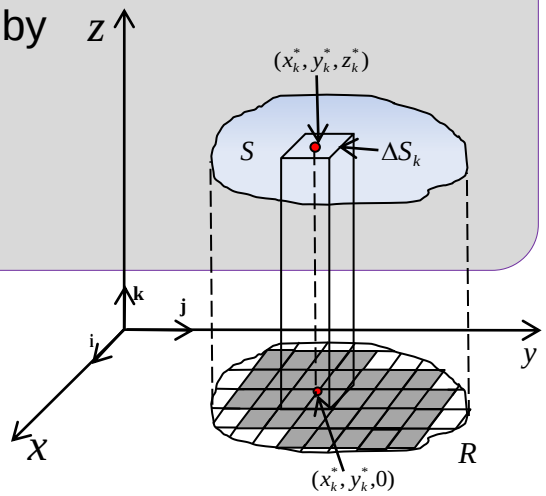
Surface Integrals

Definition 9.12

Surface Integral

Let G be a function of three variables defined over a region of space containing the surface S. then the **surface integral of G over S** is given by

$$\iint_S G(x,y,z)dS = \lim_{\|P\|\to 0}\sum_{k=1}^{n} G(x_k^*, y_k^*, z_k^*)\Delta S_k \cdots (4)$$



Method of Evaluation

If G, f, f_x *and* f_y are continuous throughout a region containing S, we can evaluate (4) by means of a double integral

$$\iint_S G(x,y,z)dS = \iint_R G(x,y,f(x,y))\sqrt{1+[f_x(x,y)]^2+[f_y(x,y)]^2}\,dA \cdots (5)$$

when $G=1$, (5) reduces to furmular (2) for surface area

Surface Integrals

Projection of S into Other Planes

If $y = g(x,z)$ is the equation of a surface *S* that projects onto a region *R* of the xz-plane, then

$$\iint_S G(x,y,z)dS = \iint_R G(x,g(x,z),z)\sqrt{1+[g_x(x,z)]^2+[g_z(x,z)]^2}\,dA$$

If $x = h(y,z)$ is the equation of a surface *S* that projects onto a region *R* of the yz-plane, then

$$\iint_S G(x,y,z)dS = \iint_R G(h(y,z),y,z)\sqrt{1+[h_y(y,z)]^2+[h_z(y,z)]^2}\,dA$$

Mass of a Surface

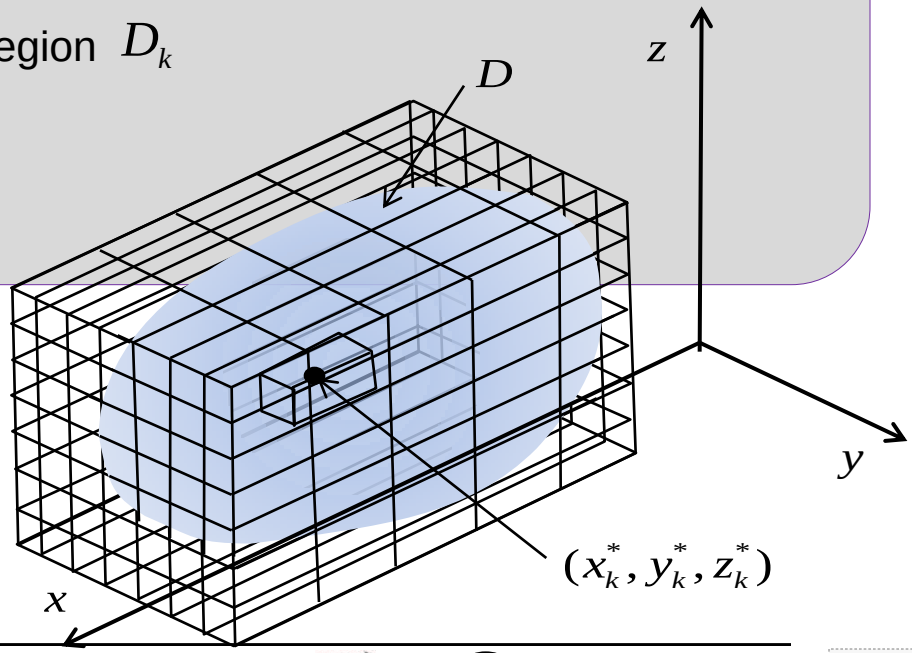
$$m = \iint_S \rho(x,y,z)dS$$

Triple Integrals

$w = F(x, y, z)$

1. Let F be defined a closed and bounded region *D* of space.
2. By means of a three- dimensional grid of vertical and horizontal planes parallel to the coordinate planes, form a partition *P* of *D* into n subregions (boxes) D_k of volumes ΔV_k that lie entirely in *D*.
3. Let $\| P \|$ be the **norm** of the partition or the length of the longest diagonal of the D_k
4. Choose a point (x_k^*, y_k^*, z_k^*) in each subregion D_k
5. Form the sum

$$\sum_{k=1}^{n} F(x_k^*, y_k^*, z_k^*)\Delta V_k$$



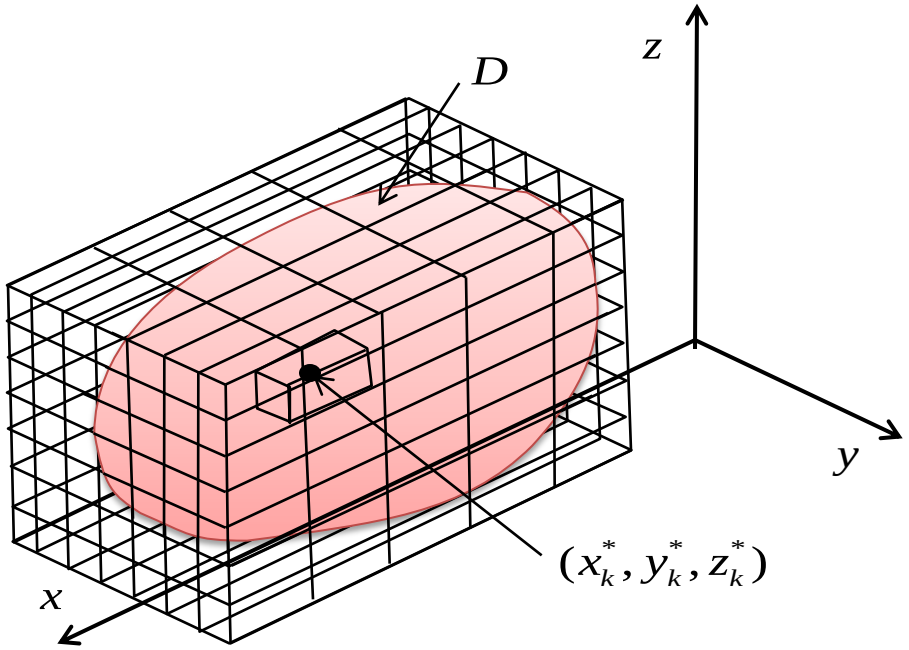
Triple Integrals

Definition 9.13

The Triple Integral

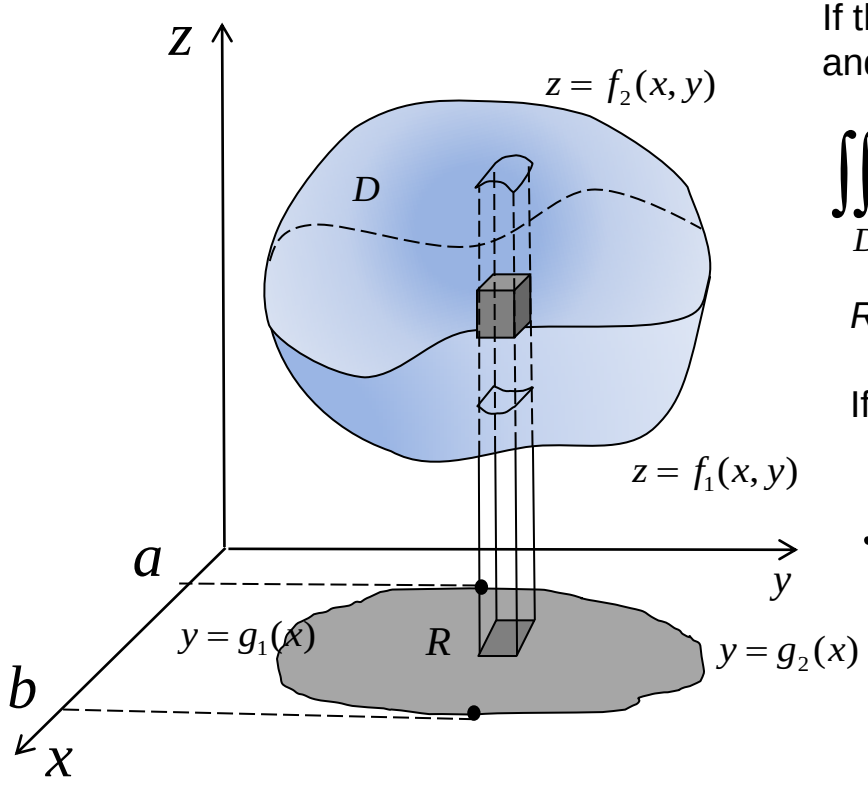
Let F be a function of three variables defined over a closed region *D* of space. Then the **triple integral of *F* over *D*** is given by

$$\iiint_D F(x,y,z)dV = \lim_{\|P\|\to 0} \sum_{k=1}^{n} F(x_k^*, y_k^*, z_k^*)\Delta V_k$$



Triple Integrals

Evaluation by Iterated Integrals



If the region *D* is bounded by above by the graph of $z = f_1(x, y)$ and bounded below by the graph of $z = f_2(x, y)$ then

$$\iiint_D F(x,y,z)dV = \iint_R \left[\int_{f_1(x,y)}^{f_2(x,y)} F(x,y,z)dz \right] dA$$

R is the orthogonal projection of *D* onto the xy-plane.

If *R* is a Type I region,

$$\iiint_D F(x,y,z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} F(x,y,z)dzdydx$$

Type I :

$R : a \le x \le b, \quad g_1(x) \le y \le g_2(x)$

Type II :

$R : c \le y \le d, \quad h_1(y) \le x \le h_2(y)$

In triple integral, there are three possible orders of integration.

$$dzdydx, \quad dzdxdy, \quad dydxdz,$$
$$dxdydz, \quad dxdzdy, \quad dydzdx,$$

The last two differentials tell the coordinate plane in which the region *R* is situated.

Triple Integrals

Volume $V = \iiint_D dV$

Mass $m = \iiint_D \rho(x, y, z)dV$

First Moments

The **first moment** of the solid about the coordinate planes indicated by the subscripts are given by

$$M_{xy} = \iiint_D z\rho(x, y, z)dV \quad M_{xz} = \iiint_D y\rho(x, y, z)dV \quad M_{yz} = \iiint_D x\rho(x, y, z)dV$$

Center of Mass $\bar{x} = \frac{M_{yz}}{m}, \ \bar{y} = \frac{M_{xz}}{m}, \ \bar{z} = \frac{M_{xy}}{m}$

Centroid

If $\rho(x, y, z) =$ a constant, the center of mass is called the **centroid** of the solid

Triple Integrals

Second Moments

The **Second moment** or **moments of inertia** of D about the coordinate axes indicated by the subscripts, are given by

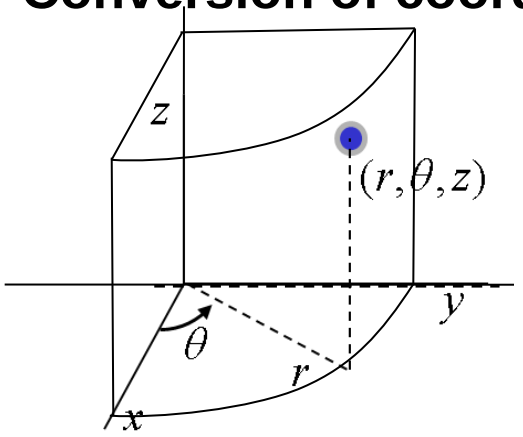
$$I_x = \iiint_D (y^2 + z^2)\rho(x, y, z)dV \qquad I_y = \iiint_D (x^2 + z^2)\rho(x, y, z)dV$$

$$I_z = \iiint_D (x^2 + y^2)\rho(x, y, z)dV$$

Radius of Gyration $R_g = \sqrt{\frac{I}{m}}$

Triple Integrals

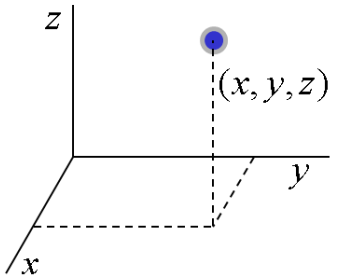
Conversion of coordinates



Cylindrical coordinates

Cylindrical → Cartesian

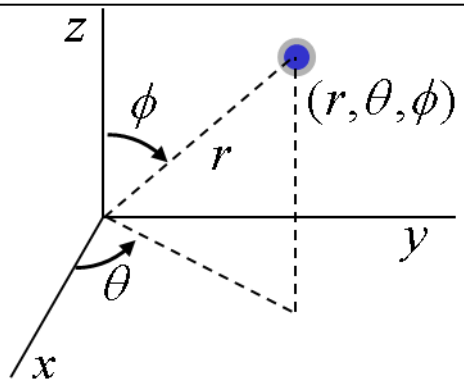
$$x = r\cos\theta \quad y = r\sin\theta \quad z = z$$



Cartesian coordinates

Cartesian → Cylindrical

$$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad z = z$$



Spherical coordinates

Spherical→ Cylindrical

$$r = \rho\sin\phi \qquad \theta = \theta \qquad z = \rho\cos\phi$$

Spherical→ Cartesian

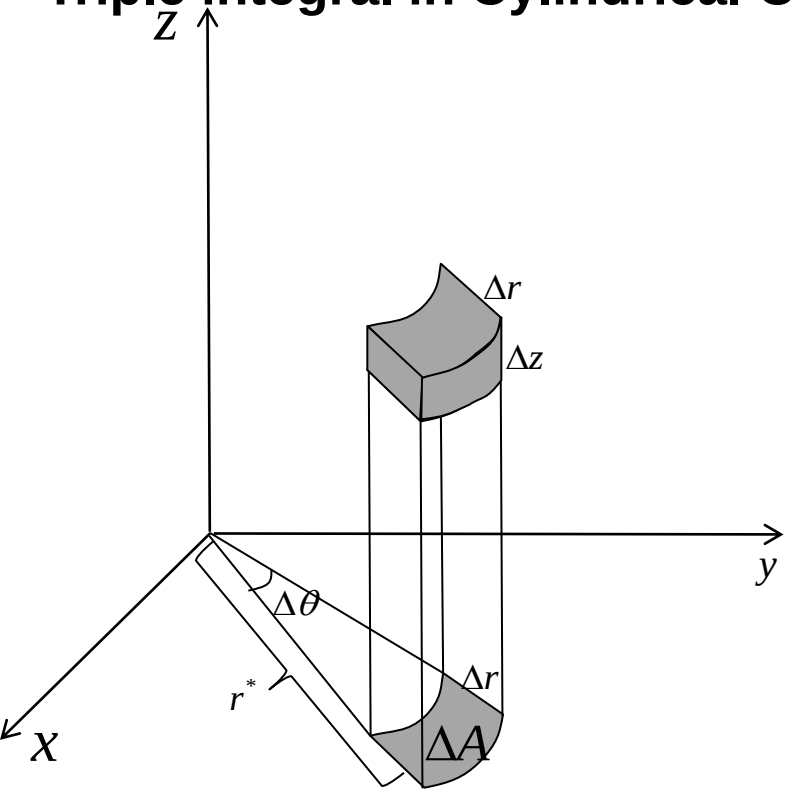
$$x = r\sin\phi\cos\theta \qquad y = r\sin\phi\sin\theta \qquad z = r\cos\phi$$

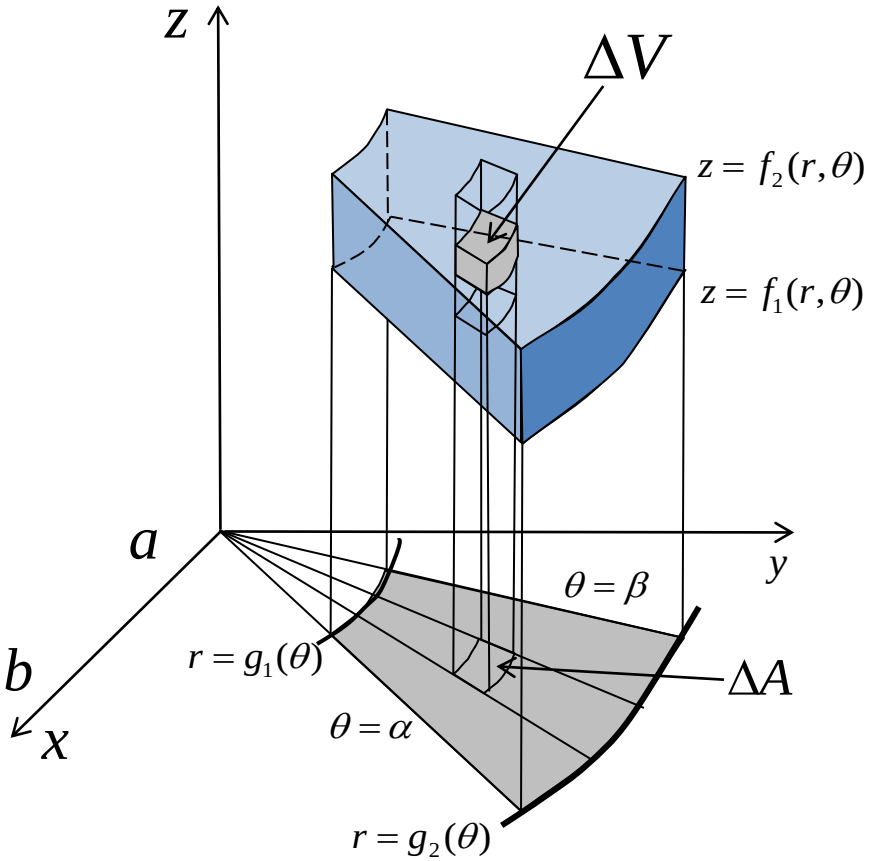
Cartesian → Spherical

$$r = \sqrt{x^2 + y^2 + z^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad \phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

Triple Integrals

Triple integral in Cylindrical Coordinates



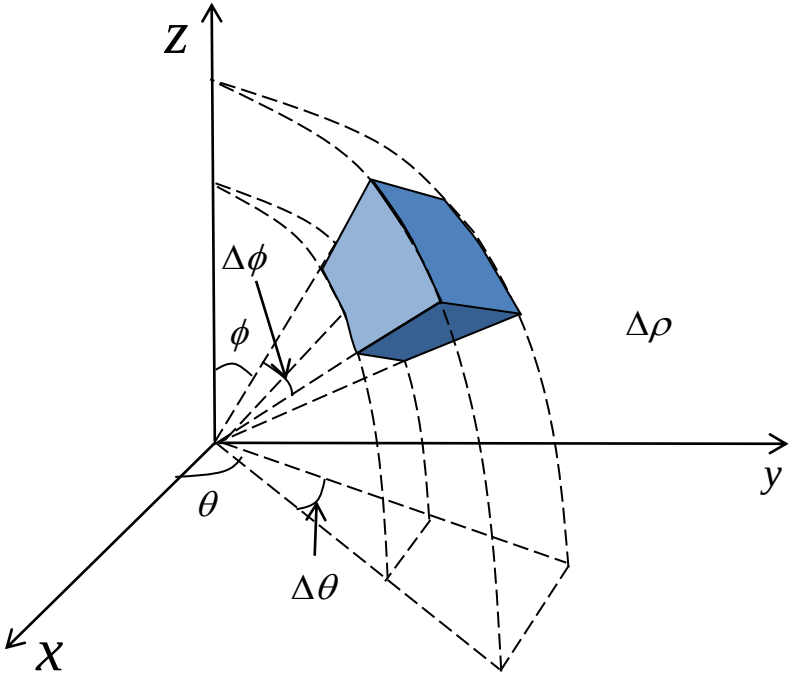


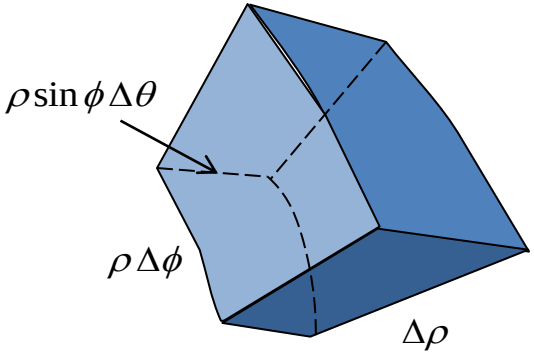
$$\Delta A = (r^* \Delta\theta) \cdot \Delta r \qquad \Delta V = \Delta A \Delta z = r^* \Delta r \Delta\theta \Delta z$$

$$\iiint_D F(r,\theta,z)dV = \iint_R \left[\int_{f_1(r,\theta)}^{f_2(r,\theta)} F(r,\theta,z)dz \right] dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{f_1(r,\theta)}^{f_2(r,\theta)} F(r,\theta,z) r\, dz\, dr\, d\theta$$

Triple Integrals

Triple integral in Spherical Coordinates





$$\Delta V \approx \rho^2 \sin\phi \, \Delta\rho \, \Delta\phi \, \Delta\theta$$

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\iiint_D F(r,\theta,z)dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{f_1(\phi,\theta)}^{f_2(\phi,\theta)} F(r,\phi,\theta)\,\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Reference slides

Line Integrals

Proof

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P dx + Q dy + R dz$$

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k} = [P, Q, R]$$

$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dx\mathbf{k} = [dx, dy, dz]$$

$$\mathbf{F} \bullet d\mathbf{r} = (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \bullet (dx\mathbf{i} + dy\mathbf{j} + dx\mathbf{k}) = P dx + Q dy + R dz$$

$$\therefore \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P dx + Q dy + R dz$$

Reference slides

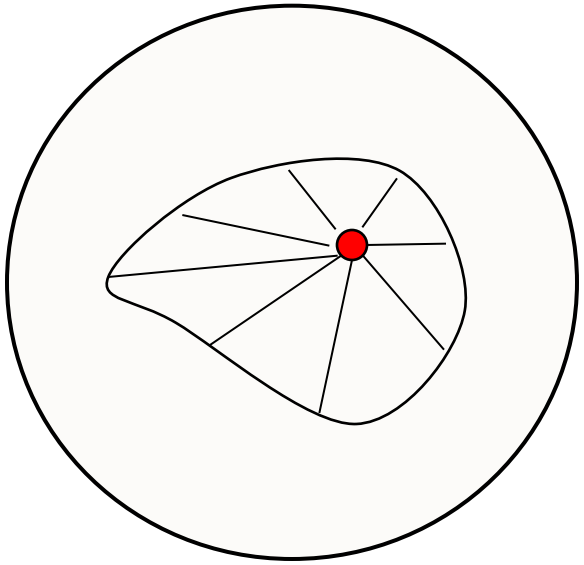
Simply Connected

Simply connected

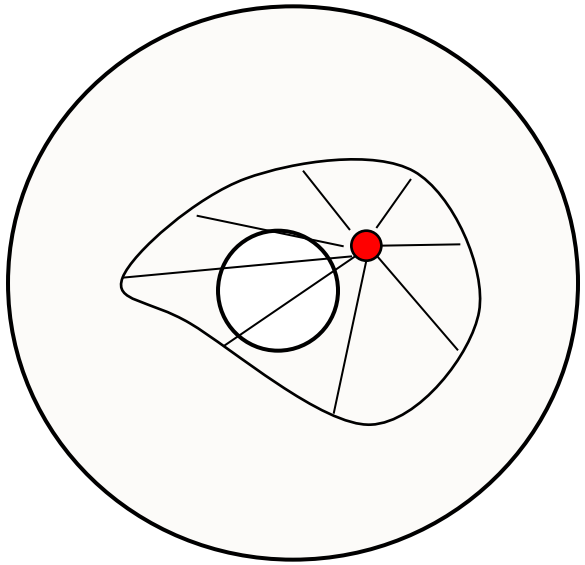
- **Simple connect**

A domain D is called simply connected if every closed curve in D can be continuously shrunk to any point in D without leaving D.

it can shrink to a point in D continuously



it can't shrink to a point in D continuously



Reference slides

Conservative Force and Mechanical Energy Conservation

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(참고) 보존력과 역학적 에너지 보존

▪ 보존력 : 모든 닫힌 경로를 따라 운동하는 입자에 어떠한 힘이 한 알짜 일이 '0'일 때 이 힘을 보존력이라 함

(*닫힌 경로: 어떤 위치를 출발하여 임의의 경로를 거쳐 다시 처음의 위치로 되돌아 올 때)

$$W = \oint \mathbf{F}_{\text{보존력}} \cdot d\mathbf{r} = 0$$

▪보존력의 예 :만유인력,중력, 탄성력, 전기력

단위 입자에 작용하는 보존력을 보존장이라 할 수 있다.

▪보존장의 예 :만유인력장,중력장, 탄성장, 전기장

보존력이 물체에 한일의 음의 값을 보존력에 의한 퍼텐셜에너지의 변화로 정의한다.

$$\Delta U = -W$$

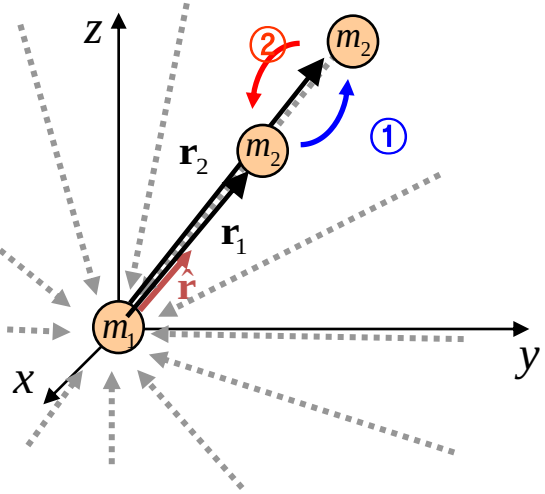
(만유인력과 탄성력 등이 보존력이 아니라면 중력 퍼텐셜에너지나 탄성 퍼텐셜에너지라는 말을 쓰지 않았을 것이다.)

만약, 물체가 보존장 속에서 운동하고, 보존장 외부에서 힘이 작용하지 않는다면 보존장 내에서의 Mechanical Energy는 보존된다.
(Mechanical Energy= Kinetic Energy+ Potential Energy)

(참고) 보존력과 역학적 에너지 보존

$$\mathbf{F}_g = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}$$

힘의 방향은일정하고 거리가 부호가 +,-로 바뀌니까 일은 0이다.



$$|\mathbf{r}_2| > |\mathbf{r}_1|$$

$$\mathbf{r}(t) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\mathbf{F}(\mathbf{r}) = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$$

$\hat{\mathbf{r}}$:unit position vector

$$r = |\mathbf{r}|$$

① 물체 m_2가 $\mathbf{r}_1$에서 $\mathbf{r}_2$까지 이동할때 만유인력 $\mathbf{F}_g$가 물체에 한 일 W_1

$$W_1 = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{r}$$

$$= \int_{r_1}^{r_2} -G\frac{m_1 m_2}{r^2}\,dr$$

$$= \left[G\frac{m_1 m_2}{r}\right]_{r_1}^{r_2}$$

$$= G\frac{m_1 m_2}{r_2} - G\frac{m_1 m_2}{r_1}$$

< 0 ,(힘의방향(−), 이동방향(+) 따라서일은 (−))

② 물체 m_2가 $\mathbf{r}_2$에서 $\mathbf{r}_1$까지 이동할때 만유인력 $\mathbf{F}_g$가 물체에 한 일 W_2

$$W_2 = \int_{\mathbf{r}_2}^{\mathbf{r}_1} \mathbf{F}\cdot d\mathbf{r}$$

$$= \int_{r_2}^{r_1} -G\frac{m_1 m_2}{r^2}\,dr$$

$$= \left[G\frac{m_1 m_2}{r}\right]_{r_2}^{r_1}$$

$$= G\frac{m_1 m_2}{r_1} - G\frac{m_1 m_2}{r_2}$$

> 0 ,(힘의방향(−), 이동방향(−) 따라서일은 (+))

③ 물체 m_2가 $\mathbf{r}_1$에서 $\mathbf{r}_2$까지 이동한후 다시 $\mathbf{r}_1$으로 돌아왔을때(닫힌 경로) 만유인력 $\mathbf{F}_g$가 물체에 한 일 W

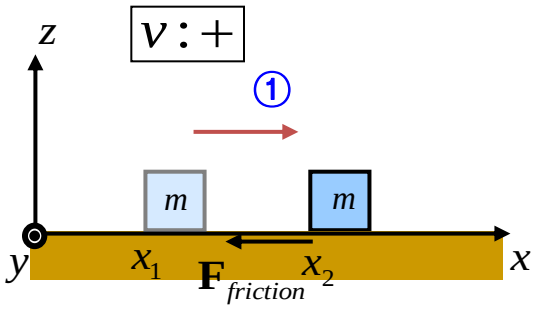
$$W = W_1 + W_2 = G\frac{m_1 m_2}{r_2} - G\frac{m_1 m_2}{r_1} + G\frac{m_1 m_2}{r_1} - G\frac{m_1 m_2}{r_2} = 0$$

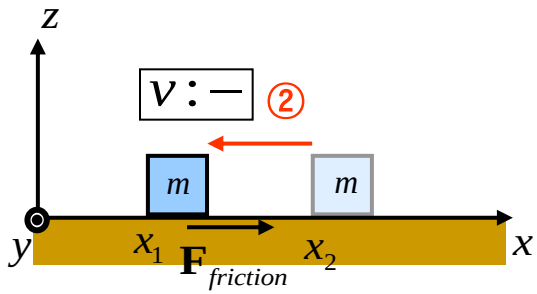
닫힌 경로를 따라 만유인력이 한 일이 0이므로 만유인력은 보존력이다.

(참고) 보존력과 역학적 에너지 보존

마찰력이 작용할때는 포텐셜에너지의 개념이 맞지않는다.

$$\mathbf{F}_{friction} = -\mu mg(\operatorname{sgn} v)\mathbf{i}$$





$x_1 < x_2$

① 물체 m이 x_1에서 x_2까지 이동할때 마찰력 $\mathbf{F}_{friction}$이 물체에 한 일 W_1

$$W_1 = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}_{friction} \cdot d\mathbf{r}$$

$$= \int_{x_1}^{x_2} -\mu mg(\sin v)dx$$

$$= -\mu mg(\sin v)\int_{x_1}^{x_2} dx$$

$$= -\mu mg\int_{x_1}^{x_2} dx$$

$$= -\mu mg(x_2 - x_1)$$

< 0 ,(**힘의방향**(−), **이동방향**(+)

따라서일은 (−))

② 물체 m이 x_2에서 x_1까지 이동할때 마찰력 $\mathbf{F}_{friction}$이 물체에 한 일 W_1

$$W_2 = \int_{\mathbf{r}_2}^{\mathbf{r}_1} \mathbf{F}_{friction} \cdot d\mathbf{r}$$

$$= \int_{x_2}^{x_1} -\mu mg(\sin v)dx$$

$$= -\mu mg(\sin v)\int_{x_2}^{x_1} dx$$

$$= \mu mg\int_{x_2}^{x_1} dx$$

$$= \mu mg(x_1 - x_2)$$

< 0 ,(**힘의방향**(+), **이동방향**(−)

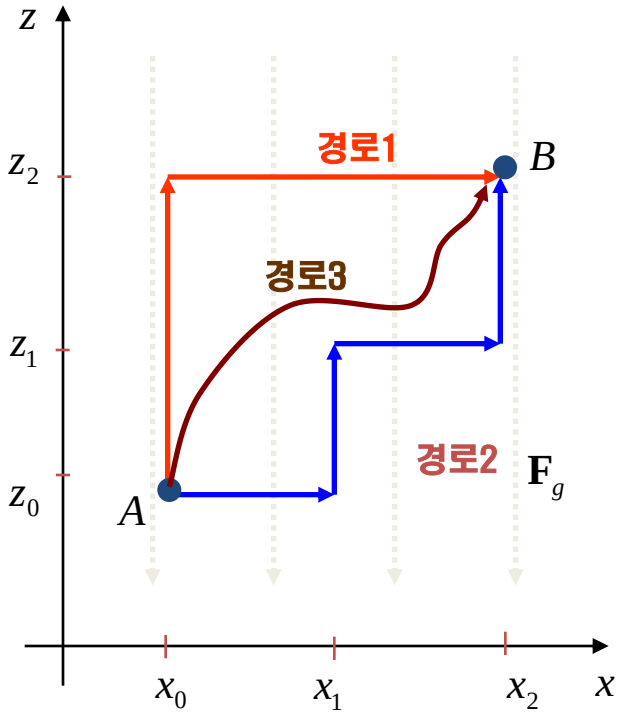
따라서일은 (−))

③ 물체 m이 x_1에서 x_2까지 이동한후 다시 x_1으로 돌아왔을때(닫힌 경로) 마찰력 $\mathbf{F}_{frition}$이 물체에 한 일 W

$$W = W_1 + W_2 = -\mu mg(x_2 - x_1) + \mu mg(x_1 - x_2) = -2\mu mg(x_2 - x_1) \neq 0$$

닫힌 경로를 따라 마찰력이 한 일이 0이 아니므로 마찰력은 비보존력이다.

(참고) 보존력과 역학적 에너지 보존



다음과 같이 힘 F_g 를 받는 중력장 (→보존장) 속에서 각각 경로 1,2,3를 따라 물체를 움직일 때 한 일을 구해보자.

일의 정의에 따라 힘과 수직으로 움직인 경로의 일은 0이다.

경로1:

$$= F_g(z_2 - z_0) + 0$$
$$= F_g(z_2 - z_0)$$

경로2: $W_2 = \int \mathbf{F}_g \cdot d\mathbf{r}$

$$= 0 + F_g(z_1 - z_0) + 0 + F_g(z_2 - z_1)$$
$$= F_g(z_2 - z_0)$$

경로3:

$$= \int_{\mathbf{r}_i}^{\mathbf{r}_f} \left(F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}\right) \cdot \left(dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}\right)$$
$$= \int_{x_i}^{x_f} F_x dx + \int_{y_i}^{y_f} F_y dy + \int_{z_i}^{z_f} F_z dz$$
$$= 0 + 0 + \int_{z_0}^{z_2} F_g dz$$
$$= F_g(z_2 - z_0)$$

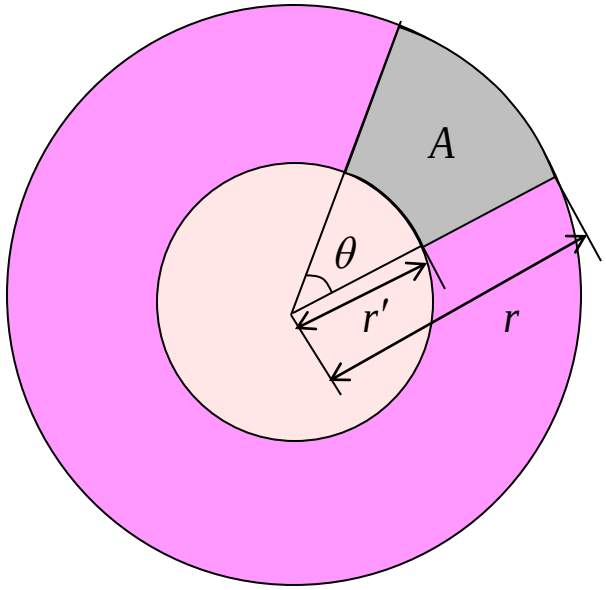
$F_x, F_y = 0$

따라서 보존력인 한 일은 경로에 무관함

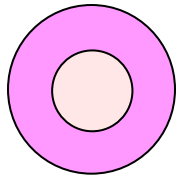
Reference slides

Area of polar rectangles

Area of polar rectangles

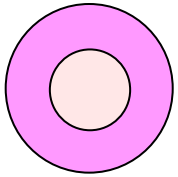


Area of outer circle minus inner circle:



$$\pi r^2 - \pi r'^2$$

Area of polar rectangle A :

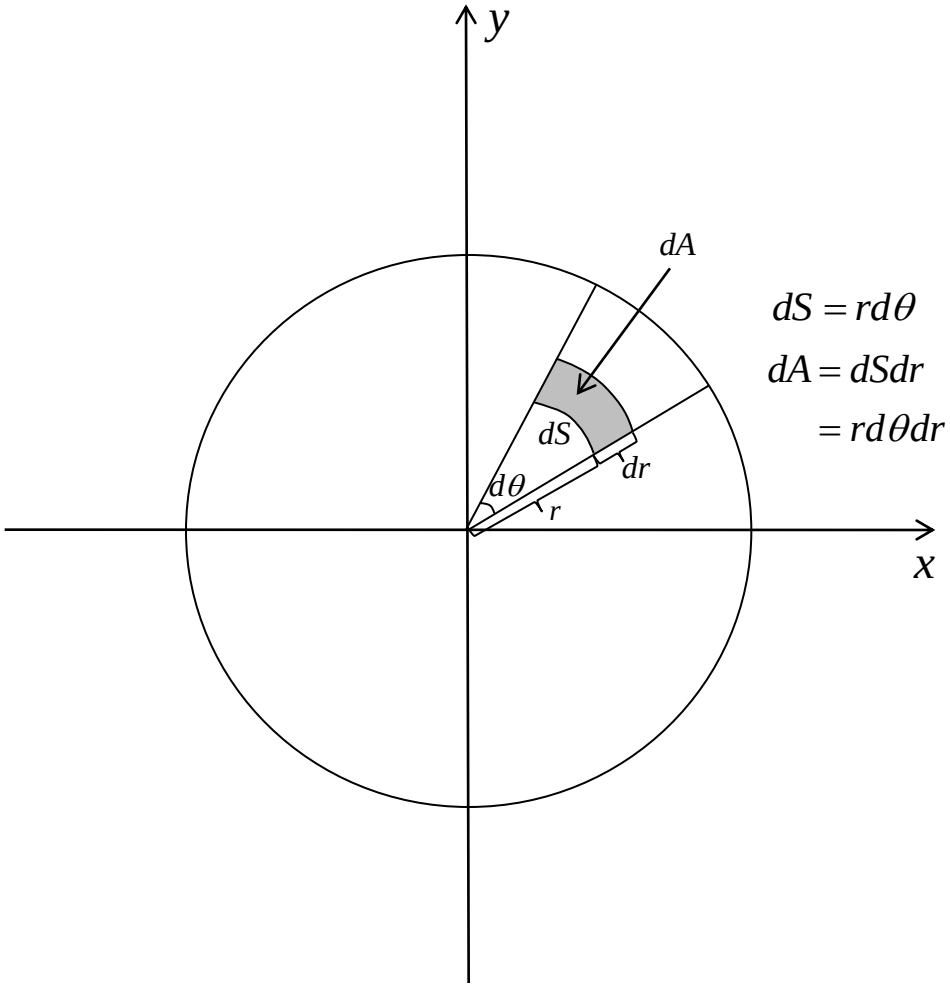
 : = $2\pi : \theta$

$$(\pi r^2 - \pi r'^2) : A = 2\pi : \theta$$

$$A = (\pi r^2 - \pi r'^2)\frac{\theta}{2\pi}$$

$$= \frac{1}{2} r^2 \theta - \frac{1}{2} r'^2 \theta$$

Double Integrals in Polar Coordinates



$$dS = rd\theta$$

$$dA = dSdr$$

$$= rd\theta dr$$

Vector Calculus (4) : Green's, Stokes' and Divergence Theorem

Green's, Theorem
Stokes' Theorem
Divergence Theorem

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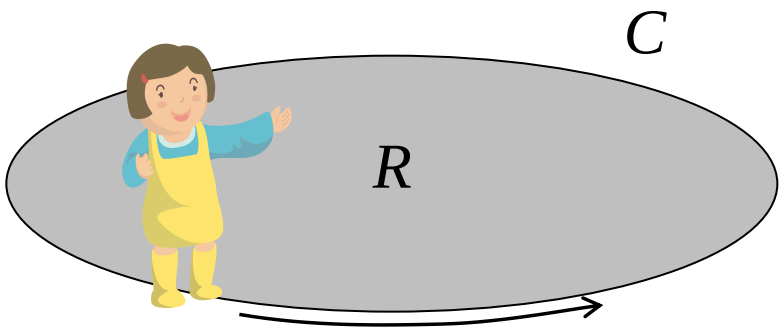
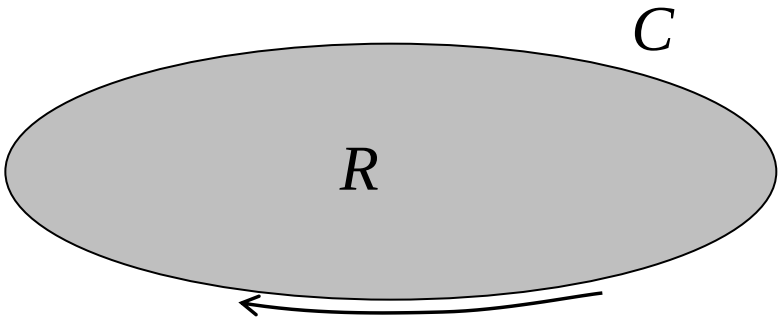
Green's Theorem



Positive direction



Negative direction

Theorem 9.13 **Green's Theorem in the Plane**

Suppose that *C* is piecewise smooth simple closed curve bounding a region *R*.

If $P, Q, \partial P/\partial y, \text{and } \partial Q/\partial x$ are continuous on R, then

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Green's Theorem

Theorem 9.13 **Green's Theorem in the Plane**

Suppose that *C* is piecewise smooth simple closed curve bounding a region *R*.

If $P, Q, \partial P/\partial y, \text{and } \partial Q/\partial x$ are continuous on R, then

$$\oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

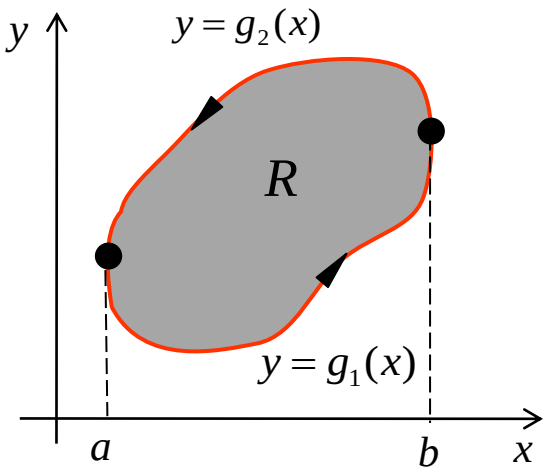
Proof) $R: g_1(x) \le y \le g_2(x), \quad a \le x \le b$

$$-\iint_R \frac{\partial P}{\partial y} dA = -\int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y} dy dx$$

$$= -\int_a^b [P(x, g_2(x)) - P(x, g_1(x))] dx$$

$$= \int_a^b P(x, g_1(x)) dx - \int_a^b P(x, g_2(x)) dx$$

$$= \oint_C P(x, y) dx$$



Green's Theorem

Theorem 9.13 **Green's Theorem in the Plane**

Suppose that C is piecewise smooth simple closed curve bounding a region R.

If $P, Q, \partial P/\partial y, \text{and } \partial Q/\partial x$ are continuous on R, then

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

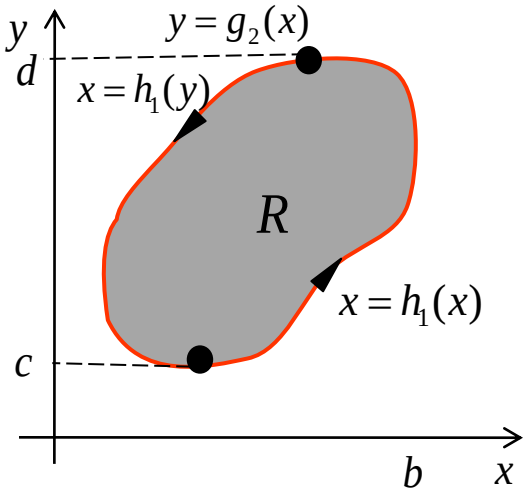
$$R : h_1(y) \le x \le h_2(y), \quad c \le y \le d$$

$$\iint_R \frac{\partial Q}{\partial y}\,dA = \int_c^d \int_{h_1(x)}^{h_2(x)} \frac{\partial Q}{\partial y}\,dx\,dy$$

$$= \int_c^d [Q(h_2(y), y) - Q(h_1(y), y)]\,dy$$

$$= \int_c^d Q(h_1(y), y)\,dy - \int_c^d Q(h_2(y), y)\,dy$$

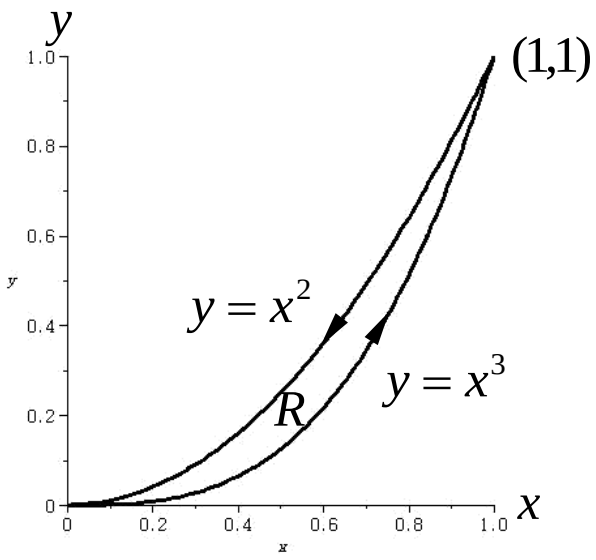
$$= \oint_C Q(x, y)\,dy$$



Green's Theorem

☑ Example 1
Using Green's Theorem

Evaluate $\oint_C (x^2 - y^2)dx + (2y - x)dy$, where C consists of the boundary of the region in the first quadrant that is bounded by the graphs of $y=x^2$ and $y=x^3$.



Solution)

$$\oint_C (x^2 - y^2)dx + (2y - x)dy$$

$$= \iint_R \left(\frac{\partial(2y - x)}{\partial x} - \frac{\partial(x^2 - y^2)}{\partial y} \right) dA$$

$$= \iint_R (-1 + 2y)dA$$

$$= \int_0^1 \int_{x^3}^{x^2} (-1 + 2y)dydx$$

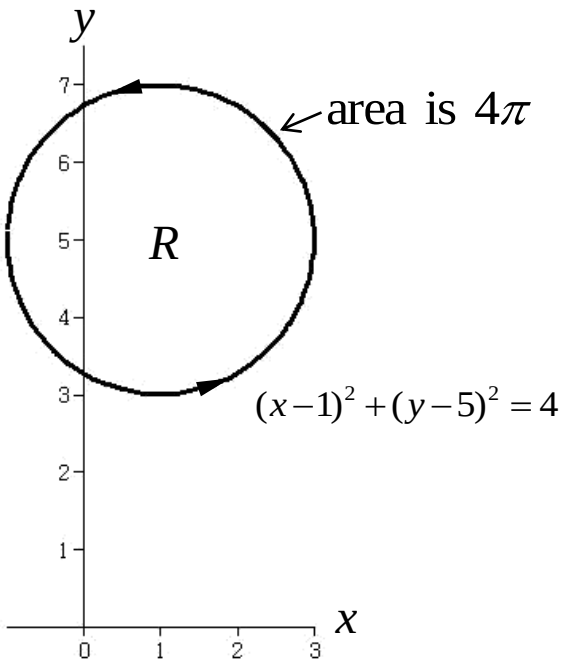
$$= \int_0^1 (-y + y^2)\Big]_{x^3}^{x^2} dx$$

$$= \int_0^1 (-x^6 + x^4 + x^3 - x^2)dx = -\frac{11}{420}$$

Green's Theorem

☑ Example 2
Using Green's Theorem

Evaluate $\oint_C (x^5+3y)dx+(2x-e^{y^3})dy$,
where C is the circle $(x\text{-}1)^2+(y\text{-}5)^2=4$.



Solution)

$$\oint_C (x^5+3y)dx+(2x-e^{y^3})dy$$

$$=\iint_R \left(\frac{\partial(2x-e^{y^3})}{\partial x}-\frac{\partial(x^5+3y)}{\partial y}\right)dA$$

$$=\iint_R (2-3)dA$$

$$=-\iint_R dA$$

$$=-4\pi$$

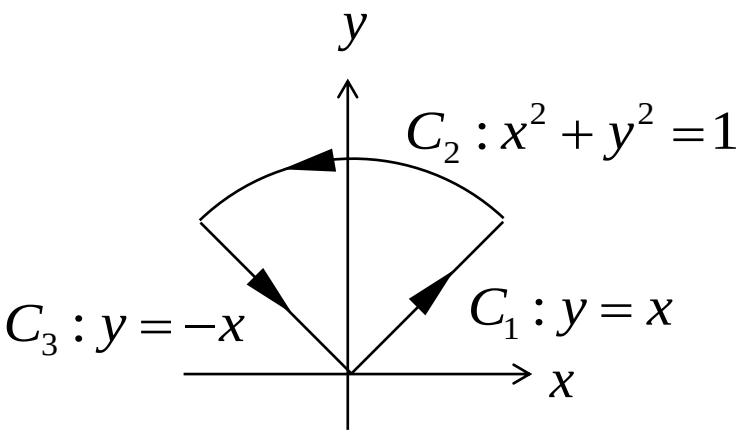
Green's Theorem

☑ Example 3 Work Done by a Force

Find the work done by the force

$$\mathbf{F} = (-16y + \sin x^2)\mathbf{i} + (4e^y + 3x^2)\mathbf{j}$$

acting along the simple closed curve C shown in Figure below.



Solution)

$$W = \oint_C \mathbf{F} \bullet d\mathbf{r}$$

$$= \oint_C (-16y \sin x^2)dx + (4e^y + 3x^2)dy$$

$$= \iint_R \left(\frac{\partial(4e^y + 3x^2)}{\partial x} - \frac{\partial(-16y \sin x^2)}{\partial y} \right) dA$$

$$= \iint_R (6x + 16)dA$$

$$R : 0 \le r \le 1, \ \frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$$

$$W = \int_{\pi/4}^{3\pi/4} \int_0^1 (6r\cos\theta + 16) r\,dr\,d\theta$$

$$= \int_{\pi/4}^{3\pi/4} (2r^3 \cos\theta + 8r^2)\Big]_0^1 d\theta$$

$$= \int_{\pi/4}^{3\pi/4} (2\cos\theta + 8)d\theta = 4\pi$$

Green's Theorem

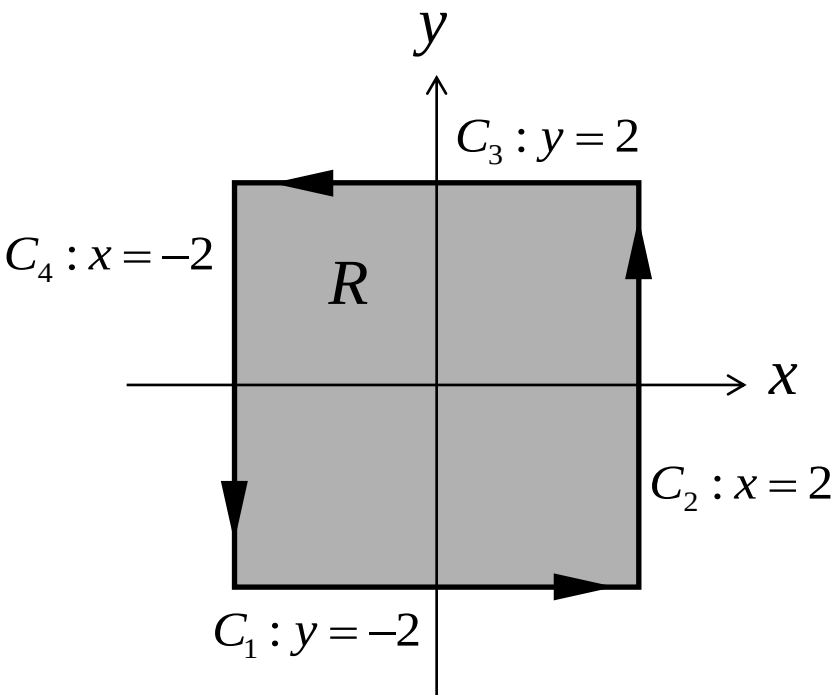
☑ Example 4

Green's Theorem Not Applicable

Let C be the closed curve consisting of the four straight line segments C_1, C_2, C_3, C_4 shown in Figure. Green's theorem is not applicable to the line integral

$$\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

since P, Q, $\partial P/\partial y$, and $\partial Q/\partial y$ are not continuous at the origin.



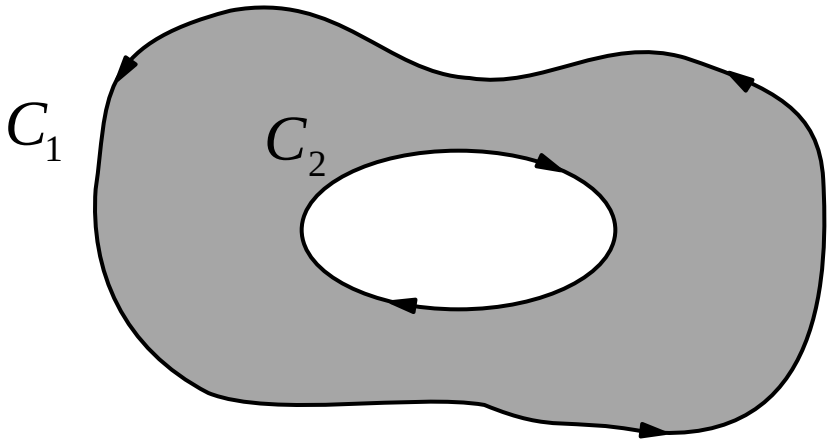
Green's Theorem

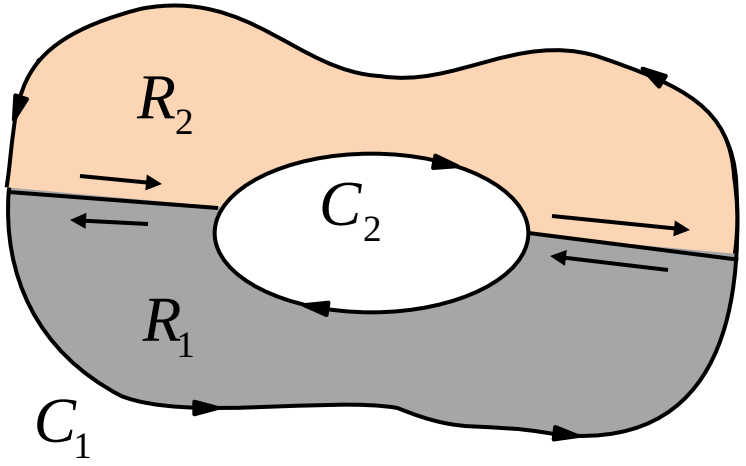
Region with Holes

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_{R_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA + \iint_{R_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \oint_{C_1} P dx + Q dy + \oint_{C_2} P dx + Q dy$$

$$= \oint_{C} P dx + Q dy \qquad (C = C_1 \cup C_2)$$





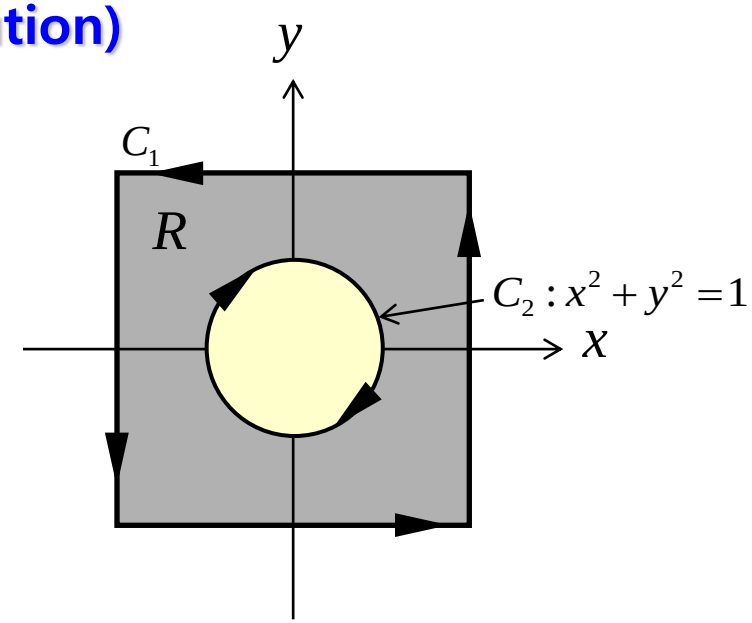
Green's Theorem

☑ Example 5

Region with a Hole in It

Evaluate $\oint_C \frac{-y}{x^2+y^2}dx+\frac{x}{x^2+y^2}dy,$ where $C=C_1 \cup C_2$ is the boundary of the shaded region R shown in Figure

Solution)



$$P(x,y)=\frac{-y}{x^2+y^2},\quad Q(x,y)=\frac{x}{x^2+y^2}$$

$$\frac{\partial P}{\partial y}=\frac{y^2-x^2}{(x^2+y^2)^2},\quad \frac{\partial Q}{\partial x}=\frac{y^2-x^2}{(x^2+y^2)^2}$$

since P, Q, $\partial P/\partial y$, and $\partial Q/\partial y$ are continuous on the region R bounded by C, it follows from the above discussion that

$$\oint_C \frac{-y}{x^2+y^2}dx+\frac{x}{x^2+y^2}dy$$

$$=\oint_{C_1} \frac{-y}{x^2+y^2}dx+\frac{x}{x^2+y^2}dy$$

$$+\oint_{C_2} \frac{-y}{x^2+y^2}dx+\frac{x}{x^2+y^2}dy$$

$$=\iint_R \left[\frac{y^2-x^2}{(x^2+y^2)^2}-\frac{y^2-x^2}{(x^2+y^2)^2}\right]dA=0.$$

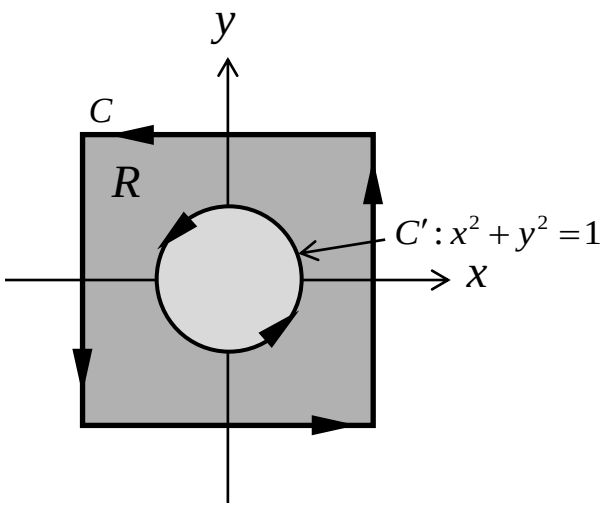
Green's Theorem

☑ Example 6
Example 4 Revisited

Evaluate the line integral in Example 4.

Example 4.

$$\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$



Solution)

$$\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$

$$P(x,y) = \frac{-y}{x^2+y^2},\quad Q(x,y) = \frac{x}{x^2+y^2}$$

$$\frac{\partial P}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2},\quad \frac{\partial Q}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$= \oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$$= \oint_{C'} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$$= \int_0^{2\pi} \left[-\sin t(-\sin t) + \cos t(\cos t)\right] dt$$

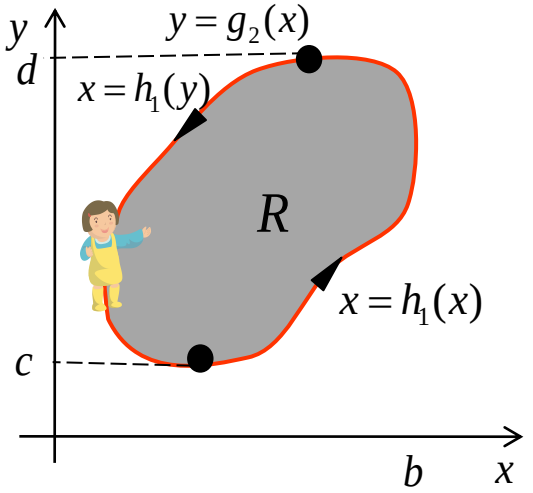
$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt$$

$$= \int_0^{2\pi} dt = 2\pi$$

Stokes' Theorem

Vector Form of Green's Theorem

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C \mathbf{F}\cdot\mathbf{T}ds = \iint_R (\text{curl}\,\mathbf{F})\cdot\mathbf{k}dA\cdots(1)$$

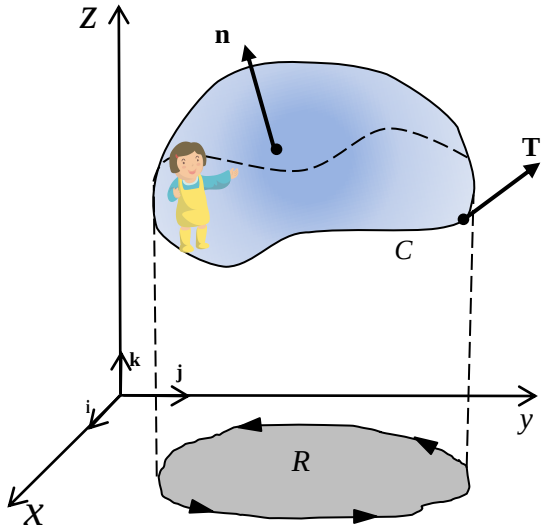


Green Theorem in 2-D space

Green's Theorem in 3-Space (Stokes' theorem)

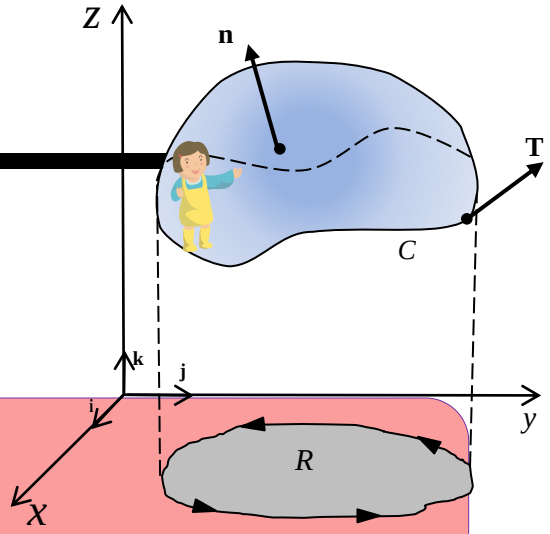
Equation (1) relate a line integral around a closed curve *C* forming the boundary of a plane region R to a double integral over *R*.

Green's theorem in 3-space relates a line integral around a closed curve C forming the boundary of a surface S with a surface integral over *S*.



Green Theorem in 3-D space

Stokes' Theorem



Theorem 9.14

Stokes' Theorem

Let S be a piecewise smooth orientable surface bounded by a piecewise smooth simple closed curve *C*. Let $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ be a vector filed for which *P,Q*, and *R* are continuous and have continuous first partial derivatives in a region of 3-space containing *S*. if *C* is traversed in the positive direction, then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C (\mathbf{F}\cdot\mathbf{T})ds = \iint_S (\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\,dS$$

Where **n** is a unit normal to *S* in the direction of the orientation of *S*.

See also
Streeter V.L., Fluid Mechanics, McGraw-Hill, 1948,
p47, '24. Stokes' Theorem'

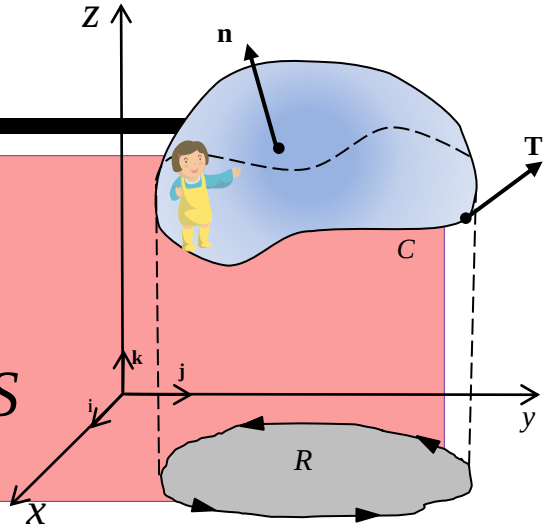
Stokes' Theorem

Theorem 9.14

Stokes' Theorem

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C (\mathbf{F}\cdot\mathbf{T})ds = \iint_S (\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\,dS$$



Partial Proof)

(R.H.S)

S is oriented upward and is defined by a function $z = f(x,y)$

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

If we write $g(x,y,z) = z - f(x,y) = 0$

$$\mathbf{n} = \frac{-\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}$$

If S is defined by g(x,y,z)=0,

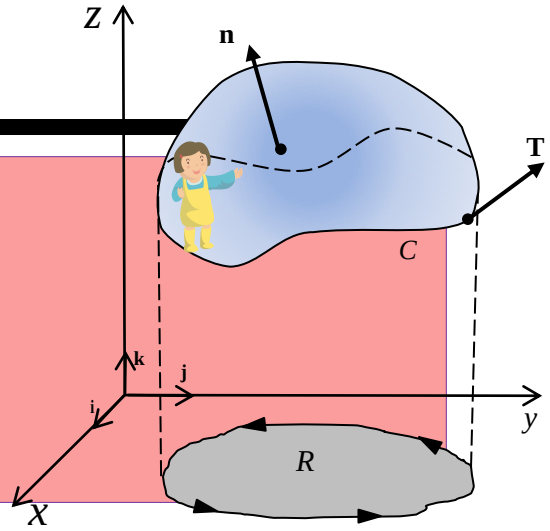
$$\mathbf{n} = \frac{1}{\|\nabla g\|}\nabla g$$

Stokes' Theorem

Theorem 9.14 **Stokes' Theorem**

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C (\mathbf{F}\cdot\mathbf{T})ds = \iint_S (\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\,dS$$



$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

$$\mathbf{n} = \frac{-\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}$$

Hence,

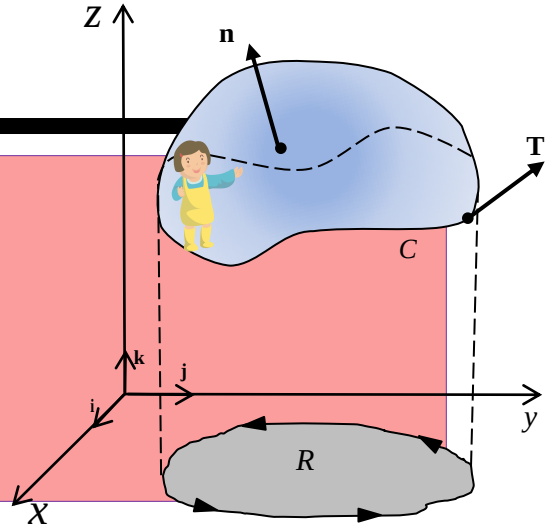
$$\iint_S (\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\,dS = \iint_R \left[-\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\frac{\partial f}{\partial x} - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\frac{\partial f}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right] dA$$

Stokes' Theorem

Theorem 9.14 **Stokes' Theorem**

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C (\mathbf{F}\cdot\mathbf{T})ds = \iint_S (\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\,dS$$



(L.H.S) C_{xy} : (Projection of C onto the xy-plane)

$$C: \quad x = x(t),\ y = y(t),\ z = f(x(t),y(t)) \quad a \le t \le b$$

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy + R\,dz$$

$$= \int_a^b \left[P\frac{dx}{dt} + Q\frac{dy}{dt} + R\left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{\partial y}{dt} \right) \right] dt \qquad \text{Chain rule}$$

$$= \oint_{C_{xy}} \left(P + R\frac{\partial f}{\partial x} \right) dx + \left(Q + R\frac{\partial f}{\partial y} \right) dy$$

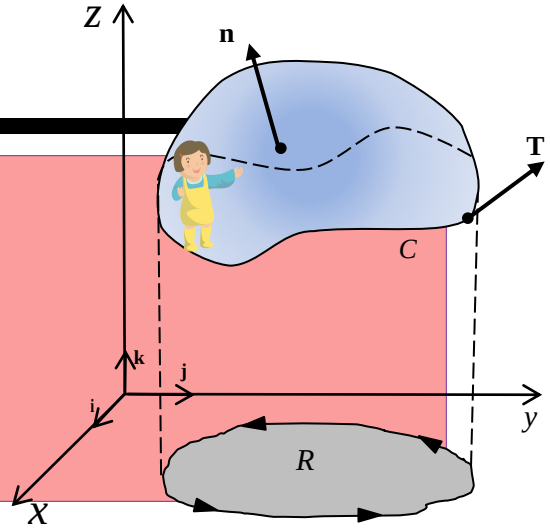
$$= \iint_R \left[\frac{\partial}{\partial x}\left(Q + R\frac{\partial f}{\partial y} \right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial f}{\partial x} \right) \right] dA \qquad \text{Green's theorem}$$

Stokes' Theorem

Theorem 9.14 **Stokes' Theorem**

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C (\mathbf{F}\cdot\mathbf{T})ds = \iint_S (\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\,dS$$



$$\frac{\partial}{\partial x}\left(Q + R\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left[Q(x,y,f(x,y)) + R(x,y,f(x,y))\frac{\partial f}{\partial y}\right]$$

$$= \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial f}{\partial x} + R\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial y}\left(\frac{\partial R}{\partial x} + \frac{\partial R}{\partial z}\frac{\partial f}{\partial x}\right)$$

Chain and Product Rules

$$= \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial f}{\partial x} + R\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial R}{\partial x}\frac{\partial f}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial f}{\partial y}\frac{\partial f}{\partial x}$$

Similarly,

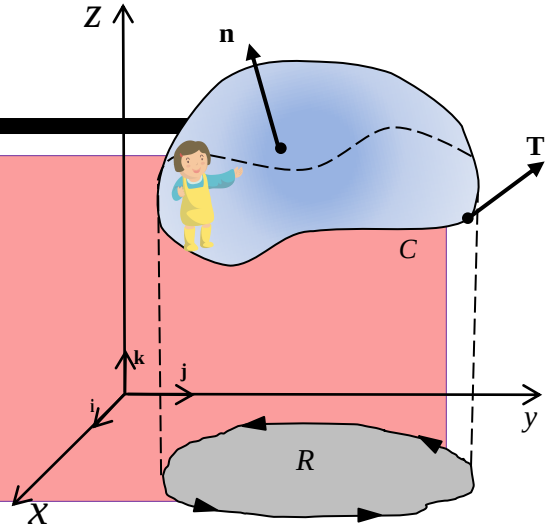
$$\frac{\partial}{\partial y}\left(P + R\frac{\partial f}{\partial x}\right) = \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial f}{\partial y} + R\frac{\partial^2 f}{\partial y \partial x} + \frac{\partial R}{\partial y}\frac{\partial f}{\partial x} + \frac{\partial R}{\partial z}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}$$

Stokes' Theorem

Theorem 9.14 **Stokes' Theorem**

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C (\mathbf{F}\cdot\mathbf{T})ds = \iint_S (\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\,dS$$



$$\frac{\partial}{\partial x}\left(Q + R\frac{\partial f}{\partial y}\right) = \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial f}{\partial x} + R\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial R}{\partial x}\frac{\partial f}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial f}{\partial y}\frac{\partial f}{\partial x}$$

$$\frac{\partial}{\partial y}\left(P + R\frac{\partial f}{\partial x}\right) = \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial f}{\partial y} + R\frac{\partial^2 f}{\partial y \partial x} + \frac{\partial R}{\partial y}\frac{\partial f}{\partial x} + \frac{\partial R}{\partial z}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}$$

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \left[\frac{\partial}{\partial x}\left(Q + R\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial f}{\partial x}\right)\right] dA$$

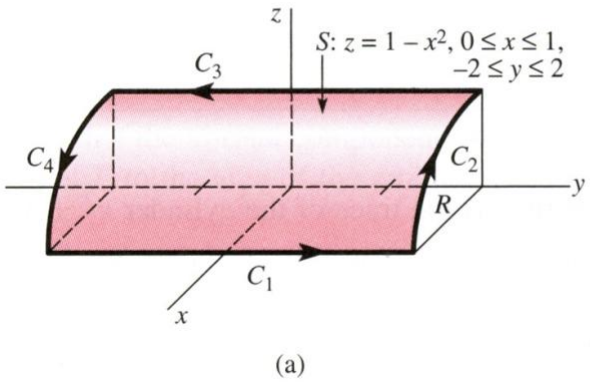
$$= \iint_R \left[-\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\frac{\partial f}{\partial x} - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\frac{\partial f}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right] dA$$

$\therefore$ **(L.H.S)=(R.H.S)**

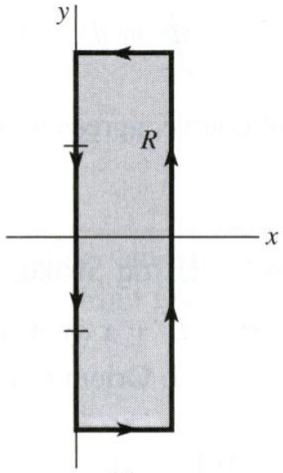
Stokes' Theorem

☑ Example 1 Verifying Stokes' Theorem

Let S be the part of the cylinder $z=1-x^2$ for $0\le x\le 1$, $-2\le y\le 2$.
Verify Stokes' theorem if $\mathbf{F}=xy\mathbf{i}+yz\mathbf{j}+xz\mathbf{k}$.



(a)



(b)

Solution) 1) Surface Integral

$$\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$$

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & yz & xz \end{vmatrix} = -y\mathbf{i} - z\mathbf{j} - x\mathbf{k}$$

$$g(x,y,z) = z + x^2 - 1 = 0$$

$$\mathbf{n} = \frac{\nabla g}{\|\nabla g\|} = \frac{2x\mathbf{i} + \mathbf{k}}{\sqrt{4x^2+1}}$$

$$\iint_S (\operatorname{curl}\mathbf{F} \bullet \mathbf{n})\,dS = \iint_S \frac{-2xy - x}{\sqrt{4x^2+1}}\,dS$$

$$= \iint_R (-2xy - x)\,dA$$

$$= \int_0^1 \int_{-2}^{2} (-2xy - x)\,dy\,dx$$

$$= \int_0^1 \left[-xy^2 - xy \right]_{-2}^{2} dx$$

$$= \int_0^1 (-4x)\,dx = -2$$

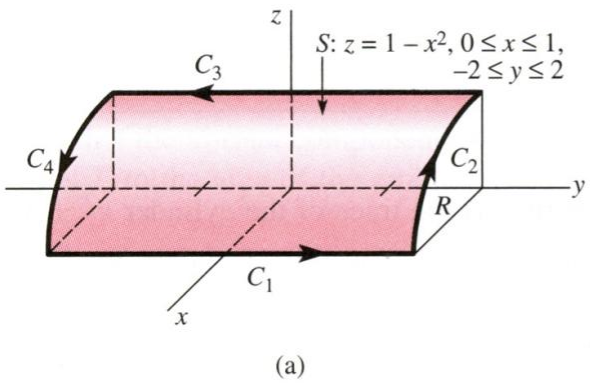
$$= \int_0^1 (\frac{1}{2} +$$

Stokes' Theorem

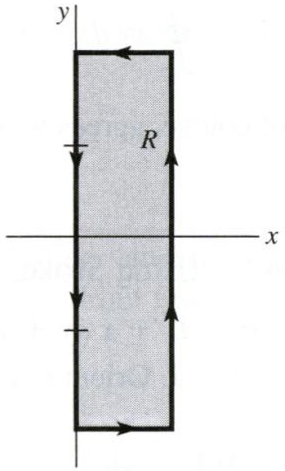
$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C P dx + Q dy + R dz$$

☑ Example 1 Verifying Stokes' Theorem

Let S be the part of the cylinder $z=1-x^2$ for $0\le x\le 1$, $-2\le y\le 2$.
Verify Stokes' theorem if $\mathbf{F}=xy\mathbf{i}+yz\mathbf{j}+xz\mathbf{k}$.



(a)



(b)

Solution) 2) Line Integral

$$\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$

$$C_1 : x=1,\ z=0,\ dx=0,\ dz=0$$

$$\int_{C_1} y(0) + y(0)dy + 0 = 0$$

$$C_2 : y=2,\ z=1-x^2,\ dy=0,\ dz=-2xdx$$

$$\int_{C_2} 2xdx + 2(1-x^2)0 + x(1-x^2)(-2xdx)$$

$$= \int_1^0 (2x - 2x^2 + 2x^4)dx = -\frac{11}{15}$$

$$C_3 : x=0,\ z=1,\ dx=0,\ dz=0$$

$$\int_{C_3} 0 + ydy + 0 = \int_2^{-2} ydy = 0$$

$$C_4 : y=-2,\ z=1-x^2,\ dy=0,\ dz=-2xdx$$

$$\int_{C_4} -2xdx - 2(1-x^2)0 + x(1-x^2)(-2x)dx$$

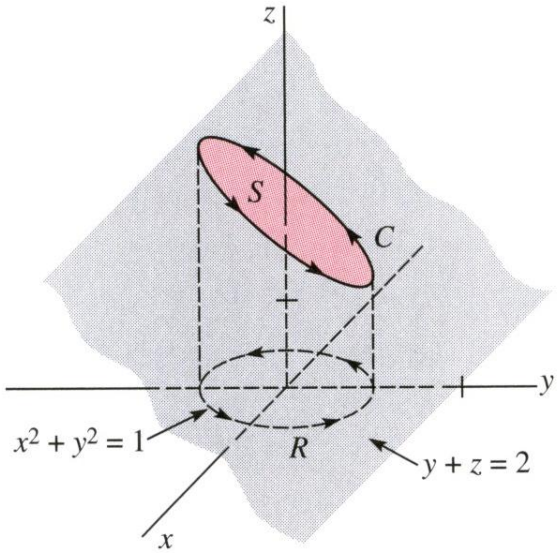
$$= \int_0^1 (-2x - 2x^2 + 2x^4)dx = -\frac{19}{15}$$

$$\therefore \oint_C xydx + yz + xzdz = 0 - \frac{11}{15} + 0 - \frac{19}{15} = -2$$

Stokes' Theorem

☑ Example 2
Using Stokes' Theorem

Evaluate $\oint_C z\,dx + x\,dy + y\,dz$,
where C is the trace of the cylinder $x^2+y^2=1$ in the plane $y+z=2$.
Orient C counterclockwise as viewed from above. See the Figure below



Solution)

$$\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$$

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$g(x,y,z) = y + z - 2 = 0$$

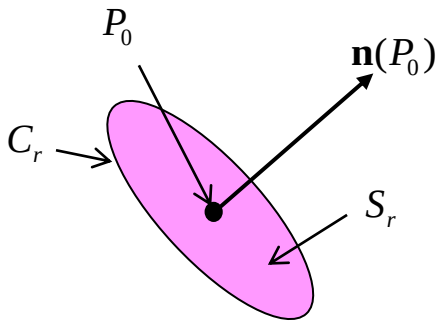
$$\mathbf{n} = \frac{\nabla g}{\|\nabla g\|} = \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$$

$$\oint_C \mathbf{F} \bullet d\mathbf{r} = \iint_S \left[(\mathbf{i}+\mathbf{j}+\mathbf{k}) \bullet \left(\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k} \right) \right] dS$$

$$= \sqrt{2}\iint_S dS = \sqrt{2}\iint_R \sqrt{2}\,dA = 2\pi$$

Stokes' Theorem

Physical Interpretation of Curl



C_r is Small circle of radius r centered at P_0

By Stokes' theorem,

$$\oint_{C_r} \mathbf{F} \cdot d\mathbf{r} = \iint_{S_r} (\text{curl } \mathbf{F}) \cdot \mathbf{n} dS$$

Because radius of circle C_r is small, then we approximate curl $\mathbf{F}(P)$ ≈ curl $\mathbf{F}(P_0)$,

$$\oint_{C_r} \mathbf{F} \cdot d\mathbf{r} \approx \iint_{S_r} (\text{curl } \mathbf{F}(P_0)) \cdot \mathbf{n}(P_0) dS$$

$$= (\text{curl } \mathbf{F}(P_0)) \cdot \mathbf{n}(P_0) \iint_{S_r} dS$$

$$= (\text{curl } \mathbf{F}(P_0)) \cdot \mathbf{n}(P_0) A_r$$

$$(\text{curl } \mathbf{F}(P_0)) \cdot \mathbf{n}(P_0) = \lim_{r \to 0} \frac{1}{A_r} \oint_{C_r} \mathbf{F} \cdot d\mathbf{r}$$

See also
Streeter V.L., Fluid Mechanics, McGraw-Hill, 1948, p49, '25. Circulation'

For a small but fixed value of r,

$$(\text{curl } \mathbf{F}(P_0)) \cdot \mathbf{n}(P_0) \approx \frac{1}{A_r} \oint_{C_r} \mathbf{F} \cdot d\mathbf{r}$$

Roughly then, the curl of **F** is the circulation of **F** per unit area.

Divergence Theorem*

$$\text{div}\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Divergence Theorem of Gauss
(Transformation Between Triple and Surface Integrals)

Let T be a closed bounded region in space whose boundary is a piecewise smooth orientable surface S.

$\mathbf{F}(x, y, z)$ **: a vector function that is continuous and has continuous first partial derivatives in T**

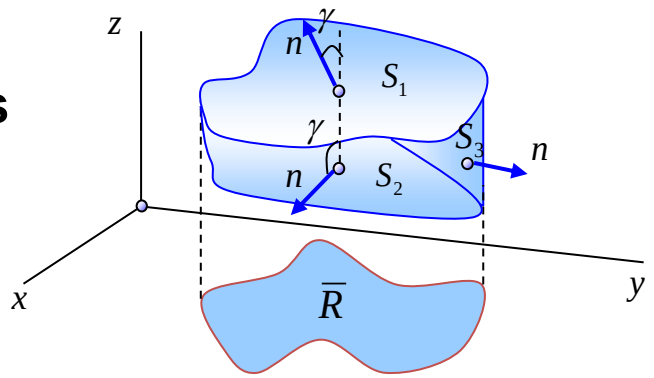


Fig. Example of a special region

$$\textbf{(2)} \quad \iiint_T \text{div}\mathbf{F}\, dV = \iint_S \mathbf{F}\cdot\mathbf{n}\, dA$$

Using component, $\mathbf{F} = [F_1, F_2, F_3], n = [\cos\alpha, \cos\beta, \cos\gamma]$

$$\textbf{(2')} \quad \iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dxdydz = \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\, dA$$

Ref. Divergence Theorem

$$= \iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy)$$

*Erwin Kreyszig, Advanced Engineering Mathematics 9th ,2006, Wiley,Ch10.7(p458~463)

Divergence Theorem

(proof) we can start with (2*)

$$\iiint_T \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)dxdydz = \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)dA$$

$$= \iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy)$$

This equation is true if and only if the integrals of each component on both sides are equal

(3) $$\iiint_T \frac{\partial F_1}{\partial x}dxdydz = \iint_S F_1\cos\alpha\, dA = \iint_S F_1 dydz$$

(4) $$\iiint_T \frac{\partial F_2}{\partial y}dxdydz = \iint_S F_2\cos\beta\, dA = \iint_S F_2 dxdz$$

(5) $$\iiint_T \frac{\partial F_3}{\partial z}dxdydz = \iint_S F_3\cos\gamma\, dA = \iint_S F_3 dxdy$$

Divergence Theorem

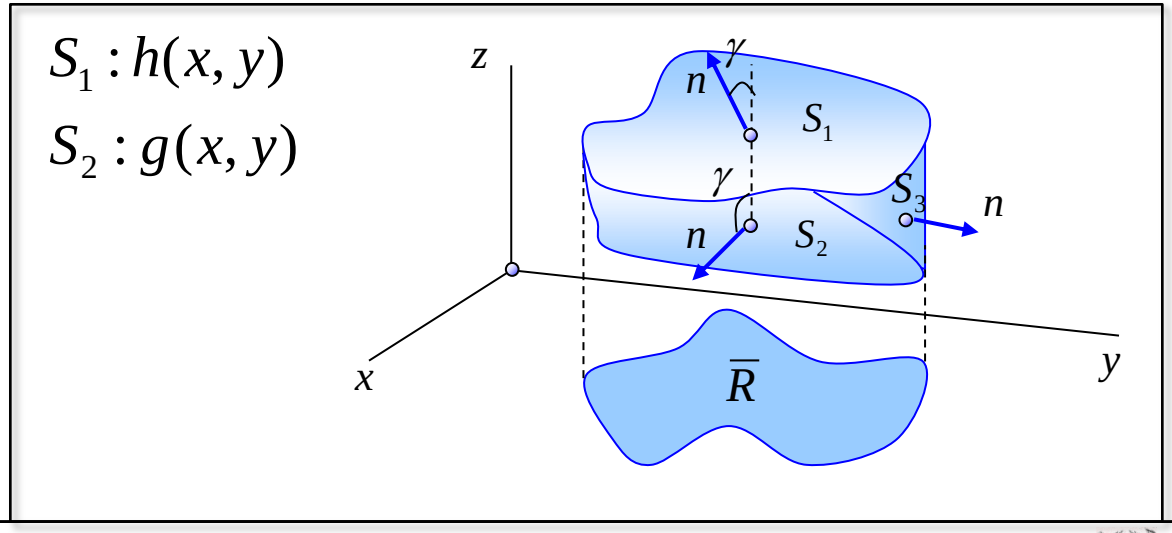
(proof continue)

(5) $$\iiint_T \frac{\partial F_3}{\partial z} dxdydz = \iint_S F_3 \cos\gamma \, dA$$

We first prove (5) for a special region T that is bounded by a piecewise smooth orientable surface S and has the property that any straight line parallel to any one of the coordinate axes and intersecting T has at most one segment (or a single point)

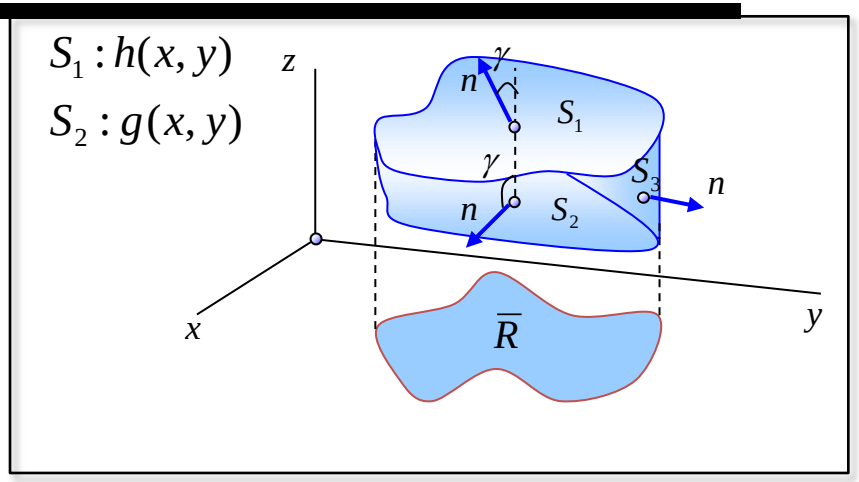
It implies that T can be represented in the form

(6) $$g(x,y) \le z \le h(x,y)$$



Divergence Theorem

(proof continue)



(5) $$\iiint_T \frac{\partial F_3}{\partial z} dxdydz = \iint_S F_3 \cos\gamma \, dA$$

$$\iiint_T \frac{\partial F_3}{\partial z} dxdydz = \iint_{\bar{R}} \left[\int_{g(x,y)}^{h(x,y)} \frac{\partial F_3}{\partial z} dz \right] dxdy$$

$$= \iint_{\bar{R}} F_3[x, y, h(x,y)] dxdy - \iint_{\bar{R}} F_3[x, y, g(x,y)] dxdy$$

We can decide the sign of the integral because $\cos\gamma < 0$ **on S_2, and** $\cos\gamma > 0$ **on S_1**

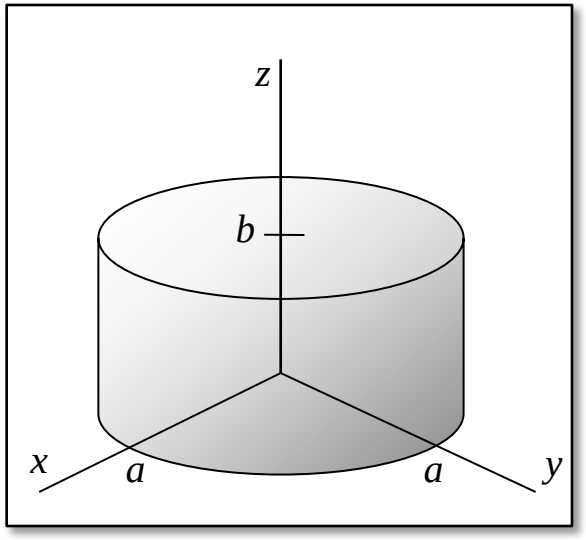
$$\iint_S F_3 \cos\gamma dA = \iint_S F_3 dxdy = \iint_{\bar{R}} F_3[x, y, h(x,y)] dxdy - \iint_{\bar{R}} F_3[x, y, g(x,y)] dxdy$$

Therefore, we prove (5). In the same manner, (3),(4) can be proven.

Divergence Theorem

(2) $\iiint_T \operatorname{div}\mathbf{F}\,dV = \iint_S \mathbf{F}\cdot\mathbf{n}\,dA$

(Example 1) Evaluate $I = \iint_S \left(x^3 dydz + x^2 ydzdx + x^2 zdxdy\right)$



$S : x^2 + y^2 = a^2 (0 \le z \le b)$

$z = 0 \;\; and \;\; z = b\,(x^2 + y^2 \le a^2)$

(sol) $F_1 = x^3, F_2 = x^2 y, F_3 = x^2 z$

$$\operatorname{div}\mathbf{F} = 3x^2 + x^2 + x^2 = 5x^2$$

Changing to the polar coordinate,

$$x = r\cos\theta, y = r\sin\theta, z = z$$

$$I = \iiint_T 5x^2 dxdydz = \int_{z=0}^{b}\int_{\theta=0}^{2\pi}\int_{r=0}^{a}\left(5r^2\cos^2\theta\right) r\,dr\,d\theta\,dz$$

$$= 5\int_{z=0}^{b}\int_{\theta=0}^{2\pi}\frac{a^2}{4}\cos^2\theta d\theta\,dz = 5\int_{z=0}^{b}\frac{a^4\pi}{4}dz = \frac{5\pi}{4}a^4 b$$

Divergence Theorem

(Example 2) Evaluate $\iint_S (7x\mathbf{i} - z\mathbf{k}) \bullet \mathbf{n}\, dA$ **over the sphere** $S : x^2 + y^2 + z^2 = 4$

(a) by Divergence Theorem $\text{div}\mathbf{F} = \text{div}[7x, 0, -z] = 7 - 1 = 6$

$$\iint_S \mathbf{F} \cdot \mathbf{n} dA = \iiint_T \text{div}\mathbf{F} dV = \iiint_T 6\, dV = 6 \times \frac{4\pi}{3} 2^3 = 64\pi$$

(b) directly

on spherical coordinate, $\mathbf{r} = [2\cos v \cos u, 2\cos v \sin v, 2\sin v]$

$\mathbf{r}_u = [-2\cos v \sin u, 2\cos v \cos u, 0]$, $\mathbf{r}_v = [-2\sin v \cos u, -2\sin v \sin u, 2\cos v]$

We shall use $\mathbf{n}\, dA = \mathbf{N} du dv$

$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = [4\cos^2 v \cos u, 4\cos^2 v \sin u, 4\cos v \sin v]$

$\mathbf{F} = [7x, 0, -z] = [14\cos v \cos u, 0, -2\sin v]$

$\mathbf{F} \bullet \mathbf{N} = 14\cos v \cos u \cdot 4\cos^2 v \cos u + (-2\sin v) \cdot 4\cos v \sin v = 56\cos^3 v \cos^2 u - 8\cos v \sin^2 v$

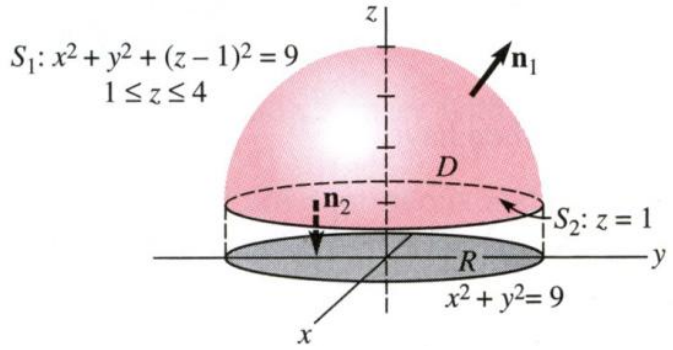
$$\iint_S (7x\mathbf{i} - z\mathbf{k}) \bullet \mathbf{n}\, dA = \int_0^{2\pi} \left[\int_0^{2\pi} \left(56\cos^3 v \cos^2 u - 8\cos v \sin^2 v\right) du \right] dv$$

$$\int_0^{2\pi} \left(\pi 56\cos^3 v - 2\pi \cdot 8\cos v \sin^2 v\right) dv = 56\pi(2 - 2/3) - 16\pi \cdot 2/3 = 64\pi$$

Divergence Theorem

☑ Example 1 Verifying Divergence Theorem

Let T be the region bounded by the hemisphere $x^2+y^2+(z-1)^2=9$, $1\le z\le 4$, and the plane $z=1$. Verify the divergence theorem if $\mathbf{F}=x\mathbf{i}+y\mathbf{j}+(z-1)\mathbf{k}$



Solution)

1) Triple Integral

$$\mathbf{F}=z\mathbf{i}+x\mathbf{j}+y\mathbf{k}$$

$$\operatorname{div}\mathbf{F}=3$$

$$\iiint_D \operatorname{div}\mathbf{F}\,dv=\iiint_D 3\,dv=3\iiint_D dv=3\cdot\frac{1}{2}\cdot\frac{4\pi 3^3}{2}=54\pi$$

2) Surface Integral

$$\iint_S=\iint_{S_1}+\iint_{S_2},\quad S_1: \text{hemisphere},\ S_2: z=1$$

S_1 : hemisphere

$$g(x,y,z)=x^2+y^2+(z-1)^2$$

$$\mathbf{n}=\frac{\nabla g}{\|\nabla g\|}=\frac{x\mathbf{i}+y\mathbf{j}+(z-1)\mathbf{k}}{\sqrt{x^2+y^2+(z-1)^2}}=\frac{x}{3}\mathbf{i}+\frac{y}{3}\mathbf{j}+\frac{z-1}{3}\mathbf{k}$$

$$\mathbf{F}\bullet\mathbf{n}=\frac{x^2}{3}+\frac{y^2}{3}+\frac{(z-1)^2}{3}=3$$

$$\iint_{S_1}(\mathbf{F}\bullet\mathbf{n})dS=\iint_R(3)\left(\frac{3}{\sqrt{9-x^2-y^2}}\,dA\right)$$

$$=9\int_0^{2\pi}\int_0^3(9-r^2)^{-1/2}r\,dr\,d\theta=54\pi$$

S_2 : $z=1$

$$\mathbf{n}=-\mathbf{k}$$

$$\mathbf{F}\bullet\mathbf{n}=-z+1$$

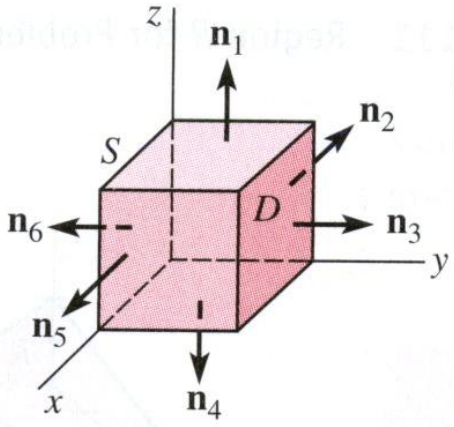
$$\iint_{S_2}(\mathbf{F}\bullet\mathbf{n})dS=\iint_{S_2}(-z+1)dS=\iint_{S_2}(0)dS=0$$

$$\therefore\iint_S(\mathbf{F}\bullet\mathbf{n})dS=54\pi$$

Divergence Theorem

☑ Example 2 Using Divergence Theorem

If $\mathbf{F}=xy\mathbf{i}+y^2z\mathbf{j}+z^3\mathbf{k}$, evaluate $\iint_S(\mathbf{F}\cdot\mathbf{n})dS$, where the unit cube defined by $0\le x\le 1,\ 0\le y\le 1,\ 0\le z\le 1$.



Solution)

$$\operatorname{div}\mathbf{F} = y + 2yz + 3z^2$$

$$\iint_S \mathbf{F}\bullet\mathbf{n}\,dS = \iiint_S (y+2yz+3z^2)\,dV$$

$$= \int_0^1\int_0^1\int_0^1 (y+2yz+3z^2)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 (y+2yz+3z^2)\,dy\,dz$$

$$= \int_0^1 \left[\left(\frac{y^2}{2} + y^2z + 3yz^2\right) \right]_0^1 dz$$

$$= \int_0^1 \left(\frac{1}{2} + z + 3z^2\right) dz$$

$$= \left[\left(\frac{1}{2}z + \frac{1}{2}z^2 + z^3\right) \right]_0^1 = 2$$

Reference slides

Divergence Theorem

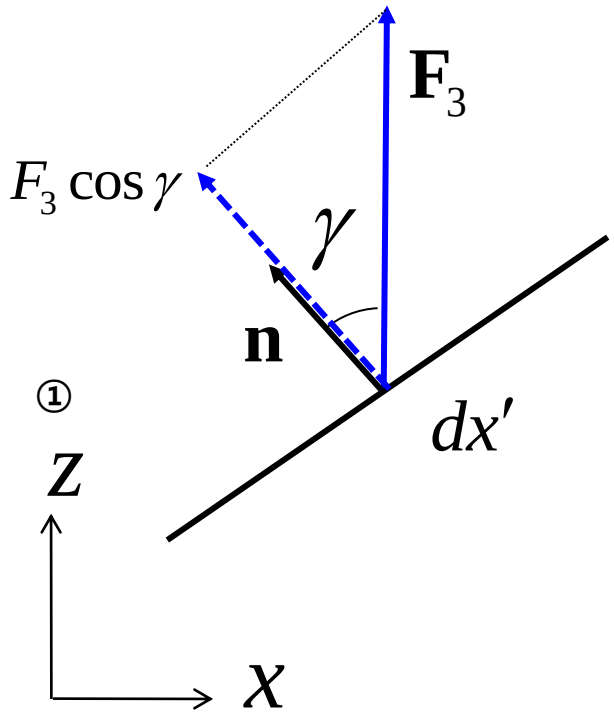
Divergence Theorem

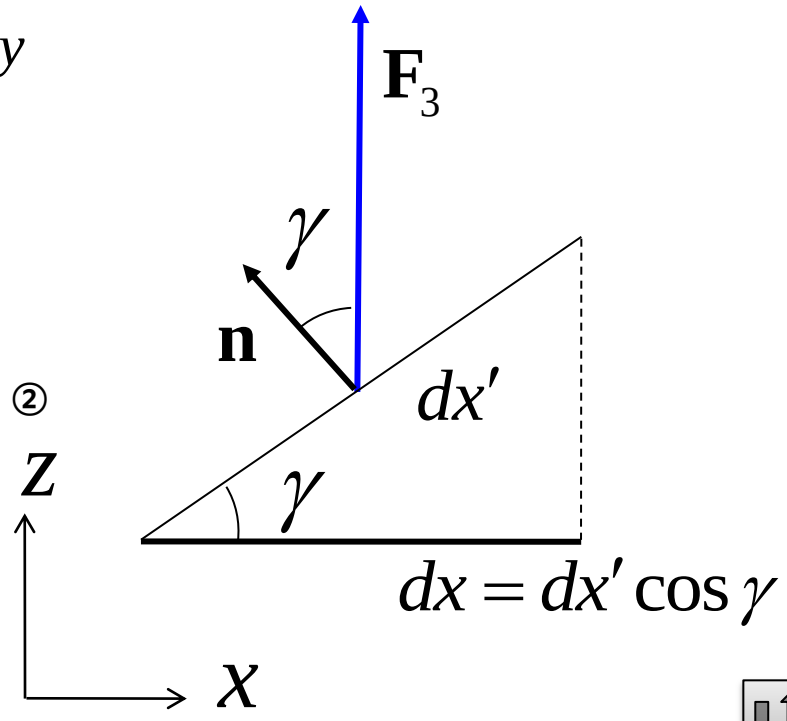
$$\iiint_T \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)dxdydz = \iint_S (F_1\cos\alpha + F_2\cos\beta + \boxed{F_3\cos\gamma)dA}\ ①$$

$$= \iint_S (F_1 dydz + F_2 dzdx + \boxed{F_3 dxdy})\ ②$$

$$F_3\cos\gamma dA = F_3\cos\gamma dx'dy$$

$$= F_3 dxdy$$





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